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An algebraic study of $L^{\mathcal{B}}$-valued general fuzzy automata: On the concept of the layers

Kh. Abolpour [1] and M. Shamsizadeh [2]

[1] *Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran*
[2] *Department of Mathematics, Behbahan Khatam Alanbia University of Technology, Khouzestan, Iran*

Abolpor_ kh@yahoo.com, shamsizadeh.m@gmail.com

Abstract

The present study aims at introducing a new concept of a layer of $L^{\mathcal{B}}$-valued general fuzzy automata ($L^{\mathcal{B}}$-valued GFA) in which $\mathcal{B}$ is said to be a set of propositions concerning the GFA, where its underlying structure is a lattice-ordered monoid. Generally, it demonstrates that the layer has a significant impact on the algebraic study of $L^{\mathcal{B}}$-valued GFA by showing the concepts of sub-automata and separated sub-automata of an $L^{\mathcal{B}}$-valued GFA in terms of its layers. In the other words, it highlights that every $L^{\mathcal{B}}$-valued general fuzzy automaton at the least demonstrates one strongly related sub-automaton. Specifically, the characterization of some algebraic concepts like sub-automaton, retrievability and connectivity of an $L^{\mathcal{B}}$-valued GFA in terms of its layers is provided. In addition, it is shown that the maximal layer of a cyclic $L^{\mathcal{B}}$-valued general fuzzy automaton as well as the minimal layer of a directable $L^{\mathcal{B}}$-valued general fuzzy automaton are found to be distinctive. Finally, we investigate the different poset structures which are connected with an $L^{\mathcal{B}}$-valued general fuzzy automaton, demonstrating some of these posets as finite upper semilattice, and introducing the isotone Galois connections between some of the pairs of the posets/finite upper semilattices introduced.

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1 Introduction

Galois connections originally appeared in the work of Ore [25] to provide a general type of correspondence between structures, and are the generalization of Galois theory introduced by É. Galois to interpret the relationship between field theory and group theory. Galois connections has offered the structure-preserving passage between two worlds of our imagination (cf., Denecke, Erní, and Wismath [10]), and these two mentioned worlds would be so diverse that the least possible connection could be seldom ever imagined (cf., García-Pardo et al., [13]). Moreover, it has been pointed out by Belohalavek [6], that Galois connections capture the very natural rules "the more objects, the less common attributes", and vice-versa. These days, Galois connections appear ubiquitary to play a vital role in human reasoning involving hierarchies. For example, some of its applications area covering situations or systems having (i) precise natures are; formal concept analysis (cf., Belohalavek and Konecny [7], Ganter and Wille [12], Wille [35]), category theory (cf., Herrlich and Husek [15], Kerkhoff [21]), logic (cf., Cornejo et al., [9]), category theory, topology and logic (cf., Denecke et al., (Eds) [10]); (ii) imprecise or uncertain natures are; mathematical morphology, category theory (cf., García et al., [14]), fuzzy transform (cf., Perfilieva [27]), Soft computing (cf., García-Pardo et al., [13]); and (iii) vagueness natures; data analysis, reasoning having incomplete information (cf., Järvinen [19]), Pawlak [26], Perfilieva [27]). Here, it is important to note that the equivalence relations based on original Pawlak's (cf., Pawlak [26]), approximation operators form isotone Galois connections and turn out to be interior and closure operators.

The Galois connections provide the important and fundamental framework to establish interrelationship between different structures involving hierarchies. The Galois connections between two sets, precisely between their power sets equipped with the inclusion order, has two perspectives, namely covariant and contravariant (cf., Birkhoff [8], Erné [10] page 1-138, García et al., [13], Herrlich and Husek [15], Ore [25]). The co-variant Galois connection between two sets is a pair of maps with order-preserving property, and therefore, the term isotone is also used for them. An order-preserving (covariant or isotone) Galois connection is also referred to as adjunctions (cf., Erne [10] page 1-138). The contravariant Galois connection between two sets is a pair of maps with order-reversing property and so the term antitone is also used for them. In Birkhoff [8], Erné [10] page 1-138, García et al., [13], the terms polarity and axiality have been used for a contravariant Galois connections and covariant Galois connections between power sets, respectively. Automata are well known mathematical models of computations studied by several authors [16, 18].

The use of Zadeh [37] fuzzy sets generalize the notion of automata to fuzzy automata, Santos [30] and Wee and Fu [34] initiated such studies and attracted researchers to develop fuzzy automata theory in several directions (cf., [24, 28, 34, 36]). Malik and Mordeson [24], established a basic framework for the algebraic aspects of the theory of fuzzy automata. Further, Ito [17], deals with the algebraic view of fuzzy automata. Recently, following the work of Ito [18], Tiwari, Yadav and Singh [32], studied fuzzy automata to explore its several algebraic properties the uniqueness of the maximal layer of cyclic automata and minimal layers of directable fuzzy automata along with relationships between upper semilattices and fuzzy automata were established.

The studies concerning algebraic automata has been carried out by several researchers in various forms (cf., eg., [4, 5, 16, 17, 18, 31, 33]). In [5], for instance, some specific notions such as separateness, connectedness and retrievability of automata have been established and investigated in detail. Further in [16], the concepts as decompositions and several products of automata have been examined. Moreover, [18] has been regarded as one of the most recent contributions in this realm which establishes the structure of an automaton. In another study [5], it has been reported that the investigations on such notions of automata obviously result in a deeper understanding of automata structure and also their applications. In [16], on the other hand, it has been emphasized that these concepts have been emerged from a desire to recognize a certain behavior of a system in an environment and its significant role in the enhancement of the fundamentals which are related to the computer science.

In [17], (i) it is shown that corresponding to every automaton there exists a poset which is isomorphic to a given poset, (ii) the sub-automata of a given automaton can be characterized by means of layers of the automaton. (iii) it has been also proved that for a given upper semilattice, there exists an automaton which induces an upper semilattice set of all sub-automata, isomorphic to the given upper semilattice, while Atani and Bazari [4], investigate conditions which determine whether for a given finite upper semilattice, there exists an automaton which induces a finite upper semilattice over class of it's all sub-automata under set inclusion, isomorphic to the given finite upper semilattice.

Concerning the above mentioned descriptions, the investigation on the algebraic fuzzy automata has been initially conducted by Malik [23] (cf., [24] for details), and subsequently a number of other studies have been performed in this area (cf., e.g., [17]). In addition, a certain study in [31] has demonstrated that it was possible to locate (fuzzy) topologies on the state-sets of fuzzy automata in some natural ways. Further, it concluded that these (fuzzy) topologies could be applied to establish some of the algebraic results of fuzzy automata which have been examined in [24] with less difficulty. From a completely different perspective, a much recent work on fuzzy automata has been carried out in [22] as well.

Doostfatemeh and Kremer [11], have explicated the notion of fuzzy automata, through which they proposed the concept of general fuzzy automata. Regarding that, the key impetus was the inadequacy of the obtain able literature to deal with certain applications which employed fuzzy automata in the form of a modeling tool which allocates membership values to active states of related fuzzy automaton. A zero-weight transition has meant no transition in all types of conventional automata. In this approach that we have employed for general fuzzy automata, however, a zero-weight transition have not necessarily required no transition. It has been the main reason that we apply [0, 1] as the fuzzy interval. The concept known as ($L^{\mathcal{B}}$-valued general fuzzy automata ($L^{\mathcal{B}}$-valued GFA) has also been established in the studies by Abolpour and Zahedi [2], in which B is considered as a set of propositions concerning the general fuzzy automata, and where its underlying structure is a complete ifinitely distiributive lattice. In addition, in their works, Zahedi and Abolpour and also a number of other researchers in the field have studied the procedures of how fuzzy automata theory have been developed [1, 2, 3, 28, 31, 32].

