Journal of Algebraic Hyperstructures and Logical Algebras

Volume 4, Number 2, (2023), pp. 131-149

Zero divisor graphs based on general hyperrings

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Abstract

This paper introduces the concepts of reproduced general hyperring and valued-orderable general hyperring and investigates some properties of these classes of general hyperrings. It presents the notions of zero divisors and zero divisor graphs, which are founded on the absorbing elements of general hyperrings. General hyperrings can have more than one zeroing element, and therefore, based on the zeroing elements, multiple zero divisors can be obtained. In this study, we discuss the isomorphism of zero divisor graphs based on the diversity of divisors of zero divisors. The non-empty intersection of the set of absorbing elements and the hyperproduct of zero divisors of general hyperrings play a significant role in producing zero divisor graphs. Indeed it investigated a type classification of zero divisor graphs based on the finite general hyperrings. We discuss the finite reproduced general hyperrings, investigate their zero divisor graphs, and show that an infinite reproduced general hyperring can have a finite zero divisor graph.

Article Information

Corresponding Author: M. Hamidi;
Received: November 2023;
Revised: December 2023;
Accepted: December 2023;
Paper type: Original.

Keywords:

(Reproduced) General hyperring, $(\alpha, 2)$-orderable general hyperring, zero divisor graph, hyperaddative absorbing-elements.

1 Introduction

The theory of Krasner hyperring as a generalization of the ring is a hyperstructure introduced by Krasner as a hyperstructure endowed with a hyperaddition and a multiplication operation that is distributive over the hyperaddition. Superring, another hyperstructure was introduced by Mittas having both additive and multiplicative hyperstructures in 1973 [16]. Later on, Vougiouklis [22] generalized the concept of superrings, introducing the hyperrings in the general sense, where addition and multiplication are hyperoperations but only the weak version of distributivity. The theory of general hyperrings is vital in real-world problems because it relates to the set of objects

https://doi.org/10.61838/KMAN.JAHLA.4.2.9

in different clustering via the systematic hyperoperations[10]. The combination of graphs and rings was interesting for many researchers and investigated. In mathematics, and more specifically in combinatorial commutative algebra, a zero-divisor graph is an undirected graph representing the zero-divisors of a commutative ring. It has elements of the ring as its vertices and pairs of elements whose product is zero as its edges. Let R be a commutative ring with 1, and let $Z(R)$ be its set of zero-divisors. It is associated a simple graph $\Gamma(R)$ to R with vertices $Z^*(R) = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z^*(R)$, the vertices x and y are adjacent if and only if $xy = 0$. The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in [6], where he was mainly interested in colourings, and this idea in a commutative ring was then continued by D. D. Anderson and M. Naseer in [7]. Later, some researchers investigated the zero divisor graphs based on the ring structures, especially zero divisor graphs based on ideals in rings, and gave important results in this regard. Some updated papers with titled graph of the ring based on the zero divisors published in several versions such as the n-zero divisor graph of a commutative semigroup[5], zero-one laws for random k-partite graphs [17], graphs derived from multirings [13], the distant graph of the projective line over a finite ring with unity [9], spectra and topological indices of comaximal graph of $\mathbb{Z}_n$ [8], on derivable tree [12], on the extended zero-divisor graph of strictly partial transformation semigroup [14], the Cayley graph of Neumann subgroups [15], zero-divisor graph of a ring with respect to an automorphism [18], frequency assignment model of zero divisor graph [19], implementation of single-valued neutrosophic soft hypergraphs on human nervous system [3], intuitionistic fuzzy hypergraphs with applications [2], intuitionistic fuzzy left k-ideals of semirings [1] and the Wiener index of the zero-divisor graph of a finite commutative ring with unity [21]. Hamidi et al. introduced the graphs based on the hyperideal of special hyperrings as an extension of graphs based on the ideals of rings. Indeed, they presented graphs based on an ideal of rings to intersection graphs based on the hyperideal of hyperrings for modelling real problems by absorbent elements of hyperrings [11].

Motivation and Advantage: In algebraic structures such as groups and rings, only two elements can be connected to one element under one or more operations at a time. However, if we want to connect two elements with more than one element simultaneously, it is necessary to extend this theory to hyperstructures such as hypergroups or hyperrings. In algebraic hyperstructures, regardless of the number of hyperoperations, we can have a more comprehensive relationship between elements, and this is an essential advantage in modeling data in the real world. On the other hand, graph theory has great importance and application in the real world and all sciences, and the extraction of graphs from regular structures adds to the importance of these sciences. Due to the limitation of the theory of rings in the simultaneous connection of more than three elements and its development to the theory of hyperrings, our motivation in extracting graphs has been created in this research. In mathematical modeling, we first convert the existing data into a rule-mod system such as general hyperrings so that there are no restrictions on the relationship of elements, and we extract graphs according to general hyperrings. Few works have been done in extracting graphs based on hyperrings, and in this research, we are dealing with graphs with zero-divisors based on general hyperrings. One of the most important advantages of this research is that the general hyperrings of our context are general, and there is no limit on the number of zeros of general hyperrings, which is more consistent with the reality of our world. Since a general hyperring can have more than one zero-divisor, we can have a variety of graphs with a zero-divisor base and even deal with the relationship between these types of graphs. In fact, for both distinct zeros in general hyperrings, we have two types of graphs with a divisible base of zeros, and considering the relationship between these two distinct zeros, we can discuss the relationship between their graphs. Another advantage of this research in the type of graphs with zero divisors is that in the case of limiting general hyperrings to other hyperrings and even rings, the same results are obtained. The aim of this research is the bring forward graphs based on zero divisors from rings to special hyperrings, as a generalized problem in hyperstructures and graphs. We have received our primary motivation for this work from problems related to graphs and algebraic structures, such as zero divisor graph based on the group, graph derived from modules, and extracted graph from ring structures. Therefore, in line with this goal, the absorbing elements concerning hyperstructures equipping hyperoperations, are considered and the properties of zero divisors via absorbing elements in general hyperrings are analyzed. We show that for any given general hyperring, we can construct some (non) isomorphic zero divisor graphs via the zero divisors that are related to hyperadditive absorbing elements. Also presented is an infinite general hyperring such that its zero divisor graph is finite. Also, we inset a class of finite hyperrings and analyzed their zero divisor graphs concerning divisors of the order of their elements. We proved that for any given infinite general hyperrings, under special conditions, such as the type of its hyperoperations or the type of zeroing elements, finite or infinite graphs can be extracted, and these graphs can be complete, Eulerian, multipartite, or have any other special characteristics.

2 Preliminaries

In this section, we introduce the important and preliminary materials and concepts that we need in our research from [4, 10, 20].

A *simple graph* G is a finite non-empty set V of objects called vertices (the singular is vertex) together with a set E of 2-element subsets of V called edges and is showed with $G = (V = V(G) = \{v_i\}_{i=1}^{n}, E = E(G) = \{e_i\}_{i=1}^{m})$. Two vertices $u, v \in V$ of a graph G are *adjacent* if there is an edge $e = \{u, v\}$ joining them and a simple graph in which each pair of distinct vertices are adjacent is a *complete* graph and denote the complete graph on n vertices by K_n. A *path* in G is a finite distinct sequence of vertices and edges of the form $x_0, e_1, x_1, e_2, x_2, \ldots, x_{n-1}, e_n, x_n$ and it is called a *cycle*, if $x_0 = x_n$. A graph is *connected* if and only if there is a path between each pair of vertices and is a *bipartite* graph if and only if each cycle of G has an even length. A graph is called *disconnected*, if it is not connected and the number of *connected components* of graph G is denoted by $t(G)$. For the simple graph G and $x, y \in V(G)$, $d(x, y)$ is defined as the *length* of a shortest path from x to y in G ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $diam(G) = sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$ and the *girth* of G, denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycles). Two graphs G and H are *isomorphic* (have the same structure) if there exists a bijective function $\varphi : V(G) \longrightarrow V(H)$ such that two vertices u and v are adjacent in G if and only if $\varphi(u)$ and $\varphi(v)$ are adjacent in H. The function φ is then called an isomorphism and G and H are isomorphic, we write $G \cong H$.

Let R be a non-empty set and $\mathcal{P}^*(R) = \{S \mid \emptyset \neq S \subseteq R\}$. Every map $\varphi : R \times R \longrightarrow \mathcal{P}^*(R)$, has been named as a *hyperoperation*, a *hyperstructure* (R, φ) is called a *hypergroupoid* and for all non-empty subsets A, B of R, $\varphi(A, B) = \bigcup\limits_{a \in A, b \in B} (\varphi(a, b))$(we identified the set $\{x\}$ with to x, so $\varphi(x, B) = \varphi(\{x\}, B)$, where $x \in R$). A hypergroupoid (R, φ) is called a *semihypergroup*, if for all $x, y, z \in R, \varphi(\varphi(x, y), z) = \varphi(x, \varphi(y, z))$ and a semihypergroup (R, φ) is called a *hypergroup*, if for all $x \in R, \varphi(x, R) = \varphi(R, x) = R$(*reproduction axiom*). A hypergroup (R, φ, θ) is called commutative, if for any $x, y \in R, \varphi(x, y) = \varphi(y, x)$. A *general hyperring* is a hyperstructure (R, φ, θ), where

(i) (R, φ) is a hypergroup,

(ii) (R, θ) is a semihypergroup and (*iii*) for any $x, y, z \in R$, $\theta(x, \varphi(y, z)) \subseteq \varphi(\theta(x, y), \theta(x, z))$ and $\theta(\varphi(x, y), z) \subseteq \varphi(\theta(x, z), \theta(y, z))$. A general hyperring (R, φ, θ) is called *commutative* (with unit element), if for all $x, y \in R$, $\theta(x, y) = \theta(y, x)$ (if there exists an element $1 \in R$ in such a way that for all $x \in R, \theta(1, x) = \theta(x, 1) = \{x\}$). A non-empty subset I of R is called a (*right*)*left hyperideal*, if $(1)(I, \varphi)$ is a subhypergroup of (R, φ) and $(2)(\theta(R, I) \subseteq I)(\theta(R, I) \subseteq I$. A hyperideal I is both a left and right hyperideal.

From now on, we use the symbols $\oplus$ and $\odot$ instead of the symbols φ and θ, respectively as hyperaddition and hypermultiplication($\varphi(x, y) = \oplus(x, y) = x \oplus y$ and $\theta(x, y) = \odot(x, y) = x \odot y$).

Theorem 2.1. [4] *Assume $p \neq 2$ is a prime number. Then $(\mathbb{Z}_p, +, \cdot)$ is a $(\oplus, \odot)$-reproduced general hyperring.*

3 Graphs based on general hyperrings

In this section, we consider an arbitrary general hyperring and introduce the concepts of zero divisor elements and zero divisor graph based on zero divisor elements via the absorbing elements. Also, we prove the conditions of how two zero divisor graphs are isomorphic.

Definition 3.1. *Let $(R, +, \cdot)$ be a ring. Then R is said to be a $(\oplus, \odot)$-reproduced general hyperring (reproduced general hyperring), if there are hyperoperations "$\oplus$" and "$\odot$", that $(R, \oplus, \odot)$ is a general hyperring and $\oplus, \odot$ are dependent to $+$ and $\cdot$, respectively (it means that for any $x, y \in R, x + y \in x \oplus y$ and $x \cdot y \in x \odot y$).*

In [4], Hamidi et al. constructed a type of general hyperring as the following theorem.

