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Pseudo-BI-algebras: Non-commutative generalization of BI-algebras

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Abstract

We define and study the *pseudo-BI-algebras* as a non-commutative generalization of BI-algebras and investigate some properties. Also, we define *distributive* pseudo-BI-algebras and construct a BI-algebra using congruences. Further, we prove that there is no proper pseudo-BI-algebra of the order under four and that every pseudo-BI-algebra of order 4 is a poset, and so is a pseudo-BH-algebra. Besides, we introduce *exchangeable* pseudo-BI-algebras and show that their class is a proper subclass of class pseudo-CI-algebras. Finally, we define the notion of (*weak*) *commutative* pseudo-BI-algebras and prove that every weak commutative pseudo-BI-algebra is a (dual) pseudo-BH-algebra. However, the converse is not true, and show that every exchangeable commutative pseudo-BI-algebra is an implication algebra.

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1 Introduction

In 1966, Imai and Iséki [11] introduced two classes of algebraic structures named BCK-algebras and BCI- algebras (see [15]). In 2006, Georgescu and Iorgulescu [10] (also see [13, 14, 12]) and independently, Kühr [21] defined the notion of a pseudo-BCK-algebra, and they discussed the basic properties of its. In 2010, Zhang and Gong [30] introduced *1-type implicative pseudo-BCK-algebra* and *2-type implicative pseudo-BCK-algebra*, and the relations among 1-type (2-type) implicative pseudo-implicative BCK-algebras and implicative BCK-algebras are discussed. In 2013, the notion of a *pseudo-BE-algebra* was given by Borzooei et al. [5], and then Ciungu proposed the *commutative pseudo-BE-algebras* and showed these coincide with commutative pseudo-BCK-algebras [7]. The notion of *normal pseudo-BCI-algebras* was studied by Dymek [9]. Then in 2015, Jun et al. [19]

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presented the notion of a *pseudo-BH-algebra*, which is a generalization of a *BH-algebra* [18] and a *pseudo-BCH-algebra* (see [16, 28, 29]). In 2018, Ciungu [8] introduced *weak pseudo-BCK-algebras* as a generalization of weak BCK-algebras. It is shown that any weak pseudo-BCK(E)-algebra is a pseudo-BE-algebra and the class of commutative weak pseudo-BCK(E)-algebras is equivalent to the class of commutative pseudo-BCK-algebras, and so is equivalent to the class of commutative pseudo-BE-algebras. In 2019, Rezaei et al. [24] investigated *pseudo-CI-algebras* and provided some conditions for a pseudo-CI-algebra to be a pseudo-BE-algebra (also see [26]). Borumand Saeid et al. [4] introduced *BI-algebras* as an extension of both a (dual) implication algebras and an implicative BCK-algebra, and they investigated some ideals and congruence relations. They showed that every implicative BCK-algebra is a BI-algebra, but the converse is not generally true. In 2019, Ahn et al. [2] discussed normal subalgebras in BI-algebras and obtained several conditions for obtaining BI-algebra on the non-negative real numbers by using an analytic method. In 2022, Rezaei et al. [25], in order to get more results, continued to study this structure and investigated some concepts such as (Bosbach) states and state-morphism operators on it.

In the preset paper we define some classes on non commutative generalization of a BI-algebra as (distributive, exchangeable, (weak) commutative) dual pseudo-BI-algebras (briefly, pseudo-BI-algebras), and the properties of these structures are investigated. We give some characterizations of pseudo-BI-algebras and show that the class of them is a proper subclass of the class pseudo-CI-algebras. Besides, we prove every weak commutative pseudo-BI-algebra is a (dual) BH-algebra. However, the converse is not valid. We show that every exchangeable commutative pseudo-BI-algebra is an implication algebra. At the end, these results are presented in a diagram.

Notice BI-algebra in this paper is indeed a dual form of the original definition, and filter is a dual form of ideal in [4].

Recall that a *BI-algebra* [4] is an algebra $(X, \to, 1)$ of type $(2, 0)$ satisfying the following axioms: for all $x, y \in X$

(B) $x \to x = 1$,

(BI) $(x \to y) \to x = x$.

It was shown that every implicative BCK-algebra [23] is a BI-algebra, but the converse may not be generally true [4].

In 1967, Abbott [1] introduced the concept of implication algebra to describe properties of logical connective "*implication*" in classical logic. Recall that a groupoid $(X, \to)$ is an "*implication algebra*", whenever it satisfies (B), (BI), and the following axioms: for all $x, y, z \in X$

(I_1) $(x \to y) \to y = (y \to x) \to x$,

(I_2) $x \to (y \to z) = y \to (x \to z)$.

Definition 1.1. [16] *An algebra* $(X, \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a *(dual) pseudo-BCH-algebra if it satisfies the following axioms: for all* $x, y, z \in X$,

(pB) $x \to x = x \rightsquigarrow x = 1$,

(pI_2) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,

(pI_3) $x \to y = y \rightsquigarrow x = 1 \Longrightarrow x = y$.

In 2015, Walendziak ([28, 29]) proposed and revised the definition of (dual) pseudo-BCH-algebra and added the following axiom: for all $x \in X$

(pI$_4$) $x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1$.

Definition 1.2. [19] *An algebra $(X, \to, \rightsquigarrow, 1)$ of type (2,2,0) is called a (dual) pseudo-BH-algebra if it satisfies (pB), (pI$_3$) and the following axiom: for all $x \in X$*

(pI$_5$) $1 \to x = 1 \rightsquigarrow x = x$.

Notice that every (dual) pseudo-BCH-algebra is a (dual) pseudo-BH-algebra, but the converse may not be valid.

In this paper, we consider Walendziak's definition and propose the following definition:

Definition 1.3. *An algebra $(X, \to, \rightsquigarrow, 1)$ of type (2,2,0) is called a dual pseudo-BH-algebra if it satisfies (pB), (pI$_3$), (pI$_4$), and (pI$_5$).*

Example 1.4. *Let $X = \{a, b, c, 1\}$ with Cayley Table 1. Then $(X, \to, \rightsquigarrow, 1)$ is a dual pseudo-BH-algebra.*

Table 1: dual pseudo-BH-algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	1	$\rightsquigarrow$	a	b	c	1
a	1	a	b	1	a	1	a	c	1
b	1	1	c	1	b	1	1	a	1
c	a	b	1	c	c	c	b	1	c
1	a	b	c	1	1	a	b	c	1

Definition 1.5. [17] *An algebra $(X, \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a dual pseudo-Q-algebra if it satisfies (pB), (pI$_2$), and (pI$_5$).*

Definition 1.6. [24] *An algebra $(X, \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a pseudo-CI-algebra if it satisfies (pB), (pI$_2$), (pI$_4$), and (pI$_5$).*

Notice that every pseudo-CI-algebra is a dual pseudo-Q-algebra in a natural way, and every dual pseudo-Q-algebra satisfying (pI$_4$) is a pseudo-CI-algebra. So, pseudo-CI-algebras are categorically isomorphic with dual pseudo-Q-algebras with (pI$_4$).

Definition 1.7. [5] *A pseudo-CI-algebra $(X, \to, \rightsquigarrow, 1)$ is called a pseudo-BE-algebra if it satisfies the following axiom: for all $x \in X$*

(pI$_6$) $x \to 1 = x \rightsquigarrow 1 = 1$.

