Journal of Algebraic Hyperstructures and Logical Algebras

Volume 4, Number 2, (2023), pp. 17-35

L-fuzzy algebraic substructures

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Abstract

This article aims to provide a method for defining L-fuzzy algebraic substructures on general algebras. In addition, the properties of L-fuzzy sets are first reviewed, and their representations are then provided. Then, algebraic substructures are generalized as the closure systems on the power set of the algebra, and the properties of the prime and maximal elements in the above closure system are investigated. By using on these facts, L-fuzzy algebraic substructures concerning the closure system are defined and studied. Two equivalence characterizations of the sup property of the ordered set L are provided using L-fuzzy substructures. Similarly, some properties of L-fuzzy prime and maximal substructures concerning the closure system are discussed. Finally, to demonstrate the broad applicability of the theory of L-fuzzy algebraic substructures, the theory is applied to some specific algebraic structures, such as groups and pseudo MV-algebras.

Article Information

Corresponding Author:
X.L. Xin;
Received: October 2023;
Revised: November 2023;
Accepted: November 2023;
Paper type: Original.

Keywords:

L-fuzzy set, closure system, L-fuzzy prime and maximal substructure, pseudo MV-algebra.

1 Introduction

Zadeh [20] first proposed fuzzy sets in 1965. Then, Goguen [12] generalized fuzzy sets to L-fuzzy sets in 1967. In 1971, Rosenfeld [17] applied Zadeh's fuzzy theory to groups and proposed fuzzy subgroups, which led to the subsequent study of fuzzy structures on groups and rings.

In recent years, logical algebras have been widely used in artificial intelligence. As a result, many scholars have conducted extensive research on various logical algebras and their substructures, such as ideals and filters. Fuzzification of these substructures has also been proposed and studied. For instance, Hoo [13] proposed fuzzy ideals of MV-algebras and BCI-algebras. Later, Jun and Dymek [9, 10, 14] proposed fuzzy, fuzzy prime, and fuzzy maximal ideals of pseudo MV-algebras. In another pseudo algebraic structure–pseudo hoops, the theory of fuzzy filters was established by Borzooei [1]. Further, Borzooei [5] introduced the concept of Multipolar fuzzy a-ideals in BCI-algebra. Jun [18] introduced a new type of hesitant fuzzy subalgebras and ideals in BCK/BCI-algebras. Jun [2] studied the lattice structure of fuzzy A-ideals in an MV-module. Recently, Xin [19] proposed fuzzy filters on Heyting algebras. Zhan [22] studied fuzzy ideals of L-algebras.

Although the fuzzifications of substructures in various algebraic structures have been widely studied, their definitions heavily depend on the algebraic structures themselves. On the contrary, we propose a method for fuzzifying the substructure, which uses closure systems instead of operations on algebras. Consequently, a unified L-fuzzification method is established.

This paper is organized as follows. Section 2 presents some basic concepts and properties used throughout the paper, while Section 3 presents the concept of L-fuzzy sets and their properties. Moreover, L-fuzzy sets are used to characterize the sup property of the ordered set L. Section 4 provides a represents for L-fuzzy sets to prove that L-fuzzy sets L^X and L-nested systems $LN(X)$ are isomorphic. Moreover, a more straightforward representation is obtained for all L-fuzzy sets by using our theorem. In Section 5, algebraic substructures are generalized as closure systems on the power set of an algebra and prime and maximal elements are discussed in a closure system. The concept of L-fuzzy substructures is introduced concerning a closure system, and their properties are explored. It is proved that L-fuzzy substructures form a complete lattice when L is complete, and some equivalent characterizations of the sup property of L are provided using L-fuzzy substructures. Furthermore, some concrete forms of L-fuzzy substructures are found when the closure systems satisfy some conditions. Section 6 introduces the concept of L-fuzzy prime and maximal substructures for a closure system and gives some properties. Equivalent characterizations of L-fuzzy maximal substructures are also provided to the closure system with some conditions. Section 7 applies the theory of L-fuzzy substructures to some algebras, such as groups and pseudo MV-algebras. It is proved that some related definitions in [7, 9, 10, 16, 21] are equivalent to the concept of L-fuzzy substructures in our paper when the appropriate closure system and L are chosen. Several results presented in these papers can be considered corollaries of our theory.

2 Preliminaries

Definition 2.1. [6] Suppose that A is set. Recall that a binary relation R on A is a subset of $A \times A$. A binary relation R on A is called an equivalence relation on A if it satisfies, for any $a, b, c \in A$,
$(E1)$ $(a, a) \in R$;
$(E2)$ $(a, b) \in R$ implies $(b, a) \in R$;
$(E3)$ $(a, b), (b, c) \in R$ implies $(a, c) \in R$.
Denote $Eq(A)$ as the set of all equivalence relations on A.

Definition 2.2. [6] Let A be an algebra of type $\mathscr{F}$. Denote $r(f)$ as the arity of f for any operation $f \in \mathscr{F}$. A subset B of A is a subuniverse of A if B is closed under every operations in $\mathscr{F}$, i.e., $f(a_1, \cdots, a_{r(f)}) \in B$ for any $a_1, \cdots, a_{r(f)} \in B$ and $f \in \mathscr{F}$.

Definition 2.3. [4] Let A be a nonempty set and R a binary relation on A. R is called an order on A if it satisfies, for any $x, y, z \in A$,
(1) $(x, x) \in R$;
(2) $(x, y), (y, x) \in R$ implies $x = y$;
(3) $(x, y), (y, z) \in R$ implies $(x, z) \in R$.

We say that the pair (A, R) is an ordered set if R is an order on A. A subset B of an ordered set A with the order of A restricted on B is said to be a sub-ordered set of A. A sub-ordered set B of an ordered set A is said to be an upset if any element of A is greater than some element of B is in B. By the duality, we have the concept of a down set. If an ordered set A has a supremum x, we say that x is the greatest element or top element of A. Similarly, if A has an infimum x, we say that x is the lowest element or bottom element of A. An ordered set is a chain if every pair of elements in it are comparable. An ordered set is an antichain if every pair of distinct elements in it is incomparable. If x and y have a supremum; we use the notion $x \vee y$ to denote the supremum of x and y and call $x \vee y$ the join of x and y; if x, y have an infimum, then we use the notion $x \wedge y$ to denote the infimum of x and y and call $x \wedge y$ the meet of x and y.

Definition 2.4. [4] An ordered set A is called a $\vee$-semilattice if every pair of elements in A has a supremum. Similarly, A is called a $\wedge$-semilattice if every pair of elements in A has an infimum. An ordered set is said to be a lattice if it is both a $\vee$-semilattice and a $\wedge$-semilattice.

A $\wedge$-semilattice L is considered $\wedge$-complete if it has arbitrary nonempty meets. Similarly, a $\vee$-semilattice L is considered $\vee$-complete if it has arbitrary nonempty joins. A lattice is complete if it is $\vee$-complete and $\wedge$-complete.

Theorem 2.5. [4] A $\wedge$-complete ordered set is a complete lattice if and only if it has a top element.

Definition 2.6. [4] Let A and B be two ordered sets. A and B are order isomorphic if there exists a mapping f from A to B such that f is a bijection and both f and f^{-1} are isotone. Two lattices M and N are considered to be isomorphic if they are isomorphic as ordered sets.

3 L-fuzzy sets

Definition 3.1. [12] Let X be a nonempty set and L be an ordered set. An L-fuzzy set of X is a mapping $\mu : X \to L$. We denote the set of all L-fuzzy sets of X as L^X.

Note: Let X be a nonempty set and $(L, \leq)$ be an ordered set. If we define a binary relation on L^X as $\mu \leq v$ if $\mu(x) \leq v(x)$ for any $x \in X$, then $\leq$ is an order on L^X. We have the following obvious conclusions.

Lemma 3.2. *Suppose that X is a nonempty set and $(L, \leq)$ is an ordered set. Then, the following statements hold:*
(1) *L^X is a lattice if and only if L is a lattice;*
(2) *L^X is a complete lattice if and only if L is a complete lattice.*

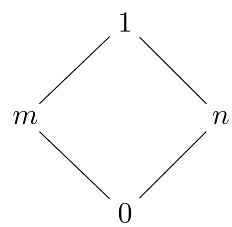
Lemma 3.3. *Let X be a nonempty set and L an ordered set. For each $\mu \in L^X$ and $t \in L$, we denote $\mu_t = \{x \in X \mid \mu(x) \geq t\}$. Then*
(1) *for any $m, n \in L$ and $\mu \in L^X$, $m \geq n$ implies $\mu_m \subseteq \mu_n$;*
(2) *for any $\mu, v \in L^X$, $\mu \leq v$ if and only if $\mu_t \subseteq v_t$ for all $t \in L$;*
(3) *for any $\mu, v \in L^X$, $\mu = v$ if and only if $\mu_t = v_t$ for all $t \in L$;*
(4) *if L is a lattice, then for any $\mu, v \in L^X$, and $t \in L$, $(\mu \wedge v)_t = \mu_t \cap v_t$ and $\mu_t \cup v_t \subseteq (\mu \vee v)_t$;*
(5) *if L is a complete lattice, then for any $\mu^i, i \in \Lambda$, and $t \in L$,*

$$(\bigwedge_{i \in \Lambda} \mu^i)_t = \bigcap_{i \in \Lambda} (\mu^i)_t \text{ and } (\bigvee_{i \in \Lambda} \mu^i)_t \supseteq \bigcup_{i \in \Lambda} (\mu^i)_t.$$

(6) *if L is a lattice, then for any $\mu \in L^X$ and $m, n \in L$, $\mu_{m \vee n} = \mu_m \cap \mu_n$.*

In the following, we will demonstrate that $\mu_t \cup v_t$ may not be equal to $(\mu \vee v)_t$ when L is just a lattice, and also illustrate that $\cup(\mu^i)_t$ may not be equal to $(\vee \mu^i)_t$ even when L is a complete chain.