This present study aims at investigating the algebraic properties of $L^{\mathcal{B}}$-valued general fuzzy automata and characterizing isotone Galois connections between some pairs of posets/upper semilattices induced by given $L^{\mathcal{B}}$-valued general fuzzy automaton. This study is therefore structured as follows:

In Section 2, the concepts such as separateness, connectedness and retrievability of $L^{\mathcal{B}}$-valued general fuzzy automata are examined. Section 3 is towards the description of algebraic concepts of $L^{\mathcal{B}}$-valued GFA concerning to its layers. In Section 4, the isotone Galois Connections between the finite upper semilattices $(S, \sqsubseteq)$ and $(Ls(E), \subseteq)$; finite upper semilattices $(Ls(E), \subseteq)$ and $(G(Ls(S)), \subseteq)$; and posets $(E, \preceq)$ and $(G(Ls(E)), \subseteq)$ associated with a given $L^{\mathcal{B}}$-valued general fuzzy automaton are established and characterized.

2 Preliminaries

In the following section, some essential concepts which are related to $L^{\mathcal{B}}$-valued general fuzzy automata, lattice-ordered monoid and isotone Galois connection, which have been recalled from [1, 2, 20, 29, 32, 33] are introduced and characterized.

Definition 2.1. *Given a poset $(S, \preceq)$ and $x, y \in S$, $x \neq y$, we call x the predecessor of y, and y the successor of x if $x \preceq z \preceq y$ and $z \in S \Rightarrow z = x$ or $z = y$, and subsequently denote this relation as $< x, y >$. Given $x, y \in S$, the element $z = x \vee y \in S$ is called the least upper bound or supremum of x and y if $x \preceq z$ and $y \preceq z$ and $z \preceq w \in S$ whenever $x \preceq w$ and $y \preceq w$ for every $w \in S$. The greatest lower bound or infimum $x \wedge y$ is defined in a similar way.*

Definition 2.2. *A poset is called a lattice if $\forall x, y \in S$, $\exists$ both a least upper bound and a greatest lower bound of x and y and an upper semilattice, if for all $x, y \in S$, $\exists$ supremum of x and y.*

Definition 2.3. *We denote by $D(S)$ the directed graph of poset $(S, \preceq)$ having its vertices as elements of S, if x, y are two distinct vertices, then there is an edge (x, y) from vertex x to vertex y if and only if $< x, y >$, for an vertex y its in-degree is defined as $deg^{-}y =$ number of edges going to y.*

Definition 2.4. *Given a poset $(S, \preceq)$, a non-empty subset A of S is called a lower set, if for $x \in S$ and $y \in A$ and $x \preceq y$ implies $x \in A$. Further, for every $x \in S$, the set defined as $< x >= \{y \in S : y \preceq x\}$ is called the principle lower set.*
The family of all lower sets of a poset $(S, \preceq)$ is denoted by $Ls(S)$.

Proposition 2.5. *Let $(S, \preceq)$ be a poset, then $(S, \preceq) \cong (G(Ls(S)), \subseteq)$.*

Definition 2.6. *Let $(R, \leq_R)$ and $(S, \leq_S)$ be posets. A pair (φ, ψ) of mappings $\varphi : R \to S, \psi : S \to R$ is called isotone Galois connection between R and S if the following equivalence is satisfied for all $r \in R$ and $s \in S$, $\varphi(r) \leq_S s$ if and only if $r \leq_R \psi(s)$.*
This notion is also called adjunction. The mapping φ is called a lower (or left) adjoint of ψ, and the mapping ψ is called an upper (or right) adjoint of φ.

Proposition 2.7. *Let $\varphi : R \to S$ and $\psi : S \to R$ be two maps between the posets $(R, \leq_R)$ and $(S, \leq_S)$. The pair (φ, ψ) is an isotone Galois connection if and only if (i) ψ and φ are order-preserving; (ii) $r \leq_R \psi(\varphi(r))$, for all $r \in R$; (iii) $\varphi(\psi(r)) \leq_S s$, for all $s \in S$.*

Definition 2.8. *An algebra $\mathcal{L} = (L, \leq, \wedge, \vee, \otimes, 0, 1)$ is regarded as a lattice-ordered monoid if*
1) *$\mathcal{L} = (L, \leq, \wedge, \vee, \otimes, 0, 1)$ has been a lattice with the least element 0 and the greatest element 1.*
2) *$(L, \otimes, e)$ has been a monoid with identity $e \in L$ such that for all $a, b, c \in L$*
(i) $a \otimes 0 = 0 \otimes a = 0$,
(ii) $a \leq b \Rightarrow \forall x \in L, a \otimes x \leq b \otimes x$ *and* $x \otimes a \leq x \otimes b$,
(iii) $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$ *and* $(b \vee c) \otimes a = (b \otimes a) \vee (c \otimes a)$.

Definition 2.9. *A monoid $(L, \otimes, e)$ is considered as a monoid without zero divisors if for all $a, b \in L$, $a \neq 0, b \neq 0 \Rightarrow a \otimes b \neq 0$.*

Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. In the case we fix an input $b_k \in \Sigma$ at time t the proposition $\gamma|_{b_k}$ can be calculated by $\mu^t(q)$ if the general fuzzy automaton $\tilde{\mathcal{F}}$ is in the state q at time t otherwise $\gamma|_{b_k}$ is 0 if $\tilde{\mathcal{F}}$ is not in the active state q. Consequently, for each state $q \in \mathcal{Q}$ it is possible to examine the truth value of $\gamma|_{b_k}$, it is designated by $\gamma|_{b_k}(q)$. As it has

been explicated before $\gamma|_{b_k}(q) \in [0,1]$. This section therefore aims at establishing the B, which is a set of propositions about the general fuzzy automaton $\tilde{\mathcal{F}}$.
We can assign the order $\leq$ on B as follows:
For $\gamma, \eta \in B$, $\gamma \leq \eta$ if and oly if $\gamma(q) \leq \eta(q)$ for all $q \in \mathcal{Q}$. Also, we define $\gamma \otimes \eta = \min(\gamma(q), \eta(q))$ for all $\gamma, \eta \in B$ and $q \in \mathcal{Q}$. One can immediately check that the contradiction, i.e., the proposition with constant truth value 0, is the least element and the tautology, i.e., the proposition with constant truth value 1 is the greatest component of B. Thus, $\mathcal{B} = (B, \leq, \wedge, \vee, \otimes, 0, 1)$ is a lattice-ordered monoid.
We can characterize $L^{\mathcal{B}}$-valued subset of $\mathcal{Q} \times \Sigma \times \mathcal{Q}$, i.e., a map $\delta : \mathcal{Q} \times \Sigma \times \mathcal{Q} \to L^{\mathcal{B}}$. The range set $L^{\mathcal{B}}$ allows us to deduce $L^{\mathcal{B}}$ as a map which assigns each (q_i, a_k, q_j) to $\delta(q_i, a_k, q_j) : \mathcal{B} \to L$. This interpretation of transition map δ will permit us to signify it as the family $\{\delta^\alpha | \alpha \in \mathcal{B}\}$ of L-valued sets $\delta^\alpha \in L^{\mathcal{Q} \times \Sigma \times \mathcal{Q}}$ of $\mathcal{Q} \times \Sigma \times \mathcal{Q}$ which is ordered by the elements of $\mathcal{B}$, in which the L-valued sets δ^α have been characterized by

$$\delta^\alpha(q_i, a_k, q_j) = \delta(q_i, a_k, q_j)(\alpha) = \begin{cases} \alpha|_{a_k}(q_i) \vee \alpha|_{a_k}(q_j), & \text{if } q_i, q_j \in \mathcal{Q}_{act}(t_i) \text{ upon input } a_k \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.10. *An $L^{\mathcal{B}}$-valued general fuzzy automaton is a 8-tuple $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, \tilde{\delta}, w, F_1, F_2)$, where $\tilde{\delta}$ is an $L^{\mathcal{B}}$-valued subset of $(\mathcal{Q} \times L) \times \Sigma \times \mathcal{Q}$, i.e., a map $\tilde{\delta} : (\mathcal{Q} \times L) \times \Sigma \times \mathcal{Q} \to L^{\mathcal{B}}$ such that:*

$$\tilde{\delta}^\alpha((q, \mu^t(q)), a_k, p) = F_1(\mu^t(q), \delta^\alpha(q, a_k, p)).$$

