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Ideals of roughness in L-algebras

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Abstract

Rough is an exceptional mathematical tool for effectively analyzing and addressing the complexities of vague action descriptions in decision problems. This paper explores the concept of an L-algebra, which leads to the introduction of lower and upper approximations. The properties of these approximations are also discussed and elucidated. Furthermore, it is proved that the lower and upper approximations serve as interior and closure operators, respectively. Additionally, by employing A-lower and A-upper approximations, this paper presents and examines conditions for a nonempty subset to be definable. Furthermore, we investigated the circumstances under which the A-lower and A-upper approximations can be rough ideals. Finally, we define an operation $\curvearrowright$ on the set of all upper approximations of L and prove that it is made an L-algebra.

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1 Introduction

The rough sets theory, initially introduced by Pawlak in [21], has consistently demonstrated its effectiveness as a powerful mathematical tool for analyzing the inherent vagueness in describing objects, precisely actions, in decision problems. Rough sets theory has proven to apply to a wide range of issues. In recent years, mathematicians have explored the idea of roughness in various fields of mathematics. For instance, Iwinski presented an algebraic perspective on rough sets in [14]. The application of rough set theory to semigroups and groups has been investigated in [17, 18]. In 1994, Biswas and Nanda introduced and examined the concept of rough groups and rough subgroups in [6]. Jun applied rough set theory to BCK-algebras in [7, 15, 20]. More recently, Rasouli introduced and studied the notion of roughness in MV-algebras in [22].

The quantum Yang-Baxter equation (QYBE), formulated independently by Zhenning Yang and

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R.J. Baxter in 1967 and 1972, respectively, stands as the cornerstone in the field of mathematical physics [10]. This equation serves as a fundamental principle linking various mathematical structures, including quantum binomial algebras [11, 12], I-type semigroups, and Bieberbach groups [13], plane curves, dyeing of bijective 1-type cocycles [10], semi-munipolar small triangular Hopf algebra [26], dynamic systems [9], geometric crystals [8], and more. While numerous early solutions to the QYBE have been discovered and extensively studied about their algebraic structures, these solutions primarily consist of variations of the standard identity solutions. Therefore, the search for non-trivial solutions becomes imperative. Drinfeld proposed investigating of a specific class of "set theory solutions" involving linear operators [8]. These operators act on vector spaces V generated by a set $\mathbb{L}$ and induced by a mapping $\mathbb{L} \times \mathbb{L} \to \mathbb{L} \times \mathbb{L}$. In 2005, W. Rump studied the algebraic solution of the quantum Yang-Baxter equation. He pointed out that $\mathbb{L}$ is a set with a binary operation $\curvearrowright$. The equation

$$(\mathfrak{b} \curvearrowright \mathfrak{p}) \curvearrowright (\mathfrak{b} \curvearrowright \mathfrak{w}) = (\mathfrak{p} \curvearrowright \mathfrak{b}) \curvearrowright (\mathfrak{p} \curvearrowright \mathfrak{w}), \qquad (L)$$

is a true statement of propositional logic and some of its generalizations [25]. On the other hand, (L) is closely related to the quantum Yang-Baxter equation. In algebraic logic, new aspects were recently found in [23]. We say that an element $1 \in \mathbb{L}$ is a logical unit [23] if $\mathfrak{b} \curvearrowright \mathfrak{b} = \mathfrak{b} \curvearrowright 1 = 1$ and $1 \curvearrowright \mathfrak{b} = \mathfrak{b}$ holds for all $\mathfrak{b} \in \mathbb{L}$. In the presence of (L), a logical unit defines a quasi-ordering $\mathfrak{b} \leq \mathfrak{p}$ if and only if $\mathfrak{b} \curvearrowright \mathfrak{p} = 1$, and if this is a partial order, we call $\mathbb{L}$ an L-algebra [23]. For the theory of L-algebras, the reader is referred to [1, 2, 3, 4, 5, 23, 24, 19].

Rough is an exceptional mathematical tool for analyzing vague action descriptions in decision problems. This paper discusses L-algebras, which introduce lower and upper approximations and examine their properties. Furthermore, it is proven that the lower and upper approximations serve as interior and closure operators, respectively. Additionally, by employing A-lower and A-upper approximations, this paper presents and examines conditions for a nonempty subset to be definable. Furthermore, we investigate the circumstances under which the A-lower and A-upper approximations can be rough ideals. Finally, we define an operation $\curvearrowright$ on the set of all upper approximations of L and prove that it is made an L-algebra.

2 Preliminaries

This section provides a list of the default contents known to be used later.