Theorem 3.2. *Let $(R, +, 0, \cdot)$ be a ring and $\mid R \mid \geq 2$. Then $(R, +, 0, \cdot)$ is a $(\oplus, \odot)$-reproduced general hyperring, where $\oplus(x, y) = \{x, y, x + y\}$ and $\odot(x, y) = \{x \cdot y, 0\}$.*

Corollary 3.3. *Any ring is a reproduced general hyperring.*

Definition 3.4. *Let $(R, \oplus, \odot)$ be a general hyperring. We say that*

(i) *$\alpha \in R$ is a $(\oplus)$-absorbing element of R, if for all $r \in R, r \in \oplus(\alpha, r) \cap \oplus(r, \alpha)$;*

(ii) *$\alpha \in R$ is a $(\odot)$-absorbing element of R, if for all $r \in R$, $\alpha \in \odot(\alpha, r) \cap \odot(r, \alpha)$;*

(iii) *$\alpha \in R$ is an absorbing element of R, if it is both $(\oplus)$-absorbing element and $(\odot)$-absorbing element of R.*

From now on, we set the all $(\oplus)$-absorbing elements of R by $\mathcal{O}_R^{(\oplus)}$, all $(\odot)$-absorbing elements of R by $\mathcal{O}_R^{(\odot)}$ and absorbing elements of general hyperring R by $\mathcal{O}_R$. It is clear that $\mathcal{O}_R = \mathcal{O}_R^{(\oplus)} \cap \mathcal{O}_R^{(\odot)}$.

Definition 3.5. *Assume $(R, \oplus, \odot)$ is a general hyperring and $\alpha \in \mathcal{O}_R^{(\oplus)}$. Then*

(i) *an element $x \in R \setminus \{\alpha\}$ is called a zero divisor, if there exists $y \in R \setminus \{\alpha\}$ such that $\alpha \in x \odot y$ and $\alpha \in y \odot x$. We will denote the set of all zero divisors by $Z^{(\alpha)}(R)$.*

(ii) *The zero divisor graph is a simple graph $G^{(\alpha)}(R) = (V = V(Z^{(\alpha)}(R) \setminus \{\alpha\}), E = E(G^{(\alpha)}(R))$ in such a way that for any given distinct $x, y \in Z^{(\alpha)}(R) \setminus \{\alpha\}$, the vertices x and y are adjacent if and only if $\mathcal{O}_R^{(\oplus)} \cap \odot(x, y) \cap \odot(y, x) \neq \emptyset$.*

Let $(R, \oplus, \odot)$ be a general hyperring. If $\mathcal{O}_R^{(\oplus)} = \emptyset$, then $G^{(\alpha)}(R)$ is not defined, so on all the following general hyperrings R, $\mathcal{O}_R^{(\oplus)} \neq \emptyset$. In the following, we present some examples for clarifying the definition of zero divisor graph of general hyperrings.

$\oplus$	2	4	6	8	10
2	2	$\{2,4\}$	$\{2,6\}$	$\{8,2\}$	$\{2,10\}$
4	$\{2,4\}$	4	$\{4,6\}$	$\{8,4\}$	$\{4,10\}$
6	$\{2,6\}$	$\{4,6\}$	2	$\{8,6\}$	$\{6,10\}$
8	$\{8,2\}$	$\{8,4\}$	$\{8,6\}$	4	$\{8,10\}$
10	$\{2,10\}$	$\{4,10\}$	$\{6,10\}$	$\{8,10\}$	2

$\odot$	2	4	6	8	10
2	4	2	4	2	4
4	2	4	2	4	2
6	4	2	4	2	4
8	2	4	2	4	2
10	4	2	4	2	4

.

Table 1: Hypergroups $(R,\oplus),(R,\odot)$ and general hyperring $(R,\oplus,\odot)$.

Example 3.6. *Let $R=\{2,4,6,8,10\}$. Then $(R,\oplus,\odot)$ is a general hyperring in Table 1. Clearly, $\mathcal{O}_R^{(\oplus)}=\{2,4\}$ and for any $\alpha\in\mathcal{O}_R^{(\oplus)}$, $Z^{(\alpha)}(R)\setminus\{\alpha\}=R\setminus\{\alpha\}$. Hence $G^{(2)}(R)\cong G^{(4)}(R)\cong K_4$ are shown in Figure 1.*

(a) $G^{(2)}(R)$ (b) $G^{(4)}(R)$

Figure 1: Zero divisor graphs $G^{(2)}(R)$ and $G^{(4)}(R)$

Let $m\in\mathbb{N}$ and $(R,\oplus,\odot)$ be a general hyperring. Define for all $x\in R, 1x=\{x\}, mx=\oplus(\underbrace{x,\ldots,x}_{m-times})$, $x^1=\{x\}$ and $x^m=\odot(\underbrace{x,\ldots,x}_{m-times})$.

Theorem 3.7. *Let $(R,\oplus,\odot)$ be a general hyperring and $\alpha\in\mathcal{O}_R^{(\oplus)}$.*

(i) *If $Z^{(\alpha)}(R)\setminus\{\alpha\}\neq\emptyset$, then $|Z^{(\alpha)}(R)\setminus\{\alpha\}|\geq 2$.*

(ii) *If $(R,\odot)$ is a commutative hypergroup and $\alpha\notin\mathcal{O}_R^{(\odot)}$, then $\mid Z^{(\alpha)}(R)\setminus\{\alpha\}\mid\geq 2$.*

Proof. (i) Let $x\in Z^{(\alpha)}(R)\setminus\{\alpha\}$. Then there exists $y\in R\setminus\{\alpha\}$ in such a way that $\alpha\in\odot(x,y)\cap\odot(y,x)$. It follows that $y\in Z^{(\alpha)}(R)\setminus\{\alpha\}$ and so $\mid Z^{(\alpha)}(R)\setminus\{\alpha\}\mid\geq 2$.

(ii) Let $\alpha\in R$. Since $\alpha\notin\mathcal{O}_R^{(\odot)}$, there exists $a,b\in R\setminus\{\alpha\}$ in such a way that $\alpha\in\odot(a,b)\cap\odot(b,a)$, because of reproduction axiom and commutativity of hypergroup R. It follows that $Z^{(\alpha)}(R)\setminus\{\alpha\}\neq\emptyset$, and so by item (i), $\{a,b\}\subseteq Z^{(\alpha)}(R)\setminus\{\alpha\}$. □

Example 3.8. *Let $R=\{0,1,2,3,4,5\}$. Then by Theorem 3.2, $(R,\oplus,\odot)$ is a general hyperring as Table 4. Clearly, $\mathcal{O}_R^{(\oplus)}=R$, $Z^{(0)}(R)\setminus\{0\}=R\setminus\{0\}, Z^{(1)}(R)\setminus\{1\}=Z^{(3)}(R)\setminus\{3\}=Z^{(5)}(R)\setminus\{5\}=\emptyset, Z^{(2)}(R)\setminus\{2\}=\{4,5\}$ and $Z^{(4)}(R)\setminus\{4\}=\{2,5\}$. Hence,*

$G^{(0)}(R)\cong G^{(3)}(R)\cong K_5, G^{(2)}(R)\cong G^{(4)}(R)\cong K_2$. We see that $(R,\odot)$ is not a hypergroup based on the Table 3, and $\mathcal{O}_R^{(\odot)}=\{0\}$, while $\mid Z^{(0)}(R)\setminus\{0\}\mid=\mid R\mid -1$. It shows that the converse of Theorem 3.7, is not necessarily true.

Definition 3.9. *Let $n\in\mathbb{N}$, $(H,\oplus,\odot)$ be a hyperring and $\alpha\in\mathcal{O}_R^{(\oplus)}$. We will call*

$\oplus$	0	1	2	3	4	5
0	0	$\{1,0\}$	$\{0,2\}$	$\{0,3\}$	$\{0,4\}$	$\{0,5\}$
1	$\{0,1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,3,4\}$	$\{1,4,5\}$	$\{1,5,0\}$
2	$\{0,2\}$	$\{1,2,3\}$	$\{2,4\}$	$\{2,3,5\}$	$\{2,4,0\}$	$\{2,5,1\}$
3	$\{0,3\}$	$\{1,3,4\}$	$\{5,2,3\}$	$\{3,0\}$	$\{3,4,1\}$	$\{3,5,2\}$
4	$\{0,4\}$	$\{1,4,5\}$	$\{0,2,4\}$	$\{1,3,4\}$	$\{4,2\}$	$\{4,5,3\}$
5	$\{0,5\}$	$\{1,5,0\}$	$\{1,2,5\}$	$\{5,3,2\}$	$\{5,4,3\}$	$\{5,4\}$

Table 2: $(R,\oplus)$

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\{0,4\}$	$\{5,0\}$
2	0	$\{0,2\}$	$\{0,4\}$	0	$\{2,0\}$	$\{0,4\}$
3	0	$\{0,3\}$	0	$\{3,0\}$	0	$\{3,0\}$
4	0	$\{0,4\}$	$\{0,2\}$	0	$\{4,0\}$	$\{0,2\}$
5	0	$\{5,0\}$	$\{0,4\}$	$\{0,3\}$	$\{0,2\}$	0

Table 3: $(R,\odot)$

Table 4: Hypergroups $(R,\oplus),(R,\odot)$ and general hyperring $(R,\oplus,\odot)$ of order 6.

(i) *$x\in R$ is an $(\alpha,2)$-orderable, if $\alpha\in x^2$.*

(ii) *$(H,\oplus,\odot)$ is a locally $(\alpha,2)$-orderable, if has some $(\alpha,2)$-orderable elements and is an $(\alpha,2)$-orderable, if all its elements are $(\alpha,2)$-orderable elements,*

(iii) *$(H,\oplus,\odot)$ is free of $(\alpha,2)$-orderable, if it is not a locally $(\alpha,2)$-orderable hyperring.*

Example 3.10. (i) *Consider the hyperring $(R,\oplus,\odot)$ in Table 4. One can see that $(R,\oplus,\odot)$ is a $(0,2)$-orderable, while it is a locally $(1,2)$-orderable, a locally $(3,2)$-orderable and a locally $(4,2)$-orderable.*

(ii) *Consider the hyperring $(R,\oplus,\odot)$ in Table 12. One can see that $(R,\oplus,\odot)$ is a $(0,2)$-orderable, while it is a locally $(1,2)$-orderable and a locally $(4,2)$-orderable.*

Theorem 3.11. *Let $(R,\oplus,\odot)$ be a finite general hyperring, $\mathcal{O}_R^{(\oplus)}=R$ and $\alpha\in\mathcal{O}_R^{(\oplus)}\setminus\mathcal{O}_R^{(\odot)}$.*

(i) *If $(R,\odot)$ is a commutative hypergroup and $(R,\oplus,\odot)$ is an $(\alpha,2)$-orderable, then $|Z^{(\alpha)}(R)\setminus\{\alpha\}|=|R|-1$.*

(ii) *If $(R,\odot)$ is a commutative hypergroup and $(R,\oplus,\odot)$ is free of $(\alpha,2)$-orderable, then $G^{(\alpha)}(R)$ is a connected graph.*

Proof. (i) Since $\mathcal{O}_R^{(\oplus)}=R$, $\alpha\in\mathcal{O}_R^{(\oplus)}\setminus\mathcal{O}_R^{(\odot)}$ and $(R,\odot)$ is a finite commutative hypergroup, applying Theorem 3.7 (ii), for all $\alpha\in R$, there exists $x\in R$ in such a way that $x\in Z^{(\alpha)}(R)\setminus\{\alpha\}$. Since $(R,\odot)$ is a finite commutative $(\alpha,2)$-orderable, for any $x\in R\setminus\{\alpha\},x^2\cap Z^{(\alpha)}(R)\setminus\{\alpha\}\neq\emptyset$. It follows that $|Z^{(\alpha)}(R)\setminus\{\alpha\}|=|R|-1$.

