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Fuzzy n-fold fantastic filters on hoop algebras

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Abstract

In this paper, first we introduce the concept of fuzzy n-fold positive implicative filters on hoop algebras and study some of properties. Next we define and study fuzzy n-fold fantastic filters on hoop algebras. Also, the relationship between fuzzy n-fold positive implicative filter and some other fuzzy filters likeness fuzzy n-fold implicative filter and fuzzy n-fold fantastic filters are investigated for example every fuzzy n-fold implicative filter is a fuzzy n-fold positive implicative filter on hoop algebras. Then we obtain some condition equivalent with fuzzy n-fold fantastic filters and every fuzzy filter on chain hoop algebra is a fuzzy n-fold fantastic filter. Finally, we show that the quotient of fuzzy n-fold fantastic filters is a Boolean algebra.

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1 Introduction

In [7, 8] Bosbach introduced hoop algebras naturally ordered commutative residuated integral monoids.

It is well known that in various logical systems, the theory of ideals and filters play a fundamental role, ideals or filters correspond to sets of provable formulas and closed with respect to modus ponens. This is to say that ideals and filters are not just abstract concepts, but are mathematically deep and significant concepts with applications in various areas. Fuzzy filters are useful tool to obtain results on classical filters on hoop algebra. Many of these results have a strong impact with fuzzy logic. several researchers characterizations of filters, implicative, positive implicative and prime filters are derived. Furthermore, the relation between these filters on hoop algebras is established [11]. In [1, 3], fuzzy filters on hoop algebras were studied. In particular, several types of fuzzy filters such as fuzzy implicative filters, fuzzy positive implicative filters, fuzzy Boolean filters and fuzzy fantastic filters were introduced [4, 6]. Several researchers investigated the theory of filter and n-fold filters on hoop algebra, in 2017, C. Luo, X.Xin, P. He, study n-fold filters theory on hoop algebras and several characterizations of n-fold filters, implicative and positive implicative are investigated. In the fuzzy approach, fuzzification ideas have been applied to some fuzzy logical algebras. Nowadays filters are tools of extreme importance in many areas of classical mathematics. For example, in topology they enhance the concept of convergence and in measure theory, prime filters can be interpreted as basic components of probability measures and in fuzzy mathematics, filters have been conceived in various manners. Considering that the notations of (n-fold) obstinate filters. In [10] A. Borumand Saeid and A. Namdar investigated fuzzy obstinate filters, fuzzy n-fold obstinate filters, fuzzy prime filter and fuzzy n-fold implicative filter on hoop algebras. They studied the relationship between fuzzy obstinate filter and some other fuzzy filters likeness fuzzy prime and fuzzy positive implicative filters are investigated and they show that the quotient of this structure is a Boolean algebra and obtained some condition equivalent with fuzzy n-fold implicative filter. This paper continues the study of fuzzy folding theory on hoop algebras for example fuzzy n-fold positive implicative filters and fuzzy n-fold fantastic filters on hoop algebras. In Section 3, some basic concepts and properties are recalled, and some new notions about the thresholds are introduced to represent fuzzy filters like fuzzy n-fold implicative filters which are convenient to study the properties of fuzzy folding filters. Next, we also analyze the relation between various fuzzy filters on hoop algebra. We study fuzzy n-fold implicative positive filters on finally ordered hoop algebra and show that every fuzzy n-fold implicative filters is fuzzy n-fold implicative positive filters. In Section 4, the relationship between fuzzy n-fold fantastic filters and other types of fuzzy filters on hoop algebras is established and we obtained the condition equivalent to the fuzzy n-fold fantastic filter. Then we study fuzzy n-fold fantastic filters on chain hoop algebra and show that every fuzzy filter is fuzzy n-fold fantastic filters. Also we show that if every element is dense, then every fuzzy n-fold fantastic filters is constant. We study properties of them and we show that quotient of hoop algebra with respect to fuzzy n-fold fantastic filter is a Boolean algebra. Finally we give the relation diagram between various fuzzy n-fold (positive) implicative filters and fuzzy n-fold fantastic filters on hoop algebras and we obtained condition equivalent for them.

2 Preliminaries

In this section, we recollect some definitions and results which will be used, not cite them every time they are used.

Definition 2.1. [2] *A* hoop algebra *or* hoop *is an algebra* $(A, \odot, \to, 1)$ *of type* $(2, 2, 0)$ *such that, for all* $x, y, z \in A$*:*
(HP1) $(A, \odot, 1)$ *is a commutative monoid,*
(HP2) $x \to x = 1$,
(HP3) $(x \odot y) \to z = x \to (y \to z)$,
(HP4) $x \odot (x \to y) = y \odot (y \to x)$.

On hoop A, we define $x \leq y$ if and only if $x \to y = 1$. It is easy to see that $\leq$ is a partial order relation on A. A hoop A is *bounded* if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$. In this case, we define a negation "$'$" on A by, $x' = x \to 0$, for all $x \in A$. If $(x')' = x$, for all $x \in A$, then the bounded hoop A is said to have the *double negation property*, or (DNP), for short. An element a in A is called *dense* if and only if $a' = 0$. The *order* of $1 \neq a \in A$, in symbols $ord(a)$

is the smallest $n \in \mathbb{N}$ such that $a^n = 0$. If no such n exists, then $ord(a) = \infty$. A hoop is called *locally finite* if for every $1 \neq a \in A$, $ord(a) < \infty$.

Definition 2.2. [3] *Let A be a bounded hoop and for any $x, y \in A$, we define $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. If $\vee$ is the join operation on A, then A is called a* $\vee$-hoop.

Proposition 2.3. [7, 8] *Let A be a bounded hoop. Then the following properties hold, for all $x, y, z \in A$:*
(i) $(A, \leq)$ *is a meet-semilattice with* $x \wedge y = x \odot (x \to y)$,
(ii) $x \leq (x \to y) \to y$, $x \leq y \to x$, $x \leq (y \to x) \to x$,
(iii) $1 \to x = x$, $x \odot y \leq x, y$, $x \to x = 1$,
(iv) *if* $x \leq y$, *then* $z \to x \leq z \to y$, $y \to z \leq x \to z$ *and* $x \to y = 1$,
(v) $x \to y \leq (z \to x) \to (z \to y)$, $x \to y \leq (y \to z) \to (x \to z)$,
(vi) $x \to (y \to z) = y \to (x \to z)$,
(vii) $x \odot y \leq z$ *if and only if* $y \leq x \to z$,
$(viii)$ $(x \to y) \wedge (y \to z) \leq (x \to z)$.