Theorem 1.8. [5] *In a pseudo-BE-algebra X, the following hold:*

(E_1) *$x \le y \to x$ and $x \le y \rightsquigarrow x$,*

(E_2) *$x \le (x \to y) \rightsquigarrow y$ and $x \le (x \rightsquigarrow y) \to y$,*

(E_3) *if $x \le y \to z$, then $y \le x \rightsquigarrow z$,*

(E_4) *if $x \le y \rightsquigarrow z$, then $y \le x \to z$,*

(E_5) *if $x \le y$, then $x \le z \to y$ and $x \le z \rightsquigarrow y$,*

(E_6) *if $x \to y = z$ or $x \rightsquigarrow y = z$, then $y \to z = y \rightsquigarrow z = 1$,*

(E_7) *if* $x \to y = y$ *and* $x \rightsquigarrow y = z$, *then* $x \to z = z$ *and*

$$x \to (y \to z) = (x \to y) \to (x \to z) = x \to (y \rightsquigarrow z) = (x \to y) \rightsquigarrow (x \to z) = 1,$$

(E_8) *if* $x \to y = z$ *and* $x \rightsquigarrow y = t$, *then* $x \rightsquigarrow z = x \to t$,

for all $x, y, z \in X$, *where*

$$x \leq y \iff x \to y = 1 \iff x \rightsquigarrow y = 1.$$

Theorem 1.9. [21] *Every finite commutative pseudo-BE-algebra is a BE-algebra.*

2 On pseudo-BI-algebras

All symbols $\to$, $\diamond$, $*$ and $\rightsquigarrow$ are used for binary operations. Some researchers have used $*$ and $\diamond$ (see [5, 9, 18, 19, 16, 28, 29]) and others have used $\to$ and $\rightsquigarrow$ (see [8, 7, 12, 13, 14, 21, 24, 26, 30]). Usually, the symbols $\to$ and $\rightsquigarrow$ are used in a dual form $*$ and $\diamond$. For example, when we consider pseudo-BCK-algebras with two implications, $\to$ and $\rightsquigarrow$ with one constant 1, that is the greatest element, thus such pseudo-BCK-algebras are in the "negative cone". Notice that at the first pseudo-BCK-algebras were introduced by G. Georgescu and A. Iorgulescu in [10] as algebras with "two difference", a left- and right- difference, instead of one $*$ with a constant element 0 as the last element. Besides, $\to$ is used as residuum. In order to overcome this misunderstanding, we used the famous symbols $\to$ and $\rightsquigarrow$ in the sequel. This section is a continuation of the [4] and [30], where the properties of BI-algebras and implicative pseudo-BCK-algebras as an extension of (implicative) BCK-algebras were studied.

Definition 2.1. *An algebra* $(X, \to, \rightsquigarrow, 1)$ *of type* $(2, 2, 0)$ *is called a pseudo-BI-algebra if it satisfies (pB), (pI$_4$), and the following axiom: for all* $x, y \in X$,

(pBI) $(x \to y) \rightsquigarrow x = x = (x \rightsquigarrow y) \to x$.

From now on, we will denote pseudo-BI-algebra $(X, \to, \rightsquigarrow, 1)$ *by* X, *unless otherwise stated.*
It is evident that X *is a BI-algebra if and only if* $\to = \rightsquigarrow$. *We say* X *is proper if* $x \to y \neq x \rightsquigarrow y$, *for some* $x, y \in X$. *Also, every implication algebra is a pseudo-BI-algebra naturally.*

Example 2.2. (i) *Let* $X = \{a, b, c, 1\}$ *with Cayley Table 2. Then* $(X, \to, \rightsquigarrow, 1)$ *is a pseudo-BI-algebra.*

Table 2: pseudo-BI-algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	1	$\rightsquigarrow$	a	b	c	1
a	1	c	c	1	a	1	b	b	1
b	a	1	c	1	b	c	1	c	1
c	b	b	1	1	c	a	b	1	1
1	a	b	c	1	1	a	b	c	1

(ii) *Let* X *be a set with* $|X| \geq 4$, *and let* 1, x_0, y_0 *and* z *be elements of* X. *Define binary operations "$\to$" and "$\rightsquigarrow$", for all* $x, y \in X$ *as follows:*

$$x \to y = \begin{cases} 1 & \text{if } x = y, \\ z & \text{if } x = y_0 \text{ and } y = x_0, \\ y & \text{otherwise,} \end{cases}$$

$$x \rightsquigarrow y = \begin{cases} 1 & \text{if } x = y, \\ z & \text{if } x = x_0 \text{ and } y = y_0, \\ y & \text{otherwise.} \end{cases}$$

Then $(X, \to, \rightsquigarrow, 1)$ *is a proper pseudo-BI-algebra.*

(iii) *Let* $(X, *, e)$ *be a group. Define binary operations "*$\to$*" and "*$\rightsquigarrow$*" on* X *by*

$$x \to y = x * y^{-1} \text{ and } x \rightsquigarrow y = y^{-1} * x,$$

for all $x, y \in X$*. Then* $(X, \to, \rightsquigarrow, e)$ *is a pseudo-BCH-algebra* [28]*, but not a pseudo-BI-algebra, since* $(x \to y) \rightsquigarrow x = (x * y^{-1}) \rightsquigarrow x = x^{-1} * (x * y^{-1}) = (x^{-1} * x) * y^{-1} = y^{-1} \neq x$.

Example 2.3 shows that (pB), (pBI), and (pI_4) are independent.

Example 2.3. *Let* $X = \{a, b, c, 1\}$.

(i) *The algebra* $(X, \to, \rightsquigarrow, 1)$ *with Cayley Table 3 satisfies (pBI), and (*pI_4*) but does not satisfy (pB), since* $1 \to 1 \neq 1$.

Table 3: Algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	1	$\rightsquigarrow$	a	b	c	1
a	1	b	1	1	a	1	b	1	1
b	a	1	c	1	b	a	1	c	1
c	b	b	1	1	c	b	b	1	1
1	a	b	c	b	1	a	b	c	c

(ii) *Consider algebra* $(X, \to, \rightsquigarrow, 1)$ *with Cayley Table 4. It satisfies (pB), and (*pI_4*), but does not satisfy (pBI), since* $(c \to a) \rightsquigarrow c = a \rightsquigarrow c = 1 \neq c$.

Table 4: Algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	1	$\rightsquigarrow$	a	b	c	1
a	1	1	1	1	a	1	1	1	1
b	1	1	1	1	b	1	1	1	1
c	1	a	b	1	c	a	a	1	1
1	a	b	c	1	1	a	b	c	1

(iii) *The algebra* $(X, \to, \rightsquigarrow, 1)$ *with Cayley Table 5 satisfies (pB), (*pI_4*), and for all* $x, y \in X$*, we have* $(x \to y) \rightsquigarrow x = x$*, but does not satisfy (pBI), since* $(b \rightsquigarrow a) \to b = a \to b = 1 \neq b$.

Table 5: Algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	1	$\rightsquigarrow$	1	a	b	c
a	1	1	1	1	a	1	1	1	1
b	c	1	c	1	b	a	1	c	1
c	1	b	1	1	c	1	b	1	1
1	a	b	c	1	1	a	b	c	1

Table 6: Algebra $(X, \rightarrow, \rightsquigarrow, 1)$

$\rightarrow$	a	b	c	1	$\rightsquigarrow$	a	b	c	1
a	1	c	c	1	a	1	1	1	1
b	c	1	c	1	b	c	1	c	1
c	b	b	1	1	c	a	b	1	1
1	a	b	c	1	1	a	b	c	1

$(b \rightsquigarrow a) \rightarrow b = a \rightarrow b = 1 \neq b$.

(iv) *Consider algebra $(X, \rightarrow, \rightsquigarrow, 1)$ with Cayley Table 6. Then $(X, \rightarrow, \rightsquigarrow, 1)$ satisfies (pB), and (pBI), but does not satisfy (pI_4), since $a \rightarrow b \neq 1$ and $a \rightsquigarrow b = 1$.*

Theorem 2.4. *Let $(X, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BI-algebra and X_0 be a set such that $X_0 \cap X = \emptyset$. Then $(X_1, \rightarrow_*, \rightsquigarrow_*, 1)$ is a pseudo-BI-algebra, where $X_1 = X \cup X_0$ and the operations "$\rightarrow_*$" and "$\rightsquigarrow_*$" defined as follows:*

$$x \rightarrow_* y = \begin{cases} 1 & \text{if } x = y, \\ x \rightarrow y & \text{if } x, y \in X, \\ y & \text{otherwise}, \end{cases}$$

$$x \rightsquigarrow_* y = \begin{cases} 1 & \text{if } x = y, \\ x \rightsquigarrow y & \text{if } x, y \in X, \\ y & \text{otherwise}. \end{cases}$$

Proof. It is clear that $(X_1, \rightarrow_*, \rightsquigarrow_*, 1)$ satisfies (pB), and (pI_4). To prove (pBI) consider the following cases:

Case 1. If $x = y$, then $(x \rightarrow_* y) \rightsquigarrow_* x = 1 \rightsquigarrow_* x = x$. By a similar argument $(x \rightsquigarrow_* y) \rightarrow_* x = x$.