(ii) Using Theorem 3.7, $\mid Z^{(\alpha)}(R)\setminus\{\alpha\}\mid\geq 2$. Since $(R,\oplus,\odot)$ is free of $(\alpha,2)$-orderable, for all $x\in H,\alpha\notin x^2$. It follows that there exist $y\neq z\in R\setminus\{x\}$ such that $\alpha\notin\odot(y,z)\cap\odot(z,y)$ and so x is adjacent with other vertices and so $G^{(\alpha)}(R)$ is a connected graph. □

Theorem 3.12. *Let $(R,\oplus,\odot)$ be a finite general hyperring. If $(R,\oplus,\odot)$ is locally $(\alpha,2)$-orderable and $|Z^{(\alpha)}(R)\setminus\{\alpha\}|$ is an odd, then $G^{(\alpha)}(R)$ is a disconnected graph.*

Proof. Let $Z^{(\alpha)}(R)\setminus\{\alpha\}=\{x_1,x_2,\ldots,x_n\}$, which n is an odd. Since $(R,\oplus,\odot)$ is locally $(\alpha,2)$-orderable, there exists a unique $1\leq j\leq n$ such that $\alpha\in x_j^2$. It follows that for any $1\leq j\neq k\leq n$, x_j is not adjacent with x_k and so $G^{(\alpha)}(R)$ is a disconnected graph. □

In what follows, for any $k\leq n\in\mathbb{N}$, the number of *combinations* of k objects from n objects is denoted by $\binom{n}{k}$ and for any $r\in\mathbb{R}$, the *ceiling* function of r by $\lceil r\rceil$ and the *floor* function of r by $\lfloor r\rfloor$.

Theorem 3.13. *Let $(R, \oplus, \odot)$ be a commutative general hyperring, $\alpha \in \mathcal{O}_R^{(\oplus)}$ and $\mid Z^{(\alpha)}(R)\setminus\{\alpha\} \mid = n$. If $(R, \oplus, \odot)$ is free of $(\alpha, 2)$-orderable, then*

(i) $\mid E(G^{(\alpha)}(R)) \mid \geq \lceil \frac{n}{2} \rceil$.

(ii) *For $n \geq 3$ is an odd, then there exists $x \in R$ in such a way that $deg(x) \geq 2$.*

Proof. (i) Let $Z^{(\alpha)}(R) \setminus \{\alpha\} = \{x_1, x_2, \ldots, x_n\}$. Then for any $1 \leq i \leq n$, there exists at least $1 \leq j \leq n$ in such a way that $j \neq i$ and $\alpha \in \odot(x_i, x_j) \cap \odot(x_i, x_j)$ and so $\odot(x_i, x_j) \cap \odot(x_i, x_j) \cap \mathcal{O}_R^{(\oplus)} \neq \emptyset$, because $\alpha \in \mathcal{O}_R^{(\oplus)}$ and $(R, \oplus, \odot)$ is free of $(\alpha, 2)$-orderable. It follows that there exists $\lceil \frac{n}{2} \rceil$ numbers of edges as $\{x_i, x_j\}$ in such a way that $\{x_i, x_j\} \in E(G^{(\alpha)}(R))$.

(ii) Let $n \in \mathbb{N}$ be an odd and $Z^{(\alpha)}(R) \setminus \{\alpha\} = \{x_1, x_2, \ldots, x_n\}$. In a similar way of item (i), for any $1 \leq j \leq n$, there exists at least $1 \leq i \leq n$ in such a way that $j \neq i$ and $\alpha \in \odot(x_i, x_j) \cap \odot(x_i, x_j)$ and so $\odot(x_i, x_j) \cap \odot(x_i, x_j) \cap \mathcal{O}_R^{(\oplus)} \neq \emptyset$. Since for all $x_i \in Z^{(\alpha)}(R) \setminus \{\alpha\}, \alpha \notin x_i^2$, there exists at least distinct $x_i, x_j, x_k \in Z^{(\alpha)}(R) \setminus \{\alpha\}$ in such a way that $\alpha \in \odot(x_i, x_j) \cap \odot(x_i, x_j) \cap \odot(x_i, x_k) \cap \odot(x_k, x_i)$ and so $deg(x_i) \geq 2$. □

Definition 3.14. *Let $(R, \oplus, \odot)$ be a general hyperring and $\emptyset \neq \Delta \subseteq R$. Then we say R is a Δ-general hyperring, if for all $x, y \in R, x \odot y = y \odot x = \Delta$.*

Theorem 3.15. *Let $(R, \oplus, \odot)$ be a finite Δ-general hyperring and $\alpha \in \Delta \subseteq \mathcal{O}_R^{(\oplus)}$. Then*

(i) $Z^{(\alpha)}(R) \setminus \{\alpha\} = R \setminus \{\alpha\}$,

(ii) $\mid E(G^{(\alpha)}(R)) \mid = \binom{|R| - 1}{2}$,

(iii) $G^{(\alpha)}(R) \cong K_{|R|-1}$.

Proof. (i) Let $x \in R$. Then $x \in Z^{(\alpha)}(R) \setminus \{\alpha\}$ if and only if $x \neq \alpha$ and there exists $\alpha \neq y \in R$ in such a way that $\odot(x, y) \cap \odot(y, x) \cap \mathcal{O}_R^{(\oplus)} \neq \emptyset$. Thus $x \in Z^{(\alpha)}(R) \setminus \{\alpha\}$ if and only if $x \in R \setminus \{\alpha\}$, because of $\odot(x, y) \cap \odot(y, x) \cap \mathcal{O}_R^{(\oplus)} = \Delta \cap \Delta \cap \mathcal{O}_R^{(\oplus)} = \Delta \neq \emptyset$.

(ii) Let $R = \{x_1, x_2, \ldots, x_{n-1}, x_n\}$ and $\Delta = \{x_1, x_2, \ldots x_m\} \subseteq \mathcal{O}_R^{(\oplus)}$. Applying the item (i), for any $1 \leq i \leq m, Z^{(x_i)}(R) \setminus \{x_i\} = R \setminus \{x_i\}$. Then for any $1 \leq k \leq n$, there exists at least $1 \leq j \leq n$ in such a way that $i \neq k$ and $x_i \in \odot(x_k, x_j) \cap \odot(x_j, x_k)$ and so $\odot(x_k, x_j) \cap \odot(x_j, x_k) \cap \mathcal{O}_R^{(\oplus)} \neq \emptyset$, because $(R, \oplus, \odot)$ is a Δ-general hyperring. Thus, for all $x_i, x_j \in Z^{(\alpha)}(R) \setminus \{\alpha\}, \odot(x_i, x_j) \cap \odot(x_i, x_j) \cap \mathcal{O}_R^{(\oplus)} \neq \emptyset$. It follows that $E(G^{(\alpha)}(R)) = (R \setminus \{x_i\}) \times (R \setminus \{x_i\})$ and so $\mid E(G^{(\alpha)}(R)) \mid = \binom{|R| - 1}{2}$.

(iii) It is clear by item (ii). □

Example 3.16. *Let $R = \{1, 3, 5, 7, 9\}$. Then $(R, \oplus, \odot)$ is a Δ-general hyperring, where $\Delta = \{1, 3\}$ in Table 5. Clearly, $\mathcal{O}_R^{(\oplus)} = R$ and for any $\alpha \in \mathcal{O}_R^{(\oplus)} \setminus \{5, 7, 9\}$, $Z^{(\alpha)}(R) \setminus \{\alpha\} = R \setminus \{\alpha\}$. Hence $G^{(1)}(R) \cong G^{(3)}(R) \cong K_4$, where is shown in Figure 2.*

Theorem 3.17. *Let $(R, \oplus, \odot)$ be a finite Δ-general hyperring and $\alpha, \beta \in \Delta \subseteq \mathcal{O}_R^{(\oplus)}$. Then $G^{(\alpha)}(R) \cong G^{(\beta)}(R)$.*

Proof. Let $\alpha, \beta \in \Delta \subseteq \mathcal{O}_R^{(\oplus)}$. Then $\mid \mathcal{O}_R^{(\oplus)} \mid \geq 2$ and there exist $x, y \in R$ in such a way that $\alpha \in \odot(\beta, x)$ and $\beta \in \odot(\alpha, x)$ and so $\beta \in Z^{(\alpha)}(R) \setminus \{\alpha\}, \beta \in Z^{(\alpha)}(R) \setminus \{\alpha\}$. In addition there exist $z, w, z', w' \in R$ in such a way that $\alpha \in \odot(z, w)$ and $\beta \in \odot(z', w')$, since $(R, \odot)$ is a commutative

$\oplus$	1	3	5	7	9
1	1	$\{1,3\}$	$\{1,5\}$	$\{7,1\}$	$\{1,9\}$
3	$\{1,3\}$	3	$\{3,5\}$	$\{7,3\}$	$\{3,9\}$
5	$\{1,5\}$	$\{3,5\}$	5	$\{7,5\}$	$\{5,9\}$
7	$\{7,1\}$	$\{7,3\}$	$\{7,5\}$	7	$\{7,9\}$
9	$\{1,9\}$	$\{3,9\}$	$\{5,9\}$	$\{7,9\}$	9

,

$\odot$	1	3	5	7	9
1	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$
3	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$
5	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$
7	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$
9	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$

.

Table 5: Hypergroups $(R,\oplus),(R,\odot)$ and Δ-general hyperring $(R,\oplus,\odot)$.

(a) $G^{(1)}(R)$ (b) $G^{(3)}(R)$

Figure 2: Zero divisor graphs of $(R,\oplus,\odot)$.

hypergroup. It follows that $(Z^{(\alpha)}(R)\setminus\{\alpha\})\setminus(Z^{(\beta)}(R)\setminus\{\beta\})=\{\beta\}$ and $(Z^{(\beta)}(R)\setminus\{\beta\})\setminus(Z^{(\alpha)}(R)\setminus\{\alpha\})=\{\alpha\}$. Now, define a map $f:Z^{(\alpha)}(R)\setminus\{\alpha\}\to Z^{(\beta)}(R)\setminus\{\beta\}$ by $f(x)=\begin{cases}\alpha & \text{if } x=\beta\\ x & \text{if } x\neq\beta\end{cases}$. Using Theorem 3.15, f is a bijection, because α and β are't adjacent, we get that f is a homomorphism and so $G^{(\alpha)}(R)\cong G^{(\beta)}(R)$. □

Theorem 3.18. *Let $(R,\oplus,\odot)$ be a finite commutative general hyperring and $\alpha,\beta\in\mathcal{O}_R^{(\oplus)}$. If for all $x\in R, x\odot\alpha=x\odot\beta=\mathcal{O}_R^{(\oplus)}$, then $G^{(\alpha)}(R)\cong G^{(\beta)}(R)$.*

Proof. Let $\alpha,\beta\in\mathcal{O}_R^{(\oplus)}$. Since for all $x\in R, x\odot\alpha=x\odot\beta=\mathcal{O}_R^{(\oplus)}$, $\beta\in Z^{(\alpha)}(R)\setminus\{\alpha\},\beta\in Z^{(\alpha)}(R)\setminus\{\alpha\}$. In addition there exist $z,w,z',w'\in R$ in such a way that $\alpha\in\odot(z,w)$ and $\beta\in\odot(z',w')$, since $(R,\odot)$ is a commutative hypergroup. It follows that $(Z^{(\alpha)}(R)\setminus\{\alpha\})\setminus(Z^{(\beta)}(R)\setminus\{\beta\})=\{\beta\}$ and $(Z^{(\beta)}(R)\setminus\{\beta\})\setminus(Z^{(\alpha)}(R)\setminus\{\alpha\})=\{\alpha\}$. Now, define a map $f:Z^{(\alpha)}(R)\setminus\{\alpha\}\to Z^{(\beta)}(R)\setminus\{\beta\}$ by $f(x)=\begin{cases}\alpha & \text{if } x=\beta\\ x & \text{if } x\neq\beta\end{cases}$. Using Theorem 3.15, f is a bijection, because α and β are't adjacent, we get that f is a homomorphism and so $G^{(\alpha)}(R)\cong G^{(\beta)}(R)$. □

Corollary 3.19. *Let $(R,\oplus,\odot)$ be a finite commutative general hyperring and $\alpha,\beta\in\mathcal{O}_R^{(\oplus)}$. If for all $x\in R, x\odot\alpha=x\odot\beta\supseteq\mathcal{O}_R^{(\oplus)}$, then $G^{(\alpha)}(R)\cong G^{(\beta)}(R)$.*

3.1 Zero divisor graph on $(\oplus,\odot)$-reproduced general hyperring $(\mathbb{Z}_n,+,\cdot)$

In this subsection, we consider the finite $(\oplus,\odot)$-reproduced general hyperrings $(\mathbb{Z}_n,+,\cdot)$ and compute them zero divisor graphs.