Definition 2.1. [23] *An algebraic structure* $(\mathbb{L}; \curvearrowright, 1)$ *is referred to as an* L-algebra *if it satisfies the following conditions for any* $\mathfrak{b}, \mathfrak{p}, \mathfrak{w} \in \mathbb{L}$:
(L_1) $\mathfrak{b} \curvearrowright \mathfrak{b} = \mathfrak{b} \curvearrowright 1 = 1$ *and* $1 \curvearrowright \mathfrak{b} = \mathfrak{b}$,
(L_2) $(\mathfrak{b} \curvearrowright \mathfrak{p}) \curvearrowright (\mathfrak{b} \curvearrowright \mathfrak{w}) = (\mathfrak{p} \curvearrowright \mathfrak{b}) \curvearrowright (\mathfrak{p} \curvearrowright \mathfrak{w})$,
(L_3) *if* $\mathfrak{b} \curvearrowright \mathfrak{p} = \mathfrak{p} \curvearrowright \mathfrak{b} = 1$, *then* $\mathfrak{b} = \mathfrak{p}$.

Note. We must note that each logical unit in $\mathbb{L}$ is always unique. If $\mathbb{L}$ has the smallest element 0, it is a bounded L-algebra. In an L-algebra with 0, for any element $\mathfrak{b}$, we define an operation $'$ on $\mathbb{L}$ such that $\mathfrak{b}' = \mathfrak{b} \curvearrowright 0$. A bounded L-algebra $\mathbb{L}$ is said to have a 'negation' if the mapping $\mathfrak{b} \mapsto \mathfrak{b}'$ is bijective. The inverse mapping is $\mathfrak{b} \mapsto \overline{\mathfrak{b}}$. If $\overline{\mathfrak{b}} = \mathfrak{b}'$, then $\mathbb{L}$ is known as an L-algebra with double negation.

If we consider the operation $\curvearrowright$ as a logical implication, then we can define a partial order on $\mathbb{L}$ using the following:

$$\mathfrak{b} \leq \mathfrak{p} \text{ if and only if } \mathfrak{b} \curvearrowright \mathfrak{p} = 1. \qquad (2.1)$$

You can see the proof in [23].

Proposition 2.2. *Consider an L-algebra* $(\mathbb{L}, \curvearrowright, 1)$. *If* $\mathfrak{b} \leq \mathfrak{p}$, *then it follows that* $\mathfrak{w} \curvearrowright \mathfrak{b} \leq \mathfrak{w} \curvearrowright \mathfrak{p}$ *for any* $\mathfrak{b}, \mathfrak{p}, \mathfrak{w} \in \mathbb{L}$.

Proposition 2.3. *The following conditions are equivalent for an L-algebra* $(\mathbb{L}, \curvearrowright, 1)$:
(i) $\mathfrak{b} \leq \mathfrak{p} \curvearrowright \mathfrak{b}$,
(ii) *if* $\mathfrak{b} \leq \mathfrak{p}$, *then* $\mathfrak{p} \curvearrowright \mathfrak{w} \leq \mathfrak{b} \curvearrowright \mathfrak{w}$,
(iii) $((\mathfrak{b} \curvearrowright \mathfrak{p}) \curvearrowright \mathfrak{w}) \curvearrowright \mathfrak{w} \leq ((\mathfrak{b} \curvearrowright \mathfrak{p}) \curvearrowright \mathfrak{w}) \curvearrowright ((\mathfrak{p} \curvearrowright \mathfrak{b}) \curvearrowright \mathfrak{w})$, *for any* $\mathfrak{b}, \mathfrak{p}, \mathfrak{w} \in \mathbb{L}$.

Definition 2.4. [23] *A KL-algebra is defined as an L-algebra* $(\mathbb{L}, \curvearrowright, 1)$ *that satisfies the condition:*

$$\mathfrak{b} \curvearrowright (\mathfrak{p} \curvearrowright \mathfrak{b}) = 1, \qquad (K)$$

for any $\mathfrak{b}, \mathfrak{p} \in \mathbb{L}$.

Notation. If $(\mathbb{L}, \curvearrowright, 1)$ is a KL-algebra, then the equivalent statements of Proposition 2.3 hold true.

Definition 2.5. [23] *A CKL-algebra is an L-algebra* $(\mathbb{L}, \curvearrowright, 1)$ *that satisfies the condition:*

$$\mathfrak{b} \curvearrowright (\mathfrak{p} \curvearrowright \mathfrak{w}) = \mathfrak{p} \curvearrowright (\mathfrak{b} \curvearrowright \mathfrak{w}), \qquad (C)$$

for any $\mathfrak{b}, \mathfrak{p}, \mathfrak{w} \in \mathbb{L}$.