Case 2. If $x, y \in X$, then the proof is trivial.

Case 3. If $x \in X$ and $y \notin X$ or if $x \notin X$ and $y \in X$, then $(x \rightarrow_* y) \rightsquigarrow_* x = y \rightsquigarrow_* x = x$ also $(x \rightsquigarrow_* y) \rightarrow_* x = y \rightarrow_* x = x$.

Therefore, $(X, \rightarrow_*, \rightsquigarrow_*, 1)$ is a pseudo-BI-algebra. □

Notice that by Theorem 2.4, and Example 2.2(ii), we have a proper pseudo-BI-algebra of any arbitrary order more than 3.

Throughout this paper, we define a binary relation "$\leq$" on a pseudo-BI-algebra, X by

$$x \leq y \iff x \rightarrow y = 1 \iff x \rightsquigarrow y = 1, \quad \forall x, y \in X.$$

We note that by (pB), "$\leq$" is reflexive.

Theorem 2.5. *The following hold, for all $x, y, z \in X$:*

(P_1) $1 \rightarrow x = 1 \rightsquigarrow x = x$,

(P_2) $x \rightarrow 1 = x \rightsquigarrow 1 = 1$,

(P_3) $x \rightarrow y = x \rightarrow (x \rightarrow y)$ *and* $x \rightsquigarrow y = x \rightsquigarrow (x \rightsquigarrow y)$,

(P_4) *if* $x \rightarrow y = x$ *or* $x \rightsquigarrow y = x$, *then* $x = 1$,

(P_5) *if* $x \rightsquigarrow y = z$, *then* $z \to x = x$ *and* $x \rightsquigarrow z = z$,

(P_6) *if* $x \to y = z$, *then* $z \rightsquigarrow x = x$ *and* $x \to z = z$,

(P_7) *if* $0 \in X$ *and* $0 \leq x$, *for all* $x \in X$, *then* $x \to 0 \notin \{0, x\}$ *and* $x \rightsquigarrow 0 \notin \{0, x\}$, *where* $x \neq 1$,

(P_8) *if* $(x \circ y) \diamond z = (x \diamond z) \diamond y$, *for all* $x, y, z \in X$, *then* $X = \{1\}$, *where* $\circ, \diamond \in \{\to, \rightsquigarrow\}$,

(P_9) *if* $(x \circ y) \diamond (z \diamond u) = (x \diamond z) \circ (y \diamond u)$, *for all* $x, y, z, u \in X$, *then* $X = \{1\}$, *where* $\circ, \diamond \in \{\to, \rightsquigarrow\}$,

(P_{10}) *if* X *is right distributive* (*it means* $(x \circ y) \diamond z = (x \diamond z) \circ (y \diamond z)$, *for all* $x, y, z \in X$), *then* $X = \{1\}$, *where* $\circ, \diamond \in \{\to, \rightsquigarrow\}$,

(P_{11}) *if* X *is proper and* $y \neq 1$, *then* $x \to y \neq x$ *and* $x \rightsquigarrow y \neq x$.

Proof. (P_1) Using (pB), and (pBI), we get $x = (x \rightsquigarrow x) \to x = 1 \to x$. Similarly, $1 \rightsquigarrow x = x$.

(P_2) $x \to 1 = (1 \rightsquigarrow x) \to 1 = 1$. Similarly, $x \rightsquigarrow 1 = 1$.

(P_3) By (pBI), $x \to y = ((x \to y) \rightsquigarrow x) \to (x \to y) = x \to (x \to y)$. Similarly, $x \rightsquigarrow y = x \rightsquigarrow (x \rightsquigarrow y)$.

(P_4) Let $x \to y = x$, for $x, y \in X$. Then by (pBI), $x = (x \to y) \rightsquigarrow x = x \rightsquigarrow x = 1$. By a similar way, if $x \rightsquigarrow y = x$, for $x, y \in X$, then $x = 1$.

(P_5) Let $x \rightsquigarrow y = z$. By (pBI), $z \to x = (x \rightsquigarrow y) \to x = x$.

(P_6) The proof is similar to (P_5).

(P_7) Assume $0 \in X$ and $0 \leq x$, for all $x \in X$. Then $0 \to x = 0 \rightsquigarrow x = 1$, for all $x \in X$. By (P_4), $x \to 0 \neq x$ and $x \rightsquigarrow 0 \neq x$. If $x \to 0 = 0$, then by (pBI), $(x \to 0) \rightsquigarrow x = 0 \rightsquigarrow x = 1 \neq x$, which is a contradiction.

(P_8) Assume $(x \to y) \rightsquigarrow z = (x \rightsquigarrow z) \rightsquigarrow y$, for all $x, y, z \in X$. Then by assumption for all $x \in X$, we have

$$1 = (1 \to x) \rightsquigarrow 1 = (1 \rightsquigarrow 1) \rightsquigarrow x = 1 \rightsquigarrow x = x.$$

Thus, $X = \{1\}$. The proof of other cases are similar.

(P_9) Let $x \in X$. If $y = x = u$, $z = 1$, then

$$x = 1 \rightsquigarrow x = (x \to x) \rightsquigarrow (1 \rightsquigarrow x) = (x \rightsquigarrow 1) \to (x \rightsquigarrow x) = 1 \to 1 = 1.$$

Hence $X = \{1\}$.

(P_{10}) Let $(x \to x) \rightsquigarrow x = (x \rightsquigarrow x) \to (x \rightsquigarrow x)$, for all $x \in X$.Then by (P_1), and (pB) we have

$$x = 1 \rightsquigarrow x = (x \to x) \rightsquigarrow x = (x \rightsquigarrow x) \to (x \rightsquigarrow x) = 1.$$

Thus, $X = \{1\}$. By a similar way, by other cases we conclude that $X = \{1\}$.

(P_{11}) By (P_4), the proof is clear. □

Definition 2.6. *A pseudo-BI-algebra is called left distributive or briefly distributive if it satisfies the following axioms: for all* $x, y, z \in X$

(pI_7) $z \to (x \rightsquigarrow y) = (z \to x) \rightsquigarrow (z \to y)$,

(pI_8) $z \rightsquigarrow (x \to y) = (z \rightsquigarrow x) \to (z \rightsquigarrow y)$.

Example 2.7. (i) *Let* $X = \{a, b, c, d, 1\}$ *with Cayley Table 7. Then* $(X, \to, \rightsquigarrow, 1)$ *is distributive.*

(ii) *According to Example 2.2(ii),* $(X, \to, \rightsquigarrow, 1)$ *is not distributive. Since*

$$z \rightsquigarrow (y_0 \to x_0) = z \rightsquigarrow z = 1 \neq z = y_0 \to x_0 = (z \rightsquigarrow y_0) \to (z \rightsquigarrow x_0).$$

Table 7: distributibe pseudo-BI-algebra $(X, \rightarrow, \rightsquigarrow, 1)$

$\rightarrow$	a	b	c	d	1	$\rightsquigarrow$	a	b	c	d	1
a	1	b	b	1	1	a	1	c	c	1	1
b	d	1	1	d	1	b	a	1	1	a	1
c	a	1	1	a	1	c	d	1	1	d	1
d	1	c	c	1	1	d	1	b	b	1	1
1	a	b	c	d	1	1	a	b	c	d	1

Open problem 1. Find examples which (pI_7), and (pI_8) are independent, i.e. there exists a pseudo-BI-algebra verifying (pI_7), and not verifying (pI_8) and there exists a pseudo-BI-algebra verifying (pI_8) and not verifying (pI_7).