In [4], Hamidi et al. constructed a type of $(\oplus,\odot)$-reproduced general hyperring as follows.

Theorem 3.20. *Let $n\in\mathbb{N}$ be an even. Then $(\mathbb{Z}_n,+,\cdot)$ is a $(\oplus,\odot)$-reproduced general hyperring, which $\oplus(\overline{x},\overline{y})=\overline{x}+_{\overline{a}}\overline{y}=\{\overline{x+y},\overline{x+y+a}\},\odot(\overline{x},\overline{y})=\overline{x}\cdot\overline{y}=\{\overline{xy},\overline{xy+a}\}$ and $\overline{a}\neq\overline{0},\overline{x},\overline{y}\in\mathbb{Z}_n$.*

Example 3.21. *Consider the general hyperring $(\mathbb{Z}_2,\oplus,\odot)$, based on Theorem 3.20. It is clear that* $\mathcal{O}^{(\oplus)}_{\mathbb{Z}_2}=\{\overline{0},\overline{1}\}, Z^{(\overline{0})}(\mathbb{Z}_2)\backslash\{\overline{0}\}=\{\overline{1}\}, Z^{(\overline{1})}(\mathbb{Z}_2)\backslash\{\overline{1}\}=\{\overline{0}\}, G^{(\overline{0})}(\mathbb{Z}_2,\oplus,\odot)=K_1$ *and* $G^{(\overline{1})}(\mathbb{Z}_2,\oplus,\odot)=K_1$.

Based on the above hyperoperations in Theorem 3.20, we have the following results.

Theorem 3.22. *Let $3\leq n\in\mathbb{N}$ be an even. Then*

(i) $\mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}=\{\overline{0},\overline{\lfloor\frac{n}{2}\rfloor}\}$.

(ii) *For any $\alpha\in\mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}$, $Z^{(\alpha)}(\mathbb{Z}_n)\setminus\{\alpha\}=\mathbb{Z}_n\setminus\{\alpha\}$.*

(iii) $G^{(\overline{0})}(\mathbb{Z}_n,\oplus,\odot)\cong G^{(\overline{\lfloor\frac{n}{2}\rfloor})}(\mathbb{Z}_n,\oplus,\odot)\cong K_{1,n-2}$.

Proof. (i) Let $\overline{x},\overline{y}\in\mathbb{Z}_n$ and $\alpha\in\mathbb{Z}_n$. Since $\oplus(\overline{x},\overline{y})=\overline{x}+_{\overline{\lfloor\frac{n}{2}\rfloor}}\overline{y}=\{\overline{x+y},\overline{x+y+\lfloor\frac{n}{2}\rfloor}\}$, we get that $\alpha\in\mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}$ if and only if $\alpha=\overline{0}$ or $\alpha=\overline{\lfloor\frac{n}{2}\rfloor}$. Hence $\mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}=\{\overline{0},\overline{\lfloor\frac{n}{2}\rfloor}\}$.

(ii) Let $\overline{x},\overline{y}\in\mathbb{Z}_n$ and $\alpha\in\mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}$. Then $\odot(\overline{x},\overline{y})=\overline{x}\cdot_{\overline{\lfloor\frac{n}{2}\rfloor}}\overline{y}=\{\overline{xy},\overline{xy+\lfloor\frac{n}{2}\rfloor}\}$ and so for any $\alpha\in\mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}$ and $\overline{x}\in\mathbb{Z}_n$, we get that $\overline{x}\in Z^{(\alpha)}(\mathbb{Z}_n)\setminus\{\alpha\}$, if there exists $\overline{x}\neq\overline{y}\in\mathbb{Z}_n$ in such a way that $\alpha\in\{\overline{xy},\overline{xy+\lfloor\frac{n}{2}\rfloor}\}$. If $\alpha=\overline{0}$, then $\overline{x}$ is a divisor of $\mathbb{Z}_n$ or $n\mid(xy+\lfloor\frac{n}{2}\rfloor)$. If $\alpha=\overline{\lfloor\frac{n}{2}\rfloor}$, then in a way similar, we get $\overline{x}$ is a divisor of $\mathbb{Z}_n$ or $n\mid(xy+\lfloor\frac{n}{2}\rfloor)$. It follows that $Z^{(\alpha)}(\mathbb{Z}_n)\setminus\{\alpha\}=\{\overline{x}\mid\exists\,\overline{y}\in\mathbb{Z}_n \text{ in such a way that } n\mid xy \text{ or } n\mid(xy+\lfloor\frac{n}{2}\rfloor)\}=\mathbb{Z}_n\setminus\{\alpha\}$.

(iii) Using Theorem 3.18, $G^{(\overline{0})}(\mathbb{Z}_n,\oplus,\odot)\cong G^{(\overline{\lfloor\frac{n}{2}\rfloor})}(\mathbb{Z}_n,\oplus,\odot)$, because of

$$\odot(\overline{0},\overline{x})=\odot(\overline{x},\overline{0})=\odot(\overline{\lfloor\frac{n}{2}\rfloor},\overline{x})=\odot(\overline{x},\overline{\lfloor\frac{n}{2}\rfloor})=\{\overline{0},\overline{\lfloor\frac{n}{2}\rfloor}\}=\mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}.$$

Let $\overline{x},\overline{y}\in\mathbb{Z}_n$ and $\alpha\neq\beta\in\mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}$. Indeed, for any $\overline{x}\in\mathbb{Z}_n,\odot(\overline{x},\overline{0})=\odot(\overline{0},\overline{x})=\{\overline{0},\overline{\lfloor\frac{n}{2}\rfloor}\}$ and $\odot(\overline{x},\overline{\lfloor\frac{n}{2}\rfloor})=\odot(\overline{\lfloor\frac{n}{2}\rfloor},\overline{x})=\{\overline{0},\overline{\lfloor\frac{n}{2}\rfloor}\}$. Then for any $\overline{x}\in Z^{(\alpha)}(\mathbb{Z}_n)\setminus\{\alpha\}$, there exists $\beta\neq\overline{y}\in\mathbb{Z}_n$ in such a way that $\alpha\in\odot(\overline{x},\beta)\cap\odot(\beta,\overline{x})$ and for any $\beta\neq\overline{y},\alpha\notin\odot(\overline{x},\overline{y})\cap\odot(\overline{y},\overline{x})$. Now, define a map $f:Z^{(\overline{\lfloor\frac{n}{2}\rfloor})}(R)\setminus\{\overline{\lfloor\frac{n}{2}\rfloor}\}\to V(K_{1,n-2})$ by $f(x)=\begin{cases}x_0 & \text{if } x=\overline{0}\\ x_i & \text{if } x=\overline{x_i}\neq\overline{0}\end{cases}$, which $V(K_{1,n-2})=V_1\cup V_2, V_1=\{x_0\}$ and $V_2=\{x_1,x_2,\ldots,x_{n-1}\}\backslash\{x_{\lfloor\frac{n}{2}\rfloor}\}$ and $Z^{(\overline{\lfloor\frac{n}{2}\rfloor})}(R)\backslash\{\overline{\lfloor\frac{n}{2}\rfloor}\}=\{\overline{1},\overline{2},\ldots,\overline{n-1}\}\backslash\{\overline{\lfloor\frac{n}{2}\rfloor}\}$. Clearly, f is a bijection, since $\overline{0}$ and $\overline{\lfloor\frac{n}{2}\rfloor}$ are't adjacent, we get that f is a homomorphism and so $G^{(\overline{\lfloor\frac{n}{2}\rfloor})}(\mathbb{Z}_n,\oplus,\odot)\cong K_{1,n-2}$. □

Corollary 3.23. *Let $3\leq n\in\mathbb{N}$ be an even. Then*

(i) $diam(G^{(\overline{0})}(\mathbb{Z}_n,\oplus,\odot)=diam(G^{(\overline{\lfloor\frac{n}{2}\rfloor})}(\mathbb{Z}_n,\oplus,\odot)=2$.

(ii) $gr(G^{(\overline{0})}(\mathbb{Z}_n,\oplus,\odot)=gr(G^{(\overline{\lfloor\frac{n}{2}\rfloor})}(\mathbb{Z}_n,\oplus,\odot)=\infty$.

Example 3.24. *By Theorem 3.20, $(\mathbb{Z}_6,+,\cdot)$ is a $(\oplus,\odot)$-reproduced general hyperring by the Table 8. Clearly $\mathcal{O}^{(\oplus)}_{\mathbb{Z}_6}=\{\overline{0},\overline{3}\}$ and for any $\alpha\in\mathcal{O}^{(\oplus)}_{\mathbb{Z}_6}$, $Z^{(\alpha)}(\mathbb{Z}_6)\setminus\{\alpha\}=\mathbb{Z}_6\setminus\{\alpha\}$. Hence $G^{(\overline{0})}(\mathbb{Z}_6)\cong G^{(\overline{3})}(\mathbb{Z}_6)\cong K_{1,4}$ are shown in Figure 3.*

Let $\overline{x},\overline{y}\in\mathbb{Z}_p$. Define
$\overline{x}+_{\{p,\overline{a}\}}\overline{y}=\begin{cases}\overline{0} & \overline{x}=\overline{y}=\overline{0}\\ \{\overline{x+y},\overline{x+y+a}\} & \text{otherwise}\end{cases}$ and $\overline{x}\cdot_{\{p,\overline{a}\}}\overline{y}=\begin{cases}\overline{0} & \overline{x}=\overline{0}\text{ or }\overline{y}=\overline{0}\\ \mathbb{Z}_p & \text{otherwise}\end{cases}$, so have the following results.

$\oplus$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$
$\overline{1}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$	$\{\overline{0},\overline{3}\}$
$\overline{2}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$
$\overline{3}$	$\{\overline{3},\overline{0}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$
$\overline{4}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$
$\overline{5}$	$\{\overline{5},\overline{2}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{1},\overline{4}\}$

Table 6: $(\mathbb{Z}_6,\oplus)$

$\odot$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{3}\}$
$\overline{1}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$
$\overline{2}$	$\{\overline{0},\overline{3}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{4},\overline{1}\}$
$\overline{3}$	$\{\overline{3},\overline{0}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{0},\overline{3}\}$
$\overline{4}$	$\{\overline{0},\overline{3}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{2},\overline{5}\}$
$\overline{5}$	$\{\overline{0},\overline{3}\}$	$\{\overline{5},\overline{2}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{1},\overline{4}\}$

Table 7: $(\mathbb{Z}_6,\odot)$

Table 8: Hypergroups $(\mathbb{Z}_6,\oplus),(\mathbb{Z}_6,\odot)$ and general hyperring $(\mathbb{Z}_6,\oplus,\odot)$.

(a) $G^{(\overline{0})}(\mathbb{Z}_6)$ (b) $G^{(\overline{3})}(\mathbb{Z}_6)$

Figure 3: Zero divisor graphs $G^{(\overline{0})}(\mathbb{Z}_6)$ and $G^{(\overline{3})}(\mathbb{Z}_6)$.

Theorem 3.25. *Let $p\in\mathbb{N}$ be a prime and $\overline{a}\in(\mathbb{Z}_p,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}})$.*

(i) $\mathcal{O}_{\mathbb{Z}_p}^{(+_{\{p,\overline{a}\}})}=\{\overline{0},\overline{p-a}\}$.

(ii) *For any $\alpha\in\mathcal{O}_{\mathbb{Z}_p}^{(\oplus)}$, $Z^{(\alpha)}(\mathbb{Z}_p)\setminus\{\alpha\}=\mathbb{Z}_p\setminus\{\alpha\}$.*

(iii) $G^{(\overline{0})}(\mathbb{Z}_p,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}})\cong G^{(\overline{p-1})}(\mathbb{Z}_p,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}})\cong K_{p-1}$.