Example 3.4. *Let $X = \{a, b, c, d\}$ and L be a lattice as shown in the following Hasse graph. We define $\mu, v \in L^X$ as follows:*



$$\mu(x) = \begin{cases} n & x = a \\ m & x = b \\ m & x = c \\ 0 & x = d \end{cases} \qquad \mu_t = \begin{cases} X & t = 0 \\ \{b, c\} & t = m \\ \{a\} & t = n \\ \varnothing & t = 1 \end{cases}$$

$$v(x) = \begin{cases} m & x = a \\ n & x = b \\ 0 & x = c \\ 0 & x = d \end{cases} \qquad v_t = \begin{cases} X & t = 0 \\ \{a\} & t = m \\ \{b\} & t = n \\ \varnothing & t = 1 \end{cases}$$

$$(\mu \vee v)(x) = \begin{cases} 1 & x = a \\ 1 & x = b \\ m & x = c \\ 0 & x = d \end{cases} \qquad (\mu \vee v)_t = \begin{cases} X & t = 0 \\ \{a, b, c\} & t = m \\ \{a, b\} & t = n \\ \{a, b\} & t = 1 \end{cases}$$

Apparently, $\mu_1 \cup v_1 \neq (\mu \vee v)_1$.

Example 3.5. *Let X be a nonempty set and L be the real interval $[0, 1]$. Define $\mu^n \in L^X$ as $\mu^n(x) = 1 - \frac{1}{n}$ $(\forall x \in X)$, $n \in \{1, 2, 3, \dots\}$. It is clear that $(\bigvee_{n \in N} \mu^n)_1 = X \neq \varnothing = \bigcup_{n \in N} (\mu^n)_1$.*

The following two theorems give the characterizations for the inclusion in Lemma 3.3 (4) and (5) being equal.

Theorem 3.6. *Let X be a nonempty set and L be a lattice. Then for any $\mu, v \in L^X$, $\mu_t \cup v_t = (\mu \vee v)_t (\forall t \in L)$ if and only if L is a chain.*

Proof. Suppose that L is a chain, $\mu, v \in L^X$ and $t \in L$. Then we have $x \in (\mu \vee v)_t \Leftrightarrow (\mu(x) \vee v(x)) \geq t \Leftrightarrow max\{\mu(x), v(x)\} \geq t \Leftrightarrow \mu(x) \geq t$ or $v(x) \geq t \Leftrightarrow x \in \mu_t \cup v_t$. and hence $\mu_t \cup v_t = (\mu \vee v)_t$. Conversely, if L is not a chain, then there exist two elements m, n of L such that $m \vee n$ is strictly greater than m and n. Define $\mu, v \in L^X$ as $\mu(x) = m$ and $v(x) = n$ $(\forall x \in X)$. It is apparent that $(\mu \vee v)_{m \vee n} = \{x \in X \mid (\mu \vee v)(x) \geq m \vee n\} = X \neq \varnothing = \mu_{m \vee n} \cup v_{m \vee n}$, which is a contradiction. □

Theorem 3.7. *Let X be a nonempty set and L a complete lattice. Then the following statements are equivalent.*
(i) *For any $\mu^i \in L^X$, $(i \in \Lambda)$, $(\bigvee_{i \in \Lambda} \mu^i)_t = \bigcup_{i \in \Lambda} (\mu^i)_t$ $(\forall t \in L)$.*
(ii) *L has the sup property (The sup property of a complete lattice L means that $\bigvee R \in R$ for any R $(\neq \varnothing) \subseteq L$).*

Proof. Suppose that L has the sup property. Then, we have

$$x \in (\underset{i\in\Lambda}{\vee} \mu^i)_t \ \Rightarrow \underset{i\in\Lambda}{\vee} \mu^i(x) \geq t \Rightarrow \exists\ i_0 \in \Lambda,$$

such that

$$\mu^{i_0}(x) = \underset{i\in\Lambda}{\vee} \mu^i(x) \geq t \Rightarrow x \in (\mu^{i_0})_t \Rightarrow x \in \underset{i\in\Lambda}{\cup} (\mu^i)_t,$$

and thus $(\underset{i\in\Lambda}{\vee} \mu^i)_t \subseteq \underset{i\in\Lambda}{\cup} (\mu^i)_t$. The converse inclusion follows from Lemma 3.3 and hence $(\underset{i\in\Lambda}{\vee} \mu^i)_t = \underset{i\in\Lambda}{\cup} (\mu^i)_t$. Conversely, suppose $R = \{a^i \in L \mid i \in \Lambda\}$ and $a = \underset{i\in\Lambda}{\vee} a^i$ and define $\mu^i(x) = a^i$ $(\forall x \in X)$. Since $X = (\underset{i\in\Lambda}{\vee} \mu^i)_a = \{x \in X \mid (\underset{i\in\Lambda}{\vee} \mu^i)(x) \geq a\} = \underset{i\in\Lambda}{\cup} (\mu^i)_a$, there exists an index $i_0 \in \Lambda$ such that $(\mu^{i_0})_a \neq \varnothing$. So $a \geq a^{i_0} = \mu^{i_0}(x_0) \geq a$ for some x_0 in X, which implies $a = a^{i_0} \in R$ and hence L has the sup property. □

4 Representation of L-fuzzy sets

Belohlavek [3] proved that there exists a bijection between L^X and L-nested systems when L is a complete residuated lattice. Similarly, we proved that the same result holds when L is an ordered set. Moreover, there is a one-to-one correspondence between L^X and a special subset of C-nested systems, where C is a complete lattice that L can be embedded in.

The definition of L-nested systems was given in [3], where L is a complete residuated lattice. In the following, the index L is considered as an ordered set.

Definition 4.1. [3] Let X be a nonempty set and L an ordered set. An L-indexed system $\mathscr{A} = \{A_a \subseteq X \mid a \in L\}$ is called an L-nested system if
(1) $a \leq b$ implies $A_b \subseteq A_a$ for any $a, b \in L$;
(2) for any $x \in X$, the set $\{a \in L \mid x \in A_a\}$ is nonempty and has the greatest element.

Let $LN(X)$ denote the collection of all L-nested systems on X. We say that the two L-indexed systems $\{A_a \subseteq X \mid a \in L\}$ and $\{B_a \subseteq X \mid a \in L\}$ are equal if $A_a = B_a$ for every $a \in L$.

Lemma 4.2. *Suppose that X is a nonempty set and L is an ordered set. Then the L-indexed system $\{\mu_a \subseteq X \mid a \in L\}$ is L-nested for any $\mu \in L^X$.*

Proof. By Lemma 3.3, $m \geq n$ implies $\mu_m \subseteq \mu_n$. Suppose $x \in X$ and denote $x_L^\mu = \{a \in L \mid x \in \mu_a\}$. Since $\mu(x) \in x_L^\mu$ and $\mu(x) \geq a$ for each $a \in x_L^\mu$, $\mu(x) = max\{a \in L \mid x \in \mu_a\}$, $\{\mu_a \subseteq X \mid a \in L\}$ is L-nested. □

Lemma 4.3. *Let X be a nonempty set, L an ordered set and $\mathscr{A} = \{A_a \subseteq X \mid a \in L\} \in LN(X)$. Then $\mathscr{A}$ admits an L-fuzzy set $\mu^{\mathscr{A}}$, where $\mu^{\mathscr{A}}(x) = max\{a \in L \mid x \in A_a\}$ $(\forall x \in X)$ and $\mu_t^{\mathscr{A}} = A_t$ for every $t \in L$.*

Proof. The mapping $\mu^{\mathscr{A}} : X \to L$ is well defined since $\{a \in L \mid x \in A_a\}$ has the greatest element for every $x \in X$. Suppose $t \in L$ and $A_t = \varnothing$. If $\mu_t^{\mathscr{A}} \neq \varnothing$, then, we have

$$\mu_t^{\mathscr{A}} \neq \varnothing \Rightarrow \exists x_0 \in \mu_t^{\mathscr{A}} \Rightarrow \mu^{\mathscr{A}}(x_0) = max\{a \in L \mid x_0 \in A_a\} \geq t.$$

Denote $a_0 = max\{a \in L \mid x_0 \in A_a\}$, then it follows from the definition of L-nested systems that $x_0 \in A_{a_0} \subseteq A_t$, which contradicts $A_t = \varnothing$. Therefore, $A_t = \mu_t^{\mathscr{A}} = \varnothing$. Suppose $t \in L$ and $A_t \neq \varnothing$. Then

$$x \in A_t \Rightarrow t \in \{a \in L \mid x \in A_a\} \Rightarrow \mu^{\mathscr{A}}(x) = max\{a \in L \mid x \in A_a\} \geq t \Rightarrow x \in \mu_t^{\mathscr{A}},$$

which implies $\mu_t^{\mathscr{A}} \neq \varnothing$ and $A_t \subseteq \mu_t^{\mathscr{A}}$. Conversely,

$$x \in \mu_t^{\mathscr{A}} \Rightarrow t \leq \mu^{\mathscr{A}}(x) = max\{a \in L \mid x \in A_a\}.$$

By the definition of L-nested systems, we have $\mu^{\mathscr{A}}(x) \in \{a \in L \mid x \in A_a\}$ and $A_{\mu^{\mathscr{A}}(x)} \subseteq A_t$, which implies $x \in A_{\mu^{\mathscr{A}}(x)} \subseteq A_t$ and $\mu_t^{\mathscr{A}} \subseteq A_t$. Therefore, $A_t = \mu_t^{\mathscr{A}}$ for any $t \in L$. □

The following theorem shows that L-fuzzy sets of X can be represented by L-nested systems, and also, the order structure of L^X is determined by L-nested systems.