Definition 2.4. [5, 9] *Let F be a non-empty subset of A such that $1 \in F$. Then for any $x, y, z \in A$:*
(i) *F is called an* n-fold positive implicative filter *of A, if $x^n \to (y \to z) \in F$ and $x^n \to y \in F$, then $x^n \to z \in F$.*
(ii) *F is called an* n-fold fantastic filter *of A, if $x \to y \in F$, then $((y^n \to x) \to x) \to y \in F$.*

If A is a bounded hoop, then a filter is proper if and only if it is not containing 0.

Definition 2.5. [3] *Let $\mu(x) \leq \mu(1)$. Then a fuzzy subset μ of A is called:*
(i) fuzzy filter *if* $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$.
(ii) fuzzy n-fold implicative filter *if* $\mu((x^n \to y) \to x) \leq \mu(x)$.

Theorem 2.6. [3, 9] (i) *Let μ be a fuzzy filter of A and $x \leq y$. Then $\mu(x) \leq \mu(y)$.*
(ii) *Let μ be a fuzzy n-fold implicative filter of A. Then $\mu((x^n \to y) \to y) = \mu((y^n \to x) \to x)$, for any $x, y \in A$.*
(iii) *If μ is a fuzzy n-fold implicative filter, then $\mu(((x^n)' \to x) \to x) = \mu(1)$.*
(iv) *Let μ be a fuzzy filter. Then $\mu(x \odot y) \leq \mu(x) \wedge \mu(y)$.*
(v) *μ is a fuzzy filter if and only if $\mu_\alpha \neq \emptyset$ is a filter of A for any $\alpha \in [0, 1]$.*

Theorem 2.7. [1] *Let μ be a fuzzy filter of A and we define a relation $\sim_{\mu_{\mu(1)}}$ on A as follows:*

$$x \sim_{\mu_{\mu(1)}} y \quad if \ and \ only \ if \quad x \to y \in \mu_{\mu(1)}, \ y \to x \in \mu_{\mu(1)}.$$

Then $\sim_{\mu_{\mu(1)}}$ is a congruence relation on A. Also $A/\mu = (A/\mu, \wedge, \odot, \to, \mu^1)$ is a hoop algebra.

Definition 2.8. [3] *Let μ be a fuzzy filter of A and $A/\mu = \{\mu^x \mid x \in A\}$. For all $\mu^x, \mu^y \in A/\mu$, define $\mu^x \to \mu^y = \mu^{x \to y}$, $\mu^x \odot \mu^y = \mu^{x \odot y}$ and $\mu^x : A \to [0, 1]$ which is defined by $\mu^x(y) = \mu(x \to y) \wedge \mu(y \to x)$.*

Notation: From now one, we let $(A, \odot, \to, 1)$ or A is a hoop algebra, unless otherwise state.

3 Fuzzy n-fold positive implicative filters

In this section, we introduce the notion of fuzzy n-fold positive implicative filter on hoop algebra and investigate some properties of them.

Definition 3.1. *A fuzzy set μ of A is called a fuzzy n-fold positive implicative filter if for any $x, y, z \in A$,*

$$\mu(x) \leq \mu(1) \quad \text{and} \quad \mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z).$$

In the following example, we show that any fuzzy filter may not be a fuzzy n-fold positive implicative filter.

Example 3.2. (i) *Let $(A = \{0, a, b, c, 1\}, \leq)$ be a poset with $0 < c < a, b < 1$, but a, b are incomparable. Define the operations $\odot$ and $\to$ on A as follows:*

→	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

⊙	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop algebra. Let μ be a fuzzy filter such that

$$\mu(0) = 1/2, \mu(c) = \mu(a) = 5/9, \ \mu(b) = 6/9, \mu(1) = 8/9.$$

Clearly, μ is a fuzzy n-fold positive implicative filter for any $n \in \mathbb{N}$.
(ii) *Define for all $x, y \in [0, 1]$, $x \odot y = \min\{x, y\}$,*

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x \end{cases}$$

and

$$\mu(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2) \\ 1 & \text{if } x \in [1/2, 1] \end{cases}$$

It is clear that $A = ([0, 1], \odot, \to, 0, 1)$ is a hoop algebra and μ is a fuzzy filter of A. Let $x = y = 1/3$ and $z = 1/10$. Then $\mu((1/3)^2 \to (1/3 \to 1/10)) \wedge \mu((1/3)^2 \to 1/3) = \mu((1/3)^3 \to 1/10) \wedge \mu(1) \nleq \mu((1/3)^2 \to 1/10) = \mu(1/10)$. Therefore, μ is not a fuzzy n-fold positive implicative filter for any $n \in \mathbb{N}$.

Proposition 3.3. *Every fuzzy n-fold positive implicative filter of A is a fuzzy filter.*

Proof. Let μ be a fuzzy n-fold positive implicative filter of A. Then $\mu(x) \leq \mu(1)$. Also

$$\mu(1^n \to (x \to y)) \wedge \mu(1^n \to x) \leq \mu(1^n \to y).$$

By Proposition 2.3(iii), $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$. Thus μ is a fuzzy filter of A. □

Example 3.4. *In Example 3.2(ii), we show that the converse of Proposition 3.3, is not true, in general.*

Theorem 3.5. *Every fuzzy subset μ of A is a fuzzy n-fold positive implicative filter if and only if $\mu_\alpha = \{x \in A \mid \alpha \leq \mu(x)\}$ is either empty or is an n-fold positive implicative filter of A for any $\alpha \in [0, 1]$.*

Proof. Let μ be a fuzzy n-fold positive implicative filter of A and $\alpha \in [0,1],\ x \in \mu_\alpha,\ \mu(x) \geq \alpha$. Since μ is a fuzzy filter, then $\mu(x) \leq \mu(1)$. Hence $1 \in \mu_\alpha$.
Let $x^n \to (y \to z) \in \mu_\alpha$ and $x^n \to y \in \mu_\alpha$, for $x, y, z \in A$. Then

$$\alpha \leq \mu(x^n \to (y \to z)), \quad and \quad \alpha \leq \mu(x^n \to y).$$