Let Σ^* be a monoid generated by a nonempty set Σ. Define a map $\tilde{\delta}^* : (\mathcal{Q} \times L) \times \Sigma^* \times \mathcal{Q} \to L^{\mathcal{B}}$ such that:

$$\tilde{\delta}^{*\alpha}((q, \mu^t(q)), \wedge, p) = \begin{cases} 1, & \text{if } q = p \\ 0, & \text{otherwise} \end{cases}, \text{ and}$$

$\forall q, p \in \mathcal{Q}, \forall u \in \Sigma^*, \forall x \in \Sigma$ *and* $\forall \alpha \in \mathcal{B}$

$$\tilde{\delta}^{*\alpha}((q, \mu^{t_i}(q)), ux, p) = \vee\{\tilde{\delta}^{*\alpha}((q, \mu^{t_i}(q)), u, q') \otimes \tilde{\delta}^\alpha((q', \mu^{t_j}(q')), x, p) | q' \in \mathcal{Q}_{pred}(p, x)\}.$$

To simplify notation; $\tilde{\delta}^*$ is also denoted by $\tilde{\delta}$.

Definition 2.11. *Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued general fuzzy automaton, $\alpha \in \mathcal{B}$ and $\mathcal{Q}' \subseteq \mathcal{Q}$. The successor and the predecessor of $\mathcal{Q}'$ are, respectively, the sets:*

$$S^\alpha(\mathcal{Q}') = \{p \in \mathcal{Q} | \tilde{\delta}^\alpha((q, \mu^t(q)), x, p) > 0 \text{ for some } x \in \Sigma \text{ and } q \in \mathcal{Q}'\},$$

$$P^\alpha(\mathcal{Q}') = \{q \in \mathcal{Q} | \tilde{\delta}^\alpha((q, \mu^t(q)), x, p) > 0 \text{ for some } x \in \Sigma \text{ and } p \in \mathcal{Q}'\}.$$

Definition 2.12. *An $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}}' = (\mathcal{Q}', \Sigma, \tilde{R}', Z, \tilde{\delta}', w', F_1, F_2)$ is called a sub-automaton of an $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ if $\mathcal{Q}' \subseteq \mathcal{Q}$, $q_0 \in \mathcal{Q}'$, $w' = w|_{\mathcal{Q}'}$, $S^\alpha(\mathcal{Q}') = \mathcal{Q}'$ and $\tilde{\delta}' = \tilde{\delta}|_{(\mathcal{Q}' \times L) \times \Sigma \times \mathcal{Q}'}$. Further, this sub-automaton regarded as separated if $S^\alpha(\mathcal{Q} - \mathcal{Q}') \cap \mathcal{Q}' = \emptyset$.*

Definition 2.13. *An $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}}$ is called*
(i) *strongly connected if $\forall p, q \in \mathcal{Q}$, $q \in S^\alpha(p)$,*
(ii) *connected if $\tilde{\mathcal{F}}$ has no separated proper sub-automaton,*
(iii) *retrievable if $\tilde{\delta}^\alpha((q, \mu^{t_i}(q)), a, p) > 0$, for some $(q, a, p) \in \mathcal{Q} \times \Sigma \times \mathcal{Q}$, then*

$$\tilde{\delta}^\alpha((p, \mu^{t_j}(p)), b, q) > 0 \text{ for some } b \in \Sigma.$$

Definition 2.14. *A homomorphism from an $L^{\mathcal{A}}$-valued GFA $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ to an $L^{\mathcal{B}}$-valued GFA $\tilde{\mathcal{F}}' = (\mathcal{Q}', \Sigma, \tilde{R}', Z, w', \tilde{\delta}', F_1, F_2)$ is a pair (h, k) of maps, where $h : \mathcal{Q} \to \mathcal{Q}'$ and $k : \mathcal{A} \to \mathcal{B}$ are functions such that:*
(i) $\tilde{\delta}'^{k(\alpha)}((h(q), \mu^t(h(q))), u, h(p)) \geq \tilde{\delta}^{\alpha}((q, \mu^t(q)), u, p)$,
(ii) $w(q) = z \Leftrightarrow w(h(q)) = z$,
(iii) $h(q_0) = q_0'$.

Proposition 2.15. *Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued general fuzzy automaton and $\mathcal{Q}' \subseteq \mathcal{Q}$. Then $S^{\alpha}(\mathcal{Q} - \mathcal{Q}') = \mathcal{Q} - \mathcal{Q}'$ if and only if $P^{\alpha}(\mathcal{Q}') = \mathcal{Q}'$.*

Definition 2.16. *An $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ is called cyclic if for all $p \in \mathcal{Q}$, there exists $q \in \mathcal{Q}$ and $u \in \Sigma^*$ such that $\tilde{\delta}^{\alpha}((q, \mu^t(q)), u, p) > 0$.*

3 Layers of $L^{\mathcal{B}}$-valued general fuzzy automata

In what follows, the notion of a layer of an $L^{\mathcal{B}}$-valued general fuzzy automaton is presented. It is shown that the layer has a great impact on the algebraic study of $L^{\mathcal{B}}$-valued GFA by explicating some significant concepts such as sub-automata and separated sub-automata of an $L^{\mathcal{B}}$-valued GFA in terms of its respected layers. It is also demonstrated that every cyclic $L^{\mathcal{B}}$-valued GFA encompasses a unique maximal layer and every directable $L^{\mathcal{B}}$-valued GFA encompasses a unique minimal layer. In the following, first, the concept of layers of an $L^{\mathcal{B}}$-valued GFA is introduced.
Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, \tilde{\delta}, w, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued general fuzzy automaton. Define a relation R on $\mathcal{Q}$ as:
$(p, q) \in R$ if and only if $\tilde{\delta}^{\alpha}((p, \mu^{t_i}(p)), u, q) > 0$, and $\tilde{\delta}^{\alpha}((q, \mu^{t_j}(q)), v, p) > 0$, for some $u, v \in \Sigma^*$.
Subsequently, R is an equivalent relation on $\mathcal{Q}$. For $p \in \mathcal{Q}$, we call the set $L_p = \{q \in \mathcal{Q} | (p, q) \in R\}$ a layer of $\tilde{\mathcal{F}}$. For two layers L_p and L_q of $\mathcal{Q}$, define $L_p \preceq L_q$ if $\tilde{\delta}^{\alpha}((q, \mu^{t_i}(q)), u, p) > 0$, for some $u \in \Sigma^*$. Now, it will not be difficult to observe that $\preceq$ is a partial order. By E, we mean $(\{L_p | p \in \mathcal{Q}\}, \preceq)$, which is certainly a poset.