Theorem 4.4. *Let X be a nonempty set and L an ordered set. Then there is a bijection between L^X and $LN(X)$. Furthermore, if we define an order $\leq$ on $LN(X)$, for any $\mathscr{A} = \{A_a \subseteq X \mid a \in L\}, \mathscr{B} = \{B_a \subseteq X \mid a \in L\} \in LN(X)$, $\mathscr{A} \leq \mathscr{B}$ if and only if $A_a \subseteq B_a$ for any $a \in L$, then L^X and $LN(X)$ are order isomorphic. If L is a lattice, then L^X and $LN(X)$ are lattice isomorphic.*

Proof. Define $\varphi : L^X \to LN(X)$ as $\mu \mapsto \{\mu_a \mid a \in L\}$ $(\forall \mu \in L^X)$, $\psi : LN(X) \to L^X$ as $\mathscr{A} = \{A_a \subseteq X \mid a \in L\} \mapsto \mu^{\mathscr{A}}$ $(\forall \mathscr{A} \in LN(X))$, where $\mu^{\mathscr{A}}(x) = max\{a \in L \mid x \in A_a\}(\forall x \in X)$. Then φ and ψ are both well-defined due to Lemma 4.2 and Lemma 4.3.

Suppose $\mu \in L^X$. By Lemma 4.3, we have $\psi(\varphi(\mu))_t = \mu_t$ for each $t \in L$, and hence $\psi(\varphi(\mu)) = \mu$ by Lemma 3.3 (3). Conversely, suppose $\mathscr{A} = \{A_a \subseteq X \mid a \in L\} \in LN(X)$. Then by Lemma 4.3, we have $\varphi(\psi(\mathscr{A})) = \mathscr{A}$. Therefore, φ is a bijection and $\varphi^{-1} = \psi$.

Suppose that μ, v are in L^X and $\mu \leq v$. By Lemma 3.3, $\varphi(\mu) \leq \varphi(v)$. Conversely, suppose that $\mathscr{A} = \{A_a \subseteq X \mid a \in L\}$ and $\mathscr{B} = \{B_a \subseteq X \mid a \in L\} \in LN(X)$ with $\mathscr{A} \leq \mathscr{B}$. It follows from Lemma 4.3 that $\psi(\mathscr{A})_t = A_t \subseteq B_t = \psi(\mathscr{B})_t$ for each $t \in L$. This implies $\psi(\mathscr{A}) \leq \psi(\mathscr{B})$ and hence L^X and $LN(X)$ are order isomorphic.

Next, we will prove that L^X and $LN(X)$ are lattice isomorphic when L is a lattice. Since it has been proven that L^X and $LN(X)$ are order isomorphic, we only need to prove that $LN(X)$ is also a lattice.

Denote $\mathscr{A} \wedge \mathscr{B} = \{(\psi(\mathscr{A}) \wedge \psi(\mathscr{B}))_t \mid t \in L\}$ for all $\mathscr{A} = \{A_a \subseteq X \mid a \in L\}$ and $\mathscr{B} = \{B_a \subseteq X \mid a \in L\} \in LN(X)$. Since $\psi(\mathscr{A}) \wedge \psi(\mathscr{B}) \in L^X$, it follows directly from Lemma 4.2 that $\mathscr{A} \wedge \mathscr{B} \in LN(X)$. By Lemma 3.3 (4), we have $(\psi(\mathscr{A}) \wedge \psi(\mathscr{B}))_t = (\psi(\mathscr{A})_t \cap (\psi(\mathscr{B}))_t = A_t \cap B_t$ for each $t \in L$. It is easy to see that $\mathscr{A} \wedge \mathscr{B}$ is indeed the infimum of $\mathscr{A}$ and $\mathscr{B}$ in $LN(X)$.

Define $\mathscr{A} \vee \mathscr{B} = \{(\psi(\mathscr{A}) \vee \psi(\mathscr{B}))_t \mid t \in L\}$. Similarly, we have $\mathscr{A} \vee \mathscr{B} \in LN(X)$. By Lemma 3.3 (4), we have $(\psi(\mathscr{A}) \vee \psi(\mathscr{B}))_t \supseteq \psi(\mathscr{A})_t \cup \psi(\mathscr{B})_t = A_t \cup B_t$ for any $t \in L$, which implies that $\mathscr{A} \vee \mathscr{B}$ is an upper bound of $\mathscr{A}$ and $\mathscr{B}$. Suppose $\mathscr{C} = \{C_a \subseteq X \mid a \in L\}$ is an upper bound of $\mathscr{A}$ and $\mathscr{B}$, i.e., $(\psi(\mathscr{C}))_t = C_t \supseteq A_t = (\psi(\mathscr{A}))_t$ and $(\psi(\mathscr{C}))_t = C_t \supseteq B_t = (\psi(\mathscr{B}))_t$ for any $t \in L$. It follows from Lemma 3.3 that $\psi(\mathscr{C}) \geq \psi(\mathscr{A})$ and $\psi(\mathscr{C}) \geq \psi(\mathscr{B})$, which implies $\psi(\mathscr{C}) \geq \psi(\mathscr{A}) \vee \psi(\mathscr{B})$. Since L^X and $LN(X)$ are order isomorphic, we have $\mathscr{C} = \varphi(\psi(\mathscr{C})) \geq \varphi(\psi(\mathscr{A}) \vee \psi(\mathscr{B})) = \mathscr{A} \vee \mathscr{B}$ and $\mathscr{A} \vee \mathscr{B}$ is indeed the supremum of $\mathscr{A}$ and $\mathscr{B}$. Hence $(LN(X), \wedge, \vee)$ is a lattice and L^X and $LN(X)$ are lattice isomorphic. □

Remark: When L is an ordered set, any subset of L is also an ordered set. For an arbitrary L-fuzzy set $\mu \in L^X$, μ can be viewed as a mapping from X to the ordered set $Im(\mu)$–the image of μ, which implies that μ can be determined by $\{\mu_a \subseteq X \mid a \in Im(\mu)\}$ and $\mu(x) = max\{a \in Im(\mu) \mid x \in \mu_a\}(\forall x \in X)$.

Next, we will show a simple fact and a very famous theorem proved in [15].

Lemma 4.5. *Let X be a nonempty set and M and L be two ordered sets. Then M^X is a sub-ordered set of L^X if and only if M is a sub-ordered set of L.*

Lemma 4.6. [15] (Dedekind–MacNeille) For any ordered set E, there exists a complete lattice L such that E can be embedded into L and the meets and joins that exist in E are preserved in L.

Corollary 4.7. *Suppose that X is a nonempty set, L is an ordered set, C is a complete lattice, and L can be embedded into C. Then L^X is a sub-ordered set of C^X. Similarly, L is a lattice, implying that L^X is a sublattice of C^X.*

Corollary 4.7 implies that when L is an ordered set or lattice, L^X can be viewed as a sub-ordered set or sublattice of C^X. However, it remains a question whether C^X is the smallest complete lattice that L^X can be embedded in, when C is the smallest complete lattice where L can be embedded. The following example demonstrates that the answer may not be positive.

Example 4.8. *Let $X = \{a, b\}$ and $L = \{m, n\}$ be an antichain. Then $L^X = \{A^{mm}, A^{nn}, A^{mn}, A^{nm}\}$ is an antichain, where*

$$A^{mm}(x) = \begin{cases} m & x = a \\ m & x = b \end{cases} \qquad A^{nn}(x) = \begin{cases} n & x = a \\ n & x = b \end{cases}$$

$$A^{mn}(x) = \begin{cases} m & x = a \\ n & x = b \end{cases} \qquad A^{nm}(x) = \begin{cases} n & x = a \\ m & x = b \end{cases}$$

Denote $M = \{0, A^{mm}, A^{nn}, A^{mn}, A^{nm}, 1\}$, where 0 and 1 are the infimum and the supremum of M and $\{A^{mm}, A^{nn}, A^{mn}, A^{nm}\}$ is an anti-chain. Obviously, M is a complete lattice that L^X can be embedded in. Put $C = \{0, m, n, 1\}$. Clearly, C is the smallest complete lattice which contains L. Since C^X has 16 elements, which is far more than the number of elements in M, C^X is not the smallest complete lattice that L^X can be embedded in.

Note: When C is the smallest complete lattice where L can be embedded, although C^X may not be the smallest complete lattice that L^X can be embedded in, C is indeed the smallest complete lattice that guarantees that C^X is a complete lattice and $L^X \subseteq C^X$.

5 L-fuzzy substructures

In many algebras, there special substructures exist such as subgroups in groups, ideals in rings or lattices. These substructures play a fundamental role in studying the structures of the algebras themselves. For instance, normal subgroups, first introduced by Galois, are essential for defining quotient groups and the homomorphism theorem between groups. Some special substructures of algebras are closely related to congruences on the algebras. For example, filters and congruences on MV-algebras are equivalent. Since these substructures are often associated with the closure systems and closure operators on the power set of the algebra, in this section, we will delve into closure systems and closure operators on the power set of an algebra, and introduce the concept of L-fuzzy substructures to closure systems.

Definition 5.1. [8] Suppose that M is an ordered set. The mapping $c : M \to M$ is called a closure operator on M if
(1) for all $x, y \in M$, $x \leq y$ implies $c(x) \leq c(y)$;
(2) for all $x \in M$, $c(x) \geq x$;
(3) for all $x \in M$, $c(c(x)) = c(x)$.

Definition 5.2. [8] Let M be a complete lattice. $A \subseteq M$ is called a closure system if
(1) $1 \in A$, where 1 is the top element of M;
(2) A is closed under arbitrary nonempty meet.
We use the notation $\mathscr{C}(M)$ to represent the set of all closure systems on M when M is complete.