Since μ is a fuzzy n-fold positive implicative filter thus

$$\alpha \leq \mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z).$$

So $x^n \to z \in \mu_\alpha$ and μ_α is an n-fold positive implicative filter of A.
Conversely, if μ_α is an n-fold positive implicative filter of A, then we show that μ is a fuzzy n-fold positive implicative filter. Let $\alpha = \mu(x)$. Then $x \in \mu_{\mu(x)}$, because $\mu(x) \leq \mu(x)$. So $\mu_{\mu(x)} \neq \emptyset$, thus $\mu_{\mu(x)}$ is an n-fold positive implicative filter and $1 \in \mu_{\mu(x)}$. Hence $\mu(x) \leq \mu(1)$ for any $x \in A$. Let $x, y, z \in A$, we show that

$$\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z).$$

Let there exists $a, b, c \in A$ such that

$$\mu(a^n \to c) < \mu(a^n \to (b \to c)) \wedge \mu(a^n \to b).$$

Then with $\alpha_0 = 1/2(\mu(a^n \to c) + (\mu(a^n \to (b \to c)) \wedge \mu(a^n \to b)))$, we have:

$$\mu(a^n \to c) < \alpha_0 < \mu(a^n \to (b \to c)) \wedge \mu(a^n \to b).$$

Hence $a^n \to c \notin \mu_{\alpha_0}$, but $a^n \to (b \to c) \in \mu_{\alpha_0}$ and $a^n \to b \in \mu_{\alpha_0}$, which is a contradiction since μ_{α_0} is an n-fold positive implicative filter of A. Therefore, μ is a fuzzy n-fold positive implicative filter. □

Corollary 3.6. *Any non empty subset F of A is an n-fold positive implicative filter if and only if the characteristic function χ_F is a fuzzy n-fold positive implicative filter.*

Proposition 3.7. *Let μ be a fuzzy filter of A. The following conditions are equivalent for any $x, y, z \in A$;*
(i) μ is a fuzzy n-fold positive implicative filter of A.
(ii) $\mu(x^{n+1} \to y) \leq \mu(x^n \to y)$.
(iii) $\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z))$

Proof. $(i) \Rightarrow (ii)$ Let $x, y \in A$. Since μ is a fuzzy n-fold positive implicative filter of A, we have

$$\mu(x^n \to (x \to y)) \wedge \mu(x^n \to x) \leq \mu(x^n \to y).$$

On the other hand $x^n \to (x \to y) = x^{n+1} \to y$ and $x^n \to x = 1$.
Therefore, $\mu(x^{n+1} \to y) \leq \mu(x^n \to y)$.
$(ii) \Rightarrow (iii)$ By Proposition 2.3(v), $y \to z \leq (x^n \to y) \to (x^n \to z)$. Hence

$$x^n \to (y \to z) \leq x^n \to ((x^n \to y) \to (x^n \to z)).$$

On the other hand,

$$x^n \to ((x^n \to y) \to (x^n \to z)) = x^n \to (x^n \to ((x^n \to y) \to z)) = x^{2n} \to ((x^n \to y) \to z).$$

Since μ is a fuzzy filter, then

$$\mu(x^n \to (y \to z)) \leq \mu(x^{2n} \to ((x^n \to y) \to z)).$$

By hypothesis,

$$\mu(x^{n+1} \to ((x^n \to y) \to z)) \leq \mu(x^n \to ((x^n \to y) \to z)).$$

For n time, we obtain

$$\mu(x^{2n} \to ((x^n \to y) \to z)) \leq \mu(x^n \to ((x^n \to y) \to z)).$$

Hence

$$\mu(x^n \to (y \to z)) \leq \mu(x^n \to ((x^n \to y) \to z)).$$

On the other hand

$$x^n \to ((x^n \to y) \to z) = (x^n \to y) \to (x^n \to z).$$

Therefore,

$$\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z)).$$

$(iii) \Rightarrow (i)$ By hypothesis

$$\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z)).$$

Since μ is a fuzzy filter, we have

$$\mu((x^n \to y) \to (x^n \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z).$$

Hence

$$\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z).$$

Therefore, μ is a fuzzy n-fold positive implicative filter of A. □

Corollary 3.8. *Let μ be a fuzzy n-fold positive implicative filter of bounded hoop A. Then $\mu((x^{n+1})') = \mu((x^n)')$ for any $x \in A$.*

Proposition 3.9. *Let μ be a fuzzy subset of A. Then μ is a fuzzy n-fold positive implicative filter of A if and only if for any $x, y, z \in A$,*

$$\mu(x) \wedge \mu(x \to (y^{n+1} \to z)) \leq \mu(y^n \to z).$$

Proof. Let μ be a fuzzy n-fold positive implicative filter. Then μ is a fuzzy filter and

$$\mu(x) \wedge \mu(x \to (y^{n+1} \to z)) \leq \mu(y^{n+1} \to z).$$

Since μ is a fuzzy n-fold positive implicative filter, we have by Proposition 3.7(ii),

$$\mu(y^{n+1} \to z) \leq \mu(y^n \to z).$$

Hence

$$\mu(x) \wedge \mu(x \to (y^{n+1} \to z)) \leq \mu(y^n \to z).$$

Conversely, by hypothesis, if $y = z = 1$, then $\mu(x) \leq \mu(1)$. Also

$$\mu(x) \wedge \mu(x \to (1^{n+1} \to y)) \leq \mu(1^n \to y).$$

On the other hand $y = 1^n \to y$ and $x \to (1^{n+1} \to y) = x \to y$. Hence

$$\mu(x) \wedge \mu(x \to y) \leq \mu(y).$$

we conclude that μ is a fuzzy filter. To prove that μ is a fuzzy n-fold positive implicative filter, we show that

$$\mu(x^{n+1} \to y) \leq \mu(x^n \to y).$$

By hypothesis, we have

$$\mu(x^{n+1} \to y) = \mu(1) \wedge \mu(1 \to (x^{n+1} \to y)) \leq \mu(x^n \to y).$$