Theorem 2.8. *Let X be distributive. Then the following hold:*

(d_1) $y \leq x \rightsquigarrow y$

(d_2) $y \leq x \rightarrow y$,

(d_3) $y \leq (x \rightarrow y) \rightsquigarrow y$,

(d_4) $y \leq (x \rightarrow y) \rightarrow y$,

(d_5) $y \leq (x \rightsquigarrow y) \rightsquigarrow y$,

(d_6) $x \leq (x \rightarrow y) \rightsquigarrow y$,

(d_7) $x \leq (x \rightsquigarrow y) \rightarrow y$,

(d_8) $x \rightsquigarrow y \leq (z \rightarrow x) \rightsquigarrow (z \rightarrow y)$,

(d_9) $x \rightarrow y \leq (z \rightsquigarrow x) \rightarrow (z \rightsquigarrow y)$,

(d_{10}) $(X, \leq)$ *is transitive and reflexive,*

(d_{11}) *if* $x \leq y$, *then* $z \rightsquigarrow x \leq z \rightsquigarrow y$ *and* $z \rightarrow x \leq z \rightarrow y$,

(d_{12}) $(x \rightsquigarrow y) \rightarrow z \leq x \rightsquigarrow (y \rightarrow z)$,

(d_{13}) $(x \rightarrow y) \rightsquigarrow z \leq x \rightarrow (y \rightsquigarrow z)$,

(d_{14}) *if* $x \rightarrow y = z$, *then* $x \rightsquigarrow z = x \rightsquigarrow y$ *and* $y \leq z$,

(d_{15}) *if* $x \rightsquigarrow y = z$, *then* $x \rightarrow z = x \rightarrow y$ *and* $y \leq z$,

(d_{16}) *if* $x \rightarrow y = y$ *and* $x \rightsquigarrow y = z$, *then* $x \rightarrow z = y$, $z \leq y$ *and* $y \leq z$,

(d_{17}) *if* $x \rightsquigarrow y = y$ *and* $x \rightarrow y = z$, *then* $x \rightsquigarrow z = y$, $z \leq y$ *and* $y \leq z$,

(d_{18}) $x \rightarrow y \leq x \rightsquigarrow y$ *and* $x \rightsquigarrow y \leq x \rightarrow y$,

for all $x, y, z \in X$.

Proof. Assume $x, y, z \in X$. Then

(d_1) By (pI_7), (B), and (P_2),

$$y \to (x \rightsquigarrow y) = (y \to x) \rightsquigarrow (y \to y) = (y \to x) \rightsquigarrow 1 = 1,$$

and so $y \leq x \rightsquigarrow y$.

(d_2) The proof is similar to (d_1).

(d_3) By (pI_7), (B), and (P_2),

$$y \to ((x \to y) \rightsquigarrow y) = (y \to (x \to y)) \rightsquigarrow (y \to y) = (y \to (x \to y)) \rightsquigarrow 1 = 1.$$

(d_4), and (d_5) The proof is similar to (d_3).

(d_6) By (pI_7), (P_3), and (B),

$$x \to ((x \to y) \rightsquigarrow y) = (x \to (x \to y)) \rightsquigarrow (x \to y) = (x \to y) \rightsquigarrow (x \to y) = 1.$$

(d_7) The proof is similar to (d_7).

(d_8) By (d_1), $x \rightsquigarrow y \leq z \to (x \rightsquigarrow y)$. By ($pI_7$), $x \rightsquigarrow y \leq (z \to x) \rightsquigarrow (z \to y)$.

(d_9) The proof is similar to (d_8).

(d_{10}) By (pB), $(X, \leq)$ is reflexive. Let $x \leq y$ and $y \leq z$. Then by (d_8),

$$y \rightsquigarrow z \leq (x \to y) \rightsquigarrow (x \to z) \Longrightarrow 1 \leq 1 \rightsquigarrow (x \to z).$$

Hence $x \to z = 1$ and $x \leq z$. Therefore, X is transitive.

(d_{11}) If $x \leq y$, then

$$1 = z \to 1 = z \to (x \rightsquigarrow y) = (z \to x) \rightsquigarrow (z \to y).$$

Thus $z \to x \leq z \to y$. By a similar way, $z \rightsquigarrow x \leq z \rightsquigarrow y$.

(d_{12}) By (d_1), $z \leq x \rightsquigarrow z$. By ($d_{10}$), $(x \rightsquigarrow y) \to z \leq (x \rightsquigarrow y) \to (x \rightsquigarrow z)$. By ($pI_8$), $(x \rightsquigarrow y) \to z \leq x \rightsquigarrow (y \to z)$.

(d_{13}) The proof is similar to (d_{12}).

(d_{14}) Let $x \to y = z$. Then by (pI_8), (B), and (P_1),

$$x \rightsquigarrow z = x \rightsquigarrow (x \to y) = (x \rightsquigarrow x) \to (x \rightsquigarrow y) = 1 \to (x \rightsquigarrow y) = x \rightsquigarrow y.$$

Using (pB), (P_2), and (pI_8), we obtain

$$y \rightsquigarrow z = y \rightsquigarrow (x \to y) = (y \rightsquigarrow x) \to (y \rightsquigarrow y) = (y \rightsquigarrow x) \to 1 = 1.$$

Hence $y \leq z$.

(d_{15}) The proof is similar to (d_{14}).

(d_{16}) Let $x \to y = y$ and $x \rightsquigarrow y = z$. Then by (d_{15}), $y \leq z$. Also,

$$x \to z = x \to (x \rightsquigarrow y) = (x \to x) \rightsquigarrow (x \rightsquigarrow y) = 1 \rightsquigarrow z = z.$$

Using $x \to z = y$ and (d_{14}), $z \leq y$.

(d_{17}) The proof is similar to (d_{16}).

(d_{18}) Let $x \to y = z$ and $x \rightsquigarrow y = t$. Then by (P_6), $x \to z = z$. Also,

$$x \rightsquigarrow z = x \rightsquigarrow (x \to y) = (x \rightsquigarrow x) \to (x \rightsquigarrow y) = 1 \to t = t.$$

Using (d_{16}), $x \to z = z$ and $x \rightsquigarrow z = t$, we obtain $z \leq t$ and $t \leq z$. Thus, $x \to y \leq x \rightsquigarrow y$ and $x \rightsquigarrow y \leq x \to y$. □

By (d_{10}), the operation "$\leq$" is transitive and reflexive. In the following example, we show that $(X, \leq)$ is not an order in generality.

Example 2.9. (i) *According distributive pseudo-BI-algebra $X = \{a, b, c, d, 1\}$ with Cayley Table 7. Since $b \leq c$ and $c \leq b$, "$\leq$" is not antisymmetric and so is not an order.*

(ii) *Let $X = \{a, b, c, 1\}$ with Cayley Table 2. Then "$\leq$" is an order on pseudo-BI-algebra $(X, \rightarrow, \rightsquigarrow, 1)$.*

Definition 2.10. *Let "$\sim$" be an equivalence relation on X. The relation "$\sim$" is called:*

(i) *Left congruence relation on X if $x \sim y$ implies $u \rightarrow x \sim u \rightarrow y$ and $u \rightsquigarrow x \sim u \rightsquigarrow y$, for all $u \in X$.*

(ii) *Right congruence relation on X if $x \sim y$ implies $x \rightarrow v \sim y \rightarrow v$ and $x \rightsquigarrow v \sim y \rightsquigarrow v$, for all $v \in X$.*

(iii) *Congruence relation on X if has the substitution property with respect to "$\rightarrow$" and "$\rightsquigarrow$", that is, for any $x \sim y$, $u \sim v$ we have*

$$(x \rightarrow u) \sim (y \rightarrow v) \text{ and } (x \rightsquigarrow u) \sim (y \rightsquigarrow v).$$

Theorem 2.11. *Let X be distributive and relation "$\sim$" be defined as follows:*

$$x \sim y \text{ if and only if } x \leq y \text{ and } y \leq x.$$

Then

(i) *"$\sim$" is an equivalence relation on X,*

(ii) $[1] = \{1\}$,

(iii) *"$\sim$" is a left congruence relation on X.*

Proof. By (B), (d_{10}), (P_1), and (d_{11}), the proof is routine. □

Computer Check: For all distributive pseudo-BI-algebra of the order under 6, "$\sim$" is a congruence relation.

Open problem 2. Find condition/conditions in which the relation $\sim$ is a congruence relation.