Remark:[8] When M is a complete lattice, we can observe that a closure system A of M is a $\wedge$-complete $\wedge$-sublattice of M, and $(A, \wedge_M, \vee_A)$ forms a complete lattice. Both $C(M)$ and $\mathscr{C}(M)$ are complete lattices and are dual isomorphic, where the isomorphic mapping from $C(M)$ to $\mathscr{C}(M)$ is given by $c \mapsto Im(c) = \{c(x) \mid x \in X\}$ $(\forall c \in C(M))$, and the inverse mapping is $A \mapsto c^A$ $(\forall A \in \mathscr{C}(M))$, where $c^A(x) = \wedge\{a \in A \mid x \leq a\}$ $(\forall x \in M)$.

Definition 5.3. [4] Let X be a nonempty set and $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$. $M \in \mathscr{A}$ where $M \neq \varnothing$ is said to be prime in $\mathscr{A}$ if M is proper and $A \cap B \subseteq M$ implies $A \subseteq M$ or $B \subseteq M$ for any $A, B \in \mathscr{A}$. $M \in \mathscr{A}$ is said to be maximal in $\mathscr{A}$ if M is proper and $M \subseteq A$ implies $A = M$ or $A = X$ for any $A \in \mathscr{A}$. We denote by $Prim(\mathscr{A})$ and $Max(\mathscr{A})$ the set of all prime elements and maximal elements in $\mathscr{A}$, respectively.

Remark: In fact, if we regard $\mathscr{A}$ as a lattice, then the definition of prime elements in $\mathscr{A}$ is similar to that in lattice theory, so we still use $Prim(L)$ to denote the set of all prime elements in lattice or $\wedge$-semilattice L.

Proposition 5.4. *Let X be a nonempty set, $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$ and $P_1, P_2 \in Prim(\mathscr{A})$. Then*
(1) $P_1 \cap P_2 \in Prim(\mathscr{A})$ if and only if $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$;
(2) $Prim(\mathscr{A})$ is $\cap$-closed if and only if $(Prim(\mathscr{A}), \subseteq)$ is a chain.

Proof. (1) If $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$, then $P_1 \cap P_2 = P_1$ or P_2 and hence $P_1 \cap P_2 \in Prim(\mathscr{A})$. Conversely, if $P_1 \cap P_2 \in Prim(\mathscr{A})$, then $P_1 \cap P_2 \subseteq P_1 \cap P_2$ implies $P_1 \subseteq P_1 \cap P_2$ or $P_2 \subseteq P_1 \cap P_2$, and hence $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ and $(Prim(\mathscr{A}), \subseteq)$ is a chain.

(2) The result follows directly from (1). □

Definition 5.5. *Let X be a nonempty set, $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$ and L an ordered set. $\mu \in L^X$ is called an L-fuzzy substructure to $\mathscr{A}$ if μ_t $(\mu_t \neq \varnothing) \in \mathscr{A}$ for any $t \in L$. We call μ an $L(\mathscr{A})$ substructure for short and denote $L(\mathscr{A})$ as the set of all the $L(\mathscr{A})$ substructures.*

Proposition 5.6. *Let X be a nonempty set, $\mathscr{A}$ a closure system of X and L a complete lattice. Then $L(\mathscr{A})$ is a closure system.*

Proof. Obviously, $L(\mathscr{A})$ has the greatest element—the constant mapping which sends every element of X to 1. Suppose $\mu^i \in L(\mathscr{A})$, $i \in \Lambda$. Since $\mu^i_t \in \mathscr{A}$ $(\forall t \in L, i \in \Lambda)$, we have $\bigcap_{i\in\Lambda} \mu^i_t \in \mathscr{A}$. It follows from Lemma 3.3 that $(\bigwedge_{i\in\Lambda} \mu^i)_t = \bigcap_{i\in\Lambda} \mu^i_t \in \mathscr{A}$ $(\forall t \in L)$, which implies $\bigwedge_{i\in\Lambda} \mu^i \in L(\mathscr{A})$, and hence $L(\mathscr{A})$ is a closure system. □

Note: Proposition 5.6 shows that $L(\mathscr{A})$ is a complete lattice for any $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$ when L is complete. When L is a lattice, it is evident that $L(\mathscr{A})$ forms a $\wedge$-semilattice for any $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$. However, whether $L(\mathscr{A})$ forms a lattice for any $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$ remains a question.

The following theorem shows the order relations between the set of all closure systems $\mathscr{C}(\mathrm{P}(X))$ and the set of all L-fuzzy substructures concerning these closure systems. We reveal that $\mathscr{C}(\mathrm{P}(X))$ and the set of all L-fuzzy substructures for these closure systems are lattice isomorphic under inclusion order, i.e., $L(\mathscr{A}^1 \cap \mathscr{A}^2) = L(\mathscr{A}^1) \cap L(\mathscr{A}^2)$ and $L(\mathscr{A}^1 \vee \mathscr{A}^2) = L(\mathscr{A}^1) \vee L(\mathscr{A}^2)$, where $\mathscr{A}^1 \cap \mathscr{A}^2$ and $\mathscr{A}^1 \vee \mathscr{A}^2$ are the infimum and the supremum of $\mathscr{A}^1$ and $\mathscr{A}^2$ in $\mathscr{C}(\mathrm{P}(X))$, respectively. $L(\mathscr{A}^1) \cap L(\mathscr{A}^2)$ and $L(\mathscr{A}^1) \vee L(\mathscr{A}^2)$ are the infimum and the supremum of $L(\mathscr{A}^1)$ and $L(\mathscr{A}^2)$ in $\{L(\mathscr{A}) \mid \mathscr{A} \in \mathscr{C}(\mathrm{P}(X))\}$, respectively.

Theorem 5.7. *Let X be a nonempty set and L an ordered set. Then the mapping f, $\mathscr{A} \mapsto L(\mathscr{A})$, is a lattice isomorphism between $\mathscr{C}(\mathrm{P}(X))$ and the image $Im(f)$ of f and $Im(f)$ is a complete lattice, where the order on $\mathscr{C}(\mathrm{P}(X))$ and $Im(f)$ is inclusion order.*

Proof. We first prove that f is injective. Suppose that $\mathscr{A}^1$, $\mathscr{A}^2 \in \mathscr{C}(\mathrm{P}(X))$ and $\mathscr{A}^1 \neq \mathscr{A}^2$. We may assume that there exists an element E of $\mathscr{A}^1$ such that $E \notin \mathscr{A}^2$. Define $\mu^E : X \to L$ as

$$\mu^E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases} .$$

It's clear that $\mu^E \in f(\mathscr{A}^1)$ and $\mu^E \notin f(\mathscr{A}^2)$, which implies $f(\mathscr{A}^1) \neq f(\mathscr{A}^2)$ and hence f is injective. f is order-preserving. Suppose $\mathscr{B}^1, \mathscr{B}^2 \in Im(f)$ with $\mathscr{B}^1 \subseteq \mathscr{B}^2$. Assume that $\mathscr{A}^1 = f^{-1}(\mathscr{B}^1)$ and $\mathscr{A}^2 = f^{-1}(\mathscr{B}^2)$ and $\mathscr{A}^1 \nsubseteq \mathscr{A}^2$, then $\exists E \in \mathscr{A}^1$ such that $E \notin \mathscr{A}^2$. It follows directly that $\mu^E \in f(\mathscr{A}^1) = \mathscr{B}^1$ and $\mu^E \notin f(\mathscr{A}^2) = \mathscr{B}^2$, which is a contradiction. Therefore, $\mathscr{A}^1 \subseteq \mathscr{A}^2$, i.e., f^{-1} is isotone and hence $\mathscr{C}(\mathrm{P}(X))$ and $Im(f)$ are order isomorphic.

Let $\mathscr{B}^i \in Im(f), i \in \Lambda$ and $f(\mathscr{A}^i) = \mathscr{B}^i$. It's easy to see that $\bigcap_{i\in\Lambda} \mathscr{A}^i \in \mathscr{C}(\mathrm{P}(X))$. Next, we will prove that $f(\bigcap_{i\in\Lambda} \mathscr{A}^i) = \bigcap_{i\in\Lambda} \mathscr{B}^i$. Since f is order preserving, we have $f(\bigcap_{i\in\Lambda} \mathscr{A}^i) \subseteq \bigcap_{i\in\Lambda} f(\mathscr{A}^i) = \bigcap_{i\in\Lambda} \mathscr{B}^i$. Conversely, for any $\mu \in \bigcap_{i\in\Lambda} \mathscr{B}^i$, we have $\mu_t \in \mathscr{A}^i$ ($\forall i \in \Lambda, t \in L$), which implies $\mu_t \in \bigcap_{i\in\Lambda} \mathscr{A}^i$ ($\forall t \in L$) and hence $\mu \in f(\bigcap_{i\in\Lambda} \mathscr{A}^i)$. Therefore, $\bigcap_{i\in\Lambda} \mathscr{B}^i \subseteq f(\bigcap_{i\in\Lambda} \mathscr{A}^i)$ and so $f(\bigcap_{i\in\Lambda} \mathscr{A}^i) = \bigcap_{i\in\Lambda} \mathscr{B}^i$, i.e., $\bigcap_{i\in\Lambda} \mathscr{B}^i \in Im(f)$. Apparently, $\bigcap_{i\in\Lambda} \mathscr{B}^i$ is indeed, the infimum of $\{\mathscr{B}^i, i \in \Lambda\}$, which implies that $Im(f)$ is $\cap$-complete. Since $Im(f)$ has the greatest element L^X, $Im(f)$ is a complete lattice. Since $\mathscr{C}(\mathrm{P}(X))$ and $Im(f)$ are both complete lattices and are order isomorphic, they are lattice isomorphic. □

Since every closure system in $\mathscr{C}(\mathrm{P}(X))$ admits a closure operator on $\mathrm{P}(X)$, by the above proof, the set of all closure operators in $\mathrm{P}(X)$ can be embedded in the set of all closure operators in $\mathrm{P}(L^X)$ when L is a complete lattice.