(iv) $diam(G^{(\overline{0})}(\mathbb{Z}_p,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}})=diam(G^{(\overline{p-1})}(\mathbb{Z}_p,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}}))=1$, *which* $p\geq 5$.

(v) $gr(G^{(\overline{0})}(\mathbb{Z}_p,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}}))=gr(G^{(\overline{p-1})}(\mathbb{Z}_p,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}}))=3$, *which* $p\geq 5$.

Proof. (i) Let $\overline{x},\overline{y}\in\mathbb{Z}_p$ and $\alpha\in\mathbb{Z}_p$. Since $\overline{x}+_{\{p,\overline{a}\}}\overline{y}=\begin{cases}\overline{0} & \overline{x}=\overline{y}=\overline{0}\\ \{\overline{x+y},\overline{x+y+a}\} & \text{otherwise}\end{cases}$, we get that $\alpha\in\mathcal{O}_{\mathbb{Z}_p}^{(+_{\{p,\overline{a}\}})}$ if and only if $\alpha=\overline{0}$ or $\alpha=\overline{p-a}$. Hence $\mathcal{O}_{\mathbb{Z}_p}^{(+_{\{p,\overline{a}\}})}=\{\overline{0},\overline{p-a}\}$.

(ii) Let $\overline{x},\overline{y}\in\mathbb{Z}_p$ and $\alpha\in\mathcal{O}_{\mathbb{Z}_p}^{(+_{\{p,\overline{a}\}})}$. Then $\overline{x}\cdot_{\{p,\overline{a}\}}\overline{y}=\begin{cases}\overline{0} & \overline{x}=\overline{0}\text{ or }\overline{y}=\overline{0}\\ \mathbb{Z}_p & \text{otherwise}\end{cases}$ and so for any $\alpha\in\mathcal{O}_{\mathbb{Z}_p}^{(+_{\{p,\overline{a}\}})}$ and $\overline{x}\in\mathbb{Z}_p$, we get that $\overline{x}\in Z^{(\alpha)}(\mathbb{Z}_p)\setminus\{\alpha\}$, if there exists $\overline{x}\neq\overline{y}\in\mathbb{Z}_p$ in such a way that $\alpha\in\{\overline{0},\mathbb{Z}_p\}$. It follows that $Z^{(\alpha)}(\mathbb{Z}_p)\setminus\{\alpha\}=\mathbb{Z}_p\setminus\{\alpha\}$.

(iii) Using Theorem 3.18, $G^{(\overline{0})}(\mathbb{Z}_n,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}})\cong G^{(\overline{\lfloor\frac{n}{2}\rfloor})}(\mathbb{Z}_n,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}})$, because by definition, we have

$$\cdot_{\{p,\overline{a}\}}(\overline{0},\overline{x})=\cdot_{\{p,\overline{a}\}}(\overline{x},\overline{0})=\{\overline{0}\},\cdot_{\{p,\overline{a}\}}(\overline{p-a},\overline{x})=\cdot_{\{p,\overline{a}\}}(\overline{x},\overline{p-a})\supseteq\{\overline{0},\overline{p-a}\}.$$

Let $\overline{x},\overline{y}\in\mathbb{Z}_p$ and $\alpha\in\mathcal{O}_{\mathbb{Z}_p}^{(+_{\{p,\overline{a}\}})}$. Indeed, $\overline{x}\cdot_{\{p,\overline{a}\}}\overline{y}=\overline{y}\cdot_{\{p,\overline{a}\}}\overline{x}\subseteq\mathcal{O}_{\mathbb{Z}_p}^{(+_{\{p,\overline{a}\}})}$ and so for any $\overline{x}\in Z^{(\alpha)}(\mathbb{Z}_p)\setminus\{\alpha\}$, there exists $\overline{x}\neq\overline{y}\in\mathbb{Z}_p$ in such a way that $\alpha\in(\overline{x}\cdot_{\{p,\overline{a}\}}\overline{y})\cap(\overline{y}\cdot_{\{p,\overline{a}\}}\overline{x})$. It follows that any $\overline{x},\overline{y}\in\mathbb{Z}_p$ are adjacent and so $G^{(\overline{0})}(\mathbb{Z}_p,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}})\cong G^{(\overline{p-1})}(\mathbb{Z}_p,+_{\{p,\overline{a}\}},\cdot_{\{p,\overline{a}\}})\cong K_{p-1}$. □

Example 3.26. *By Theorem 2.1,* $(\mathbb{Z}_5,+_{\{\overline{1},5\}},\cdot_{\{\overline{1},5\}})$ *is a general hyperring by Table 9. Clearly*

$+_{\{\overline{1},5\}}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{0}$	$\{\overline{0}\}$	$\{\overline{1},\overline{2}\}$	$\{\overline{2},\overline{3}\}$	$\{\overline{3},\overline{4}\}$	$\{\overline{4},\overline{0}\}$
$\overline{1}$	$\{\overline{1},\overline{2}\}$	$\{\overline{2},\overline{3}\}$	$\{\overline{3},\overline{4}\}$	$\{\overline{4},\overline{0}\}$	$\{\overline{0},\overline{1}\}$
$\overline{2}$	$\{\overline{2},\overline{3}\}$	$\{\overline{3},\overline{4}\}$	$\{\overline{4},\overline{0}\}$	$\{\overline{0},\overline{1}\}$	$\{\overline{1},\overline{2}\}$
$\overline{3}$	$\{\overline{3},\overline{4}\}$	$\{\overline{4},\overline{0}\}$	$\{\overline{0},\overline{1}\}$	$\{\overline{1},\overline{2}\}$	$\{\overline{2},\overline{3}\}$
$\overline{4}$	$\{\overline{4},\overline{0}\}$	$\{\overline{0},\overline{1}\}$	$\{\overline{1},\overline{2}\}$	$\{\overline{2},\overline{3}\}$	$\{\overline{3},\overline{4}\}$

,

$\cdot_{\{\overline{1},5\}}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{0}$	$\{\overline{0}\}$	$\{\overline{0}\}$	$\{\overline{0}\}$	$\{\overline{0}\}$	$\{\overline{0}\}$
$\overline{1}$	$\{\overline{0}\}$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$
$\overline{2}$	$\{\overline{0}\}$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$
$\overline{3}$	$\{\overline{0}\}$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$
$\overline{4}$	$\{\overline{0}\}$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_5$

.

Table 9: Hypergroups $(\mathbb{Z}_5,+_{\{\overline{1},5\}}),(\mathbb{Z}_5,\cdot_{\{\overline{1},\overline{5}\}})$ and general hyperring $(\mathbb{Z}_5,+_{\{\overline{1},5\}},\cdot_{\{\overline{1},\overline{5}\}})$.

$\mathcal{O}_{\mathbb{Z}_5}^{(+_{\{p,\overline{a}\}})}=\{\overline{0},\overline{4}\}$ *and for any* $\alpha\in\mathcal{O}_{\mathbb{Z}_5}^{(+_{\{p,\overline{a}\}})}$, $Z^{(\alpha)}(\mathbb{Z}_5)\setminus\{\alpha\}=\mathbb{Z}_5\setminus\{\alpha\}$. *Hence* $G^{(\overline{0})}(\mathbb{Z}_5)\cong G^{(\overline{4})}(\mathbb{Z}_5)\cong K_4$ *are shown in Figure 4.*

(a) $G^{(\overline{0})}(\mathbb{Z}_5)$ (b) $G^{(\overline{4})}(\mathbb{Z}_5)$

Figure 4: Zero divisor graph $G^{(\overline{0})}(\mathbb{Z}_5)$ and $G^{(\overline{4})}(\mathbb{Z}_5)$.

It is clear to see that $(\mathbb{Z}_n,+,0,\cdot)$ is a $(\oplus,\odot)$-reproduced general hyperring, where $\oplus(\overline{x},\overline{y})=\overline{x}+'\overline{y}=\{\overline{x},\overline{y},\overline{x+y}\}$ and $\odot(\overline{x},\overline{y})=\overline{x}\cdot'\overline{y}=\{\overline{xy},\overline{0}\}$. From now on, based the above hyperoperations, we have the following results.

Theorem 3.27. *Let* $4\leq n\in\mathbb{N}$ *and consider* $(\mathbb{Z}_n,\oplus,\odot)$.

(i) $\mathcal{O}_{\mathbb{Z}_n}^{(\oplus)}=\mathbb{Z}_n$.

(ii) *For* $\alpha=\overline{0}$, $Z^{(\alpha)}(\mathbb{Z}_n)\setminus\{\alpha\}=\mathbb{Z}_n\setminus\{\alpha\}$.

(iii) $G^{(\overline{0})}(\mathbb{Z}_n,\oplus,\odot)\cong K_{n-1}$.

(iv) $diam(G^{(\overline{0})}(\mathbb{Z}_n,\oplus,\odot)=1$.

(v) $gr(G^{(\overline{0})}(\mathbb{Z}_n,\oplus,\odot)=3$.

Proof. (i) Let $\overline{x},\overline{y}\in\mathbb{Z}_n$ and $\alpha\in\mathbb{Z}_n$. Since $\oplus(\overline{x},\overline{y})=\{\overline{x},\overline{y},\overline{x+y}\}$, we get that $\alpha\in\mathcal{O}_{\mathbb{Z}_n}^{(\oplus)}$ if and only if $\alpha\in\mathbb{Z}_n$, hence $\mathcal{O}_{\mathbb{Z}_n}^{(\oplus)}=\mathbb{Z}_n$.

(ii) Let $\overline{x},\overline{y}\in\mathbb{Z}_n$ and $\alpha=\overline{0}$. Then $\odot(\overline{x},\overline{y})=\{\overline{xy},\overline{0}\}$ and so for any $\overline{x}\in\mathbb{Z}_n$, we get that $\overline{x}\in Z^{(\alpha)}(\mathbb{Z}_n)\setminus\{\alpha\}$, if and only if there exists $\overline{x}\neq\overline{y}\in\mathbb{Z}_n$ in such a way that $\alpha\in\{\overline{0},\overline{xy}\}$. It follows that $Z^{(\alpha)}(\mathbb{Z}_n)\setminus\{\alpha\}=\mathbb{Z}_n\setminus\{\alpha\}$.

(iii) Let $\overline{x}, \overline{y} \in \mathbb{Z}_n$ and $\alpha = \overline{0}$. Indeed, $\odot(\overline{x}, \overline{y}) = \odot(\overline{y}, \overline{x}) = \{\overline{0}, \overline{xy}\}$ and so for any $\overline{x} \in Z^{(\alpha)}(\mathbb{Z}_n) \setminus \{\alpha\}$, there exists $\overline{x} \neq \overline{y} \in \mathbb{Z}_n$ in such a way that $\alpha \in \odot(\overline{x}, \overline{y}) \cap \odot(\overline{y}, \overline{x})$. It follows that $V(G^{(\overline{0})}(\mathbb{Z}_n, \oplus, \odot))) = \mathbb{Z}_n \setminus \{\overline{0}\}$. Since for any $\overline{x}, \overline{y} \in \mathbb{Z}_n, \overline{0} \in \odot(\overline{x}, \overline{y})$, get that $\overline{x}, \overline{y}$ are adjacent and so $E(G^{(\overline{0})}(\mathbb{Z}_n, \oplus, \odot))) = (\mathbb{Z}_n \setminus \{\overline{0}\}) \times (\mathbb{Z}_n \setminus \{\overline{0}\})$, thus $G^{(\overline{0})}(\mathbb{Z}_n, \oplus, \odot)) \cong K_{n-1}$. □

Let $m \in \mathbb{N}$. Then in the general hyperring $(\mathbb{Z}_n, \oplus, \odot)$, for $\overline{x} \in \mathbb{Z}_n$, we have $m\overline{x} = \{\overline{x}, \overline{mx}\}$ and $\overline{x}^m = \{\overline{0}, \overline{x^m}\}$.