Note. It is evident that every CKL-algebra is also a KL-algebra. This is because for any $\mathfrak{b}, \mathfrak{p} \in \mathbb{L}$, we have the following:

$$\mathfrak{b} \curvearrowright (\mathfrak{p} \curvearrowright \mathfrak{b}) = \mathfrak{p} \curvearrowright (\mathfrak{b} \curvearrowright \mathfrak{b}) = \mathfrak{p} \curvearrowright 1 = 1. \qquad (2.2)$$

Proposition 2.6. [4] *Assuming* $(\mathbb{L}, \curvearrowright, 1)$ *is a CKL-algebra, then the following properties hold for any* $\mathfrak{b}, \mathfrak{p}, \mathfrak{w} \in \mathbb{L}$:
(i) *if* $\mathfrak{b} \leq \mathfrak{p}$, *then* $\mathfrak{w} \curvearrowright \mathfrak{b} \leq \mathfrak{w} \curvearrowright \mathfrak{p}$,
(ii) $\mathfrak{b} \curvearrowright (\mathfrak{p} \curvearrowright \mathfrak{b}) = 1$, *i.e.,* $\mathfrak{b} \leq \mathfrak{p} \curvearrowright \mathfrak{b}$,
(iii) $\mathfrak{b} \leq (\mathfrak{b} \curvearrowright \mathfrak{p}) \curvearrowright \mathfrak{p}$,
(iv) $\mathfrak{b} \leq \mathfrak{p} \curvearrowright \mathfrak{w}$ *if and only if* $\mathfrak{p} \leq \mathfrak{b} \curvearrowright \mathfrak{w}$,
(v) *if* $\mathfrak{b} \leq \mathfrak{p}$, *then* $\mathfrak{p} \curvearrowright \mathfrak{w} \leq \mathfrak{b} \curvearrowright \mathfrak{w}$,
(vi) $((\mathfrak{b} \curvearrowright \mathfrak{p}) \curvearrowright \mathfrak{w}) \curvearrowright \mathfrak{w} \leq ((\mathfrak{b} \curvearrowright \mathfrak{p}) \curvearrowright \mathfrak{w}) \curvearrowright ((\mathfrak{p} \curvearrowright \mathfrak{b}) \curvearrowright \mathfrak{w})$,
(vii) $\mathfrak{w} \curvearrowright \mathfrak{p} \leq (\mathfrak{p} \curvearrowright \mathfrak{b}) \curvearrowright (\mathfrak{w} \curvearrowright \mathfrak{b})$,
$(viii)$ $\mathfrak{w} \curvearrowright \mathfrak{p} \leq (\mathfrak{b} \curvearrowright \mathfrak{w}) \curvearrowright (\mathfrak{b} \curvearrowright \mathfrak{p})$,
If $\mathbb{L}$ *has a least element as* 0, *then*
(ix) *if* $\mathfrak{b} \leq \mathfrak{p}$, *then* $\mathfrak{p}' \leq \mathfrak{b}'$,
(x) $\mathfrak{b} \leq \mathfrak{b}''$, *and* $\mathfrak{b}' = \mathfrak{b}'''$,
(xi) $\mathfrak{b}' \leq \mathfrak{b} \curvearrowright \mathfrak{p}$,
(xii) $((\mathfrak{b} \curvearrowright \mathfrak{p}) \curvearrowright \mathfrak{p}) \curvearrowright \mathfrak{p} = \mathfrak{b} \curvearrowright \mathfrak{p}$,
$(xiii)$ $(\mathfrak{n} \curvearrowright \mathfrak{p}) \curvearrowright (\mathfrak{b} \curvearrowright \mathfrak{p}) \leq (\mathfrak{n} \curvearrowright \mathfrak{x}) \curvearrowright [(\mathfrak{x} \curvearrowright \mathfrak{p}) \curvearrowright (\mathfrak{b} \curvearrowright \mathfrak{p})]$.
(xiv) *If* $\mathbb{L}$ *has double negation, then* $\mathfrak{b} \curvearrowright \mathfrak{p} = \mathfrak{p}' \curvearrowright \mathfrak{b}'$.