Theorem 5.8. *Suppose that X is a nonempty set and L is a complete lattice. Then the following statements are equivalent.*
(1) L has the sup property.
(2) For any $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$, $\mu^i \in L(\mathscr{A}), i \in \Lambda$ and $t \in L$, $(\bigvee_{i\in\Lambda} \mu^i)_t = \bigvee_{i\in\Lambda} \mu^i_t$, where $\bigvee_{i\in\Lambda} \mu^i$ is the supremum of μ^i in $L(\mathscr{A})$ and $\bigvee_{i\in\Lambda} \mu^i_t$ is the supremum of μ^i_t in $\mathscr{A}$.
(3) For any $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$, $i \in \Lambda, \mu^i \in L(\mathscr{A}), t \in L$ and $I \in \mathscr{A}$, if $\bigvee_{i\in\Lambda} (\mu^i)_t \subseteq I$, then there exists an L-fuzzy set υ in $L(\mathscr{A})$ such that $\bigvee_{i\in\Lambda} \mu^i \leq \upsilon$ and $\upsilon_t = I$.

Proof. (1) $\Rightarrow$ (2) Let $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$, $\mu^i \in L(\mathscr{A}), i \in \Lambda$, and $t \in L$. Apparently, $\mu^i_t \subseteq (\bigvee_{i\in\Lambda} \mu^i)_t$ and so $\bigvee_{i\in\Lambda} (\mu^i)_t \subseteq (\bigvee_{i\in\Lambda} \mu^i)_t$. Conversely, by Lemma 3.3 (1), since $m \leq n$ implies $\mu^i_n \subseteq \mu^i_m$ ($\forall i \in \Lambda$), we have $\bigvee_{i\in\Lambda} \mu^i_n \subseteq \bigvee_{i\in\Lambda} \mu^i_m$. Since L has the sup property, we have $\vee\{a \in L \mid x \in \bigvee_{i\in\Lambda} \mu^i_a\} \in \{a \in L \mid x \in \bigvee_{i\in\Lambda} \mu^i_a\}$ ($\forall x \in X$) and so $\{\bigvee_{i\in\Lambda} \mu^i_a \mid a \in L\}$ is an L-nested system. By Theorem 4.4, $\mu^j_t \subseteq \bigvee_{i\in\Lambda} \mu^i_t = [\psi(\{\bigvee_{i\in\Lambda} \mu^i_a \mid a \in L\})]_t$ ($\forall j \in \Lambda, t \in L$), which implies $\mu^j \leq \psi(\{\bigvee_{i\in\Lambda} \mu^i_a \mid a \in L\})$ ($\forall j \in \Lambda$) and hence $\bigvee_{i\in\Lambda} \mu^i \leq \psi(\{\bigvee_{i\in\Lambda} \mu^i_a \mid a \in L\})$. It follows that $(\bigvee_{i\in\Lambda} \mu^i)_t \subseteq [\psi(\{\bigvee_{i\in\Lambda} \mu^i_a \mid a \in L\})]_t = \bigvee_{i\in\Lambda} \mu^i_t$ and so $(\bigvee_{i\in\Lambda} \mu^i)_t = \bigvee_{i\in\Lambda} \mu^i_t$.

(2) ⇒ (1) Since $L(\mathscr{A}) = L^X$ when $\mathscr{A} = \mathrm{P}(X)$, the result follows directly from Theorem 3.7.

(2) ⇒ (3) Put

$$\lambda_I(x) = \begin{cases} t & x \in I \\ 0 & x \notin I \end{cases}$$

then it is clear that $\lambda_I \in L(\mathscr{A})$. Put $\upsilon = \underset{i\in\Lambda}{\vee} \mu^i \vee \lambda_I$, then $\underset{i\in\Lambda}{\vee} \mu^i \leq \upsilon$ and

$$\upsilon_t = (\underset{i\in\Lambda}{\vee} \mu^i \vee \lambda_I)_t = \underset{i\in\Lambda}{\vee} (\mu^i)_t \vee (\lambda_I)_t = I.$$

(3) ⇒ (2) Let $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$, $\mu^i \in L(\mathscr{A}), i \in \Lambda$, and $t \in L$. Since $\underset{i\in\Lambda}{\vee} (\mu^i)_t \subseteq \underset{i\in\Lambda}{\vee} (\mu^i)_t$, there exists an L-fuzzy set ω in $L(\mathscr{A})$ with $\underset{i\in\Lambda}{\vee} \mu^i \leq \omega$ and $\omega_t = \underset{i\in\Lambda}{\vee} (\mu^i)_t$. Hence $(\underset{i\in\Lambda}{\vee} \mu^i)_t \subseteq \omega_t = \underset{i\in\Lambda}{\vee} (\mu^i)_t$ and so $(\underset{i\in\Lambda}{\vee} \mu^i)_t = \underset{i\in\Lambda}{\vee} (\mu^i)_t$. □

Remark: According to Theorem 4.4 and Theorem 5.8, it's easy to see the following result holds. If L has the sup property, then for any $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X)), i \in \Lambda$, $\mu^i \in L(\mathscr{A})$, and $x \in X$, $(\underset{i\in\Lambda}{\vee} \mu^i)(x) = \vee\{a \in L \mid x \in \underset{i\in\Lambda}{\vee} \mu^i_a\}$. Theorem 5.8 gives the join formula of two L-fuzzy substructures to the closure system and proves the existence of the L-fuzzy substructures satisfying certain properties when L has the sup property. To be interesting, the above three are equivalent. The result in Theorem 5.11 can be widely used. When we deal with problems by using fuzzy sets in the real world, it is no need to use the real integer $[0, 1]$; a finite subset of $[0, 1]$ will be sufficient, and any finite subset of $[0, 1]$ has the sup property.

In the following of part, we discuss the characters of L-fuzzy substructures for the closure system with special features and give several equivalent characterizations for the L-fuzzy sets equipped with special properties.

Theorem 5.9. *Suppose that X and L are two ordered sets and $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$. Recall that a map μ from X to L is said to be isotone if $x \leq y$ implies $\mu(x) \leq \mu(y)$ for any x, y in X. Similarly, μ is said to be anti-tone if $x \leq y$ implies $\mu(x) \geq \mu(y)$. Then, the following statements hold.*
(1) μ is anti-tone for any $\mu \in L(\mathscr{A})$ if and only if every element ($\neq \varnothing$) of $\mathscr{A}$ is downset;
(2) μ is isotone for any $\mu \in L(\mathscr{A})$ if and only if every element ($\neq \varnothing$) of $\mathscr{A}$ is upset.

Proof. (1) Suppose that A is a nonempty element in $\mathscr{A}$. Define an L-fuzzy set λ_A as

$$\lambda_A(x) = \begin{cases} t & x \in A \\ s & x \notin A \end{cases}$$

where $s < t$. Apparently, λ_A is in $L(\mathscr{A})$. Let $y \in A$ and $x \in X$ with $x \leq y$. Since λ_A is anti-tone, we have $\lambda_A(x) \geq \lambda_A(y) = t$, i.e., $x \in (\lambda_A)_t = A$. Therefore, A is down set. Conversely, suppose $\mu \in L(\mathscr{A})$ and $x, y \in X$ with $x \leq y$. We may denote n as $\mu(y)$, apparently, $y \in \mu_n$ and $\mu_n \in \mathscr{A}$. Since μ_n is downset, we have $x \in \mu_n$, which implies $\mu(x) \geq n = \mu(y)$ and hence μ is anti-tone.

(2) The proof for (2) is similar to (1). □

Lemma 5.10. *Let X be a nonempty set, L an ordered set and $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$. If there exists a constant element c that belongs to every element ($\neq \varnothing$) of $\mathscr{A}$, then $\mu(c) \geq \mu(x)$ for all $\mu \in L(\mathscr{A})$ and $x \in X$.*

Proof. Suppose $\mu \in L(\mathscr{A})$ and $x \in X$. Since $\mu(x) \in \mu_{\mu(x)}$ implies $\mu_{\mu(x)} \neq \varnothing$, $c \in \mu_{\mu(x)}$, i.e., $\mu(c) \geq \mu(x)$. □

Corollary 5.11. *Let X and L be two ordered sets, $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$ and $\mu \in L(\mathscr{A})$. Then*
(1) if every element ($\neq \varnothing$) of $\mathscr{A}$ is down set and X has the lowest element 0, then $\mu(0) \geq \mu(x)$ ($\forall x \in X$);
(2) if every element ($\neq \varnothing$) of $\mathscr{A}$ is up set and X has the greatest element 1, then $\mu(1) \geq \mu(x)$ ($\forall x \in X$).

Theorem 5.12. *Let X be an algebra of type $\mathscr{F}$, L a lattice and $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$. Then every element ($\neq \varnothing$) of $\mathscr{A}$ is a subuniverse of X if and only if for any $\mu \in L(\mathscr{A})$ and $f \in \mathscr{F}$,*

$$\mu(f(x_1, x_2, \cdots, x_{r(f)})) \geq \mu(x_1) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_{r(f)}),$$

for any $x_1, x_2, \cdots, x_{r(f)} \in X$.