Theorem 3.28. *Let $3 \leq p \in \mathbb{N}$ be a prime and consider $(\mathbb{Z}_p, \oplus, \odot)$. Then*

(i) $|Z^{(\alpha)}(\mathbb{Z}_p) \setminus \{\alpha\}| = p - 2$.

(ii) $G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot))$ *is a disconnected graph.*

(iii) $diam(G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot)) = \infty$.

(iv) $gr(G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot)) = \infty$.

Proof. (i) By Theorem 3.27, $\overline{1} \in \mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}$. Let $\overline{x} \in \mathbb{Z}_p$ and $\alpha = \overline{1}$. Because p is a prime, there exists $\overline{y} \in \mathbb{Z}_p$ in such a way that $\alpha \in \odot(\overline{x}, \overline{y}) \cap \odot(\overline{y}, \overline{x})$. In addition, for all $\overline{x} \in \mathbb{Z}_p$,

$$\alpha \in \overline{x}^2 = \odot(\overline{x}, \overline{x}) \Leftrightarrow p \mid (x^2 - 1) \Leftrightarrow \overline{x} \in \{\overline{1}, \overline{p-1}\}.$$

Hence for all $\overline{x} \in \mathbb{Z}_p$, there exists $\overline{x} \neq \overline{y} \in \mathbb{Z}_p$ in such a way that $\alpha \in \odot(\overline{x}, \overline{y}) \cap \odot(\overline{y}, \overline{x})$. Thus $Z^{(\alpha)}(\mathbb{Z}_p) \setminus \{\alpha\} = \mathbb{Z}_p \setminus \{\alpha, \overline{0}\}$ and so $|Z^{(\alpha)}(\mathbb{Z}_p) \setminus \{\alpha\}| = p - 2$.

(ii) By item (i), $|Z^{(\alpha)}(\mathbb{Z}_p) \setminus \{\alpha\}| = p - 2$ is an odd, since p is a prime. Applying Theorem 3.12, $G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot))$ is a disconnected graph.

(iii, iv) Since $G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot))$ is a disconnected graph, by definition $diam(G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot)) = \infty$ and $gr(G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot)) = \infty$. □

Theorem 3.29. *Let $3 \leq p \in \mathbb{N}$ be a prime and consider $(\mathbb{Z}_p, \oplus, \odot)$. Then $t(G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot))) = \lfloor \frac{p}{2} \rfloor$.*

Proof. Applying Theorem 3.28, $G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot))$ is a disconnected. Let $\overline{x} \in \mathbb{Z}_p$ and $\alpha = \overline{1}$. Because p is a prime, there exists a unique $\overline{y} \in \mathbb{Z}_p$ in such a way that $\alpha \in \odot(\overline{x}, \overline{y}) \cap \odot(\overline{y}, \overline{x})$. In addition, $\overline{1} \in \overline{p-1}^2$, thus, $\overline{p-1}$ is an isolated vertex. Assume that $Z^{(\overline{1})}(\mathbb{Z}_p) \setminus \{\overline{1}\} = \{x_1, x_2, \ldots, x_{p-3}, x_{p-2}\}$, because of $|Z^{(\alpha)}(\mathbb{Z}_p) \setminus \{\alpha\}| = p - 2$ by Theorem 3.28. Hence, $G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot))$ has $\lceil \frac{p-1}{2} \rceil$ subgraphs $G_i = (V_i, E_i)$, such as

$G_1 = (\{x_1, x_2\}, \{x_1, x_2\}), G_2 = (\{x_3, x_4\}, \{x_3, x_4\}) \ldots G_{\lceil \frac{p-2}{2} \rceil - 1} = (\{x_{p-4}, x_{p-3}\}, \{x_{p-4}, x_{p-3}\})$ and $G_{\lceil \frac{p-2}{2} \rceil} = (\{x_{p-2}\}, \{x_{p-2}\})$ by a rearrangement. Then $t(G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot))) = \lfloor \frac{p}{2} \rfloor$. □

Example 3.30. *Let $R = \{0, 1, 2, 3, 4\}$. Then by Theorem 3.2, $(R, \oplus, \odot)$ is a general hyperring as Table 12. Clearly, $\mathcal{O}^{(\oplus)}_R = R$, $Z^{(0)}(R) \setminus \{0\} = R \setminus \{0\}, Z^{(1)}(R) \setminus \{1\} = \{2, 3, 4\}, Z^{(2)}(R) \setminus \{2\} = \{3, 4\}, Z^{(3)}(R) \setminus \{3\} = \{2, 4\}$, and $Z^{(4)}(R) \setminus \{4\} = \{2, 3\}$. Hence,*

$G^{(0)}(R) \cong K_4, G^{(2)}(R) \cong G^{(3)}(R) \cong G^{(4)}(R) \cong K_2$ and $G^{(1)}(R) \cong G$ as shown in Figure 5. We see that $(R, \odot)$ is a commutative hypergroup based on the Table 11, and $\mathcal{O}^{(\odot)}_R = \{0\}$.

In what follows, for any $x, y \in \mathbb{Z}$, we consider the congruence modulo p, by $x \overset{p}{\equiv} y$ or $x \cong y$ $(mod\ p)$.

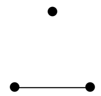

Figure 5: Graph $G^{(1)}(R)$

$\oplus$	0	1	2	3	4
0	0	$\{1,0\}$	$\{0,2\}$	$\{0,3\}$	$\{0,4\}$
1	$\{0,1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,3,4\}$	$\{1,4,0\}$
2	$\{0,2\}$	$\{1,2,3\}$	$\{2,4\}$	$\{2,3,0\}$	$\{2,4,1\}$
3	$\{0,3\}$	$\{1,3,4\}$	$\{0,2,3\}$	$\{3,1\}$	$\{3,4,2\}$
4	$\{0,4\}$	$\{1,4,0\}$	$\{1,2,4\}$	$\{2,3,4\}$	$\{4,3\}$

Table 10: $(R,\oplus)$

$\odot$	0	1	2	3	4
0	0	0	0	0	0
1	0	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\{0,4\}$
2	0	$\{0,2\}$	$\{0,4\}$	$\{0,1\}$	$\{3,0\}$
3	0	$\{0,3\}$	$\{0,1\}$	$\{4,0\}$	$\{0,2\}$
4	0	$\{0,4\}$	$\{0,3\}$	$\{0,2\}$	$\{1,0\}$

Table 11: Hypergroup $(R,\odot)$

Table 12: General hyperring $(R,\oplus,\odot)$ of order 5 and graph $G^{(1)}(R)$.

Theorem 3.31. *Let $p\in\mathbb{N}$ be a prime, consider $(\mathbb{Z}_p,\oplus,\odot),\overline{x},\overline{y}\in\mathbb{Z}_p$ and $\alpha\in\mathcal{O}^{(\oplus)}_{\mathbb{Z}_p}$.*

(i) *If $\overline{0}\in\oplus(\overline{x},\overline{y})\cap\oplus(\overline{y},\overline{x})$, then $x^2\overset{p}{\equiv}y^2$.*

(ii) *If $\alpha\in\overline{x}^2$, then $\alpha\in\{\overline{0},\overline{1},\overline{4},\overline{9},\dots,\overline{(\lfloor\frac{p}{2}\rfloor)^2+2-p},\overline{(\lfloor\frac{p}{2}\rfloor)^2}\}$.*

(iii) *If $J_\alpha=\{\overline{x}\mid\overline{x}^2\overset{p}{\equiv}\alpha,\overline{x}\in\mathbb{Z}_p\}$, then $|J_\alpha|\leq 2$.*

Proof. (i) Let $\overline{x},\overline{y}\in\mathbb{Z}_p$. If $\overline{x}=\overline{0}$ or $\overline{y}=\overline{0}$, then it is clear. If $\overline{0}\notin\{\overline{x},\overline{y}\}$, then $\overline{0}\in\oplus(\overline{x},\overline{y})\cap\oplus(\overline{y},\overline{x})$ if and only if $\overline{0}\in\{\overline{x},\overline{y},\overline{x+y}\}$ if and only if $y=p-x$. It follows that $\{\overline{0},\overline{x}^2\}=\overline{x}^2=\odot(\overline{x},\overline{x})=\odot(\overline{y+p}),(\overline{y+p}))=\{\overline{0},\overline{(y+p)(y+p)}\}=\{\overline{0},\overline{y}^2\}$ and so $\overline{x}^2=\overline{y}^2$ or $x^2\overset{p}{\equiv}y^2$.

(ii) Let $\overline{x}\in\mathbb{Z}_p$. By definition there exists a unique $\overline{y}\in\mathbb{Z}_p$ in such a way that $\overline{0}\in\oplus(\overline{x},\overline{y})$. If $\mathcal{I}=\{\overline{x}\mid\exists\ \overline{y}$ in such a way that $\overline{0}\in\oplus(\overline{x},\overline{y}),\overline{x},\overline{y}\in\mathbb{Z}_p\}$, then $\mathcal{I}\neq\emptyset(|\mathcal{I}|=\overline{\lfloor\frac{p}{2}\rfloor}+1)$ and by item (i), for any $\overline{x}\in\mathcal{I}$, there exists $\overline{y}\in\mathbb{Z}_p$, in such a way that $x^2\overset{p}{\equiv}y^2$. If $\overline{a_n}:\overline{0},\overline{1\cdot 1},\overline{2\cdot 2},\dots,\overline{(p-1)\cdot(p-1)}$, then we obtain that $\overline{a_n}\overset{p}{\equiv}n^2$ and it follows that for any $\overline{x}\in\mathcal{I}$, we get that $\overline{x}^2\subseteq\{\overline{0},\overline{1},\overline{4},\overline{9},\dots,\overline{(\lfloor\frac{p}{2}\rfloor)^2+2-p},\overline{(\lfloor\frac{p}{2}\rfloor)^2}\}$.

(iii) If $\alpha=\overline{0}$, then by items $(i),(ii)$, we get that $\overline{x}^2\overset{p}{\equiv}\alpha$, it implies that $\overline{x}=\alpha$ and so $|J_\alpha|=1$. If $\alpha\neq\overline{0}$, then by items $(i),(ii)$, we get that $\overline{x}^2\overset{p}{\equiv}\alpha$, it implies that there exists $\overline{k}\in\mathbb{Z}_p$ in such a way that $\overline{k}^2\overset{p}{\equiv}\alpha$ and $\overline{p-k}^2\overset{p}{\equiv}\alpha$ and so $|J_\alpha|=2$. □

Theorem 3.32. *Let $p\in\mathbb{N}$ be a prime and $\alpha\in\mathcal{O}^{(\oplus)}_{\mathbb{Z}_p}\setminus\{\overline{0},\overline{1}\}$.*

(i) *If $J_\alpha=\emptyset$, then $|Z^{(\alpha)}(\mathbb{Z}_p)\setminus\{\alpha\}|=p-3$, which $p\geq 5$.*

(ii) *If $J_\alpha\neq\emptyset$, then $|Z^{(\alpha)}(\mathbb{Z}_p)\setminus\{\alpha\}|=p-5$, which $p\geq 7$.*

Proof. (i) Since $J_\alpha=\emptyset$, for all $\overline{k}\in\mathbb{Z}_p,k^2\overset{p}{\not\equiv}\alpha$ and $(p-k)^2\overset{p}{\not\equiv}\alpha$. Hence in similar to Theorem 3.28 and using Theorem 3.31, we get that $Z^{(\alpha)}(\mathbb{Z}_p)\setminus\{\alpha\}=\mathbb{Z}_p\setminus\{\alpha,\overline{0},\overline{1}\}$ and for all $\overline{x}\in\mathbb{Z}_p$ there exists $\overline{x}\neq\overline{y}\in\mathbb{Z}_p$ in such a way that $\alpha\in\odot(\overline{x},\overline{y})\cap\odot(\overline{y},\overline{x})$. It follows that $|Z^{(\alpha)}(\mathbb{Z}_p)\setminus\{\alpha\}|=p-3$.

