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Strongly regular relations derived from fundamental relations

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Abstract

We introduce a new regular relation δ on a given group G and show that δ is a congruence relation on G, concerning module the commutator subgroup of G. Then we show that the effect of this relation on the fundamental relation β is equal to the fundamental relation γ. We conclude that, if ρ is an arbitrary strongly regular relation on the hypergroup H, then the effect of δ on ρ, results in a strongly regular relation on H such that its quotient is an abelian group.

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1 Introduction

A particular type of equivalence relations, called the fundamental relations, plays an essential role in algebraic hyperstructures theory. The fundamental relations are one of the most important and interesting concepts in algebraic hyperstructures that ordinary algebraic structures are derived from algebraic hyperstructures by them. The fundamental relation β^*, which is the transitive closure of β was defined by M. Koskas [11], P. Corsini [6], D. Ferni [7, 8] and T. Vogiouklis [13]. Then D. Ferni introduced the fundamental relation γ^* which is the transitive closure of γ and is the smallest relation such that H/γ^* is an abelian group. He also showed that on hypergroups, $\beta = \beta^*$ and $\gamma = \gamma^*$. T. Vogiouklis generalized the fundamental relations in [13] for use on hyperrings, and then R. Ameri explained them based on polynomials [2]. In 2011, A. Connes and C. Consani introduced hyperrings corresponding to adele classes and studied algebraic geometry based on hyperrings [4]. Then, they utilized hyperrings to prove certain propositions in number theory [5]. In 2018, J. Jun also investigated tropical varieties on hyperrings [10]. In addition, in his doctoral thesis, he studied algebraic geometry on hyperstructures and proved important propositions in the field of algebraic geometry on hyperfields [9].
In recent years, the study of hyperstructures from the perspective of category theory has gained attention. Through this, important categories such as hypergroups, hyperrings, hypermodules, ect., have been investigated, and their relationships with classical categories have been studied [1,

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3, 12]. Our aim is to identify specific subsets of the set of strongly regular relations on hypergroups based on a particular property. If this property is commutativity, then we will perform this task using the relation δ, which will be introduced in Section 3. This is a part of a larger goal related to algebraic geometry, wherein concepts of algebraic geometry based on hyperstructures are transformed into classical algebraic geometry concepts using fundamental relations.

2 Preliminaries

Let $(H, \circ)$ be a semi-hypergroup and ρ be an equivalence relation on H. If A and B are non-empty subsets of H then $A\overline{\rho}B$ means that for each $a \in A$, there exists $b \in B$ such that $a\rho b$ and for each $b' \in B$ there exists $a' \in A$ such that $a'\rho b'$. Also $A\overline{\overline{\rho}}B$ means that for each $a \in A$ and $b \in B$ we have $a\rho b$ [6].

Definition 2.1. [6] *Let $(H, \circ)$ be a semi-hypergroup. The equivalence relation ρ is called:*
1- Regular on the right (on the left)*, if for all $x \in H$, from $a\rho b$, it follows that $(a\circ x)\overline{\rho}(b\circ x)$ $((x\circ a)\overline{\rho}(x\circ b)$ respectively);*
2- Strongly regular on the right (on the left)*, if for all $x \in H$, from $a\rho b$, it follows that $(a\circ x)\overline{\overline{\rho}}(b\circ x)$ $((x\circ a)\overline{\overline{\rho}}(x\circ b)$ respectively);*
3- Regular (strongly regular)*, if it is regular (strongly regular) on the right and on the left.*

Theorem 2.2. [6] *Let $(H, \circ)$ be a semi-hypergroup and ρ be an equivalence relation on H;*
If ρ is regular, then H/ρ is a semi-hypergroup with respect to the hyperoperation $\bar{x}\otimes\bar{y} = \{\bar{z}; z \in x\circ y\}$;
If the above hyperoperation is well defined on H/ρ, then ρ is regular.

Corollary 2.3. [6] *If $(H, \circ)$ is a hypergroup and ρ is an equivalence relation on H, then ρ is regular (strongly regular) if and only if $(H/\rho, \otimes)$ is a hypergroup (group).*

Theorem 2.4. [6] *Let $(H, \circ)$ be a semi-hypergroup and ρ be an equivalence relation on H.*
If ρ is strongly regular, then H/ρ is a semi-group with respect to the operation $\bar{x}\otimes\bar{y} = \bar{z}$, for all $z \in x\circ y$;
If the above operation is well defined on H/ρ, then ρ is strongly regular.