Proof. ($\Rightarrow$) Suppose $\mu \in L(\mathscr{A})$, $f \in \mathscr{F}$ and $x_1, x_2, \cdots, x_{r(f)} \in X$ and assume $\mu(x_i) = m_i$, $i = 1, 2, \cdots, r(f)$. It's clear that $\mu(x_i) = m_i \geq \bigwedge_{j=1,2,\cdots,r(f)} m_j$ ($i = 1, 2, \cdots, r(f)$). Denote m as $\bigwedge_{j=1,2,\cdots,r(f)} m_j$, then we have $x_i \in \mu_m$. Since μ_m is a subuniverse of X, $f(x_1, x_2, \cdots, x_{r(f)}) \in \mu_m$ and hence

$$\mu(f(x_1, x_2, \cdots, x_{r(f)})) \geq m = \bigwedge_{j=1,2,\cdots,r(f)} m_j = \mu(x_1) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_{r(f)}).$$

($\Leftarrow$) Suppose A ($\neq \varnothing$) $\in \mathscr{A}$, $f \in \mathscr{F}$ and $x_1, x_2, \cdots, x_{r(f)} \in A$ and define an L-fuzzy set λ_A as

$$\lambda_A(x) = \begin{cases} t & x \in A \\ s & x \notin A \end{cases}$$

where $s < t$. It clear that $\lambda_A \in L(\mathscr{A})$ and so we have

$$\lambda_A(f(x_1, x_2, \cdots, x_{r(f)})) \geq \lambda_A(x_1) \wedge \lambda_A(x_2) \wedge \cdots \wedge \lambda_A(x_{r(f)}) = t,$$

which implies $f(x_1, x_2, \cdots, x_{r(f)}) \in (\lambda_A)_t = A$. Therefore, A is a subuniverse of X. □

6 L-fuzzy prime substructures and maximal substuctures

Definition 6.1. *Let X be a nonempty, L an ordered set and $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$. Then $\mu \in L(\mathscr{A})$ is said to be prime if μ is not constant and μ_t ($\mu_t \neq \varnothing, X$) $\in Prim(\mathscr{A})$ for all $t \in L$, and μ is called an L-fuzzy prime substructure concerning $\mathscr{A}$. We may denote $PL(\mathscr{A})$ as the set of all the L-fuzzy prime substructures to $\mathscr{A}$.*

Proposition 6.2. *Suppose that X is a nonempty set, L is a lattice, $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$ and $\mu \in PL(\mathscr{A})$. Then $\mu_m \cap \mu_n \neq \varnothing$ implies $\mu_m \subseteq \mu_n$ or $\mu_n \subseteq \mu_m$ for all $m, n \in L$.*

Proof. Let $m, n \in L$ with $\mu_m \cap \mu_n \neq \varnothing$. If $\mu_m = X$ or $\mu_n = X$, then result is obvious. Otherwise, we have $\mu_m, \mu_n \in Prim(\mathscr{A})$. Since $\mu_{m \vee n} = \mu_m \cap \mu_n$ (by Lemma 3.3 (6)), we have $\mu_{m \vee n} \in Prim(\mathscr{A})$ and it follows from Proposition 5.4 that $\mu_m \subseteq \mu_n$ or $\mu_n \subseteq \mu_m$. □

Prime substructures are necessary for algebras. In the following, we give a sufficient condition for an L-fuzzy substructure to be a prime L-fuzzy substructure when L is a lattice. Every prime element of L-fuzzy substructures $L(\mathscr{A})$ is a prime L-fuzzy substructure for $\mathscr{A}$, and we show that the inverse result may not valid in Example 6.4.

Theorem 6.3. *Let X be a nonempty set, L a lattice with a bottom element 0 and $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$. Then every prime element in $L(\mathscr{A})$ is an L-fuzzy prime substructures concerning $\mathscr{A}$, i.e.,*

$$Prim(L(\mathscr{A})) \subseteq PL(\mathscr{A}).$$

Proof. Let $\omega \in Prim(L(\mathscr{A}))$ and $t \in L$ with $\omega_t \neq \varnothing$ and $\omega_t \neq X$. For any $A, B \in \mathscr{A}$ with $A \cap B \subseteq \omega_t$, define two L-fuzzy sets λ_A and λ_B as

$$\lambda_A(x) = \begin{cases} t & x \in A \\ 0 & x \notin A \end{cases} \quad \text{and} \quad \lambda_B(x) = \begin{cases} t & x \in B \\ 0 & x \notin B \end{cases}$$

then it follows directly that $\lambda_A, \lambda_B \in L(\mathscr{A})$ and $\lambda_A \wedge \lambda_B \leq \omega$. Since $\omega \in Prim(L(\mathscr{A}))$, we have $\lambda_A \leq \omega$ or $\lambda_B \leq \omega$. By Lemma 3.3(2), we have $A = (\lambda_A)_t \subseteq \omega_t$ or $B = (\lambda_B)_t \subseteq \omega_t$ and hence $\omega_t \in Prim(\mathscr{A})$, which implies $\omega \in PL(\mathscr{A})$ and hence $Prim(L(\mathscr{A})) \subseteq PL(\mathscr{A})$. □

In general, the inverse inclusion in Theorem 6.3 may not hold, and the following example will show that.

Example 6.4. *Let $X = \{a, b, c\}$, L a lattice as shown in Example 3.4 and $\mathscr{A} = \mathrm{P}(X)$. It can be easily checked that $Prim(\mathscr{A}) = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. Put*

$$\omega(x) = \begin{cases} m & x = a \\ m & x = b \\ 0 & x = c \end{cases}$$

Apparently, ω is in $PL(\mathscr{A})$ and if we put

$$\mu(x) = \begin{cases} m & x = a \\ 1 & x = b \\ m & x = c \end{cases}, \quad \nu(x) = \begin{cases} 1 & x = a \\ m & x = b \\ n & x = c \end{cases}$$

it's easy to check that $\mu, \nu \in L(\mathscr{A})$, $\mu \wedge \nu \leq \omega$, $\mu \nleq \omega$ and $\nu \nleq \omega$, which implies $\omega \notin Prim(L(\mathscr{A}))$.

Definition 6.5. *Let X be a nonempty set, L an ordered set and $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$. Then $\mu \in L(\mathscr{A})$ is said to be maximal if μ is not constant and μ_t $(\mu_t \neq \varnothing, X) \in Max(\mathscr{A})$ for all $t \in L$, and then μ is called an L-fuzzy maximal substructure to $\mathscr{A}$. Denote $ML(\mathscr{A})$ as the set of all the L-fuzzy maximal substructures for $\mathscr{A}$.*

Theorem 6.6 gives two characterizations of maximal L-fuzzy substructures concerning closure $\mathscr{A}$. We find the formula of maximal L-fuzzy substructures and use the set $\mu_{\mu(c)}$ to characterize it. This result will be helpful to readers who want to define maximal L-fuzzy substructures on a specific algebra, such as MV-algebra (see Example 7.8).

Theorem 6.6. *Let X be a nonempty set, L an ordered set, $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$ and $\mu \in L(\mathscr{A})$. If there exists an element c that belongs to every element $(\neq \varnothing)$ of $\mathscr{A}$. Then the following statements are equivalent*

(1) $\mu \in ML(\mathscr{A})$;

(2) $\mu_{\mu(c)} \in Max(\mathscr{A})$;

(3) $\mu(x) = \begin{cases} n & x \in A \\ m & x \notin A \end{cases}$ *for some $A \in Max(\mathscr{A})$ and $m, n \in L$ with $m < n$.*

Proof. (1) $\Rightarrow$ (2) Suppose $\mu \in ML(\mathscr{A})$. By Lemma 5.10, we have $\mu(c) \geq \mu(x)$ for all $x \in X$. Since μ is not constant, there exists an element x_0 of X such that $\mu(x_0) \neq \mu(c)$, i.e., $\mu(x_0) < \mu(c)$. It follows that $x_0 \notin \mu_{\mu(c)}$. Since $c \in \mu_{\mu(c)}$, we have $\mu_{\mu(c)} \neq \varnothing$ and $\mu_{\mu(c)} \neq X$ and hence $\mu_{\mu(c)} \in Max(\mathscr{A})$.

(2) $\Rightarrow$ (3) Suppose $\mu_{\mu(c)} \in Max(\mathscr{A})$. For any $x \in \mu_{\mu(c)}$, we have $\mu(c) \geq \mu(x)$ (by Lemma 5.10) and $\mu(x) \geq \mu(c)$, which implies $\mu(x) = \mu(c)$ for all $x \in \mu_{\mu(c)}$. Since μ is not constant, there is an element x_1 of X with $\mu(x_1) \neq \mu(c)$ (i.e., $\mu(c) > \mu(x_1)$). Assume that there exists another element $x_2 \in X$ with $\mu(x_2) \neq \mu(x_1)$ and $\mu(x_2) \neq \mu(c)$. If $\mu(x_1) < \mu(x_2)$ or $\mu(x_2) < \mu(x_1)$, then $\mu_{\mu(c)} \subsetneqq \mu_{\mu(x_2)} \subsetneqq \mu_{\mu(x_1)}$ or $\mu_{\mu(c)} \subsetneqq \mu_{\mu(x_1)} \subsetneqq \mu_{\mu(x_2)}$, which contradicts $\mu_{\mu(c)} \in Max(\mathscr{A})$. If $\mu(x_1)$ and $\mu(x_2)$ are incomparable, then $x_1 \notin \mu_{\mu(x_2)}$ and so $\mu_{\mu(c)} \subsetneqq \mu_{\mu(x_2)} \subsetneqq X$, which is a contradiction. Therefore,

$$\mu(x) = \begin{cases} \mu(c) & x \in \mu_{\mu(c)} \\ \mu(x_1) & x \notin \mu_{\mu(c)} \end{cases}$$

where $\mu_{\mu(c)} \in Max(\mathscr{A})$ and $\mu(x_1) < \mu(c)$.

(3) $\Rightarrow$ (1) Suppose

$$\mu(x) = \begin{cases} n & x \in A \\ m & x \notin A \end{cases}$$

for some $A \in Max(\mathscr{A})$ and $m, n \in L$ with $m < n$. Apparently, $\mu_t \in \{A, X, \varnothing\}$ $(\forall t \in L)$ and hence $\mu \in ML(\mathscr{A})$. $\square$

The following example reveals that the condition that there exists a constant element c that belongs to every element ($\neq \varnothing$) of $\mathscr{A}$ is necessary for Theorem 6.6.

Example 6.7. *Let $X = \{a, b, c\}$, L a lattice as shown in Example 3.4 and*

$$\mathscr{A} = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}.$$

Clearly, $Max(\mathscr{A}) = \{\{a\}, \{b\}, \{c\}\}$. Denote

$$\mu(x) = \begin{cases} m & x = a \\ n & x = b \\ 0 & x = c \end{cases}$$

It can be seen that μ is in $ML(\mathscr{A})$ without being the form in Theorem 6.6.