Therefore, by Proposition 3.7, μ is a fuzzy n-fold positive implicative filter. □

Proposition 3.10. (i) *Every fuzzy n-fold positive implicative filter is a fuzzy $(n+1)$-fold positive implicative filter.*
(ii) *Every fuzzy n-fold positive implicative filter is a fuzzy $(n+k)$-fold positive implicative filter for any $k \geq 0$.*

Proof. (i) Let μ be a fuzzy n-fold positive implicative filter. Then by Proposition 3.7(ii),

$$\mu(x^{n+1} \to (x \to y)) \leq \mu(x^n \to (x \to y)).$$

On the other hand,

$$x^n \to (x \to y) = x^{n+1} \to y \quad and \quad x^{n+1} \to (x \to y) = x^{n+2} \to y.$$

Hence

$$\mu(x^{n+2} \to y) \leq \mu(x^{n+1} \to y).$$

Therefore, μ is a fuzzy $(n+1)$-fold positive implicative filter.
(ii) By part (i), it is clear. □

In the following example, we show that the converse of Proposition 3.10, is not true, in general.

Example 3.11. *Let $(A = \{0, a, b, 1\}, \leq)$ be a chain. Define the operations $\odot$ and $\to$ on A as follows:*

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $(A, \odot, \to, 1, 0)$ is a bounded hoop algebra.
Let $\alpha, \beta \in [0, 1]$ that $\beta < 1/2 < \alpha$ and μ be a fuzzy subset on A such that $\mu(b) = \mu(1) = \alpha$, $\mu(0) = \mu(a) = \beta$. It is clear that for any $n \geq 2$, μ is a fuzzy n-fold positive implicative filter, but it is not 1-fold positive implicative filter. Because $\alpha = \mu(1) = \mu(a \to (0 \to a)) \wedge \mu(a \to 0) \nleq \mu(a \to a) = \beta$.

Proposition 3.12. [3] *Let F and G be two filters of A such that $F \subseteq G$. If F is an n-fold positive implicative filter, then G is an n-fold positive implicative filter of A.*

Theorem 3.13. *Let μ and λ be two fuzzy filters of A such that $\mu \subseteq \lambda$ and $\mu(1) = \lambda(1)$. If μ is a fuzzy n-fold positive implicative filter, then λ is a fuzzy n-fold positive implicative filter.*

Proof. Let μ be a fuzzy n-fold positive implicative filter. Then we show that for any $\alpha \in [0,1]$, the α- level subset $\lambda_\alpha = \{x \in A : \lambda(x) \geq \alpha\}$ is an n-fold positive implicative filter of A when $\lambda_\alpha \neq \emptyset$. If μ_α is a n-fold positive implicative filter and $\mu_\alpha \subseteq \lambda_\alpha$, then by Proposition 3.12, λ_α is a n-fold positive implicative filter. Therefore, by Proposition 3.7, λ is a fuzzy n-fold positive implicative filter. □

Proposition 3.14. (i) *If A is a locally finite for $n \in \mathbb{N}$, then any fuzzy filter of A is a fuzzy n-fold positive implicative filter of A.*
(ii) *If for any element of A, $x^2 = x$, then any fuzzy filter μ of A is a fuzzy n-fold positive implicative filter of A.*

Proof. (i) Let m be a maximal $ord(x)$ in A, Then for any $n \geq m$, $\mu(x^{n+1} \to y) \leq \mu(x^n \to y)$ and fuzzy filter μ is a fuzzy n-fold positive implicative filter.
(ii) If $x^2 = x$, then $\mu(x^{n+1} \to y) \leq \mu(x^n \to y)$. Therefore, by Proposition 2.3(ii), μ is a fuzzy n-fold positive implicative filter of A. □

Theorem 3.15. (i) *Every fuzzy n-fold implicative filter of A is a fuzzy n-fold positive implicative filter.*
(ii) *Let μ be a fuzzy n-fold positive implicative filter of A and $\mu((x^n \to y)^n \to y) \leq \mu((y^n \to x)^n \to x)$ and $x \leq (x^n \to y) \to y$ for all $x, y \in A$. Then μ is a fuzzy n-fold implicative filter.*

Proof. (i) By Proposition 2.3(v),(vi)

$$\begin{aligned}
(x^{n+1} \to y)^n \to (x^n \to y) &= ((x^{n+1} \to y)^{n-1} \odot (x^{n+1} \to y)) \to (x^n \to y) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^n \to y)) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^{n-1} \to (x \to y))) \\
&= (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x^{n+1} \to y) \to (x \to y))) \\
&= (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x \to (x^n \to y)) \to (x \to y))) \\
&\geq (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x^n \to y) \to y)) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^n \to y) \to (x^{n-1} \to y)) \\
&= (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)).
\end{aligned}$$

Also

$$\begin{aligned}
(x^n \to y) \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1} &= (x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot (x^{n+1} \to y) \odot x \odot x^{n-2} \\
&= (x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2} \odot x \odot (x^{n+1} \to y).
\end{aligned}$$

Since $x^{n+1} \to y \leq x^{n+1} \to y = x \to (x^n \to y)$, then $x \odot (x^{n+1} \to y) \leq x^n \to y$. So

$$(x^n \to y) \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1} \leq (x^n \to y)^2 \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2}.$$

Hence

$$((x^n \to y)^2 \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2}) \to y \leq ((x^n \to y) \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1}) \to y,$$

and so

$$(x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)) \leq (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)).$$

Now, we have

$$(x^{n+1} \to y)^n \to (x^n \to y) \geq (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)),$$

and

$$(x^{n+1} \to y)^n \to (x^n \to y) \geq (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)).$$

Hence, by repeating the process n times we get

$$\begin{aligned}(x^{n+1} \to y)^n \to (x^n \to y) &\geq (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y))\\ &\geq \\ &\vdots \\ &\geq (x^n \to y)^n \to ((x^{n+1} \to y)^0 \to (x^0 \to y))\\ &= (x^n \to y)^n \to (1 \to (1 \to y))\\ &= (x^n \to y)^n \to y.\end{aligned}$$