Let X be distributive, and the relation $\sim$ be defined as in Theorem 2.11. By $[x]$, we mean

$$[x] = \{y \in X \mid x \sim y\}.$$

Put $\frac{X}{\sim} = \{[x] \mid x \in X\}$. Define two operations, "$\rightarrow$", and "$\rightsquigarrow$" on $\frac{X}{\sim}$ as follows:

$$[x] \rightarrow [y] = [x \rightarrow y] \text{ and } [x] \rightsquigarrow [y] = [x \rightsquigarrow y].$$

It is routine to prove that two operations, "$\rightarrow$" and "$\rightsquigarrow$" are well-defined.

Theorem 2.12. *If X is distributive and "$\sim$" is a right congruence relation, then $(\frac{X}{\sim}, \rightarrow, \rightsquigarrow, [1])$ is a distributive BI-algebra.*

Proof. Assume "$\sim$" be a right congruence relation. By Theorem 2.11(iii), "$\sim$" is a congruence relation and so it is routine to prove that two operations, "$\to$" and "$\rightsquigarrow$" on $\frac{X}{\sim}$ are well-defined. Let $x, y, z \in X$. Then

$$[x] \to [x] = [x \to x] = [1],$$
$$([x] \to [y]) \rightsquigarrow [x] = [x \to y] \rightsquigarrow [x] = [(x \to y) \rightsquigarrow x] = [x],$$

$$\begin{aligned} [z] \to ([x] \rightsquigarrow [y]) &= [z] \to ([x \rightsquigarrow y]) = [z \to (x \rightsquigarrow y)] \\ &= [(z \to x) \rightsquigarrow (z \to y)] = [z \to x] \rightsquigarrow [z \to y] \\ &= ([z] \to [x]) \rightsquigarrow ([z] \to [x]). \end{aligned}$$

Then $(\frac{X}{\sim}, \to, \rightsquigarrow, [1])$ is a distributive pseudo-BI-algebra. By Theorem 2.8(d_{18}), we get $(x \to y) \leq (x \rightsquigarrow y)$ and $(x \rightsquigarrow y) \leq (x \to y)$. Hence $[x \to y] = [x \rightsquigarrow y]$ and so $[x] \to [y] = [x] \rightsquigarrow [y]$. It follows that $(\frac{X}{\sim}, \to, [1])$ is a BI-algebra. $\square$

Example 2.13. *Consider Example 2.7(i). Then*

$$[a] = \{a, d\}, \quad [b] = \{b, c\}, \quad [1] = \{1\}.$$

We can see that $(\frac{X}{\sim}, \to, \rightsquigarrow, [1])$ *is a distributive BI-algebra with Cayley Table 8.*

Table 8: distributive BI-algebra $(\frac{X}{\sim}, \to, \rightsquigarrow, [1])$

$\to = \rightsquigarrow$	$[a]$	$[b]$	$[1]$
$[a]$	$[1]$	$[b]$	$[1]$
[b]	$[a]$	$[1]$	$[1]$
$[1]$	$[a]$	$[b]$	$[1]$

Lemma 2.14. *Let X be proper and $x \to y \neq x \rightsquigarrow y$, for some $x, y \in X$. Then*

(i) *if $x \to y = y$ or $x \rightsquigarrow y = y$, then $y \to x \neq y \rightsquigarrow x$,*

(ii) *if $x \to y = y$ or $x \rightsquigarrow y = y$, then x and y are incomparable,*

(iii) *if $x \to y = y$, then $y \to x \neq x$,*

(iv) *if $x \rightsquigarrow y = y$, then $y \rightsquigarrow x \neq x$,*

(v) *if $x \to y = z$, then $z \to x \neq x$ and $x \rightsquigarrow z \neq z$,*

(vi) *if $x \rightsquigarrow y = z$, then $z \rightsquigarrow x \neq x$ and $x \to z \neq z$.*

Proof. (i) Suppose $x \to y = y$. Using (P_6), we obtain $y \rightsquigarrow x = x$. On the contrary, if $y \to x = y \rightsquigarrow x = x$, then by ($P_5$), $x \rightsquigarrow y = y = x \to y$, which is a contradiction. Hence $y \to x \neq y \rightsquigarrow x$.

The proofs (ii), (iii), and (iv) are obvious by (i).

(v) Assume $x \to y \neq x \rightsquigarrow y$ and $x \to y = z$. Thus, $x \rightsquigarrow y \neq z$. Using ($P_5$), we get $z \to x \neq x$ and $x \rightsquigarrow z \neq z$.

(vi) The proof is similar to the proof (v). $\square$

Example 2.15. *According to Table 7, one can see that* $a \to b \neq a \rightsquigarrow b$, $a \to b = b$, *a and b are incomparable,* $b \to a \neq b \rightsquigarrow a$ *and* $b \to a \neq a$. *Also,* $c \to d \neq c \rightsquigarrow d$, $c \rightsquigarrow d = d$ *and* $d \rightsquigarrow c \neq c$. $b \to a = d$, $d \to b = c \neq b$ *and* $b \rightsquigarrow a = d \neq a$. $a \rightsquigarrow b = c$, $c \rightsquigarrow a \neq a$ *and* $a \to c \neq c$.

Corollary 2.16. *There is no proper pseudo-BI-algebra X, where* $|X| < 4$.

Proof. Assume $(X, \to, \rightsquigarrow, 1)$ is a proper pseudo-BI-algebra. Thus, for some $x, y \in X$, $x \to y \neq x \rightsquigarrow y$. By (pB), ($P_2$), and ($P_1$), $x \to x = x \rightsquigarrow x$, $1 \to x = 1 \rightsquigarrow x$, $x \to 1 = x \rightsquigarrow 1$. Hence $x, y \neq 1$ and $x \neq y$. Let $x \to y = z$, for some $z \in X$. By (P_{11}), we get that $z \neq x$. If $z = y$, then since $x \rightsquigarrow y \neq x \to y$, consequently $x \rightsquigarrow y \neq y$, and so $x \rightsquigarrow y = z$ and $z \neq 1$, $z \neq x$, $z \neq y$ (Using (pI_4), $z \neq 1$, using (P_4), $z \neq x$). Therefore, at least four distinct elements $1, x, y, z \in X$ and $|X| \geq 4$. □

Theorem 2.17. *Let $X = \{x, y, z, 1\}$ be proper. Then*

(i) $x \to y \neq x \rightsquigarrow y$ *implies* $x \to y = y$ *or* $x \rightsquigarrow y = y$,

(ii) $x \to y \neq x \rightsquigarrow y$ *implies* $x \to y = z$ *or* $x \rightsquigarrow y = z$,

(iii) $x \to y \neq x \rightsquigarrow y$ *implies* ($x \to y = y$ *and* $x \rightsquigarrow y = z$) *or* ($x \to y = z$ *and* $x \rightsquigarrow y = y$),

(iv) $x \to y \neq x \rightsquigarrow y$ *implies* $y \to x \neq y \rightsquigarrow x$,

(v) *every two distinct elements of X are incomparable,*

(vi) $(X, \leq, 1)$ *is a singular poset,*

(vii) *X is not distributive.*

Proof. Let $X = \{x, y, z, 1\}$.

Assume $x \to y \neq x \rightsquigarrow y$ and $x \to y \neq y$. Then by (pI_4), we have $x \to y \neq 1$ and by (P_4), we get $x \to y \neq x$. This shows that $x \to y = z$. Similarly, $x \rightsquigarrow y = y$.
If $x \to y \neq x \rightsquigarrow y$ and $x \to y = y$, then by (pI_4), $x \rightsquigarrow y \neq 1$, and so $x \rightsquigarrow y \neq y$. Now, by ($P_4$), we have $x \rightsquigarrow y \neq x$. Hence $x \rightsquigarrow y = z$. Then (i), (ii), and (iii) are valid.

(iv) By Lemma 2.14(i), and (i) the proof is clear.