Definition 2.5. [6] *For all $n > 1$, we define the relations β_n and γ_n on a semi-hypergroup H, as follows:*

$$a\beta_n b \Longleftrightarrow \exists(x_1, x_2, ..., x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^n x_i,$$

$$a\gamma_n b \Longleftrightarrow \exists(x_1, x_2, ..., x_n) \in H^n, \sigma \in \mathbb{S}_n : \ x \in \prod_{i=1}^n x_i \ , \ y \in \prod_{i=1}^n y_{\sigma(i)},$$

and $\beta = \bigcup_{n\geq 1} \beta_n$ and $\gamma = \bigcup_{n\geq 1} \gamma_n$ where $\beta_1 = \gamma_1 = \{(x, x); x \in H\}$. Let β^* be the transitive closure of β and γ^* be the transitive closure of γ.

The relations β^* and γ^* are strongly regular and if H is a hypergroup then $\gamma = \gamma^*$ and $\beta = \beta^*$. Also, H/β^* and β^* are called the fundamental group and fundamental relation, respectively, and β^* is the smallest strongly regular relation on H.

Corollary 2.6. [6] *The quotient H/γ^* is a commutative semi-group and if H is a hypergroup, then H/γ^* is a commutative group.*

Theorem 2.7. [6] *The relation γ^* is the smallest strongly regular relation on a semi-hypergroup (hypergroup) such that the quotient H/γ^* is a commutative semi-group (commutative group).*

3 Operator $\delta *-$ on strongly regular relations

In this section, we provide a decomposition for γ on hypergroups, which significantly aids in examining the properties of the fundamental relation γ and also serves as a strategy for studying other fundamental relations.

Definition 3.1. *We define the relations δ and α on a group G as follows:*

$$g_1 \delta g_2 \Longleftrightarrow \begin{cases} \text{there are } m\in\mathbb{N},\ (y_1,y_2,...,y_m)\in G^m \text{and } \sigma\in\mathbb{S}_m \text{ such that} \\ g_1=\prod_{i=1}^m y_i \quad \text{and} \quad g_2=\prod_{i=1}^m y_{\sigma(i)}; \end{cases}$$

$$g_1 \alpha g_2 \Longleftrightarrow g_1 g_2^{-1} \in G'.$$

Where G' is the commutator subgroup of G.

Proposition 3.2. *Let G be a group. Then the relations δ and α are congruence relations on G.*

Proof. Since G' is normal then α is a congruence relation, and it is easy to see that δ is a reflexive and symmetric relation. Let $g_1\delta g_2$ and $g_2\delta g_3$. Hence by definition δ, there will be $m, n \in \mathbb{N}$, $\sigma \in \mathbb{S}_m$, $\tau \in \mathbb{S}_n$, $(x_1, x_2, ..., x_m) \in G^m$ and $(y_1, y_2, ..., y_n) \in G^n$ such that $g_1 = \prod_{i=1}^m x_i$, $g_3 = \prod_{j=1}^n y_{\tau(j)}$ and $g_2 = \prod_{i=1}^m x_{\sigma(i)} = \prod_{j=1}^n y_j$. So

$$x_1x_2...x_my_1y_2...y_n \ \delta \ y_{\tau(1)}y_{\tau(2)}...y_{\tau(n)}x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(m)}.$$

Now we can write:

$$x_1x_2...x_my_1y_2...y_n \ g_2^{-1} \ \delta \ y_{\tau(1)}y_{\tau(2)}...y_{\tau(n)}x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(m)} \ g_2^{-1}.$$

Therefore $g_1\delta g_3$. Now suppose that $g_1\delta g_2$ and $g_3 \in G$, so there are $n \in \mathbb{N}$, $\sigma \in \mathbb{S}_n$ and $(x_1, x_2, ..., x_n) \in G^n$ such that

$$(g_1, g_2) = (x_1x_2...x_n, x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(n)}) \in \delta.$$

So $(g_3g_1, g_3g_2) = (g_3x_1x_2...x_n, g_3x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(n)}) \in \delta.$ □