Proposition 3.1. *Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, \tilde{\delta}, w, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued general fuzzy automaton. Then*
(i) *if $\tilde{\mathcal{F}}$ is retrievable, then for all $q \in \mathcal{Q}$ and $\alpha \in \mathcal{B}$, $S^{\alpha}(q)$ is a layer of $\tilde{\mathcal{F}}$, and*
(ii) *if $\tilde{\mathcal{F}}$ is strongly connected, subsequently $\mathcal{Q}$ itself is a layer of $\tilde{\mathcal{F}}$.*

Proof. It has been proved from the definition which is related to retrievable and strongly connected $L^{\mathcal{B}}$-valued general fuzzy automata. □

Proposition 3.2. *Let $E = \{L_p | p \in \mathcal{Q}\}$ be the set of all layers of an $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, \tilde{\delta}, w, F_1, F_2)$. Then $\tilde{\mathcal{F}}' = (\mathcal{Q}', \Sigma, \tilde{R}', Z, \tilde{\delta}', w', F_1, F_2)$ is a sub-automaton of $\tilde{\mathcal{F}}$ if and only if*
(i) *$\exists L_{p_1}, L_{p_2}, \ldots, L_{p_r} \in E$ such that $\mathcal{Q}' = \{q \in \mathcal{Q} | L_p \preceq L_{p_i}\}$, for some $i \in \{1, 2, \ldots, r\}$, and*
(ii) *$\tilde{\delta}'((q, \mu^t(q)), a_k, p) = \tilde{\delta}((q, \mu^t(q)), a_k, p)$, $\forall p, q \in \mathcal{Q}'$ and $\forall a_k \in \Sigma$.*

Proof. Let $\tilde{\mathcal{F}}' = (\mathcal{Q}', \Sigma, \tilde{R}', Z, \tilde{\delta}', w', F_1, F_2)$ be a sub-automaton of $\tilde{\mathcal{F}}$. Then $\mathcal{Q}' \subseteq \mathcal{Q}$, $S^{\alpha}(\mathcal{Q}') = \mathcal{Q}'$ and $\tilde{\delta}|_{(\mathcal{Q}' \times L) \times \Sigma \times \mathcal{Q}'} = \tilde{\delta}'$. Now, $S^{\alpha}(\mathcal{Q}') = \mathcal{Q}' \Rightarrow \mathcal{Q}' = \{q \in \mathcal{Q} | \tilde{\delta}^{\alpha}((p, \mu^t(p)), u, q) > 0 \text{ for some } (u, p) \in \Sigma^* \times \mathcal{Q}'\}$, or that $\exists L_{p_i} \in E' = \{L_p | p \in \mathcal{Q}'\}$ such that $\mathcal{Q}' = \{q \in \mathcal{Q} | L_q \preceq L_{p_i}\}$, i.e., $\exists L_{p_1}, L_{p_2}, \ldots, L_{p_r} \in E$ such that $\mathcal{Q}' = \{q \in \mathcal{Q} | L_q \preceq L_{p_i}, \text{ for some } i \in \{1, 2, \ldots, r\}\}$. Also, as $\tilde{\delta}' = \tilde{\delta}|_{(\mathcal{Q}' \times L) \times \Sigma \times \mathcal{Q}'}$ (ii) follows obviously.
On the contrary, let conditions as (i) and (ii) be held. To demonstrate that $\tilde{\mathcal{F}}'$ is a sub-automaton

of $\tilde{\mathcal{F}}$ it is enough to illustrate that $S^{\alpha}(\mathcal{Q}') \subseteq \mathcal{Q}'$. For this, let $q \in S^{\alpha}(\mathcal{Q}')$. Then there exist $p \in \mathcal{Q}'$ and $u \in \Sigma^*$ such that $\tilde{\delta}^{\alpha}((p, \mu^t(p)), u, q) > 0$. Now, $p \in \mathcal{Q}'$ implies that $L_p \preceq L_{p_i}$, for some $i \in \{1, 2, \ldots, r\}$, i.e., there exists $v \in \Sigma^*$ such that $\tilde{\delta}^{\alpha}((p_i, \mu^t(p_i)), v, p) > 0$. Also, $\tilde{\delta}^{\alpha}((p_i, \mu^t(p_i)), vu, q) \geq \tilde{\delta}^{\alpha}((p_i, \mu^t(p_i)), v, q) \wedge \tilde{\delta}^{\alpha}((p, \mu^t(p)), u, q) > 0$ implies that $L_q \preceq L_{p_i}$, or that $q \in \mathcal{Q}'$. Thus, $S^{\alpha}(\mathcal{Q}') \subseteq \mathcal{Q}'$. □

Proposition 3.3. *Let $E = \{L_p | p \in \mathcal{Q}\}$ be the set of all layers of an $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, \tilde{\delta}, w, F_1, F_2)$. Then $\tilde{\mathcal{F}}' = (\mathcal{Q}', \Sigma, \tilde{R}', Z, \tilde{\delta}', w', F_1, F_2)$ is a separated sub-automaton of $\tilde{\mathcal{F}}$ if and only if*
(i) *$\exists L_{p_1}, L_{p_2}, \ldots, L_{p_r} \in E$ such that $\mathcal{Q}' = \{q \in \mathcal{Q} | L_q \preceq L_{p_i}$ and $L_{p_j} \preceq L_q$, for some $i, j \in \{1, 2, \ldots, r\}\}$, and*
(ii) *$\tilde{\delta}'((q, \mu^t(q)), a_k, p) = \tilde{\delta}^{\alpha}((q, \mu^t(q)), a_k, p), \forall p, q \in \mathcal{Q}'$ and $\forall a_k \in \Sigma$.*

Proof. Based on Definition 2.12 and Proposition 2.15 and 3.2, It is only necessary to explain that $P^{\alpha}(\mathcal{Q}') = \mathcal{Q}'$ if and only if $q \in \mathcal{Q}'$ such that $L_{p_j} \preceq L_q$, for some $j \in \{1, 2, \ldots, r\}$. For this, let $P^{\alpha}(\mathcal{Q})' = \mathcal{Q}'$. Then $\mathcal{Q}' = \{q \in \mathcal{Q} | \tilde{\delta}^{\alpha}((q, \mu^t(q)), u, p) > 0$, for some $(u, p) \in \Sigma^* \times \mathcal{Q}'\}$, or that $\exists L_{p_j} \in E' = \{L_p | p \in \mathcal{Q}'\}$ such that $L_{p_j} \preceq L_q$, for some $j \in \{1, 2, \ldots, r\}$.
Conversely, let $q \in \mathcal{Q}'$ such that $L_{p_j} \preceq L_q$, for some $j \in \{1, 2, \ldots, r\}$. Also, let $p \in P^{\alpha}(\mathcal{Q})'$. Then there exist $q \in \mathcal{Q}'$ and $u \in \Sigma^*$ such that $\tilde{\delta}^{\alpha}((p, \mu^t(p)), u, q) > 0$. Now, $q \in \mathcal{Q}'$ implies that $L_{p_j} \preceq L_q$, for some $j \in \{1, 2, \ldots, r\}$, i.e., there exists $v \in \Sigma^*$ such that $\tilde{\delta}^{\alpha}((q, \mu^t(q)), v, p_j) > 0$. Also,

$$\tilde{\delta}^{\alpha}((p, \mu^t(p)), vu, p_j) \geq \tilde{\delta}^{\alpha}((p, \mu^t(p)), v, q) \wedge \tilde{\delta}^{\alpha}((q, \mu^t(q)), u, p_j) > 0,$$

implies that $L_{p_j} \preceq L_p$, or that $p \in \mathcal{Q}'$. Thus, $P^{\alpha}(\mathcal{Q})' \subseteq \mathcal{Q}'$, which together with $\mathcal{Q}' \subseteq P^{\alpha}(\mathcal{Q}')$, shows that $P^{\alpha}(\mathcal{Q}') = \mathcal{Q}'$. □

Proposition 3.4. *Every $L^{\mathcal{B}}$-valued general fuzzy automaton contains at least one strongly connected sub-automaton.*