Hence,

$$((x^n \to y)^n \to y) \to ((x^{n+1} \to y)^n \to (x^n \to y)) = 1,$$

and so

$$(x^{n+1} \to y)^n \to (((x^n \to y)^n \to y) \to (x^n \to y)) = 1.$$

By Proposition 2.3(iv),

$$(x^{n+1} \to y)^n \leq (((x^n \to y)^n \to y) \to (x^n \to y)).$$

Then

$$\mu(x^{n+1} \to y) \leq \mu((x^{n+1} \to y)^n) \leq \mu(((x^n \to y)^n \to y) \to (x^n \to y)).$$

Since μ is a fuzzy n-fold implicative filter

$$\mu(x^{n+1} \to y) \leq \mu(x^n \to y).$$

Therefore, by Proposition 3.7(ii), μ is a fuzzy n-fold positive implicative filter.
(ii) If μ is a fuzzy n-fold positive implicative filter and $x \leq (x^n \to y) \to y$, then by Proposition 2.3(iv),

$$(x^n \to y)^n \to x \leq (x^n \to y)^n \to ((x^n \to y) \to y).$$

Hence, $\mu((x^n \to y)^n \to x) \leq \mu((x^n \to y)^n \to ((x^n \to y) \to y))$. Since μ is a fuzzy n-fold positive implicative filter,

$$\mu((x^n \to y)^n \to ((x^n \to y) \to y)) \wedge \mu((x^n \to y)^n \to (x^n \to y)) \leq \mu((x^n \to y)^n \to y).$$

By assumption

$$\mu((x^n \to y)^n \to ((x^n \to y) \to y)) \leq \mu((y^n \to x)^n \to x).$$

Hence $\mu((x^n \to y)^n \to x) \leq \mu((y^n \to x)^n \to x)$. Let $\alpha = \mu((x^n \to y)^n \to x)$, and $\beta = \mu((y^n \to x)^n \to x)$. Then $((x^n \to y)^n \to x) \in \mu_\alpha$, and $((y^n \to x)^n \to x) \in \mu_\beta$. Since $\alpha \leq \beta$ thus $\mu_\beta \subseteq \mu_\alpha$ and $(y^n \to x)^n \to x \in \mu_\alpha$. By Proposition 2.3(ii),(iii)

$$y \leq x^n \to y, \quad (x^n \to y)^n \to x \leq y^n \to x.$$

So $\mu((x^n \to y)^n \to x) \leq \mu(y^n \to x)$. Let $\gamma = \mu(y^n \to x)$. Then $y^n \to x \in \mu_\gamma$. Since $\alpha \leq \gamma$, thus $\mu_\gamma \subseteq \mu_\alpha$. Hence $y^n \to x \in \mu_\gamma \subseteq \mu_\alpha$. Since $(y^n \to x)^n \to x \in \mu_\alpha$, then $x \in \mu_\alpha$ so $\alpha = \mu((x^n \to y)^n \to x) \leq \mu(x)$. By Proposition 2.3(iii), (iv), $(x^n \to y)^n \leq x^n \to y$, thus $(x^n \to y) \to x \leq (x^n \to y)^n \to x$. Hence,

$$\mu((x^n \to y) \to x) \leq \mu((x^n \to y)^n \to x) \leq \mu(x).$$

Therefore, μ is a fuzzy n-fold implicative filter. □

4 Fuzzy n-fold fantastic filters

In this section, we introduce the notion of fuzzy n-fold fantastic filter on hoop algebra and investigate some properties of them. Also we study relation between this new fuzzy n-fold filter with other some fuzzy n-fold filter for example fuzzy n-fold positive implicative filters and fuzzy n-fold implicative filters on hoop algebras.

Definition 4.1. *A fuzzy set μ of A is called a* fuzzy n-fold fantastic filter *if for any $x, y, z \in A$,*

$$\mu(x) \leq \mu(1) \quad \textit{and} \quad \mu(z \to (y \to x)) \wedge \mu(z) \leq \mu(((x^n \to y) \to y) \to x).$$

Proposition 4.2. *Every fuzzy n-fold fantastic filter μ of A is a fuzzy filter.*

Proof. Let μ be a fuzzy n-fold fantastic filter of A. Then $\mu(x) \leq \mu(1)$. By Proposition 2.3(iii), $x \to y = x \to (1 \to y)$. Thus $\mu(x \to y) = \mu(x \to (1 \to y))$. Hence

$$\mu(x) \wedge \mu(x \to y) = \mu(x) \wedge \mu(x \to (1 \to y)).$$

Since μ is a fuzzy n-fold fantastic filter, we get

$$\mu(x) \wedge \mu(x \to (1 \to y)) \leq \mu(((y^n \to 1) \to 1) \to y).$$

By Proposition 2.3(iii), $\mu(x) \wedge \mu(x \to y) \leq \mu(y)$. Therefore, μ is a fuzzy filter. □

Example 4.3. (i) *Let μ in Example 3.11. Then for any $n \in \mathbb{N}$, fuzzy filter μ is a fuzzy n-fold fantastic filter of A.*
(ii) *Let μ in Example 3.2(ii), and $y \leq x^n < x < 1/2$. Then $1 = \mu(1) = \mu(y \to x) \nleq \mu(x^n \to y) \to x) = \mu(x) = 0$. Therefore, μ is not a fuzzy n-fold fantastic filter. If $x, y \in [1/2, 1]$, then μ is a fuzzy n-fold fantastic filter.*

Theorem 4.4. *Let μ be a fuzzy filter of A. Then μ is a fuzzy n-fold fantastic filter if and only if, $\mu(y \to x)) \leq \mu(((x^n \to y) \to y) \to x)$, for any $x, y \in A$.*

Proof. If μ is a fuzzy n-fold fantastic filter and $z = 1$, then $\mu(((x^n \to y) \to y) \to x) \geq \mu(1 \to (y \to x)) \wedge \mu(1) = \mu(1 \to (y \to x))$. Hence $\mu(y \to x) \leq \mu(((x^n \to y) \to y) \to x)$.
Conversely, if μ is a fuzzy filter, then $\mu(x) \leq \mu(1)$ and $\mu(z) \wedge \mu(z \to (y \to x)) \leq \mu(y \to x)$. By assumption, $\mu(z) \wedge \mu(z \to (y \to x)) \leq \mu(((x^n \to y) \to y) \to x)$. Therefore, μ is a fuzzy n-fold fantastic filter. □