Lemma 3.3. *Let G be a group. Then $\delta = \alpha$.*

Proof. Because δ is a regular relation on G, then $(G/\delta, .)$ is a group, where $\delta(a_1).\delta(a_2) = \delta(a_1a_2)$ and $e_{G/\delta} = \delta(e_G)$ and $\delta(a)^{-1} = \delta(a^{-1})$. Since we have $a_1a_2\delta a_2a_1$ then

$$\delta(a_1).\delta(a_2) = \delta(a_1a_2) = \delta(a_2a_1) = \delta(a_2).\delta(a_1).$$

Thus $(G/\delta, .)$ is an abelian group. So $\alpha \subseteq \delta$, because α is the smallest strongly regular relation on G that G/α is an abelian group. Conversely, suppose that $a\delta b$, we must show that $ab^{-1} \in G'$. Since $a\delta b$ there will be $m \in \mathbb{N}$, $\sigma \in \mathbb{S}_m$ and $(x_1, x_2, ..., x_m) \in G^m$ that $a = \prod_{i=1}^m x_i$, and $b = \prod_{i=1}^m x_{\sigma(i)}$. We know that $x_ix_j = [x_i, x_j]x_jx_i$, where $[x_i, x_j] = x_ix_jx_i^{-1}x_j^{-1}$. So there is a natural number like k, and there are some elements of G like a_j and b_j $(1 \leq j \leq k)$ such that $x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(m)} = [a_1, b_1][a_2, b_2]...[a_k, b_k]x_1x_2...x_m$ where $g = [a_1, b_1][a_2, b_2]...[a_k, b_k] \in G^{'}$. Therefore

$$\begin{aligned} ab^{-1} &= x_1x_2...x_m(x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(m)})^{-1} \\ &= x_1x_2...x_m(gx_1x_2...x_m)^{-1} \\ &= g^{-1} \in G'. \end{aligned}$$

□

Example 3.4. (i) *Let $G = \mathbb{S}_3$, then for every $m \in \mathbb{N}$, $\sigma \in \mathbb{S}_m$ and $(x_1, x_2..., x_m) \in G^m$, $x_1x_2...x_m$ is even if and only if $x_{\sigma(1)}x_{\sigma(2)}...x_{\sigma(m)}$ is even. Therefore $(x, y) \in \delta$ if and only if $\{x, y\} \subseteq A_3$ or $\{x, y\} \cap A_3 = \emptyset$. So*

$$\delta = \Delta_{\mathbb{S}_3} \cup \{(\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_3), (\tau_2, \tau_1), (\tau_3, \tau_1), (\tau_3, \tau_2), (\sigma_1, \sigma_2), (\sigma_2, \sigma_1), (e, \sigma_1), (e, \sigma_2), (\sigma_1, e), (\sigma_2, e)\}.$$

Also, $\delta(\tau_1) = \delta(\tau_2) = \delta(\tau_3)$ and $\delta(e) = \delta(\sigma_2) = \delta(\sigma_3)$. This can be generalized as follows:

$$(x, y) \in \delta \Leftrightarrow \{x, y\} \subseteq A_n \text{ or } \{x, y\} \cap A_n = \emptyset.$$

(ii) *Let $G = Q_8$, then:*

$$\delta = \Delta_{Q_8} \cup \{(1, -1), (-1, 1), (i, -i), (-i, i), (j, -j), (-j, j), (k, -k), (-k, k)\}.$$

Remark 3.5. *Let ρ be a strongly regular relation on hypergroup H and σ be a congruence relation on group H/ρ. Define a relation $\sigma{*}\rho$ on H by*

$$(a, b) \in \sigma{*}\rho \Longleftrightarrow (\rho(a), \rho(b)) \in \sigma.$$

Lemma 3.6. *Let ρ be a strongly regular relation on hypergroup $(H, \circ)$ and σ be a congruence relation on group H/ρ, then $\sigma{*}\rho$ is also a strongly regular relation on H.*