(ii) Since $J_\alpha\neq\emptyset$, there exist $\overline{k}\in\mathbb{Z}_p$ in such a way that $k^2\overset{p}{\equiv}\alpha$ and $(p-k)^2\overset{p}{\equiv}\alpha$. Hence in similar to Theorem 3.28 and using Theorem 3.31, we get that $Z^{(\alpha)}(\mathbb{Z}_p)\setminus\{\alpha\}=\mathbb{Z}_p\setminus\{\alpha,\overline{0},\overline{1},\overline{k},\overline{p-k}\}$ and for all $\overline{x}\in\mathbb{Z}_p$ there exists $\overline{x}\neq\overline{y}\in\mathbb{Z}_p$ in such a way that $\alpha\in\odot(\overline{x},\overline{y})\cap\odot(\overline{y},\overline{x})$. It follows that $|Z^{(\alpha)}(\mathbb{Z}_p)\setminus\{\alpha\}|=p-5$. □

Corollary 3.33. *Let $7\leq p\in\mathbb{N}$ be a prime and $\alpha\in\mathcal{O}^{(\oplus)}_{\mathbb{Z}_p}\setminus\{\overline{0},\overline{1}\}$.*

(i) *$G^{(\alpha)}(\mathbb{Z}_p,\oplus,\odot))$ is a disconnected graph.*

(ii) $diam(G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot)) = \infty$.

(iii) $gr(G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot)) = \infty$.

Proof. The proof is similar to Theorem 3.28 and is obtained from Theorem 3.32. □

Corollary 3.34. *Let $7 \leq p \in \mathbb{N}$ be a prime and consider $(\mathbb{Z}_p, \oplus, \odot)$. Then*

(i) *If $J_\alpha = \emptyset$, then $t(G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot))) = \lfloor \frac{p-3}{2} \rfloor$.*

(ii) *If $J_\alpha \neq \emptyset$, then $t(G^{(\overline{1})}(\mathbb{Z}_p, \oplus, \odot))) = |p-5|$.*

Proof. The proof is similar to Theorem 3.29 and is obtained from Theorem 3.32. □

Proposition 3.35. *Let $3 \leq n \in \mathbb{N}$, consider $(\mathbb{Z}_n, \oplus, \odot), \overline{x} \in \mathbb{Z}_n$ and $\overline{\alpha} \in \mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}$.*

(i) *If $\overline{0} \in \oplus(\overline{x}, \overline{y}) \cap \oplus(\overline{y}, \overline{x})$, then $x^2 \overset{n}{\equiv} y^2$.*

(ii) *If $\overline{\alpha} \in \overline{x}^2$, then $\overline{\alpha} \in \{\overline{0}, \overline{1}, \overline{4}, \overline{9}, \dots, \lfloor \frac{n}{2} \rfloor^2\}$.*

(iii) *If $\gcd(\alpha, n) = 1$, then there exists $r \in \mathbb{Z}$ in such a way that $\overline{x} \in (\overline{rx} \odot \overline{\alpha}) \cap (\overline{\alpha} \odot \overline{rx})$.*

Proof. (i),(ii) Are similar to Theorem 3.31(i).

(iii) Since $\gcd(\alpha, n) = 1$, there exists $r, s \in \mathbb{Z}$ in such a way that $1 = r\alpha + sn$ and so for any $\overline{x} \in \mathbb{Z}_n$, get $\overline{x} \in (\overline{1} \odot \overline{x} \cap \overline{1} \odot \overline{x}) \subseteq (\overline{rx} \odot \overline{\alpha} \cap \overline{\alpha} \odot \overline{rx})$. □

Let $n \in \mathbb{N}$. From now on, $\varphi(n)$ is the **Euler phi-function(the indicator or totient)**. Consider $(\mathbb{Z}_n, \oplus, \odot), \overline{x} \in \mathbb{Z}_n$ and $\overline{0} \neq \overline{\alpha} \in \mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}$. We set $I_\alpha = \{\overline{x} \in \mathbb{Z}_n \mid \gcd(x, n) \mid \alpha\}, J_\alpha = \{\overline{x} \in \mathbb{Z}_n \mid x^2 \overset{n}{\equiv} \alpha\}$ and $K_\alpha = \{\overline{x} \in \mathbb{Z}_n \mid \overline{\alpha} \in \overline{x} \odot \overline{\alpha}\}$, so have the following results. In general hyperring $(\mathbb{Z}_{16}, \oplus, \odot)$, we have $I_7 = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}, \overline{9}, \overline{11}, \overline{13}, \overline{15}\}, K_7 = \{\overline{1}\}$ and $J_7 = \emptyset$.

Proposition 3.36. *Let $n \in \mathbb{N}$, consider $(\mathbb{Z}_n, \oplus, \odot)$ and $\overline{0} \neq \overline{\alpha} \in \mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}$. Then*

(i) *$|I_\alpha| \geq \varphi(n)$ and $|K_\alpha| \geq 1$.*

(ii) *If there exists $\overline{x} \in \mathbb{Z}_n$ in such a way that $\overline{\alpha} \in \overline{x}^2$, then $|J_\alpha| = 1$ or $|J_\alpha| = 2k$, where $k \in \mathbb{N}$.*

(iii) *If $n = p \geq 5$ is a prime, then $|I_\alpha| = p - 1, |K_\alpha| = 1$ and $(|J_\alpha| = 2$ or $|J_\alpha| = 0)$.*

Proof. (i) Let $n \in \mathbb{N}$. Then there exists $m < n$ in such a way that $\gcd(m, n) = 1$. Thus $|I_\alpha| \geq \varphi(n)$. In addition, $|K_\alpha| \geq 1$, because of $\overline{1} \in K_\alpha$.

(ii) Let $\overline{x} \in \mathbb{Z}_n$ and $\overline{\alpha} \in \overline{x}^2$. Then $\overline{\alpha} = \overline{x^2}$ and so $x^2 \overset{n}{\equiv} \alpha$. Hence $x^2 \overset{n}{\equiv} \alpha \overset{n}{\equiv} (n-x)$. If $n - x = x$ then $|J_\alpha| = 1$, but $n - x \neq x$ implies that $\overline{n-x} \in J_\alpha$ and so $|J_\alpha| \geq 2$. Because for any $\overline{x} \in \mathbb{Z}_n$, there exists $\overline{y} \in \mathbb{Z}_n$ in such a way that $\overline{0} \in \oplus(\overline{x}, \overline{y}) \cap \oplus(\overline{y}, \overline{x})$, by Proposition 3.35, we get that $x^2 \overset{n}{\equiv} y^2$ and so there exists $k \in \mathbb{N}$ in such a way that $|J_\alpha| = 2k$.

(iii) Immediate by items (i), (ii). □

Theorem 3.37. *Let $n \in \mathbb{N}$, consider $(\mathbb{Z}_n, \oplus, \odot)$ and $\overline{0} \neq \overline{\alpha} \in \mathcal{O}^{(\oplus)}_{\mathbb{Z}_n}$. Then*

$$|Z^{(\alpha)}(\mathbb{Z}_n) \setminus \{\alpha\}| = |I_\alpha \setminus (J_\alpha \cup K_\alpha \cup \{\alpha\})|.$$

Proof. Let $\overline{x} \in Z^{(\alpha)}(\mathbb{Z}_n)\setminus\{\alpha\}$. Then there exists $\overline{y} \notin \{\overline{x}, \overline{\alpha}\}$ in such a way that $\overline{\alpha} \in \odot(\overline{x}, \overline{y}) \cap \odot(\overline{y}, \overline{x})$. It follows that $xy \overset{n}{\equiv} \alpha$ and so $\overline{x} \in I_\alpha$. Since $\overline{y} \neq \overline{x}$ and $\overline{y} \neq \overline{\alpha}$, we get that $\overline{x} \notin J_\alpha$ and $\overline{x} \notin K_\alpha$, respectively. Hence $Z^{(\alpha)}(\mathbb{Z}_n) \setminus \{\alpha\} = I_\alpha \setminus (J_\alpha \cup K_\alpha \cup \{\alpha\})$, and so $V(G^{(\alpha)}(\mathbb{Z}_n, \oplus, \odot))) = I_\alpha \setminus (J_\alpha \cup K_\alpha \cup \{\alpha\})$. □

Corollary 3.38. *Let $3 \leq n \in \mathbb{N}$, consider $(\mathbb{Z}_n, \oplus, \odot)$ and $\overline{\alpha} \in \mathcal{O}^{(\oplus)}_{\mathbb{Z}_n} \setminus \{\overline{1}\}$. If $\gcd(\alpha, n) = 1$, then*

(i) $|I_\alpha| = \varphi(n), |J_\alpha| = 0$ *and* $|K_\alpha| = 1$.

(ii) $|Z^{(\alpha)}(\mathbb{Z}_n) \setminus \{\alpha\}| = \varphi(n) - 2$.

(iii) $G^{(\alpha)}(\mathbb{Z}_n, \oplus, \odot) \cong K_{\varphi(n)-2}$, *which* $n \leq 8$.

(iv) $diam(G^{(\alpha)}(\mathbb{Z}_n, \oplus, \odot)) = 1$ *and* $gr(G^{(\alpha)}(\mathbb{Z}_n, \oplus, \odot)) = 3$, *which* $n \leq 8$.

(v) $G^{(\alpha)}(\mathbb{Z}_n, \oplus, \odot) \cong K_{\varphi(n)-2}$ *is disconnected, which* $n \geq 9$.

(vi) $diam(G^{(\alpha)}(\mathbb{Z}_n, \oplus, \odot)) = gr(G^{(\alpha)}(\mathbb{Z}_n, \oplus, \odot)) = \infty$, *which* $n \geq 9$.

Example 3.39. *Consider the general hyperring $(\mathbb{Z}_{16}, \oplus, \odot)$.*

(i) *If $\alpha = 7$, then by Corollary 3.38, $G^{(\alpha)}((\mathbb{Z}_{16}, \oplus, \odot)) \cong G$ as depicted in Figure 7. If $\alpha = 14$, then by Theorem 3.37, $\overline{x} \in Z^{(\alpha)}(\mathbb{Z}_{16})\setminus\{\overline{14}\}$ if and only if $\gcd(x, 16) \mid 14$ if and only if $\gcd(12, 16) \in \{1, 2\}$ if and only if $x \in \{2, 3, 5, 6, 7, 10, 11, 13, 15\}$, hence $|Z^{(\overline{14})}(\mathbb{Z}_{16})\setminus\{\overline{14}\}| = 9 \neq \varphi(16)-2$, because of $gcd(14, 16) \neq 1$.*

(ii) *If $\alpha = 9$, then by Theorem 3.37, $\overline{x} \in Z^{(\alpha)}(\mathbb{Z}_{16}) \setminus \{\alpha\})$ if and only if $x \in \{3, 13, 5, 11, 7, 15\}$ (because of $x^2 \overset{16}{\equiv} 9 \overset{16}{\equiv} 25$). Hence $G^{(\alpha)}(\mathbb{Z}_{16}, \oplus, \odot) \cong G$ as depicted in Figure 7.*

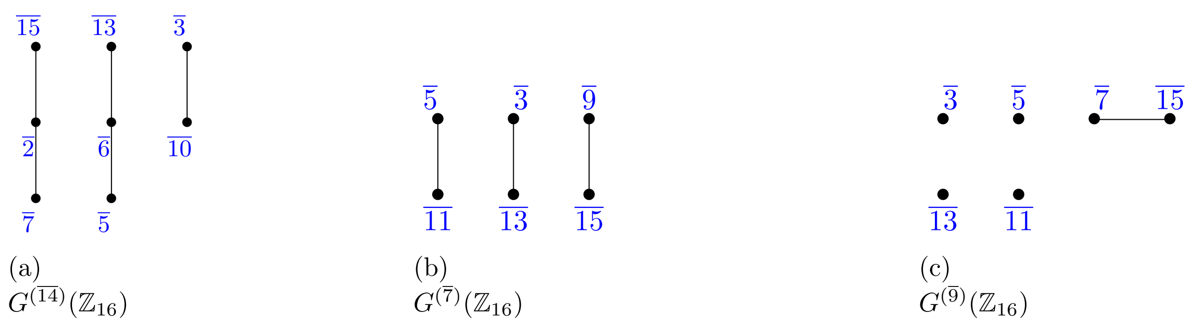

Figure 7: Zero divisor graphs $G^{(\overline{14})}(\mathbb{Z}_5), G^{(\overline{7})}(\mathbb{Z}_5)$ and $G^{(\overline{9})}(\mathbb{Z}_5)$.