Proposition 4.5. *If μ is a fuzzy filter of A, then μ is a fuzzy n-fold fantastic filter if and only if $\mu(y \to x) = \mu(((x^n \to y) \to y) \to x)$, for any $x, y \in A$.*

Proof. By Proposition 2.3(ii), $y \leq (x^n \to y) \to y$ and by Proposition 2.3(iv), $(((x^n \to y) \to y) \to x) \leq y \to x$. Then $\mu(((x^n \to y) \to y) \to x) \leq \mu(y \to x)$. Therefore, by Theorem 4.4, $\mu(y \to x) = \mu(((x^n \to y) \to y) \to x)$. □

Theorem 4.6. *Every fuzzy filter μ is a fuzzy n-fold fantastic filter of A if and only if $\mu_\alpha \neq \emptyset$ is an n-fold fantastic filter of A, for any $\alpha \in [0,1]$.*

Proof. Let μ be a fuzzy n-fold fantastic filter and $y \to x \in \mu_\alpha$. Then by Theorem 4.4, $\alpha \leq \mu(y \to x) \leq \mu(((x^n \to y) \to y) \to x)$. So $((x^n \to y) \to y) \to x \in \mu_\alpha$. Therefore, and μ_α is an n-fold fantastic filter of A. Conversely is similarly. □

Theorem 4.7. *Let μ and λ be two fuzzy filters of A such that $\mu \subseteq \lambda$ and $\mu(1) = \lambda(1)$. If μ is a fuzzy n-fold fantastic filter then λ is a fuzzy n-fold fantastic filter of A.*

Proof. Let μ and λ be two fuzzy filters of A such that $\mu \subseteq \lambda$ and μ be a fuzzy n-fold fantastic filter. Then by Theorem 4.6, $\mu_\alpha \neq \emptyset$ is an n-fold fantastic filter for all $\alpha \in [0,1]$. Since $\mu \subseteq \lambda$, so $\mu(x) \leq \lambda(x)$ for all $x \in A$. Suppose $x \in \mu_\alpha$. Then $\alpha \leq \mu(x)$ and $\alpha \leq \lambda(x)$. Thus $x \in \lambda_\alpha$ and so $\mu_\alpha \subseteq \lambda_\alpha$. Let $\mu_\alpha \neq \emptyset$. Then since $\mu(1) = \lambda(1)$, $\lambda_\alpha \neq \emptyset$. Let $y \to x \in \lambda_\alpha$, for $x, y \in A$ and for all $\alpha \in [0,1]$. Then $(y \to x) \to (y \to x) = 1$. By Proposition 2.3(vi), $y \to ((y \to x) \to x) = 1 \in \mu_\alpha$, for all $\alpha \in [0,1]$. Since μ_α is an n-fold fantastic filter,

$$((((y \to x) \to x)^n \to y) \to y) \to ((y \to x) \to x) = (y \to x) \to (((((y \to x) \to x)^n \to y) \to y) \to x) \in \mu_\alpha.$$

Since $\mu_\alpha \subseteq \lambda_\alpha$,

$$(y \to x) \to (((((y \to x) \to x)^n \to y) \to y) \to x) \in \lambda_\alpha.$$

Since $y \to x \in \lambda_\alpha$ and by Theorem 2.6(v), λ_α is a filter, thus $(((((y \to x) \to x)^n \to y) \to y) \to x) \in \lambda_\alpha$.
Let $t = (((((y \to x) \to x)^n \to y) \to y) \to x)$, we have

$$\begin{aligned} t \to (((x^n \to y) \to y) \to x) &= (((((y \to x) \to x)^n \to y) \to y) \to x) \to (((x^n \to y) \to y) \to x) \\ &\geq ((x^n \to y) \to y) \to ((((y \to x) \to x)^n \to y) \to y). \\ &\geq (((y \to x) \to x)^n \to y) \to (x^n \to y) \qquad \text{By Proposition 2.3(v)} \\ &\geq x^n \to ((y \to x) \to x)^n. \end{aligned}$$

By Proposition 2.3(ii),(iii), $x \leq (y \to x) \to x$, so $x^n \leq ((y \to x) \to x)^n$ and $x^n \to ((y \to x) \to x)^n = 1$.
Hence

$$t \to (((x^n \to y) \to y) \to x) = 1.$$

Since λ_α is a filter and $1 \in \lambda_\alpha$, we have

$$t \to (((x^n \to y) \to y) \to x) \in \lambda_\alpha.$$

On the other hands, $t \in \lambda_\alpha$ and λ_α is a filter, so

$$((x^n \to y) \to y) \to x \in \lambda_\alpha.$$

Hence λ_α is an n-fold fantastic filter for all $\alpha \in [0,1]$. Therefore, by Theorem 4.6, λ is a fuzzy n-fold fantastic filter of A. □

Proposition 4.8. *Every fuzzy n-fold fantastic filter μ of A is a fuzzy $(n+1)$-fold fantastic filter.*

Proof. By Proposition 2.3(iii),(iv), $((x^n \to y) \to y) \to x \leq ((x^{n+1} \to y) \to y) \to x$. So

$$\mu(((x^n \to y) \to y) \to x) \leq \mu(((x^{n+1} \to y) \to y) \to x).$$

Therefore, μ is a fuzzy $(n+1)$-fold fantastic filter. □

Proposition 4.9. (i) *If A is a chain, then every fuzzy filter of A is a fuzzy n-fold fantastic filter.*
(ii) *If μ is a fuzzy n-fold fantastic filter of bounded hoop A, then $\mu(((x^n)'') \to x) = \mu(1)$.*
(iii) *If every element of bounded hoop A is dense, then any fuzzy n-fold fantastic filter of A is constant.*