(v), and (vi) Since X is proper, we get $x \to y \neq x \rightsquigarrow y$, for some $x, y \in X$. Using (iv), we get $y \to x \neq y \rightsquigarrow x$. It means that $x \to y \neq 1$ and $y \to x \neq 1$. By (iii), ($x \to y = y$ and $x \rightsquigarrow y = z$) or ($x \to y = z$ and $x \rightsquigarrow y = y$). Thus, (v), and (vi) are valid.
If $x \to y = y$ and $x \rightsquigarrow y = z$. Using (P_6), we get $z \to x = x$. By Lemma 2.14(ii), z and x are incomparable, and so $z \to x \neq 1$. By a similar argument, by $y \to x \neq y \rightsquigarrow x$ and (ii), $y \to x = z$ or $y \rightsquigarrow x = z$. Let $y \to x = z$. Then by (P_6), $z \rightsquigarrow y = y$. By Lemma 2.14(ii), z and y are incomparable. Similarly, $y \rightsquigarrow x = z$, implies $z \to y \neq 1$ and $y \to z \neq 1$.
Similarly, if $x \to y = z$ and $x \rightsquigarrow y = y$, then x and y and z are incomparable.

(vii) By the contrary, let X be left distributive. Without loss of generality, let $x \to y \neq x \rightsquigarrow y$. By (iii), ($x \to y = y$ and $x \rightsquigarrow y = z$) or ($x \to y = z$ and $x \rightsquigarrow y = y$). If $x \to y = y$ and $x \rightsquigarrow y = z$, then by (P_5), and (P_6), $x \rightsquigarrow z = z$ and $y \rightsquigarrow x = x$. By (iv), $y \to x \neq y \rightsquigarrow x = x$. By (iii), $y \to x = z$. Again by (P_6), $y \to z = z$. By assumption, we have:

$$x \rightsquigarrow (y \to z) = x \rightsquigarrow z = z \neq 1 = z \to z = (x \rightsquigarrow y) \to (x \rightsquigarrow z),$$

which is a contradiction. □

The following example shows that Theorem 2.17 may not be true for pseudo-BI-algebras of order more than 4.

Example 2.18. (i) *Let* $X = \{a, b, c, d, 1\}$ *with Cayley Table 9. Then* $(X, \to, \rightsquigarrow, 1)$ *is a pseudo-BI-algebra and* $d \to b \neq d \rightsquigarrow b$, *but* $b \to d = b \rightsquigarrow d$. *Thus Theorem 2.14(iv) does not satisfy for pseudo-BI-algebra of order more than 4. Also,* $b \to d = 1$, *which means* b *and* d *are comparable. Since* $b \to c = 1$ *and* $c \to b = 1$, *"$\leq$" is not anti-symmetric and so* X *is not a poset.*

Table 9: pseudo-BI-algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	d	1	$\rightsquigarrow$	a	b	c	d	1
a	1	1	1	d	1	a	1	1	1	d	1
b	1	1	1	1	1	b	1	1	1	1	1
c	1	1	1	d	1	c	1	1	1	d	1
d	a	c	c	1	1	d	a	a	c	1	1
1	a	b	c	d	1	1	a	b	c	d	1

(ii) *According Example 2.7(i), it was shown that there exist distributive pseudo-BI-algebra of order 5.*

By Theorem 2.17(v), we have:

Corollary 2.19. *Every pseudo-BI-algebra of order 4 is a pseudo-BH-algebra.*

Example 2.20. (i) *Let* $X = \{a, b, c, d, 1\}$ *with Cayley Table 10.*

Table 10: pseudo-BI-algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	d	1	$\rightsquigarrow$	a	b	c	d	1
a	1	1	1	d	1	a	1	1	1	d	1
b	1	1	1	1	1	b	1	1	1	1	1
c	1	1	1	d	1	c	1	1	1	d	1
d	a	c	c	1	1	d	a	a	c	1	1
1	a	b	c	d	1	1	a	b	c	d	1

Then $(X, \to, \rightsquigarrow, 1)$ *is a pseudo-BI-algebra, but not a pseudo-BH-algebra, since by* $a \to b = b \rightsquigarrow a = 1$ *and* $a \neq b$, X *does not satisfy* (pI_3).

(ii) *According Table 1 in Example 1.4,* $X = \{a, b, c, 1\}$ *is a dual pseudo-BH-algebra of order 4. Since* $(b \rightsquigarrow c) \to b = a \to b = a \neq b$, X *is not a pseudo-BI-algebra.*

3 Exchangeable and commutative pseudo-BI-algebras

We introduce the class of exchangeable pseudo-BI-algebras, which is a proper subclass of the class pseudo-CI-algebras and investigate its properties,

Definition 3.1. *A pseudo-BI-algebra* X *is exchangeable if it satisfies* (pI_2).

Example 3.2. (i) *Let* $X = \{a, b, c, d, 1\}$ *with Cayley Table 11. Then* $(X, \to, \rightsquigarrow, 1)$ *is an exchangeable pseudo-BI-algebra.*

(ii) *Let* $X = \{a, b, c, d, 1\}$ *with Cayley Table 2. Then* X *is not exchangeable since*

$$a \rightsquigarrow (b \to c) = a \rightsquigarrow c = b \neq 1 = b \to b = b \to (a \rightsquigarrow c).$$

Table 11: exchangeable pseudo-BI-algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	d	1	$\rightsquigarrow$	a	b	c	d	1
a	1	d	1	d	1	a	1	b	1	d	1
b	a	1	c	1	1	b	c	1	c	1	1
c	1	b	1	d	1	c	1	b	1	d	1
d	a	1	c	1	1	d	a	1	c	1	1
1	a	b	c	d	1	1	a	b	c	d	1

Theorem 3.3. *Every exchangeable pseudo-BI-algebra is a pseudo-BE-algebra.*

Proof. Assume X is exchangeable. By (P_2) the proof is clear. □

Corollary 3.4. *Every exchangeable pseudo-BI-algebra is a pseudo-CI-algebra.*

The following example shows that the converse of Theorem 3.3 may not be generally true.

Example 3.5. *Let $X = \{a, b, c, d, 1\}$ with Cayley Table 12. Then $(X, \to, \rightsquigarrow, 1)$ is a pseudo-BE-algebra. Since $(d \to c) \rightsquigarrow d = b \rightsquigarrow d = 1 \neq d$, X does not satisfy (pBI), and is not a pseudo-BI-algebra.*

Table 12: pseudo-BE-algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	d	1	$\rightsquigarrow$	a	b	c	d	1
a	1	1	1	1	1	a	1	1	1	1	1
b	1	1	1	1	1	b	1	1	1	1	1
c	1	1	1	1	1	c	1	1	1	1	1
d	1	1	b	1	1	d	1	1	d	1	1
1	a	b	c	d	1	1	a	b	c	d	1

Theorem 3.6. *Let X be exchangeable. The following hold: for all $x, y, z \in X$*

(E_1) $x \leq y \to x$ *and* $x \leq y \rightsquigarrow x$,

(E_2) $x \leq (x \to y) \rightsquigarrow y$ *and* $x \leq (x \rightsquigarrow y) \to y$,

(E_3) *if* $x \leq y \to z$, *then* $y \leq x \rightsquigarrow z$,

(E_4) *if* $x \leq y \rightsquigarrow z$, *then* $y \leq x \to z$,

(E_5) *if* $x \leq y$, *then* $x \leq z \to y$ *and* $x \leq z \rightsquigarrow y$,

(E_6) *if* $x \to y = z$ *or* $x \rightsquigarrow y = z$, *then* $y \to z = y \rightsquigarrow z = 1$,

(E_7) *if* $x \to y = y$ *and* $x \rightsquigarrow y = z$, *then* $x \to z = z$ *and*

$$x \to (y \to z) = (x \to y) \to (x \to z) = x \to (y \rightsquigarrow z) = (x \to y) \rightsquigarrow (x \to z) = 1,$$

(E_8) *if* $x \to y = z$ *and* $x \rightsquigarrow y = t$, *then* $x \rightsquigarrow z = x \to t$,

(E_9) *if* $(x \to y) \to x = x$, *then* $(x \to y) \to (y \rightsquigarrow x) = y \rightsquigarrow x$,

(E_{10}) *if* $(x \rightsquigarrow y) \rightsquigarrow x = x$, *then* $(x \rightsquigarrow y) \rightsquigarrow (y \to x) = y \to x$.

Proof. Assume X is exchangeable and $x, y, z \in X$.