Proof. It is clear that $\sigma{*}\rho$ is an equivalence relation on H. Let $h_1, h_2, h \in H$ and $(h_1, h_2) \in \sigma{*}\rho$. Therefore $(\rho(h_1), \rho(h_2)) \in \sigma$ and $(\rho(h), \rho(h)) \in \sigma$. Given that σ is a congruence relation on H/ρ, then $\rho(h_1).\rho(h)\sigma\rho(h_2).\rho(h)$ and since ρ is a strongly regular relation, we have $\rho(h_1).\rho(h) = \rho(h_1{\circ}h) = \rho(z_1)$ and $\rho(h_2).\rho(h) = \rho(h_2{\circ}h) = \rho(z_2)$ for each $z_1 \in h_1{\circ}h$ and $z_2 \in h_2{\circ}h$. Therefore $\rho(z_1) = \rho(h_1{\circ}h)\ \sigma\ \rho(h_2{\circ}h) = \rho(z_2)$. So $z_1(\sigma{*}\rho)z_2$, for each $z_1 \in h_1{\circ}h$ and $z_2 \in h_2{\circ}h$. □

Theorem 3.7. *Let $(H, \circ)$ be a hypergroup. Then $\gamma = \delta{*}\beta$.*

Proof. By Lemma 3.6, $\delta{*}\beta$ is a strongly regular relation and threrefore $H/(\delta{*}\beta)$ will be a group. Let $h_1, h_2 \in H$, by definition of δ we have

$$\beta(h_1)\beta(h_2)\ \delta\ \beta(h_2)\beta(h_1).$$

Since β is a strongly regular relation, then $\beta(z_1) = \beta(h_1{\circ}h_2)\ \delta\ \beta(h_2{\circ}h_1) = \beta(z_2)$ for each $z_1 \in h_1{\circ}h_2, z_2 \in h_2{\circ}h_1$. It means that $\delta{*}\beta(h_1{\circ}h_2) = \delta{*}\beta(h_2{\circ}h_1)$ and since $\delta{*}\beta$ is a strongly regular relation, we have $(\delta{*}\beta)(h_1).(\delta{*}\beta)(h_2) = (\delta{*}\beta)(h_2).(\delta{*}\beta)(h_1)$.
We now know that $(H/(\delta{*}\beta), .)$ is an abelian group and because γ is the smallest relation that H/γ is an abelian group we have $\gamma \subseteq \delta{*}\beta$. Consider $h_1, h_2 \in H$ and $(h_1, h_2) \in \delta{*}\beta$. So by definition, there are $m \in \mathbb{N}$ and $\sigma \in \mathbb{S}_m$ and $(\beta(x_1), \beta(x_2), ..., \beta(x_m)) \in (H/\beta)^m$ such that $\beta(h_1) = \prod_{i=1}^{m} \beta(x_i)$ and $\beta(h_2) = \prod_{i=1}^{m} \beta(x_{\sigma(i)})$. Since β is a strongly regular relation thus:

$$h_1 \in \beta(h_1) = \beta(x_1)\beta(x_2)...\beta(x_m) = \beta(x_1{\circ}x_2{\circ}...{\circ}x_m),$$

$$h_2 \in \beta(h_2) = \beta(x_{\sigma(1)})\beta(x_{\sigma(2)})...\beta(x_{\sigma(m)}) = \beta(x_{\sigma(1)}{\circ}x_{\sigma(2)}{\circ}\cdots{\circ}x_{\sigma(m)}).$$

Let $x \in x_1{\circ}x_2{\circ}...{\circ}x_m$ and $y \in x_{\sigma(1)}{\circ}x_{\sigma(2)}{\circ}...{\circ}x_{\sigma(m)}$ then we have $x\gamma y$. Since $x \in \beta(h_1)$ and $y \in \beta(h_2)$ thus $h_1\beta x$ and $y\beta h_2$. But $\beta \subseteq \gamma$, therefor $h_1\gamma x$ and $y\gamma h_2$. This shows that $h_1\gamma x\gamma y\gamma h_2$ hence $h_1\gamma h_2$. □

Theorem 3.8. *Let ρ be a strongly regular relation on hypergroup H, then $H/\delta{*}\rho$ is an abelian group.*

Proof. Since β is the smallest strongly regular relation on H, then $\beta \subseteq \rho$, and therefore $\delta*\beta \subseteq \delta*\rho$. Now by Theorem 3.7 we have $\gamma \subseteq \delta*\rho$. $\square$