Proof. Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued GFA, $p \in \mathcal{Q}$ and $L_p \in E$ be a minimal layer (with regard to the partial order $\preceq$). Therefore for $q \in S^{\alpha}(L_p)$, there exist $u \in \Sigma^*$ and $r \in L_p$ such that $\tilde{\delta}^{\alpha}((r, \mu^t(r)), u, q) > 0$. Now, $r \in L_p$ implies that there exists $v \in \Sigma^*$ such that $\tilde{\delta}^{\alpha}((p, \mu^t(p)), v, r) > 0$. Thus, $\tilde{\delta}^{\alpha}((p, \mu^t(p)), vu, q) \geq \tilde{\delta}^{\alpha}((p, \mu^t(p)), v, r) \wedge \tilde{\delta}^{\alpha}((r, \mu^t(r)), u, q) > 0$. Also, by minimality of $L_p, L_q \preceq L_q$, which shows that $\tilde{\delta}^{\alpha}((q, \mu^t(q)), w, p) > 0$, for some $w \in \Sigma^*$. Thus, for all $q \in S^{\alpha}(L_p), q \in L_p$, or that $(L_p, \Sigma, L_{q_0}, Z, w, \tilde{\delta}|_{(L_p \times L) \times \Sigma \times L_p}, F_1, F_2)$ is a sub-automaton of $\tilde{\mathcal{F}}$. Moreover, let $q, r \in L_p$. Thus, there will be $u, v \in \Sigma^*$ such that $\tilde{\delta}^{\alpha}((p, \mu^t(p)), u, q) > 0$ and $\tilde{\delta}^{\alpha}((r, \mu^t(r)), v, p) > 0$, or that $\tilde{\delta}^{\alpha}((r, \mu^t(r)), vu, q) > 0$, i.e., $q \in S^{\alpha}(r)$, whereby the sub-automaton $(L_p, \Sigma, L_{q_0}, Z, w, \tilde{\delta}|_{(L_p \times L) \times \Sigma \times L_p}, F_1, F_2)$ is strongly connected. Hence, every $L^{\mathcal{B}}$-valued general fuzzy automaton has at least one strongly connected sub-automaton. □

Proposition 3.5. *$\tilde{\mathcal{F}}$ be a cyclic $L^{\mathcal{B}}$-valued general fuzzy automaton if and only if $\tilde{\mathcal{F}}$ has a unique maximal layer which is maximum in E.*

Proof. Let $\tilde{\mathcal{F}}$ be a cyclic $L^{\mathcal{B}}$-valued GFA and L_p be a maximal layer in E. Then, there exists $q \in \mathcal{Q}$ such that $\tilde{\delta}^{\alpha}((q, \mu^t(q)), u, p) > 0$, for some $u \in \Sigma^*$; and therefore $L_p \preceq L_q$. Also, $L_p = L_q$, because $L_p \neq L_q$ implies that $L_p \prec L_q$, which contradicts the maximality of L_p. Hence, $L_p \in E$ is a unique maximal layer.
Conversely, Let L_p be a unique maximal layer in E. Then for all $q \in \mathcal{Q}$ we have $L_q \preceq L_p$, i.e., $\tilde{\delta}^{\alpha}((p, \mu^t(p)), u, q) > 0$ for some $u \in \Sigma^*$. Hence, $\tilde{\mathcal{F}}$ is a cyclic $L^{\mathcal{B}}$-valued general fuzzy automaton. □

Before starting the next part, we first establish the notion of a directable $L^{\mathcal{B}}$-valued general fuzzy automaton, which in turn it generalizes the concept of a directable automaton which has been examined in [17].

Definition 3.6. *An $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}}$ is regarded as directable if for all $p, q \in \mathcal{Q}$ there exist $r \in \mathcal{Q}$ and $u \in \Sigma^*$ such that $\tilde{\delta}^{\alpha}((p, \mu^t(p)), u, r) > 0$ and $\tilde{\delta}^{\alpha}((q, \mu^t(q)), u, r) > 0$.*

Example 3.7. *Consider the GFA in Figure 1, it is specified as $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where $\mathcal{Q} = \{q_0, q_1, q_2\}$ is the set of states, $\Sigma = \{a, b\}$ is the set of input symbols, $\tilde{R} = \{(q_0, 1)\}, Z = \emptyset$ and ω is not applicable. We check operation of the GFA in Example 3.7 upon input "a^2b^2".*

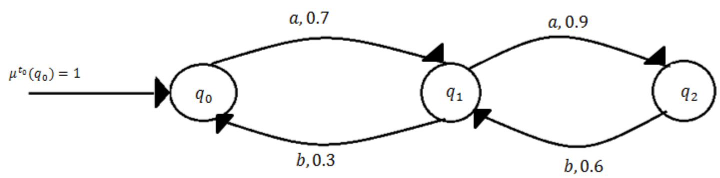

Fig 1. The GFA of Example 3.7

If we choose $F_1(\mu, \delta) = \delta, F_2(\mu, \delta) = \mu^{t+1}(q_m) = \wedge_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))$, then we have:
$\mu^{t_0}(q_0) = 1$,
$\mu^{t_1}(q_1) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) = \delta(q_0, a, q_1) = 0.7$,
$\mu^{t_2}(q_2) = F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_2)) = \delta(q_1, a, q_2) = 0.9$,
$\mu^{t_3}(q_1) = F_1(\mu^{t_2}(q_2), \delta(q_2, b, q_1)) = \delta(q_2, b, q_1)) = 0.6$,
$\mu^{t_4}(q_0) = F_1(\mu^{t_3}(q_1), \delta(q_1, b, q_0) = \delta(q_1, b, q_0) = 0.3$.

The set $B = \{0, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, 1\}$ of possible propositions concerning the general fuzzy au-

Table 1: Active states and their membership values (mv) at different times in Example 3.7

time	t_0	t_1	t_2	t_3	t_4
input	$\wedge$	a	a	b	b
$Q_{act}(t_i)$	q_0	q_1	q_2	q_1	q_0
mv	1	0.7	0.9	0.6	0.3

tomaton $\tilde{\mathcal{F}}$ is as follows:
-0 means that the GFA is not in active states of Q,
-α_0 means that the GFA has not been in active states at time t_0,
-α_1 means that the GFA has been in active states at time t_1,
-α_2 means that the GFA has been in active states at time t_2,
-α_3 means that the GFA has been in active states at time t_3,
-α_4 means that the GFA has been in active states at time t_4,
-1 means that the GFA has been in at least one active state at time t_i for any $i \geq 0$.
Here, $\alpha(q_i)$ is the maximum membership value of active states at time t_i for any $i \geq 0$. Then we have $0 = (0, 0, 0), \alpha_0 = (1, 0, 0), \alpha_1 = (0, 0.7, 0), \alpha_2 = (0, 0, 0.9), \alpha_3 = (0, 0.6, 0), \alpha_4 = (0.3, 0, 0), 1 =$

$(1, 0.7, 0.9)$.