7 Applications

Firstly, we will discuss the application of our theory in defining L-fuzzy relations. Fuzzy relations were first introduced by Zadeh [20]. We use $FR(A)$ to denote the set of all fuzzy relations on a nonempty set A. Zadeh defined a fuzzy relation from X to Y as a mapping from $X \times Y$ to the real interval $[0, 1]$, and he also introduced fuzzy equivalence relations [21]. Another particular type of fuzzy relations, namely fuzzy congruence relations, was introduced by V. Murali [16].

Next, we will apply our theory to some specific algebraic structures to define their fuzzy substructures. We will provide examples of fuzzy substructures that have been defined, as well as fuzzy prime substructures and fuzzy maximal substructures that have been introduced by some authors.

In the following section, we will present their definitions and show how they can be characterized by L-fuzzy substructures when we choose the appropriate set X, ordered set L and $\mathscr{A} \in \mathscr{C}(\mathrm{P}(X))$.

Example 7.1. *The definition of fuzzy equivalence relations defined by Zadeh* [21] *is presented as follows.*

Let A be a nonempty set and $[0,1]$ a real interval. A binary fuzzy relation θ is called a fuzzy equivalence relation if
(1) $\theta(x,x)=1$ *for any* $x\in A$;
(2) $\theta(x,y)=\theta(y,x)$ *for any* $x,y\in A$;
(3) $\theta(x,z)\geq\theta(x,y)\wedge\theta(y,z)$ *for any* $x,y,z\in A$.
Denote $FEq(A)$ as the set of all the fuzzy equivalence relations on A.

Example 7.2. *The definition of fuzzy congruence relations defined by V. Murali* [16] is shown as follows.

Let A be an algebra of type $\mathscr{F}$ and $[0,1]$ a real interval. A fuzzy equivalence relation (defined in Example 7.1) θ is called a fuzzy congruence if θ has the substitution property. θ with the substitution property means that $\widetilde{f}(\theta,\theta,\cdots,\theta)\leq\theta$ for any $f\in\mathscr{F}$, where $\widetilde{f}$ is an operation on $FR(A)$ induced by f and $r(\widetilde{f})=r(f)$. $\widetilde{f}$ is defined as $\widetilde{f}(\theta_1,\theta_2,\cdots,\theta_{r(f)})(x,y)=\underset{x,y}{sup}(\underset{1\leq i\leq r(f)}{min}(\theta_i(x_i,y_i)))$, where the $\underset{x,y}{sup}$ stands for the supremum being taken over all representations of x and y of the form $x=f(x_1,x_2,\cdots,x_{r(f)})$ and $y=f(y_1,y_2,\cdots,y_{r(f)})$. If no such representation for x and y exist, then $\widetilde{f}(\theta_1,\theta_2,\cdots,\theta_{r(f)})(x,y)=0$. Denote $FCon(A)$ as the set of all fuzzy congruences on A.

In fact, $Eq(A)$ and $Con(A)$ are closure systems on $\mathrm{P}(A^2)$.

Theorem 7.3. *Let A be a nonempty set, $L=[0,1]$ and θ be a binary fuzzy relation. Then*
(1) *θ is a fuzzy equivalence relation if and only if $\theta_t\in Eq(A)$ for any $t\in L$;*
(2) *If A is an algebra of type $\mathscr{F}$, then θ is a fuzzy congruence if and only if $\theta_t\in Con(A)$ for any $t\in L$.*

Proof. We only prove (2). V. Murali [16] had proven that if θ is a fuzzy congruence, then $\theta_t\in Con(A)$ for any $t\in L$. Conversely, suppose $\theta_t\in Con(A)$ for any $t\in L$. Since θ_1 is a congruence relation on A, $(x,x)\in\theta_1$ for any $x\in A$, i.e., $\theta(x,x)=1$ for any $x\in A$. Suppose $(x,y)\in A^2$ with $\theta(x,y)=t$. Since $(x,y)\in\theta_t$ and θ_t is a congruence, we have $(y,x)\in\theta_t$ and so $\theta(y,x)\geq t=\theta(x,y)$. Similarly, we have $\theta(x,y)\geq\theta(y,x)$ and hence $\theta(x,y)=\theta(y,x)$. Suppose $(x,y),(y,z)\in A^2$ and assume $\theta(x,y)=m$, $\theta(y,z)=n$. Then $\theta(x,y)=m\geq m\wedge n$ and $\theta(y,z)=n\geq m\wedge n$ and so $(x,y),(y,z)\in\theta_{m\wedge n}$. Since $\theta_{m\wedge n}$ is a congruence on A, we have $(x,z)\in\theta_{m\wedge n}$, and hence $\theta(x,z)\geq m\wedge n=\theta(x,y)\wedge\theta(y,z)$. Therefore, θ is a fuzzy equivalent relation on A.

Next, we will prove that θ has the substitution property. Suppose $f\in\mathscr{F}$ and $(x,y)\in A^2$. If the representations for x and y to f exist and $x=f(x_1,x_2,\cdots,x_{r(f)})$ and $y=f(y_1,y_2,\cdots,y_{r(f)})$ is one of their representations. Assume that $\theta(x_i,y_i)=m_i, 1\leq i\leq r(f)$, then it follows directly that $(x_j,y_j)\in\theta_{\underset{1\leq i\leq r(f)}{min}(m_i)}(1\leq j\leq r(f))$ since $\theta(x_j,y_j)=m_j\geq\underset{1\leq i\leq r(f)}{min}(m_i)$. Since $\theta_{\underset{1\leq i\leq r(f)}{min}(m_i)}$ is a congruence on A, we have $(x,y)=(f(x_1,x_2,\cdots,x_{r(f)}),f(y_1,y_2,\cdots,y_{r(f)}))\in\theta_{\underset{1\leq i\leq r(f)}{min}(m_i)}$. It follows that $\theta(x,y)\geq\underset{1\leq i\leq r(f)}{min}(m_i)=\underset{1\leq i\leq r(f)}{min}(\theta_i(x_i,y_i))$ and hence

$$\theta(x,y)\geq\underset{x,y}{sup}(\underset{1\leq i\leq r(f)}{min}(\theta_i(x_i,y_i)))=\widetilde{f}(\theta,\theta,\cdots,\theta)(x,y).$$

If no such representation for x and y exist, then $\theta(x,y)\geq 0=\widetilde{f}(\theta,\theta,\cdots,\theta)(x,y)$ and so $\theta(x,y)\geq\widetilde{f}(\theta,\theta,\cdots,\theta)(x,y)$ for any $(x,y)\in A^2$. Therefore, $\theta\geq\widetilde{f}(\theta,\theta,\cdots,\theta)$, i.e., θ has the substitution property, and hence θ is a fuzzy congruence on A. $\square$

Remark: Put $X = A^2$, $\mathscr{A} = Con(A)$ and $L = [0, 1]$. According to the above proof, θ is a fuzzy congruence if and only if $\theta_t \in \mathscr{A}$ for any $t \in L$. Apparently, the definition of fuzzy congruences is stronger than the concept of L-fuzzy substructures for $\mathscr{A}$. Next, we will use our theory to give two weaker definitions of L-fuzzy equivalence relations and L-fuzzy congruence relations.

Example 7.4. (1) *Let X be a nonempty set, L an ordered set, $\mathscr{A} = Eq(X)$ and $\theta \in L^{X^2}$. Then θ is called an L-fuzzy equivalence relation on X if $\theta \in L(\mathscr{A})$.*
(2) *Let X be an algebra of type $\mathscr{F}$, L an ordered set, $\mathscr{A} = Con(X)$ and $\theta \in L^{X^2}$. Then θ is called an L-fuzzy congruence on X if $\theta \in L(\mathscr{A})$.*

Next, we will give equivalent characterizations of the above definitions when L is a lattice.

($1'$) *Let X be a nonempty set, L a lattice and $\mathscr{A} = Eq(X)$. Then $\theta \in L^{X^2}$ is an L-fuzzy equivalence relation (defined in Example 7.4) on X if and only if θ satisfies the following conditions.*
(i) $\theta(x, x) \geq \theta(y, z)$ for any $x, y, z \in X$;
(ii) $\theta(x, y) = \theta(y, x)$ for any $x, y \in X$;
(iii) $\theta(x, z) \geq \theta(x, y) \wedge \theta(y, z)$ for any $x, y, z \in X$.

($2'$) *Let X be an algebra of type $\mathscr{F}$, L a lattice and $\mathscr{A} = Con(X)$. Then $\theta \in L^{X^2}$ is an L-fuzzy congruence (defined in Example 7.4) on X if and only if θ satisfies the following conditions.*
(i) $\theta(x, x) \geq \theta(y, z)$ for any $x, y, z \in X$;
(ii) $\theta(x, y) = \theta(y, x)$ for any $x, y \in X$;
(iii) $\theta(x, z) \geq \theta(x, y) \wedge \theta(y, z)$ for any $x, y, z \in X$.
(iv) for any $f \in \mathscr{F}$ and $x_1, x_2, \cdots, x_{r(f)}, y_1, y_2, \cdots, y_{r(f)} \in X$, $\theta(f(x_1, x_2, \cdots, x_{r(f)}), f(y_1, y_2, \cdots, y_{r(f)})) \geq \theta(x_1, y_1) \wedge \theta(x_2, y_2) \wedge \cdots \wedge \theta(x_{r(f)}, y_{r(f)})$.

In fact, for any $\theta \in L^{X^2}$ satisfying the above conditions in ($2'$), $\theta_t \neq \varnothing$ implies $\theta_t \in Con(X) = \mathscr{A}$. Conversely, if θ is an L-fuzzy congruence on X, by Lemma 5.10 and Theorem 5.12, we have that (i) and (iv) hold. In addition, it is easy to prove that (ii) and (iii) hold as well.