Example 3.9. *Suppose that G is a group, and H is a normal subgroup. For each x and y of G, define $x\circ y = xyH$. We know that $(G,\circ)$ is a hypergroup. Also:*

$$\begin{aligned}(x,y) \in \beta &\Leftrightarrow x, y \in zH,\ for\ some\ z \in G\\ &\Leftrightarrow xH = yH.\end{aligned}$$

So $\beta(G) = G/H$. But $(G/H)' = G'H/H$ and therefore $\delta*\beta(G) = \gamma(G) = G/G'H$. Also $(x,y) \in \gamma$ if and only if there are $n \in \mathbb{N}$, $(z_1, z_2..., z_n) \in G^n$ and $\sigma \in \mathbb{S}_n$ such that $x \in z_1z_2...z_nH$ and $y \in z_{\sigma(1)}z_{\sigma(2)}...z_{\sigma(n)}H$. Let $z = z_1z_2...z_n$ and $w = z_{\sigma(1)}z_{\sigma(2)}...z_{\sigma(n)}$, then $xH = zH$ and $yH = wH$. Since $xG'H = zG'H$, $yG'H = wG'H$ and $zG' = wG'$, then $xG'H = yG'H$.
On the other hand, if $xG'H = yG'H$, then $xHG' = yHG'$. So there are $n \in \mathbb{N}$, $(z_1, z_2, ..., z_n) \in G^n$ and $\sigma \in \mathbb{S}_n$ such that $xH = z_1Hz_2H...z_nH$ and $yH = z_{\sigma(1)}Hz_{\sigma(2)}H...z_{\sigma(n)}H$. Thus

$$x \in xH = z_1z_2...z_nH = z_1\circ z_2\circ...\circ z_n \ and\ y \in yH = z_{\sigma(1)}z_{\sigma(2)}...z_{\sigma(n)}H = z_{\sigma(1)}\circ z_{\sigma(2)}\circ...\circ z_{\sigma(n)}.$$

Corollary 3.10. *Let $\varphi : H_1 \to H_2$ be a good homomorphism of hypergroups and ρ_1 and ρ_2 are strongly regular relations on H_1 and H_2 respectively, such that $\varphi(\rho_1(x)) \subseteq \varphi(\rho_2(x))$, for every $x \in H$. Then $\bar{\varphi} : H_1/\rho_1 \to H_2/\rho_2$ where $\bar{\varphi}(\rho_1(x)) = \rho_2(\varphi(x))$ is a homomorphism of groups.*

Proof. If $(x,y) \in \rho_1$, then $(\varphi(x), \varphi(y)) \in \rho_2$, so $\bar{\varphi}$ is well-defined. Since ρ_1 and ρ_2 are strongly regular relations and φ is homomorphism, then

$$\begin{aligned}\bar{\varphi}(\rho_1(x) + \rho_1(y)) = \bar{\varphi}(\rho_1(x+y)) &= \rho_2(\varphi(x+y)) = \rho_2(\varphi(x) + \varphi(y))\\ &= \rho_2(\varphi(x)) + \rho_2(\varphi(y)) = \bar{\varphi}(\rho_1(x)) + \bar{\varphi}(\rho_1(y)).\end{aligned}$$

$\square$

Let H be a hypergroup and $\Delta_H = \{(h,h); h \in H\}$, $\nabla_H = H^2$ and $R(H)$ be the set of all strongly regular relations on H. Also consider $R'(H) = \{\rho \in R(H);\ H/\rho\ \ is\ abelian\ group\}$ and

$$\rho\vee\sigma = \rho \cup (\rho\circ\sigma) \cup (\rho\circ\sigma\circ\rho) \cup (\rho\circ\sigma\circ\rho\circ\sigma) \cup \cdots$$

where $\rho\circ\sigma = \{(x,y) \in H^2;\ (x,z) \in \rho, (z,y) \in \sigma\ \ for\ some\ z \in H\}$.