By Theorem 3.3(i), X is a pseudo-BE-algebra, and so using Theorem 1.8, (E_1) - (E_8) are valid.

(E_9) Let $(x \to y) \to x = x$. Applying (pI_2), and assumption, we get

$$(x \to y) \to (y \rightsquigarrow x) = y \rightsquigarrow ((x \to y) \to x) = y \rightsquigarrow x.$$

(E_{10}) The proof is similar to (E_9). □

Definition 3.7. *A pseudo-BI-algebra* $(X, \to, \rightsquigarrow, 1)$ *is called weak commutative if it satisfies the following axiom: for all* $x, y \in X$,

(pI_9) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$.

A weak commutative pseudo-BI-algebra is called commutative if it satisfies the following axiom:

(pI_{10}) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$.

The following example shows that the class of (weak) commutative pseudo-BI-algebras are subclasses the class of pseudo-BI-algebras and different from the class of the exchangeable pseudo-BI-algebras.

Example 3.8. *Let* $X = \{a, b, c, d, 1\}$.

(i) *Then* X *with Cayley Table 13 is a weak commutative pseudo-BI-algebra. Since*

$$(a \rightsquigarrow b) \to b = c \to b = a \neq 1 = a \to a = (b \rightsquigarrow a) \to a,$$

X *is not commutative.*

Table 13: weak commutative pseudo-BI-algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	d	1	$\rightsquigarrow$	a	b	c	d	1
a	1	b	c	c	1	a	1	c	c	c	1
b	d	1	1	d	1	b	a	1	1	d	1
c	a	a	1	a	1	c	a	a	1	a	1
d	1	b	1	1	1	d	1	b	1	1	1
1	a	b	c	d	1	1	a	b	c	d	1

(ii) *Let* $X = \{a, b, c, d, 1\}$ *with Cayley Table 14. Then* $(X, \to, \rightsquigarrow, 1)$ *is a commutative pseudo-BI-algebra.*

Table 14: commutative pseudo-BI-algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	a	b	c	d	1	$\rightsquigarrow$	a	b	c	d	1
a	1	b	c	b	1	a	1	b	c	c	1
b	a	1	c	c	1	b	a	1	c	a	1
c	a	b	1	a	1	c	a	b	1	b	1
d	1	1	1	1	1	d	1	1	1	1	1
1	a	b	c	d	1	1	a	b	c	d	1

(iii) *Pseudo-BI-algebra* $(X, \rightarrow, \rightsquigarrow, 1)$ *in Example 2.2(ii), is not weak commutative, since*

$$(y_0 \rightarrow x_0) \rightsquigarrow x_0 = z \rightsquigarrow x_0 = x_0 \neq 1 = y_0 \rightsquigarrow y_0 = (x_0 \rightarrow y_0) \rightsquigarrow y_0.$$

(iv) *pseudo-BI-algebra* $(X, \rightarrow, \rightsquigarrow, 1)$ *with Cayley Table 11, is not weak commutative, since*

$$(a \rightarrow c) \rightsquigarrow c = c \neq a = 1 \rightsquigarrow a = (c \rightarrow a) \rightsquigarrow a.$$

Proposition 3.9. *Let X be weak commutative. Then for any $x, y, z \in X$*

(i) *(pI_3) is valid,*

(ii) *$x \leq y$ implies $(y \rightarrow x) \rightsquigarrow x = y$,*

(iii) *if X is commutative and $x \leq y$, then $(y \rightarrow x) \rightsquigarrow x = (y \rightsquigarrow x) \rightarrow x = y$,*

(iv) *if $x \rightarrow y = y \rightarrow x$, then $x = y$,*

(v) *if $x \leq y$ and $y \rightarrow x = z$, then $z \rightsquigarrow x = y$,*

(vi) *if X is commutative, $x \leq y$ and $y \rightsquigarrow x = z$, then $z \rightarrow x = y$,*

(vii) *$\leq$ is antisymmetric.*

Proof. (i) Assume $x \leq y$ and $y \leq x$. Using (p_1), and (p_6), we have

$$x = 1 \rightsquigarrow x = (y \rightarrow x) \rightsquigarrow x = (x \rightarrow y) \rightsquigarrow y = 1 \rightsquigarrow y = y.$$

(ii) Assume $x, y \in X$ and $x \leq y$. Using (p_1), and (p_6), we obtain

$$y = 1 \rightsquigarrow y = (x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x.$$

(iii) Similar to the proof of (ii).
(iv) Let $x \rightarrow y = y \rightarrow x$. Applying (pBI), and (p_6), we get

$$x = (x \rightarrow y) \rightsquigarrow x = (y \rightarrow x) \rightsquigarrow x = (x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow y = y.$$

(v) Let $x \rightarrow y = 1$ and $y \rightarrow x = z$. Using (p_1), and (p_6), we have

$$y = 1 \rightsquigarrow y = (x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x = z \rightsquigarrow x.$$

(vi) Similar to the proof of (iv).
(vii) By (iii), the proof is clear. □

Theorem 3.10. (i) *Every weak commutative pseudo-BI-algebra is a dual pseudo-BH-algebra.*

(ii) *Every exchangeable commutative pseudo-BI-algebra is a commutative pseudo-BE-algebra and commutative pseudo-CI-algebra.*

(iii) *Every finite exchangeable commutative pseudo-BI-algebra is an implication algebra.*

Proof. (i) By Proposition 3.9(iv) and Definition 1.3, the proof is clear.
(ii) By Theorem 3.3 and Corollary 3.4, the proof is clear.
(iii) Let $(X, \rightarrow, \rightsquigarrow, 1)$ be an exchangeable commutative pseudo-BI-algebra. By (ii) and Theorem 1.9, we get $x \rightarrow y = x \rightsquigarrow y$, for all $x, y \in X$. By definition of implication algebra, we can easily conclude that $(X, \rightarrow, 1) = (X, \rightsquigarrow, 1)$ is an implication algebra. □

Remark 3.11. *In 2006, R. A. Borzooei and S. Khosravi Shoar* [6] *showed that the implication algebras are equivalent to the dual implicative BCK-algebra. In 2007, H. S. Kiam and Y. H. Kim* [20] *introduced the notion of a BE-algebra, as a generalization of a BCK-algebra. In 2008, A. Walenziak* [27] *proved that any implication algebra is a BE-algebra. In 2010, B. L. Meng* [22] *showed that every (commutative, self distributive) CI-algebra is a BE-algebra. In 2013, A. Borumand Saeid* [3] *proved that a CI-algebra is a dual Q-algebra. As mentioned and using Theorem 3.10(iii), every exchangeable commutative pseudo-BI-algebra is a dual implicative BCK-algebra, BE-algebra, (commutative, self distributive) CI-algebra, and a dual Q-algebra.*

The following example shows that the converse of Theorem 3.10(i) may be not true in general.

Example 3.12. *Let* $X = \{a, b, c, 1\}$ *with Cayley Table 15. Then* $(X, \to, \rightsquigarrow, 1)$ *is a dual pseudo-BH-algebra, but not a pseudo-BI-algebra, since* $(b \to a) \rightsquigarrow b = a \rightsquigarrow b = 1 \neq b$.

Table 15: dual pseudo-BH-algebra $(X, \to, \rightsquigarrow, 1)$

$\to$	1	a	b	c	$\rightsquigarrow$	1	a	b	c
1	1	a	b	c	1	1	a	b	c
a	1	1	1	1	a	1	1	1	1
b	1	a	1	c	b	1	c	1	c
c	1	b	b	1	c	1	a	b	1

Open Problem 3. Find condition/conditions which a exchangeable weak commutative pseudo-BI-algebra is a commutative pseudo-BCK-algebra (commutative pseudo-BE-algebra), and so commutative pseudo-CI-algebra.

Open Problem 4. Find condition/conditions which a exchangeable weak commutative pseudo-BI-algebra is an implication algebra.

Theorem 3.13. *Let* X *be weak commutative. Then* $(X, \leq)$ *is a chain if and only if* $|X| = 2$.