Proof. (i) Let $x \leq y$. Then by Proposition 2.3(iii), $\mu(((x^n \to y) \to y) \to x) = \mu(y \to x)$. Therefore, by Theorem 4.6, μ is a fuzzy n-fold fantastic filter.
(ii) Let $y = 0$. Then $\mu(((x^n \to y) \to y) \to x) = \mu(y \to x)$.
(iii) Let $0 \neq x \in A$ be a dense. Then by (ii),

$$\mu((((x^n)') \to 0) \to x) = \mu(1 \to x) = \mu(x) = \mu(0 \to x) = \mu(1).$$

Therefore, $\mu(x) = \mu(1) = t$. □

Proposition 4.10. *Let μ be a fuzzy n-fold fantastic filter of A. Then $I = \{x \in A \mid \mu(x) = \mu(1)\}$ is an n-fold fantastic filter.*

Proof. Since $1 \in A$ and $\mu(1) = \mu(1)$, then $1 \in I$. Let $y \to x \in I$. Then $\mu(y \to x) = \mu(1)$. By Theorem 4.4, $\mu(1) = \mu(y \to x) \leq \mu(((x^n \to y) \to y) \to x)$. Hence $\mu(((x^n \to y) \to y) \to x) = \mu(1)$. Thus $((x^n \to y) \to y) \to x \in I$. Therefore, I is an n-fold fantastic filter. □

Lemma 4.11. *Let μ be a fuzzy filter of bounded hoop A. Then for any $x, y, z \in A$, the following conditions are equivalent:*
(i) *μ is a fuzzy n-fold implicative filter of A,*
(ii) $\mu(((z^n)' \odot x) \to y) \wedge \mu(y \to z) \leq \mu(x \to z)$,
(iii) $\mu(((y^n)' \odot x) \to y) \leq \mu(x \to y)$.

Proof. $(i) \Rightarrow (iii)$ By Proposition 2.3(ii), $y \leq x \to y$, then $y^n \leq (x \to y)^n$. By Proposition 2.3(iv),

$$(x \to y)^n \to 0 \leq y^n \to 0 = (y^n)'.$$

So, $(y^n)' \to y \leq (x \to y^n)' \to y$. Thus,

$$x \to ((y^n)' \to y) \leq x \to (((x \to y)^n)' \to y).$$

So, $(y^n)' \odot x \to y \leq (((x \to y)^n)' \odot x) \to y$. Hence, $\mu((y^n)' \odot x \to y) \leq \mu(((x \to y)^n)' \odot x \to y)$. Let $\alpha = \mu((y^n)' \odot x \to y)$ and $\beta = \mu(((x \to y)^n)' \odot x \to y)$. Then $(y^n)' \odot x \to y \in \mu_\alpha$ and $((x \to y)^n)' \odot x \to y \in \mu_\beta$. So, $\alpha \leq \beta$, thus $\mu_\beta \subseteq \mu_\alpha$. Hence $((x \to y)^n)' \odot x \to y \in \mu_\alpha$ and $((x \to y)^n)' \to (x \to y) \in \mu_\alpha$. Since μ is a fuzzy n-fold implicative filter, then μ_α is an n-fold implicative filter, thus $x \to y \in \mu_\alpha$ and $\mu((y^n)' \odot x \to y) = \alpha \leq \mu(x \to y)$.
$(iii) \Rightarrow (ii)$ By Proposition 2.3(viii),

$$(((z^n)' \odot x) \to y) \wedge (y \to z) \leq (((z^n)') \odot x) \to z.$$

So

$$\mu((((z^n)' \odot x) \to y) \wedge (y \to z)) \leq \mu((((z^n)') \odot x) \to z).$$

By hypothesis $\mu((((z^n)' \odot x) \to y) \wedge (y \to z)) \leq \mu(x \to z)$.
$(ii) \Rightarrow (iii)$ Let $z = y$ in part (ii). Then $\mu(((y^n)' \odot x) \to y) \wedge \mu(y \to y) \leq \mu(x \to y)$. By Proposition 2.3(iii), $\mu(((y^n)' \odot x) \to y) \leq \mu(x \to y)$.
$(iii) \Rightarrow (i)$ By Proposition 2.3(iii), $((x^n)' \odot ((x^n)' \to x)) \to x = 1$. Then $\mu(((x^n)' \odot ((x^n)' \to x)) \to x) = \mu(1)$. By part (iii),

$$\mu(1) = \mu(((x^n)' \odot ((x^n)' \to x)) \to x) \leq \mu(((x^n)' \to x) \to x).$$

Hence $\mu(((x^n)' \to x) \to x) = \mu(1)$. Therefore, μ is a fuzzy n-fold implicative filter of A. □

Theorem 4.12. *Let μ be a fuzzy n-fold implicative filter of bounded hoop A. Then μ is a fuzzy n-fold fantastic filter.*

Proof. By Proposition 2.3(iv), $(x^n)' \leq x^n \to y$. Hence

$$(x^n \to y) \to y \leq (x^n)' \to y \;\; and \;\; ((x^n \to y) \to y) \to ((x^n)' \to y) = 1.$$

Then $((x^n)' \odot ((x^n \to y) \to y)) \to y = 1$. So $\mu(((x^n)' \odot ((x^n \to y) \to y)) \to y) = \mu(1)$. Since μ is a fuzzy n-fold implicative filter, by Lemma 4.11(ii),

$$\mu(((x^n)' \odot ((x^n \to y) \to y)) \to y) \wedge \mu(y \to x) \leq \mu(((x^n \to y) \to y) \to x).$$

Then $\mu(y \to x) \leq \mu(((x^n \to y) \to y) \to x)$. By Theorem 4.4, μ is a fuzzy n-fold fantastic filter. □

Theorem 4.13. *Let μ be a fuzzy n-fold fantastic filter of A such that $x^2 = x$ for all $x \in A$. Then μ is a fuzzy n-fold implicative filter.*

Proof. Let μ be a fuzzy n-fold fantastic filter. Then

$$\mu((x \to y) \to x) \leq \mu(((x^n \to (x \to y)) \to (x \to y)) \to x).$$

Also

$$((x^n \to (x \to y)) \to (x \to y)) \to x = ((x^{n+1} \to y) \to (x \to y)) \to x.$$

Since $x^2 = x$, by Proposition 2.3(iii),

$$((x^2 \to y) \to (x \to y)) \to x = 1 \to x = x.$$

Hence $\mu((x^n \to y) \to x) = \mu((x \to y) \to x) \leq \mu(x)$. Therefore, μ is a fuzzy n-fold implicative filter. □