Proposition 3.11. *Let H be a hypergroup. Then $R(H)$ is a lattice.*

Proof. For every $\rho, \sigma \in R(H)$, $\rho\vee\sigma$ is the smallest equivalence relation containing ρ and σ. Let $(x,y) \in \rho\vee\sigma$ and $z \in H$ then there are $n \in \mathbb{N}$ and $z_1, ..., z_n \in H$ such that $z_1 = x$, $z_n = y$ and $(z_i, z_{i+1}) \in \rho \cup \sigma$ for every $1 \le i \le n-1$. Without loss of generality assume that $(x, z_2) \in \rho$, $(z_2, z_3) \in \sigma$,..., $(z_{n-1}, y) \in \rho$. So $z\circ x\ \overline{\overline{\rho}}\ z\circ z_2\ \overline{\overline{\sigma}}\ z\circ z_3\ ...\ z\circ z_{n-1}\ \overline{\overline{\rho}}\ z\circ y$. Hence for every $t_1 \in z\circ x$ and $t_n \in z\circ y$ there are $t_2, ..., t_{n-1} \in H$ such that $t_1\rho t_2\sigma t_3...t_{n-1}\rho t_n$ where $t_k \in z\circ z_k$. Thus $(t_1, t_n) \in \rho\vee\sigma$ and $z\circ x\ \overline{\overline{\rho\vee\sigma}}\ z\circ y$ (similarly $z\circ x\ \overline{\overline{\rho\vee\sigma}}\ z\circ y$).
Therefore $\rho\vee\sigma \in R(H)$, and the poset $(R(H), \subseteq)$ is a lattice. $\square$

Since $\bigvee_{\rho\in R(H)} \rho = \nabla_H \in R(H)$ and for every $A \subseteq R(H)$; $\bigcap_{\rho\in A} \rho \in R(H)$ then $R(H)$ is complete lattice. If $\rho \in R(H)$ and $\gamma \subseteq \rho$ then for every $x, y \in H$ we have

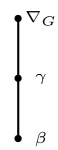
$$\rho(x)\rho(y) = \rho(xy) = \rho(\gamma(xy)) = \rho(\gamma(yx)) = \rho(yx) = \rho(y)\rho(x).$$

Hence $\rho \in R'(H)$ and since for every $\rho \in R'(H)$ we have $\gamma \subseteq \rho$ then $R'(H) = \langle\gamma\rangle \leqslant R(H)$, where $\langle\gamma\rangle$ is upper segment of lattice $R(H)$ generated by γ. Also $\bigcap_{\rho \in R'(H)} \rho = \gamma$ and $\bigvee_{\rho \in R'(H)} \rho = \nabla_H$.

Example 3.12. *Consider the hypergroup $(G, \circ)$ where $G = \{p, q, x_1, x_2, x_3, x_4, x_5\}$ and*

$\circ$	p	q	x_1	x_2	x_3	x_4	x_5
p	$\{p,q\}$	$\{p,q\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$
q	$\{p,q\}$	$\{p,q\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$
x_1	$\{x_1\}$	$\{x_1\}$	$\{p,q\}$	$\{x_4\}$	$\{x_5\}$	$\{x_2\}$	$\{x_3\}$
x_2	$\{x_2\}$	$\{x_2\}$	$\{x_5\}$	$\{p,q\}$	$\{x_4\}$	$\{x_3\}$	$\{x_1\}$
x_3	$\{x_3\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$	$\{p,q\}$	$\{x_1\}$	$\{x_2\}$
x_4	$\{x_4\}$	$\{x_4\}$	$\{x_3\}$	$\{x_1\}$	$\{x_2\}$	$\{x_5\}$	$\{p,q\}$
x_5	$\{x_5\}$	$\{x_5\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1\}$	$\{p,q\}$	$\{x_4\}$

Thus $\beta = \Delta_G \cup \{(p,q),(q,p)\}$ and $\beta(G) \cong \mathbb{S}_3$. Since β is the smallest strongly regular relation and also strongly regular relations and regular relations are the same on groups, then $R(H) = \{\beta, \gamma, \nabla_G\}$ where $\gamma = \beta \cup \{(x_i, x_j);\ i,j \in \{1,2,3\}, i \neq j\} \cup \{(x_r, x_s);\ r,s \in \{4,5\}, r \neq s\}$, $\gamma(G) \cong \mathbb{Z}_2$ and



Proposition 3.13. *Let $\rho \in R(H)$. Then $\delta * \rho = \gamma \vee \rho$.*