Example 7.5. [7] (Fuzzy subgroup) Let G be a group and $[0, 1]$ be a real interval. A fuzzy subset μ of G is called fuzzy subgroup of G if for any $x, y \in G$, μ satisfies
(1) $\mu(xy) \geq min(\mu(x), \mu(y))$;
(2) $\mu(x^{-1}) \geq \mu(x)$.
Denote $Sub(G)$ as the set of all subgroups of G and $FSub(G)$ as the set of all fuzzy subgroups of G.

The author in [7] has proven that a fuzzy set μ is a fuzzy subgroup if and only if μ_t ($\mu_t \neq \varnothing$) is a subgroup of G for any $t \in [0, 1]$. If we put $L = [0, 1]$ and $\mathscr{A} = Sub(G)$, then $\mathscr{A}$ is a closure system of $P(G)$ and $FSub(G) = L(\mathscr{A})$.

In fact, since every element ($\neq \varnothing$) of $\mathscr{A}$ is a subuniverse of G, by Theorem 5.12, μ must satisfy (1) and (2) for any $\mu \in L(\mathscr{A})$. The proposition given by the author, that $\mu(e) \geq \mu(x)$ for any $\mu \in FSub(G)$ and $x \in G$, can be seen as an inference of Lemma 5.10, since e is a constant element that belongs to every element of $Sub(G)$.

Recall that a pseudo MV-algebra [9] $A = (A, \oplus, ^{-}, ^{\sim}, 0, 1)$ is an algebra of type $(2, 1, 1, 0, 0)$ satisfying the following conditions.
$(A1)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
$(A2)$ $x \oplus 0 = 0 \oplus x = x$,

$(A3)$ $x \oplus 1 = 1 \oplus x = 1$,
$(A4)$ $1^{\sim} = 0, 1^{-} = 0$,
$(A5)$ $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-}$,
$(A6)$ $x \oplus x^{\sim} \cdot y = y \oplus y^{\sim} \cdot x = x \cdot y^{-} \oplus y = y \cdot x^{-} \oplus x$,
$(A7)$ $x \cdot (x^{-} \oplus y) = (x \oplus y^{\sim}) \cdot y$,
$(A8)$ $(x^{-})^{\sim} = x$,
where $x \cdot y = (y^{-} \oplus x^{-})^{\sim}$. If the addition $\oplus$ is commutative, the both unary operations $^{-}$ and $^{\sim}$ coincide and A is an MV-algebra. An ideal I of a pseudo MV-algebra A is a subset of A satisfying
$(I1)$ $0 \in I$,
$(I2)$ if $x, y \in I$, then $x \oplus y \in I$,
$(I3)$ if $x \in I, y \in A$ and $y \leq x$, then $y \in I$.

Example 7.6. [9] (Fuzzy ideals in pseudo MV-algebra) Let A be a pseudo MV-algebra and $[0,1]$ a real interval. A fuzzy set μ of A is called a fuzzy ideal of A if for any $x, y \in A$, it satisfies
(1) $\mu(x \oplus y) \geq \mu(x) \wedge \mu(y)$;
(2) $x \leq y$ implies $\mu(x) \geq \mu(y)$.
Denote $FI(A)$ as the set of all fuzzy ideals in A.

It can be easily proven that $Id(A)$ is a closure system of $\mathrm{P}(A)$. It was proven in [14] that a fuzzy set μ is a fuzzy ideal if and only if μ_t $(\mu_t \neq \varnothing) \in Id(A)$ for any $t \in [0,1]$. As a result, if we put $\mathscr{A} = Id(A)$ and $L = [0,1]$, then $L(\mathscr{A}) = FI(A)$. In fact, since every element $(\neq \varnothing)$ of $\mathscr{A}$ is a subuniverse of $(A, \oplus)$ and also a down set, according to Theorem 5.9 and Theorem 5.12, any $\mu \in L(\mathscr{A})$ must satisfy (1) and (2).

Example 7.7. [10] (Fuzzy prime ideals in pseudo MV-algebra) Let A be a pseudo MV-algebra and $[0,1]$ a real interval. A fuzzy ideal (defined in Example 7.6) μ is said to be fuzzy prime if it is not constant and satisfies $\mu(x \wedge y) = \mu(x) \vee \mu(y)$ for all $x, y \in A$. We denote by $PId(A)$ the set of all prime ideals of A, and by $FPId(A)$ the set of all fuzzy prime ideals of A.

Put $\mathscr{A} = Id(A)$ and $L = [0,1]$, then the definition of prime ideals in A coincides the concept of $Prim(\mathscr{A})$. Next, we will prove that $FPId(A) = PL(\mathscr{A})$. The fact that $I \in Id(A)$ is prime if and only if $x \wedge y \in I$ implies $x \in I$ or $y \in I$ had been proved in [11].

Suppose $\mu \in FPId(A)$ and $t \in L$ with $\mu_t \neq \varnothing$ and $\mu_t \neq X$. Then by Example 7.6, we have $\mu_t \in Id(A)$. For any $x, y \in A$ with $x \wedge y \in \mu_t$, we have $t \leq \mu(x \wedge y) = \mu(x) \vee \mu(y)$, which implies $x \in \mu_t$ or $y \in \mu_t$ and hence $\mu_t \in Prim(\mathscr{A})$. It follows that $\mu \in PL(\mathscr{A})$, and we have shown that $FPId(A) \subseteq PL(\mathscr{A})$. Conversely, suppose $\mu \in PL(\mathscr{A})$. By Example 7.6, we have $\mu \in FId(A)$. Since $x \wedge y \in \mu_{\mu(x \wedge y)}$, if $\mu_{\mu(x \wedge y)} = A$, then $\mu(x) \geq \mu(x \wedge y)$ and $\mu(y) \geq \mu(x \wedge y)$. It follows directly that $\mu(x) \vee \mu(y) = \mu(x \wedge y)$. If $\mu_{\mu(x \wedge y)} \neq A$, then by the premise, $\mu_{\mu(x \wedge y)} \in Prim(\mathscr{A})$, which implies $x \in \mu_{\mu(x \wedge y)}$ or $y \in \mu_{\mu(x \wedge y)}$. In other words, $\mu(x) \geq \mu(x \wedge y)$ or $\mu(y) \geq \mu(x \wedge y)$ and hence $\mu(x) \vee \mu(y) \geq \mu(x \wedge y)$. Hence $\mu(x) \vee \mu(y) = \mu(x \wedge y)$ and $\mu \in FPId(A)$. Therefore, $PL(\mathscr{A}) \subseteq FPId(A)$.

Note: The result proved in [10], that $\mu \in Prim(FId(A))$ implies $\mu \in FPId(A)$, can be seen as a corollary of Theorem 6.3.

Example 7.8. [9] (Fuzzy maximal ideals) Let A be a pseudo MV-algebra and $[0,1]$ a real interval. Then a fuzzy ideal μ (defined in Example 7.6) is said to be fuzzy maximal if A_μ is a maximal ideal of A, where $A_\mu = \mu_{\mu(0)}$.

If we denote by $MId(A)$ and $FMId(A)$ the set of all maximal ideals and fuzzy maximal ideals of A, respectively, then the maximal ideals in this paper are indeed the maximal elements of $Id(A)$.

This paper gives the following properties:
(1) $I \in Id(A)$ is maximal if and only if μ_I is fuzzy maximal, where

$$\mu_I(x) = \begin{cases} m & x \in I \\ n & x \notin I \end{cases}$$

and $m, n \in [0,1]$ with $m > n$.
(2) $I \in Id(A)$ is maximal if and only if χ_I is fuzzy maximal, where χ_I is the characteristic function of I.
(3) If $\mu \in FMId(A)$, then μ takes only two distinct values.
(4) Suppose that μ is a fuzzy subset of A and μ is not constant. Then $\mu \in FMId(A)$ if and only if $\mu_t \in MId(A)$ whenever $\mu_t \neq \varnothing$ and $\mu_t \neq A$.

Put $\mathscr{A} = Id(A)$ and $L = [0,1]$. Example 7.6 shows that $L(\mathscr{A}) = FI(A)$. Since $MId(A) = Max(\mathscr{A})$ and 0 is a constant element that belongs to every element ($\neq \varnothing$) of $\mathscr{A}$, it can easily see that the properties given above can be viewed as corollaries for Theorem 6.6.

8 Conclusions

In this paper, uses closure systems to generalize substructures in general algebra, which provides a method to establish the unified fuzzification theory for substructures. The broad applications of our theory are also illustrated. For example, if the closure system $\mathscr{A}$ is set as all equivalence relations on a set X, then the L-fuzzy substructures concerning $\mathscr{A}$ are L-fuzzy equivalence relations. Similarly, if the closure system $\mathscr{A}$ is set as all congruence relations on an algebra X, then the L-fuzzy substructures to $\mathscr{A}$ are L-fuzzy congruence relations. The definitions of L-fuzzy equivalence relations and L-fuzzy congruence relations are generalizations of Examples 7.1 and 7.2. Furthermore, our theory is also used in groups and pseudo MV-algebras. It is proved that their definitions of fuzzy substructures are equivalent to L-fuzzy substructures when appropriate closure system $\mathscr{A}$ and L are considered. Finally, it is confirmed that many results in [9], [10], and [7] can be viewed as corollaries from our theory.

Based on the closure systems, this paper only discusses some common properties of L-fuzzy substructures and several exceptional cases when the algebra X, the closure system $\mathscr{A}$ and the ordered set L have certain characters. Thus, in our future work, we expect to use our theory in fuzzy topologies and rough sets. We can also take some special ordered set L, such as a completely distributive lattice. This will provide better results as our theory is used.

Acknowledgment

This research is supported by grants of National Natural Science Foundation of China (11971384, 61976130) and Foreign Expert Program of China (Grant No. DL20230410021).

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