Proposition 4.14. *If μ is a fuzzy n-fold fantastic filter and $y \odot (x^{n+1} \to y) \leq x$ and $x^n \odot (((x^n \to y) \to y) \to x) \leq y$ for all $x, y \in A$, then μ is a fuzzy n-fold positive implicative filter.*

Proof. By hypothesis and Proposition 2.3(vii), $x^{n+1} \to y \leq y \to x$, then $\mu(x^{n+1} \to y) \leq \mu(y \to x) \leq \mu(((x^n \to y) \to y) \to x)$. By Proposition 2.3(vii), $((x^n \to y) \to y) \to x \leq x^n \to y$. Hence $\mu(((x^n \to y) \to y) \to x) \leq \mu(x^n \to y)$. Therefore, by Theorem 3.7(ii), μ is a fuzzy n-fold positive implicative filter . □

Theorem 4.15. *Let A be a bounded $\vee$-hoop with (DNP). If μ is a fuzzy n-fold fantastic filter of A such that $x^2 = x$ for all $x \in A$, then $\frac{A}{\mu}$ is a Boolean algebra.*

Proof. Since A is a bounded $\vee$-hoop, then A is a distributive lattice. By Theorem 4.13, μ is a fuzzy n-fold implicative filter and by Theorem 2.6(iii), $\mu(((x^{'})^n \to x) \to x) = \mu(1)$. By Proposition 2.6(ii), $\mu(((x^n) \to x^{'}) \to x^{'}) = \mu(1)$. By Proposition 2.3(iii),(iv),

$$(x^n \to x^{'}) \to x^{'} \leq (x \to x^{'}) \to x^{'}.$$

So

$$\mu(1) = \mu((x^n \to x^{'}) \to x^{'}) \leq \mu((x \to x^{'}) \to x^{'}),$$

and

$$((x^{'})^n \to x) \to x \leq (x^{'} \to x) \to x.$$

So

$$\mu(1) = \mu(((x^{'})^n \to x) \to x) \leq \mu((x^{'} \to x) \to x).$$

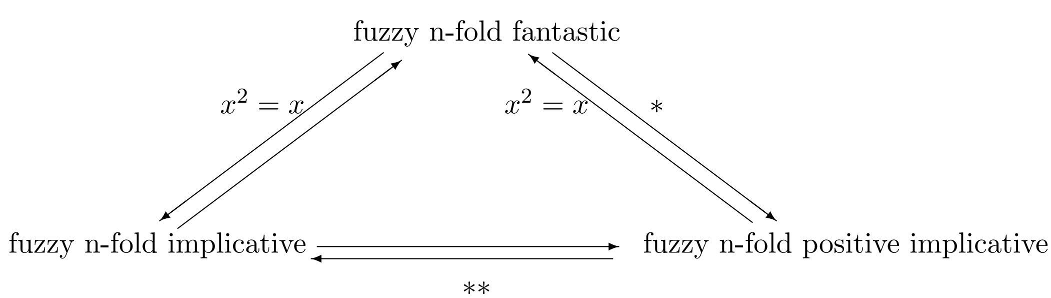
Hence

$$\mu(x \vee x^{'}) = \mu((x \to x^{'}) \to x^{'}) \wedge \mu((x^{'} \to x) \to x) = \mu(1).$$

Since $\mu(x \vee x^{'}) = \mu(1)$, then $\mu^{x \vee x^{'}} = \mu^1$. So $\mu^x \vee \mu^{x^{'}} = \mu^1$. On the other hands, $x \wedge x^{'} \leq x$ and by Proposition 2.3(iv), $x^{'} \leq (x \wedge x^{'})^{'}$. Also $x \wedge x^{'} \leq x^{'}$ and by Proposition 2.3(iv), $x^{''} \leq (x \wedge x^{'})^{'}$. By (DNP) $x \leq (x \wedge x^{'})^{'}$. Hence $(x \wedge x^{'})^{'}$ is an upper bound of $\{x, x^{'}\}$. Thus $x \vee x^{'} \leq (x \wedge x^{'})^{'}$, since $\mu(x \vee x^{'}) = \mu(1)$, then $\mu((x \wedge x^{'})^{'}) = \mu(1)$. Hence $x \wedge x^{'} \sim \mu_{\mu(1)} 0$ and $\mu^{x \wedge x^{'}} = \mu^0$. Therefore, $\frac{A}{\mu}$ is a Boolean algebra. □

In diagram (4-1), let μ be a fuzzy filter on A and $x, y \in A$. We show the relationship between fuzzy n-fold (positive) implicative and fuzzy n-fold fantastic filters on hoop algebra with by new conditions and some theorems proved in the text, where the conditions are:

$\{* : \quad y \odot (x^{n+1} \to y) \leq x, \quad x^n \odot (((x^n \to y) \to y) \to x) \leq y \ \}$
$\{** : \quad \mu((x^n \to y)^n \to y) \leq \mu((y^n \to x)^n \to x), \quad x \leq (x^n \to y) \to y \ \}.$



$diagram(4-1)$

5 Conclusion

In this article, at first we investigated fuzzy n-fold positive implicative filter on hoop algebras and studied properties of them. Then we obtained relation between fuzzy n-fold filter with fuzzy n-fold implicative filters and obtain some condition equivalent with them. We show that μ is a fuzzy n-fold positive implicative filter if and only if $\mu(x^{n+1} \to y) \leq \mu(x^n \to y)$. Hence we investigated fuzzy n-fold fantastic filter on hoop algebras and obtained relation between fuzzy n-fold fantastic filter with fuzzy n-fold (positive) implicative filters. We obtained μ is a fuzzy n-fold fantastic filter

if and only if $\mu(y \to x)) \leq \mu(((x^n \to y) \to y) \to x)$. We show that every fuzzy n-fold fantastic filter is a fuzzy filter and we investigated the following conditions equivalent for fuzzy n-fold fantastic filters. The relationship between them summarized in diagram(4-1).
In our opinion, these definitions and main results study other fuzzy n-fold filter on hoop algebra for example Dokdo filter and fuzzy(n-fold) Dokdo filter and can be extended to some other algebraic hoop such as (basic, simple, local, cancellative) hoop and etc.

Declarations

The author declares that they have no conflict of interest.

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