Proof. We know that $\delta*\rho$ is a strongly regular relation containing ρ, such that $H/\delta*\rho$ is an abelian group. Since $\gamma\vee\rho$ is the smallest strongly regular relation containing ρ, such that $H/\delta\vee\rho$ is an abelian group, then $\delta*\rho \subseteq \gamma\vee\rho$.
Let $(a,b) \in \delta*\rho$. Then $(\rho(a), \rho(b)) \in \delta$ and there are $n \in \mathbb{N}$, $\tau \in \mathbb{S}_n$ and $x_1, ..., x_n \in H$ such that

$$\rho(a) = \prod_{i=1}^{n} \rho(x_i) = \rho(\prod_{i=1}^{n} x_i);$$

$$\rho(b) = \prod_{i=1}^{n} \rho(x_{\tau(i)}) = \rho(\prod_{i=1}^{n} x_{\tau(i)}).$$

Consider $x \in \prod_{i=1}^{n} x_i$ and $y \in \prod_{i=1}^{n} x_{\tau(i)}$. Then $x\gamma y$, $a\rho x$ and $b\rho y$. So $a\ \rho\ x\ \gamma\ y\ \rho\ b$. Therefore $(a,b) \in \gamma\vee\rho$. □

So $\delta*\rho$ is the smallest strongly regular relation containing ρ in R'. Also, $*$ is associative because if $\tau \in R(H)$ and σ is a regular relation on H/τ and ρ is a regular relation on $H/\delta*\tau$, then

$$\begin{aligned}(a,b) \in \rho*(\sigma*\tau) &\Leftrightarrow ((\sigma*\tau)(a), (\sigma*\tau)(b)) \in \rho \\ &\Leftrightarrow (\sigma(\tau(a)), \sigma(\tau(b))) \in \rho \\ &\Leftrightarrow (\tau(a), \tau(a)) \in \rho*\sigma \\ &\Leftrightarrow (a,b) \in (\rho*\sigma)*\tau.\end{aligned}$$

We know that $\beta = \Delta$ on groups, so if $\rho \in R(H)$, then $(a,b) \in \beta*\rho$ if and only if $(\rho(a), \rho(b)) \in \Delta$ if and only if $(a,b) \in \rho$. Hence $\beta*\rho = \rho$.

Corollary 3.14. *Let $\rho, \sigma \in R(H)$. Then $\delta*(\rho\vee\sigma) = \delta*\rho \vee \delta*\sigma$ and $\delta*(\rho \cap \sigma) \subseteq \delta*\rho \cap \delta*\sigma$.*

Proof. The proof is clear. Note that if $\rho, \sigma \in R(H)$, then

$$\delta{*}(\rho \vee \sigma) = \gamma \vee (\rho \vee \sigma) = \gamma \vee \rho \vee \gamma \vee \sigma = \delta{*}\rho \vee \delta{*}\sigma.$$

□

4 Conclusions

It was proved above that if G is a group, then $G/G' \cong G/\delta$. So, if $\rho \in R(H)$, then

$$\frac{H/\rho}{(H/\rho)'} \cong \frac{H/\rho}{\delta}$$

but $\dfrac{H/\rho}{\delta} = \dfrac{H}{\delta{*}\rho}$. Therefore, the following diagram is commutative:

$$\begin{array}{ccc} H & \xrightarrow{\ \rho\ } & H/\rho \\ & \overset{\delta*\rho}{\searrow} & \downarrow \delta \\ & & \dfrac{H/\rho}{(H/\rho)'} \end{array}$$

Let **H-Group** denote the category of hypergroups, which its objects are hypergroups and its morphisms are good homomorphisms of hypergroups. Also by **Ab** and **Group** we denote the categories of Abelian groups and groups, respectively. Then the maps $\mathcal{F}_\gamma$: **H-Group** $\to$ **Ab**, $\mathcal{F}_\beta$: **H-Group** $\to$ **Group** and $\mathcal{F}_\delta$: **Group** $\to$ **Ab** are functors. Moreover $\mathcal{F}_\gamma = \mathcal{F}_{\delta*\beta}$ and the following diagram of functors is commutative:

$$\begin{array}{ccc} \textbf{H-Group} & \xrightarrow{\ \mathcal{F}_\beta\ } & \textbf{Group} \\ & \overset{\mathcal{F}_\gamma}{\searrow} & \downarrow \mathcal{F}_\delta \\ & & \textbf{Ab} \end{array}$$