3.2 Zero divisor graph on $(\mathbb{Z}, \oplus, \odot)$

In this subsection, we consider the infinite general hyperrings $(\mathbb{Z}, \oplus, \odot)$ and compute them zero divisor graphs based on Theorem 3.20.

Let $n \in \mathbb{N}$. Then, from now on, we set $D(n) = \{k \in \mathbb{Z} \mid k \mid n\}$ as the set of all divisors of n(positive and negative integers).

Theorem 3.40. *Let $2 \leq k \in \mathbb{N}$ and p be a prime. Then*

(i) $\mathcal{O}^{(\oplus)}_{\mathbb{Z}} = \mathbb{Z}$.

(ii) $Z^{(0)}(\mathbb{Z}) \setminus \{0\} = \mathbb{Z} \setminus \{0\}$.

(iii) $Z^{(\alpha)}(\mathbb{Z}) \setminus \{\alpha\} = D(\alpha) \setminus \{1, \alpha\}$, *where* $\alpha = p^k$.

(iv) $|Z^{(p^k)}(\mathbb{Z}) \setminus \{p^k\}| = 2k$.

Proof. (i), (ii) Let $x \in \mathbb{Z}$. Then for all $y \in \mathbb{Z}, x \in \{x, y, x+y\} = \oplus(x, y) \cap \oplus(y, x)$, so $\mathcal{O}_{\mathbb{Z}}^{(\oplus)} = \mathbb{Z}$. In addition, for all $y \neq x \neq 0, 0 \in \{0, xy\} = \odot(x, y) \cap \odot(y, x)$ imply that $Z^{(0)}(\mathbb{Z}) \setminus \{0\} = \mathbb{Z} \setminus \{0\}$.

(iii) Let $\alpha = p^k$ and $x \in Z^{(\alpha)}(\mathbb{Z}) \setminus \{\alpha\}$. Then there exists $y \notin \{1, p^k\}$ in such a way that $p^k = xy$. Hence $p^k \notin \{x, y\}$ and $x \in \{-1, \pm(p^m), -p^{2k} \mid 1 \leq m \neq k \leq 2k-1\}$, because of $x \mid p^k$, respectively. Thus $Z^{(\alpha)}(\mathbb{Z}) \setminus \{\alpha\} = D(\alpha) \setminus \{1, \alpha\}$.

(iv) It is clear by (iii). □

Theorem 3.41. *Let* $2 \leq k \in \mathbb{N}$ *and* p *be a prime.*

(i) $G^{(0)}(\mathbb{Z}, \oplus, \odot)$ *is an infinite complete graph.*

(ii) $G^{(p^{2k})}(\mathbb{Z}, \oplus, \odot)$ *is a disconnected graph.*

Proof. (i) Let $x, y \in \mathbb{Z}$. Then $0 \in \odot(x, y) \cap \odot(y, x)$ implies that $\mathcal{O}_{\mathbb{Z}}^{(\oplus)} \cap \odot(x, y) \cap \odot(y, x) \neq \emptyset$. It follows that for any given $\alpha \in \mathbb{Z}$, we get that $E(G^{(\alpha)}(\mathbb{Z}, \oplus, \odot)) = V(G^{(\alpha)}(\mathbb{Z}, \oplus, \odot)) \times V(G^{(\alpha)}(\mathbb{Z}, \oplus, \odot))$.

(ii) Since $Z^{(\alpha)}(\mathbb{Z}) \setminus \{\alpha\} = D(\alpha) \setminus \{1, \alpha\}$, for any $x \in D(\alpha) \setminus \{1, \alpha\}$ there exists a unique $y \in D(\alpha) \setminus \{1, \alpha\}$ such that $\alpha = xy$. Thus for any $z \notin \{x, y\}$, z is not adjacent to x, y and so $G^{(p^{2k})}(\mathbb{Z}, \oplus, \odot)$ is a disconnected graph. □

Corollary 3.42. *Let* $2 \leq k \in \mathbb{N}$ *and* p *be a prime. Then*

(i) *if* k *is an odd, then* $t(G^{(p^{2k})}(\mathbb{Z}, \oplus, \odot)) = k$.

(ii) *if* k *is an even, then* $t(G^{(p^{2k})}(\mathbb{Z}, \oplus, \odot)) = k + 1$.

Proof. They are obtained by Theorems 3.12 and 3.41. □

Let $n, k \in \mathbb{Z}$, and $n = \prod_{i=1}^{k} p_i^{r_i}$, where $p_1, p_2, \ldots, p_k$ are primes and $r_1, r_2, \ldots, r_k \in \mathbb{Z}^{\geq 1}$. Then from now on, will denote $\mathcal{P}(n) = \{r_1, r_2, \ldots, r_k\}$.

Theorem 3.43. *Let* $k, k' \in \mathbb{N}, p, q$ *be primes and* $\alpha, \beta \in \mathcal{O}_{\mathbb{Z}}^{(\oplus)}$. *Then*

(i) $G^{(p^{2k})}(\mathbb{Z}, \oplus, \odot) \cong G^{(p^{2k'+1})}(\mathbb{Z}, \oplus, \odot)$ *if and only if* $k - k' = 1$.

(ii) $G^{(p^k)}(\mathbb{Z}, \oplus, \odot) \cong G^{(p^{k'})}(\mathbb{Z}, \oplus, \odot)$ *if and only if* $k = k'$, *where* k, k' *are odd or* k, k' *are even.*

(iii) $G^{(\alpha)}(\mathbb{Z}, \oplus, \odot) \cong G^{(\beta)}(\mathbb{Z}, \oplus, \odot)$ *if and only if* $\mathcal{P}(\alpha) = \mathcal{P}(\beta)$.

Proof. (i) If $k - k' = 1$, then $p^{2k} = p^{2(k'+1)}$ and using Theorem 3.41, and so $G^{(p^{2k})}((\mathbb{Z}, \oplus, \odot)) \cong K_{4k'+2} \cong G^{(p^{2k'+1})}((\mathbb{Z}, \oplus, \odot))$. If $G^{(p^{2k})}((\mathbb{Z}, \oplus, \odot)) \cong G^{(p^{2k'+1})}((\mathbb{Z}, \oplus, \odot))$, then

$$| \ Z^{(p^{2k})}(\mathbb{Z}) \setminus \{p^{2k}\} \ | = | \ Z^{(p^{2k'+1})}(\mathbb{Z}) \setminus \{p^{2k'+1}\}.$$

It follows that $4k - 2 = 2(2k' + 1 + 1 - 1)$ and so $k - k' = 1$.

(ii) Let either $k \notin \mathbb{E}$(even integer) and $k' \notin \mathbb{E}$. Then by item (i), $G^{(p^k)}((\mathbb{Z}, \oplus, \odot)) \cong G^{(p^{k'})}((\mathbb{Z}, \oplus, \odot))$ if and only if $|\ Z^{(p^k)}(\mathbb{Z})\backslash\{p^k\}\ | = |\ Z^{(p^{k'})}(\mathbb{Z})\backslash\{p^{k'}\}\ |$ (because of complete graphs) if and only if $2(k+1) = 2(k'+1)$ if and only if $k = k'$.

(iii) Let $\alpha, \beta \in \mathcal{O}_{\mathbb{Z}}^{(\oplus)}, \alpha = \pm(\prod_{i=1}^{k} p_i^{r_i})$ and $\beta = \pm(\prod_{i=1}^{m} q_i^{s_i})$. Then $G^{(\alpha)}((\mathbb{Z}, \oplus, \odot)) \cong G^{(\beta)}((\mathbb{Z}, \oplus, \odot))$ if and only if $|\ Z^{(\alpha)}(\mathbb{Z}) \setminus \{p^k\}\ | = |\ Z^{(\beta)}(\mathbb{Z}) \setminus \{\beta\}\ |$ if and only if $2((\prod_{i=1}^{k}(r_i+1)) - 1) = 2((\prod_{i=1}^{m}(s_i+1)) - 1)$ if and only if $k = m$ and for any $1 \le i \le m, r_i = s_i$ if and only if $\mathcal{P}(\alpha) = \mathcal{P}(\beta)$. $\square$

Corollary 3.44. *Let $n \in \mathbb{N}, p, q$ be primes and $\alpha \in \mathcal{O}_{\mathbb{Z}}^{(\oplus)}$.*

(i) $G^{(p)}(\mathbb{Z}, \oplus, \odot) \cong G^{(p)}(\mathbb{Z}, \oplus, \odot) \cong K_2$.

(ii) $G^{(p^n)}(\mathbb{Z}, \oplus, \odot) \cong G^{(q^n)}(\mathbb{Z}, \oplus, \odot)$.

(iii) $G^{(p^{2n})}(\mathbb{Z}, \oplus, \odot) \ncong G^{(-p^{2n})}(\mathbb{Z}, \oplus, \odot)$.

(iv) $G^{(\alpha)}(\mathbb{Z}, \oplus, \odot) \cong G^{(-\alpha)}(\mathbb{Z}, \oplus, \odot)$ *if and only if* $\alpha \in \mathcal{O}_{\mathbb{Z}}^{(\oplus)} \setminus \{1, p^{2k}\}$.

(v) $G^{(p^{2n+1})}(\mathbb{Z}, \oplus, \odot) \cong G^{(-p^{2n+1})}(\mathbb{Z}, \oplus, \odot)$.

Corollary 3.45. *Let $2 \le k \in \mathbb{N}$ and p be a prime.*

(i) $diam(G^{(0)}(\mathbb{Z}, \oplus, \odot)) = \infty$ *and* $gr(G^{(0)}(\mathbb{Z}, \oplus, \odot)) = 3$.

(ii) $diam(G^{(p^{2k})}(\mathbb{Z}, \oplus, \odot) = gr(G^{(p^{2k})}(\mathbb{Z}, \oplus, \odot) = \infty$.

4 Conclusions

In this paper, we defined and considered the notion of graphs based on zero divisors of general hyperrings via absorbing elements. We try to consider the graphs based on finite general hyperrings and investigate some graphs based on infinite general hyperrings. We show that there exists infinite general hyperring in such a way that related diameters of their zero divisor graph are finite. In any general hyperrings, there can be several zeroing elements, based on which several zero divisors are created and specific graphs are produced accordingly. Also, we have counted the number of these zero divisor graphs in terms of isomorphism. Also

(i) introduced the notion of reproduced general hyperring and proved that any ring is a reproduced general hyperring.

(ii) The set of zero divisors of any general hyperring has at least two elements, in case of existence.

(iii) The concepts of locally $(\alpha, 2)$-orderable and free of $(\alpha, 2)$-orderable are introduced and is proved that If $(R, \odot)$ is a commutative hypergroup and $(R, \oplus, \odot)$ is free of $(\alpha, 2)$-orderable, then $G^{(\alpha)}(R)$ is a connected graph.

(iv) The notion of Δ-general hyperring is introduced and is shown that under some conditions, the zero divisor graphs based on Δ-general hyperring are isomorphic.

(v) We defined the hyperoperations on finite commutative rings such that their zero divisor graphs are bipartite.

(vi) It is to classify the diameters of zero divisor graphs of any given reproduced general hyperrings.

In future studies, we will try to obtain more results regarding zero divisor graphs based on fuzzy general hyperring, intersection graphs based on graded general hyperrings, zero divisor graphs based on graded general hyperrings, and fuzzy zero divisor graphs and their applications.

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