$$\begin{aligned}\tilde{\delta}^{\alpha_1}((q_1, \mu^{t_1}(q_1)), a, q_2) &= F_1(\mu^{t_1}(q_1), \delta^{\alpha_1}(q_1, a, q_2)) = \delta^{\alpha_1}(q_1, a, q_2)\\ &= \alpha_1|_a(q_1) \vee \alpha_1|_a(q_2) = 0.7 \vee 0 = 0.7,\end{aligned}$$

$$\begin{aligned}\tilde{\delta}^{\alpha_1}((q_2, \mu^{t_2}(q_2)), b, q_1) &= F_1(\mu^{t_2}(q_2), \delta^{\alpha_1}(q_2, b, q_1)) = \delta^{\alpha_1}(q_2, b, q_1)\\ &= \alpha_1|_b(q_2) \vee \alpha_1|_b(q_1) = 1 \vee 1 = 1,\end{aligned}$$

$$\begin{aligned}\tilde{\delta}^{\alpha_2}((q_0, \mu^{t_0}(q_0)), a^2, q_2) &= F_1(\mu^{t_0}(q_0), \delta^{\alpha_2}(q_0, a, q_1)) \vee F_1(\mu^{t_1}(q_1), \delta^{\alpha_2}(q_1, a, q_2))\\ &= \delta^{\alpha_2}(q_0, a, q_1) \vee \delta^{\alpha_2}(q_1, a, q_2)\\ &= 0 \vee 0.9 = 0.9,\end{aligned}$$

$$\begin{aligned}\tilde{\delta}^{\alpha_2}((q_2, \mu^{t_2}(q_2)), b^2, q_0) &= F_1(\mu^{t_2}(q_2), \delta^{\alpha_2}(q_2, b, q_1)) \vee F_1(\mu^{t_3}(q_1), \delta^{\alpha_2}(q_1, b, q_0))\\ &= \delta^{\alpha_2}(q_2, b, q_1) \vee \delta^{\alpha_2}(q_1, b, q_0)\\ &= 1 \vee 1 = 1,\end{aligned}$$

$$\begin{aligned}\tilde{\delta}^{\alpha_3}((q_0, \mu^{t_0}(q_0)), a, q_1) &= F_1(\mu^{t_0}(q_0), \delta^{\alpha_3}(q_0, a, q_1))\\ &= \delta^{\alpha_3}(q_0, a, q_1)\\ &= \alpha_3|_a(q_0) \vee \alpha_3|_a(q_1) = 1 \vee 1 = 1,\end{aligned}$$

$$\begin{aligned}\tilde{\delta}^{\alpha_3|_b}((q_1, \mu^{t_3}(q_1)), b, q_0) &= F_1(\mu^{t_3}(q_1), \delta^{\alpha_3}(q_1, b, q_0))\\ &= \delta^{\alpha_3}(q_1, b, q_0)\\ &= \alpha_3|_b(q_1) \vee \alpha_3(q_0) = 0.6 \vee 0 = 0.6.\end{aligned}$$

Then, by the Definition 3.6 $\tilde{\mathcal{F}}$ is a directable.

Proposition 3.8. *Every directable $L^{\mathcal{B}}$-valued GFA contains a unique minimal layer.*

Proof. Let $\tilde{\mathcal{F}}$ be a directable $L^{\mathcal{B}}$-valued GFA. Also, let L_p, L_q be two distinct layers of $\tilde{\mathcal{F}}$, where $p, q \in \mathcal{Q}$. Then, there dose not exist any $r \in \mathcal{Q}$ and $u \in \Sigma^*$ such that $\tilde{\delta}^{\alpha}((p, \mu^t(p)), u, r) > 0$ and $\tilde{\delta}^{\alpha}((q, \mu^t(q)), u, r) > 0$ (as $L_p \cap L_q = \emptyset$), and consequently a contradiction. Therefore, every directable $L^{\mathcal{B}}$-valued general fuzzy automaton encompasses a unique minimal layer.
The following explains the construction of an $L^{\mathcal{B}}$-valued general fuzzy automaton containing singleton which is a unique minimal layer from a given $L^{\mathcal{B}}$-valued GFA with a unique minimal layer. It is noteworthy to state that the obtained $L^{\mathcal{B}}$-valued GFA is a homomorphic image of the original $L^{\mathcal{B}}$-valued GFA.
Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued general fuzzy automaton having unique minimal layer L_p. Construct an $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}}' = (((\mathcal{Q} \backslash L_p) \cup \{r\}), \Sigma, L_{q_0}, Z, w', \tilde{\delta}', F_1, F_2)$, where r is a new state and $\tilde{\delta}' : [((\mathcal{Q} \backslash L_p) \cup \{r\}) \times L] \times \Sigma \times ((\mathcal{Q} \backslash L_p) \cup \{r\}) \to L$ is a map such that

$$\tilde{\delta}'^{\alpha}((q, \mu^t(q)), a_k, q') = \begin{cases} \tilde{\delta}^{\alpha}((q, \mu^t(q)), a_k, q'), & \text{if } q, q' \in \mathcal{Q} \backslash L_p \\ 1, & \text{otherwise.} \end{cases}$$

Then, from the definition of $\tilde{\mathcal{F}}'$, it is clear that $\{r\}$ is a unique minimal layer of $\tilde{\mathcal{F}}'$. □

Proposition 3.9. *The $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}}'$ is a homomorphic image of $\tilde{\mathcal{F}}$.*

Proof. Let $h : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}'$ be a map such that $\forall q \in \mathcal{Q}$,

$$h(q) = \begin{cases} q, & \text{if } q \in \mathcal{Q}\backslash L_p \\ r, & \text{otherwise.} \end{cases}$$

Then for, cases will arise.
Case 1. If $q, q' \in \mathcal{Q}\backslash L_p$, then $\tilde{\delta}'^{\alpha}((h(q), \mu^t(h(q))), a_k, h(q')) = \tilde{\delta}^{\alpha}((q, \mu^t(q)), a_k, q')$.
Case 2. If $q, q' \in L_p$, then $\tilde{\delta}'^{\alpha}((h(q), \mu^t(h(q))), a_k, h(q')) = \tilde{\delta}^{\alpha}((r, \mu^t(r)), a_k, r) = 1$
$\geq \tilde{\delta}^{\alpha}((q, \mu^t(q)), a_k, q')$.
Case 3. If $q \in \mathcal{Q}\backslash L_p, q' \in L_p$, then $\tilde{\delta}'^{\alpha}((h(q), \mu^t(h(q))), a_k, h(q')) = \tilde{\delta}^{\alpha}((q, \mu^t(q)), a_k, r) = 1 \geq \tilde{\delta}^{\alpha}((q, \mu^t(q)), a_k, q')$.
Case 4. If $q' \in \mathcal{Q}\backslash L_p, q \in L_p$, then $\tilde{\delta}'^{\alpha}((h(q), \mu^t(h(q))), a_k, h(q')) = \tilde{\delta}^{\alpha}((r, \mu^t(r)), a_k, q') = 1 \geq \tilde{\delta}^{\alpha}((q, \mu^t(q)), a_k, q')$.
Thus, $\forall((q, \mu^t(q)), a_k, q') \in (\mathcal{Q} \times L) \times \Sigma \times \mathcal{Q}$, $\tilde{\delta}'^{\alpha}((h(q), \mu^t(h(q))), a_k, h(q')) \geq \tilde{\delta}^{\alpha}((q, \mu^t(q)), a_k, q')$.
Also, from the definition of f, it is clear that f is onto. Hence, $\tilde{\mathcal{F}}'$ is a homomorphic image of $\tilde{\mathcal{F}}$. □

4 Galois connections between lattices associated with an $L^{\mathcal{B}}$-valued GFA

As it has been shown in the previous section, the set of all layers of an $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}}$ together with a partial order defined on it, has been a poset which is indicated by $(E, \preceq)$. We denote the family of all sub-automata of $\tilde{\mathcal{F}}$ by S. The notion $\mathcal{M} \sqsubseteq \tilde{\mathcal{F}}$ denotes $\mathcal{M}$ is a sub-automaton of an $L^{\mathcal{B}}$-valued GFA $\tilde{\mathcal{F}}$, the relation $\sqsubseteq$ is a partial order on S and $(S, \sqsubseteq)$ is a poset. We shall establish isotone Galois connection between different pairs of posets/finite upper semilattices induced by a given $L^{\mathcal{B}}$-valued general fuzzy automaton by using of its layers.