Let $\{H_i\}_{i\in I}$ and $\{G_i\}_{i\in I}$ be, respectively, a family of groups and a family of hypergroups. Since $(\prod_{i\in I} G_i)' = \prod_{i\in I} G_i'$, then γ maps every product in the category **H-Group** to a product in the category **Ab**, as:

$$\gamma(\textstyle\prod_{i\in I} H_i) = \delta{*}\beta(\prod_{i\in I} H_i) = \delta(\prod_{i\in I} \beta(H_i)) = \prod_{i\in I} \delta{*}\beta(H_i) = \prod_{i\in I} \gamma(H_i).$$

For the following research, one can investigate the mentioned property for other categorical concepts, especially coproducts, using the relation $\mathcal{F}_{\delta*\beta} = \mathcal{F}_\delta \circ \mathcal{F}_\beta$.
Additionally, we will explore the issue that if σ and τ are the smallest regular relations such that, for any arbitrary group G, the groups $\sigma(G)$ and $\tau(G)$ are solvable and nilpotent respectively, then $\sigma{*}\beta$ and $\tau{*}\beta$ are strongly regular relations, such that for any arbitrary hypergroup H, the groups $\sigma{*}\beta(H)$ and $\tau{*}\beta(H)$ are solvable and nilpotent respectively. Consequently:

$$\mathcal{F}_{\sigma*\beta} = \mathcal{F}_\sigma \circ \mathcal{F}_\beta;$$
$$\mathcal{F}_{\tau*\beta} = \mathcal{F}_\tau \circ \mathcal{F}_\beta.$$

References

[1] R. Ameri, *On categories of hypergroups and hypermodules*, Journal of Discrete Mathematical Sciences and Cryptography, 6(2-3) (2003), 121–132. https://doi.org/10.1080/09720529.2003.10697969

[2] R. Ameri, T. Nozari, *A new characterization of fundamental relation on hyperrings*, International Journal of Contemporary Mathematical Sciences, 5(15) (2010), 721–738.

[3] R. Ameri, H. Shojaei, *Projective and injective Krasner hypermodules*, Journal of Algebra and Its Applications, 20(10) (2021). https://doi.org/10.1142/S0219498821501863

[4] A. Connes, C. Consani, *The hyperring of adéle classes*, Journal of Number Theory, 131(2) (2011), 159–194. https://doi.org/10.1016/j.jnt.2010.09.001

[5] A. Connes, C. Consani, *Universal thickening of the field of real numbers*, In Advances in the Theory of Numbers, Springer, (2015), 11–74. https://doi.org/10.1007/978-1-4939-3201-6_2

[6] P. Corsini, *Prolegomena of hypergroup theory*, Aviani editor, 1994.

[7] D. Freni, *A new characterization of the derived hypergroup via strongly regular equivalences*, Communications in Algebra, 30(8) (2002), 3977-3989. https://doi.org/10.1081/AGB-120005830

[8] D. Freni, *Strongly transitive geometric spaces: Applications to hypergroups and semigroups theory*, Communications in Algebra, 32(3) (2004), 969-988. https://doi.org/10.1081/AGB-120027961

[9] J. Jun, *Algebraic geometry over semi-structures and hyper-structures of characteristic one*, PhD thesis, Johns Hopkins University, 2015.

[10] J. Jun, *Algebraic geometry over hyperrings*, Advances in Mathematics, 323 (2018), 142–192. https://doi.org/10.1016/j.aim.2017.10.043

[11] M. Koskas, D. Freni, *Groupoids, semi-hypergroups and hypergroups*, Journal de Mathématiques Pures et Appliquées, 2 (1970), 155.

[12] M. Norouzi, R. Ameri, V. Leoreanu-Fotea, *Normal hyperideals in Krasner (m,n)-hyperrings*, Analele stiintifice ale Universitatii Ovidius Constanta, Seria Mathematica, 26(3) (2018), 197-212. https://doi.org/10.2478/auom-2018-0041

[13] T. Vougiouklis, *The fundamental relation in hyperrings, the general hyperfield*, Fourth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1990), World Scientific, 203-211.