Proof. Assume $X = \{1, a\}$. Using (p_2), we have $a \to 1 = 1$, and so $a \leq 1$.
Conversely, let $|X| > 2$. Then there are $x, y \in X \setminus \{1\}$ such that $x \neq y$. Since $(X; \leq)$ is a chain, we get $x < y$ or $y < x$. Without the loss of generality, let $x < y$. Applying Proposition 3.9(ii), we obtain $y = (y \to x) \rightsquigarrow x$. Now, consider the following cases:
Case 1. If $y \to x = x$, then $y = (y \to x) \rightsquigarrow x = x \rightsquigarrow x = 1$, which is against the assumption,
Case 2. If $y \to x = y$, which is a contradiction with Theorem 2.5(P_4).
Case 3. If $y \to x = 1$, then $x = y$ by Proposition 3.9(vi).
Case 4. If $y \to x = z \neq 1$, then $z \rightsquigarrow y = y$ and $y \to z = z$, by (P_5). Thus, $z \not\leq y$ and $y \not\leq z$, which is a contradiction. Therefor, $|X| = 2$. □

Corollary 3.4 follows that:

Theorem 3.14. *Every commutative exchangeable pseudo-BI-algebra is a commutative pseudo-CI-algebra.*

Discussion

In 2016, L. C. Ciungu [7] proved that the class of commutative pseudo-CI-algebras is term equivalent to the class of commutative pseudo-BCK-algebras. In 2019, A. Rezaei et al. [24, Theorem 3.10] proved that commutative pseudo-CI-algebras are categorically isomorphic to commutative pseudo-BE-algebras. L. C. Ciungu [7, Proposition 3.9] proved that every commutative pseudo-BE-algebra $(X, \rightarrow, \rightsquigarrow, 1)$ is a join-semilattice, where

$$x \vee y = (x \rightarrow y) \rightsquigarrow y = (x \rightsquigarrow y) \rightarrow y.$$

By Theorem 3.10(ii), it follows that every commutative exchangeable pseudo-BI-algebra $(X, \rightarrow, \rightsquigarrow, 1)$ is a join-semilattice, also by Theorem 3.10(iii), is an implication algebra. Georgescu et al. [10, 13] proved that pseudo MV-algebras are categorically isomorphic to bounded commutative pseudo-BCK-algebras and so by Theorem 3.14, these are categorically isomorphic to bounded commutative exchangeable pseudo-BI-algebra. Furthermore, L. C. Ciungu [7, Proposition 3.7] proved that there are no proper finite commutative pseudo-BE-algebras. Hence, there are no proper finite commutative exchangeable pseudo-BI-algebras. In 2010, X. Zhang and H. Gong, discussed 1-type implicative pseudo-BCK-algebra $(X, \rightarrow, \rightsquigarrow, 1)$, (i.e. A pseudo-BCK-algebra with the following axiom: for all $x, y \in X$

(pI$_1$) $(x \rightarrow y) \rightarrow x = x = (x \rightsquigarrow y) \rightsquigarrow x,$

and proved that there is no proper 1-type implicative pseudo-BCK-algebra (for detail see [30, Theorem 2.3]). Also, they considered 2-type implicative pseudo-BCK-algebra $(X, \rightarrow, \rightsquigarrow, 1)$, (i.e. A pseudo-BCK-algebra with the following axiom: for all $x, y \in X$

(pBI) $(x \rightarrow y) \rightsquigarrow x = x = (x \rightsquigarrow y) \rightarrow x,$

and proved that there are no proper 2-type implicative pseudo-BCK-algebra (for detail see [30, Theorem 2.7]).

Now, consider algebra $(X, \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ with (pB), (pI$_4$) and (pBI), we have:

(i) $1 \rightarrow x = 1 \rightsquigarrow x = x,$

(ii) $x \rightarrow 1 = x \rightsquigarrow 1 = 1,$

(iii) $x \rightarrow (x \rightarrow y) = x \rightarrow y$ and $x \rightsquigarrow (x \rightsquigarrow y) = x \rightsquigarrow y,$

(iv) $(x \rightarrow (x \rightarrow y)) \rightarrow x = x$ and $(x \rightsquigarrow (x \rightsquigarrow y)) \rightsquigarrow x = x,$

(v) if $(X, \rightarrow)$ or $(X, \rightsquigarrow)$ is a right zero semigroup (i.e. $x \rightarrow y = x$ or $x \rightsquigarrow y = x$, for all $x, y \in X$), then $X = \{1\}$,

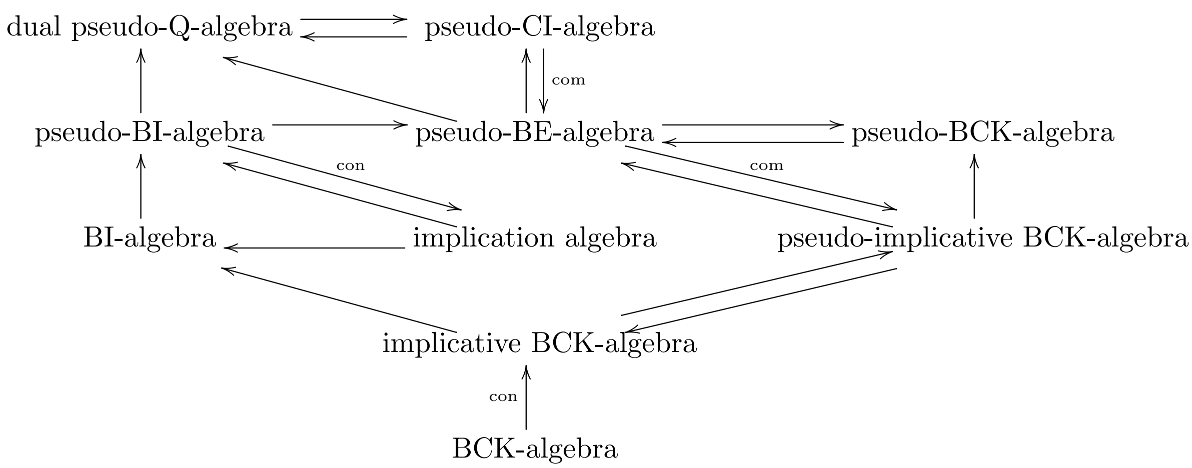
(vi) if satisfies (pI$_2$), then it is a pseudo-BE-algebra,

(vii) if satisfies (pI$_2$), (pI$_9$), and (pI$_{10}$), then it is a commutative pseudo-BE-algebra (by [7, Theorem 3.3] is a commutative pseudo-BCK-algebra),

(viii) if satisfies (pI$_2$), (pI$_9$), and (pI$_{10}$), then $x \rightarrow y = x \rightsquigarrow y$, for all $x, y \in X$ (by [30, Theorem 2.2(15)]), and is an implication algebra.

4 Conclusions

We discussed (distributive, exchangeable, (weak) commutative) dual pseudo-BI-algebras (briefly, pseudo-BI-algebras) as new algebraic structures and the properties of these structures are investigated. It is given some characterizations of pseudo-BI-algebras and shown that their class is a proper subclass of the class pseudo-CI-algebras. The notions of (*weak*) *commutative* pseudo-BI-algebras are defined and proved every weak commutative pseudo-BI-algebra is a (dual) pseudo-BH-algebra that every exchangeable commutative pseudo-BI-algebra is an implication algebra. Now, in the following diagram we summarize the results of this paper and the past results in this filed and give the relations among (weak commutative, exchangeable) pseudo-BI-algebras, implicative pseudo-BCK-algebras, pseudo-BE-algebras, pseudo-BCH-algebras, pseudo-BH-algebras. The mark $\mathrm{A} \longrightarrow \mathrm{B}$ ($\mathrm{A} \xrightarrow{\text{ex}} \mathrm{B}$) means that A implies B (respectively, A concludes B with condition "*example*" for briefly "*ex*").



In the future, one can define and study modal and state operators on (distributive, exchangeable, (weak) commutative) pseudo-BI-algebras. Fuzzy and neutrosophic filters could be also introduced and investigated. As another direction of research, one can investigate the relationship between pseudo-BI-algebras with pseudo-Hoop-algebras, Quantum algebras and pseudo-L-algebras to get exciting properties.

Conflict of interesting

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