Definition 4.1. *Let $(E, \preceq)$ be the poset induced by an $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}}$ by using its layers and $H \subseteq E$, $H \neq \emptyset$, then H is called a lower set, if $\forall L_p \in H$ and $\forall L_q \in E$, $L_q \preceq L_p \Rightarrow L_q \in H$. Also, for any $L_p \in E$, we call the set $\langle L_p \rangle = \{L_q \in E : L_q \preceq L_p\}$ the principle lower set of E.*

We denote by Ls(E), the family of all lower sets of poset $(E, \preceq)$, which with usual inclusion relation $\subseteq$ of sets run out to be a poset. The elements of $Ls(E)$ are noting but layers of $\tilde{\mathcal{F}}$, i.e., a lower set H of poset E is nothing but union of layers of $\tilde{\mathcal{F}}$.

Proposition 4.2. *Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued GFA. $(S, \sqsubseteq)$ be a poset of all sub-automata of $\tilde{\mathcal{F}}$, $(E, \preceq)$ be a poset of all layers of $\tilde{\mathcal{F}}$, $(Ls(E), \subseteq)$ be a poset of a lower set of E and*

$$G(Ls(E)) = \{L \in Ls(E) : deg^{-}L \leq 1\},$$

then, there exists an isotone Galois connection between $(E, \preceq)$ and $(G(Ls(E)), \subseteq)$.

Proof. In order to prove the existence of isotone Galois connection between $(E, \preceq)$ and $(G(LS(E)), \subseteq)$, we define a pair (φ, ψ) of mappings, where $\varphi : E \to G(Ls(E))$ and $\psi : G(Ls(E)) \to E$ are, respectively, defined as

$$\varphi(L_p) = \{L_q \in E : L_q \preceq L_p\}, \quad \forall L_p \in E, \quad \text{and} \tag{1}$$

$$\psi(\{L_q \in E : L_q \preceq L_p\}) = L_p. \tag{2}$$

But $\{L_q \in E : L_q \preceq L_p\}$, $\forall L_p \in E$ implies that $\varphi(L_p) \in Ls(E)$, i.e., L_p is unique and maximal in $\varphi(L_p)$ and hence, $\deg^{-}\varphi(L_p) \leq 1$ whereby $\varphi(L_p) \in G(Ls(E))$. For (φ, ψ) being isotone Galois connection between $(E, \preceq)$ and $(G(Ls(E)), \subseteq)$, we need to illustrate that for all $L_r \in E$ and $\{L_t \in E : L_t \preceq L_s\} \in G(Ls(E)), \varphi(L_r) \subseteq \{L_t \in E : L_t \preceq L_s\} \Leftrightarrow L_r \preceq \psi(\{L_t \in E : L_t \preceq L_s\})$. Now, let $\varphi(L_r) \subseteq \{L_t \in E : L_t \preceq L_s\})$, then using Equations (1) and (2), we have

$$\begin{aligned}\varphi(L_r) \subseteq \{L_t \in E : L_t \preceq L_s\} &\Rightarrow \{L_q \in E : L_q \preceq L_r\} \subseteq \{L_t \in E : L_t \preceq L_s\}\\ &\Rightarrow L_r \preceq L_s\\ &\Rightarrow L_r \preceq \psi(\{L_t \in E : L_t \preceq L_s\}).\end{aligned}$$

Conversely, let $L_r \preceq \psi(\{L_t \in E : L_t \preceq L_s\})$ then we have to prove that $\varphi(L_r) \subseteq \{L_t \in E : L_t \preceq L_s\}$. Now, by using Equations (2) and (1) we get

$$\begin{aligned}L_r \preceq \psi(\{L_t \in E : L_t \preceq L_s\}) &\Rightarrow L_r \preceq L_s\\ &\Rightarrow \{L_q \in E : L_q \preceq L_r\} \subseteq \{L_t \in E : L_t \preceq L_s\}\\ &\Rightarrow \varphi(L_r) \subseteq \{L_t \in E : L_t \preceq L_s\}.\end{aligned}$$

Thus, we have proved that $\varphi(L_r) \subseteq \{L_t \in E : L_t \preceq L_s\})$ if and only if $L_r \preceq \psi(\{L_t \in E : L_t \preceq L_s\})$. Therefore, we conclude that the pair (φ, ψ) is an isotone Galois connection. Now, we have the following propositions. We have left the proof due to space limitations. □

Proposition 4.3. *The posets $(S, \sqsubseteq)$, $(Ls(E), \subseteq)$ and $(G(Ls(E)), \subseteq)$ induced by $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ are finite upper semilattice.*

Proof. It is similar to Proposition 4.2 of [32]. □

Proposition 4.4. *Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued GFA. Let $(S, \sqsubseteq)$ be a finite upper semilattice of set of all sub-automaton of $\tilde{\mathcal{F}}$, $(E, \preceq)$ be a poset of set of all layers of $\tilde{\mathcal{F}}$ and $(Ls(E), \subseteq)$ be finite upper semilattice of lower sets of E. Then there exists an isotone Galois connection between finite upper semilattices $(S, \sqsubseteq)$ and $(Ls(E), \subseteq)$.*

Proposition 4.5. *Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued general fuzzy automaton, $(S, \sqsubseteq)$ and $G(Ls(S)) = \{L \in Ls(S) : deg^{-}L \leq 1\}$ be finite upper semilattices. Then there exists an isotone Galois connection between finite upper semilattices $(Ls(E), \subseteq)$ and $(G(Ls(S)), \subseteq)$.*

The next proposition presents a characterization of isotone Galois connection between finite upper semilattices $(S, \sqsubseteq)$ and $(Ls(E), \subseteq)$ induced by an $L^{\mathcal{B}}$-valued GFA $\tilde{\mathcal{F}}$. We have left the proof due to space limitations.

Proposition 4.6. *Let $(S, \sqsubseteq)$ and $(Ls(E), \subseteq)$ be finite upper semilattices associated with an $L^{\mathcal{B}}$-valued GFA $\tilde{\mathcal{F}}$. Let φ and ψ be two maps such that*

$$\varphi : S \to Ls(E) \quad \textit{and} \quad \psi : Ls(E) \to S.$$

Then, the pair (φ, ψ) be an isotone Galois connection iff
(i) *φ and ψ are order-preserving;*
(ii) *$\mathcal{M} \sqsubseteq \psi(\varphi(\mathcal{M}))$ for all $\mathcal{M} \in S$;*
(iii) *$\varphi(\psi(E_{\mathcal{M}})) \subseteq E_{\mathcal{M}}$ for all $E_{\mathcal{M}} \in Ls(E)$.*

In the following proposition, we have provided a characterization of isotone Galois Connection between posets $(E, \preceq)$ and $(G(Ls(E)), \subseteq)$ induced by layers of an $L^{\mathcal{B}}$-valued GFA $\tilde{\mathcal{F}}$.

Proposition 4.7. *Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued general fuzzy automaton. Let $(E, \preceq)$ and $(G(Ls(E)), \subseteq)$ be two posets associated with $\tilde{\mathcal{F}}, \varphi$ and ψ be two maps such that*

$$\varphi : E \to G(Ls(E) \text{ and } \psi : G(Ls(E)) \to E.$$

Then the pair (φ, ψ) be an isotone Galois connection iff
(i) *φ and ψ are order-preserving;*
(ii) *$L_p \leq \psi(\varphi(L_p))$ for all $L_p \in E$;*
(iii) *$\varphi(\psi(\{L_q \in E : L_q \preceq L_p\})) \subseteq \{L_q \in E : L_q \preceq L_p\}$ for all $\{L_q \in E : L_q \preceq L_p\} \in G(Ls(E))$.*

Proof. Let (φ, ψ) be an isotone Galois connection, where $\varphi : E \to G(Ls(E))$ and $\psi : G(Ls(E)) \to E$ be, respectively, defined as

$$\varphi(L_p) = \{L_q \in E : L_q \preceq L_p\}, \forall L_p \in E, \tag{3}$$

clearly, $\varphi(L_p) \in Ls(E)$ with L_p is unique and maximal in $\varphi(L_p)$ implies $deg^{-}\varphi(L_p) \leq 1$, whereby $\varphi(L_p) \in G(Ls(E))$; and

$$\psi(\{L_q \in E : L_q \preceq L_p\}) = L_p. \tag{4}$$

(i) To prove φ and ψ are order-preserving. Let L_p and $L_q \in E$ be arbitrary and such that $L_p \preceq L_q$. Then $L_p \preceq L_q \Rightarrow \{L_r \in E : L_r \preceq L_p\} \subseteq \{L_r \in E : L_r \preceq L_q\} \Rightarrow \varphi(L_p) \subseteq \varphi(L_q)$. Similarly, to prove ψ is order-preserving let $\{L_r \in E : L_r \preceq L_p\}, \{L_r \in E : L_r \preceq L_q\} \in G(Ls(E))$ such that

$$\{L_r \in E : L_r \preceq L_p\} \subseteq \{L_r \in E : L_r \preceq L_q\}.$$

Then

$$\begin{aligned}
\{L_r \in E : L_r \preceq L_p\} \subseteq \{L_r \in E : L_r \preceq L_q\} &\Rightarrow L_p \preceq L_q \\
&\Rightarrow \{L_r \in E : L_r \preceq L_p\} \subseteq \{L_r \in E : L_r \preceq L_q\} \\
&\Rightarrow L_p \preceq L_q \\
&\Rightarrow \psi(\{L_r \in E : L_r \preceq L_p\}) \subseteq \psi(\{L_r \in E : L_r \preceq L_q\}).
\end{aligned}$$

Hence, φ and ψ are order-preserving.
(ii) To prove $L_p \preceq \psi(\varphi(L_p))$, for all $L_p \in E$. Let L_p be an arbitrary element of E such that $\varphi(L_p) = \{L_r \in E : L_r \preceq L_p\}$, then $\varphi(L_p) = \{L_r \in E : L_r \preceq L_p\} \Rightarrow \psi(\varphi(L_p)) = \psi(\{L_r \in E : L_r \preceq L_p\})$. But $\psi(\{L_r \in E : L_r \preceq L_p\}) = L_p$. Hence, $\psi(\varphi(L_p)) = L_p$. Since $L_p \preceq L_p$ for each $L_p \in E$, it follows that $L_p \preceq \psi(\varphi(L_p))$.
(iii) To prove $\varphi(\psi(\{L_r \in E : L_r \preceq L_p\})) \subseteq \{L_r \in E : L_r \preceq L_p\}$ for all $\{L_r \in E : L_r \preceq L_p\} \in G(Ls(E))$. Let $\{L_r \in E : L_r \preceq L_p\}$ be an arbitrary element of $G(Ls(E))$, then by definition of ψ, we have $\psi(\{L_r \in E : L_r \preceq L_p\}) = L_p$. But

$$\begin{aligned}
\psi(\{L_r \in E : L_r \preceq L_p\}) = L_p &\Rightarrow \varphi(\psi(\{L_r \in E : L_r \preceq L_p\})) = \varphi(L_p) \\
&\Rightarrow \varphi(\psi(\{L_r \in E : L_r \preceq L_p\})) = \{L_r \in E : L_r \preceq L_p\})).
\end{aligned}$$

Since $\{L_r \in E : L_r \preceq L_p\} \subseteq \{L_r \in E : L_r \preceq L_p\}$, for each $\{L_r \in E : L_r \preceq L_p\} \in G(Ls(E))$, whereby $\varphi(\psi(\{L_r \in E : L_r \preceq L_p\})) \subseteq \{L_r \in E : L_r \preceq L_p\}$.
Conversely, let φ and ψ be defined as above and satisfy the Conditions (i), (ii) and (iii). To

show (φ,ψ) be an isotone Galois connection, i. e. for each $L_p \in E$ and $\{L_r \in E : L_r \preceq L_q\} \in G(Ls(E)), \varphi(L_p) \subseteq \{L_r \in E : L_r \preceq L_q\}$ if and only if $L_p \preceq \psi(\{L_r \in E : L_r \preceq L_q\})$. Let $\varphi(L_p) \subseteq \{L_r \in E : L_r \preceq L_q\}$, then

$$\begin{aligned}
&\{L_r \in E : L_r \preceq L_p\} \subseteq \{L_r \in E : L_r \preceq L_q\}\\
&\Rightarrow \psi(\{L_r \in E : L_r \preceq L_p\}) \preceq \psi(\{L_r \in E : L_r \preceq L_q\}),\\
&(\{L_r \in E : L_r \preceq L_p\}, \{L_r \in E : L_r \preceq L_q\} \in G(Ls(E)) \text{ and } \psi \text{ is order-preserving})\\
&\Rightarrow L_p \preceq \psi(\{L_r \in E : L_r \preceq L_q\}), \text{ using Equation 4.}
\end{aligned}$$

Now, to prove another side, let $L_p \preceq \psi(\{L_r \in E : L_r \preceq L_q\})$ to prove $\varphi(L_p) \subseteq \{L_r \in E : L_r \preceq L_q\}$. But

$$\begin{aligned}
L_p \preceq \psi(\{L_r \in E : L_r \preceq L_q\}) &\Rightarrow L_p \preceq L_q, (\text{by definition}\psi)\\
&\Rightarrow \varphi(L_p) \subseteq \varphi(L_q), (L_p, L_q \in E \text{ and } \varphi \text{ is order-preserving})\\
&\Rightarrow \varphi(L_p) \subseteq \{L_r \in E : L_r \preceq L_q\}(\text{by definition of } \varphi \text{ for } L_q \in E).
\end{aligned}$$

This completes the proof. □

Proposition 4.8. *Let $\tilde{\mathcal{F}} = (\mathcal{Q}, \Sigma, \tilde{R}, Z, w, \tilde{\delta}, F_1, F_2)$ be an $L^{\mathcal{B}}$-valued general fuzzy automaton. Let $(Ls(E), \subseteq)$ and $(G(Ls(S)), \subseteq)$ be finite upper semilattices associated with $\tilde{\mathcal{F}}$, and φ and ψ be two maps such that*

$$\varphi : Ls(E) \to G(Ls(S)) \quad \text{and } \psi : G(Ls(S)) \to LS(E).$$

Then the pair (φ,ψ) be an isotone Galois connection iff
(i) *φ and ψ are order-preserving;*
(ii) *$E_{\mathcal{M}} \subseteq \psi(\varphi(E_{\mathcal{M}}))$ for all $E_{\mathcal{M}} \in Ls(E)$;*
(iii) *$\varphi(\psi(\{\mathcal{M}' \in S : \mathcal{M}' \sqsubseteq \mathcal{M}\})) \subseteq \{\mathcal{M}' \in S : \mathcal{M}' \sqsubseteq \mathcal{M}\}$ for all $\{\mathcal{M}' \in S : \mathcal{M}' \sqsubseteq \mathcal{M}\} \in G(Ls(S))$.*

5 Conclusion

This study was an attempt to enhance the algebraic study of $L^{\mathcal{B}}$-valued general fuzzy automata through utilizing the notion of their respected layers. It has been currently demonstrated that some algebraic concepts of fuzzy automata which have been associated with lattice-ordered monoids (i.e., fuzzy automata where fuzziness is explicated by lattice-ordered monoids) rely on the related monoid structure (see, e.g. [20]). It is remarkable to observe the results of this present study in the framework of the generalized version of fuzzy automata and in the direction of the research conducted by Jin et al. [20]. Moreover, similar to the studies which have been done in [31, 32], it appears that the topological notions and fuzzy topological observations which are established in [31]may also be applied in some other studies as well. As further studies, we try to conduct such investigations in the near future.

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