Definition 2.7. [23] *A subset* $\mathbb{I}$ *of an L-algebra* $(\mathbb{L}, \curvearrowright, 1)$ *is referred to as an* ideal of $\mathbb{L}$ *if it satisfies the following conditions for all* $\mathfrak{b}, \mathfrak{p} \in \mathbb{L}$:
(I_1) $1 \in \mathbb{I}$,
(I_2) *if* $\mathfrak{b} \in \mathbb{I}$ *and* $\mathfrak{b} \curvearrowright \mathfrak{p} \in \mathbb{I}$, *then* $\mathfrak{p} \in \mathbb{I}$,

(I_3) *if* $\mathfrak{b} \in \mathbb{I}$, *then* $(\mathfrak{b} \curvearrowright \mathfrak{p}) \curvearrowright \mathfrak{p} \in \mathbb{I}$,
(I_4) *if* $\mathfrak{b} \in \mathbb{I}$, *then* $\mathfrak{p} \curvearrowright \mathfrak{b} \in \mathbb{I}$ *and* $\mathfrak{p} \curvearrowright (\mathfrak{b} \curvearrowright \mathfrak{p}) \in \mathbb{I}$.

It is evident that $\{1\}$ *and* $\mathbb{L}$ *are two trivial ideals of the* L*-algebra. An ideal* $\mathbb{I}$ *of* $\mathbb{L}$ *is considered a* proper ideal *if* $\mathbb{I}$ *is not equal to* $\mathbb{L}$*. The set of all ideals of* $\mathbb{L}$ *is denoted by* $\mathcal{I}d(\mathbb{L})$*, while the set of all proper ideals of* $\mathbb{L}$ *is denoted by* $p\mathcal{I}d(\mathbb{L})$*.*

Proposition 2.8. [4] *Every ideal of* $\mathbb{L}$ *is upset.*

In the context of an L-algebra, a binary relation $\cong$ is considered a congruence relation on $\mathbb{L}$ if it satisfies the following properties:
1. It is an equivalence relation.
2. For any $\mathfrak{b}, \mathfrak{p}, \mathfrak{w} \in \mathbb{L}$, the following holds:

$$\mathfrak{b} \cong \mathfrak{p} \Leftrightarrow \ (\mathfrak{w} \curvearrowright \mathfrak{b}) \cong (\mathfrak{w} \curvearrowright \mathfrak{p}) \ \text{ and } \ (\mathfrak{b} \curvearrowright \mathfrak{w}) \cong (\mathfrak{p} \curvearrowright \mathfrak{w}).$$

Theorem 2.9. [23, Proposition 1] *Let* $(\mathbb{L}, \curvearrowright, 1)$ *be an* L*-algebra. Then every ideal* $\mathbb{I}$ *of* $\mathbb{L}$ *defines a congruence relation on* $\mathbb{L}$*, for any* $\mathfrak{b}, \mathfrak{p} \in \mathbb{L}$*, where*

$$\mathfrak{b} \cong \mathfrak{p} \ \Leftrightarrow \ \ \mathfrak{b} \curvearrowright \mathfrak{p}, \ \mathfrak{p} \curvearrowright \mathfrak{b} \in \mathbb{I}.$$

Conversely, every congruence relation $\cong$ *defines an ideal* $\mathbb{I} = \{\mathfrak{b} \in \mathbb{L} \mid \mathfrak{b} \cong 1\}$*.*

Let $\dfrac{\mathbb{L}}{\mathbb{I}} = \{[\mathfrak{b}] \mid \mathfrak{b} \in \mathbb{L}\}$, where $[\mathfrak{b}] = \{\mathfrak{p} \in \mathbb{L} \mid \mathfrak{b} \cong \mathfrak{p}\}$ and $\mathbb{I} \in \mathcal{I}(\mathbb{L})$. Then the binary relation $\leq_{\mathbb{I}}$ on $\dfrac{\mathbb{L}}{\mathbb{I}}$ which is defined by

$$[\mathfrak{b}] \leq_{\mathbb{I}} [\mathfrak{p}] \text{ if and only if } \mathfrak{b} \curvearrowright \mathfrak{p} \in \mathbb{I},$$

and $(\dfrac{\mathbb{L}}{\mathbb{I}}, \leq_{\mathbb{I}})$ is a poset (see [23, Proposition 2]).

Consider $\mathbb{L}_1$ and $\mathbb{L}_2$ are two L-algebras. A map $\mathfrak{h} : \mathbb{L}_1 \to \mathbb{L}_2$ is an *L-homomorphism* if it satisfies in the following conditions:

$$\mathfrak{h}(\mathfrak{b} \curvearrowright \mathfrak{p}) = \mathfrak{h}(\mathfrak{b}) \curvearrowright \mathfrak{h}(\mathfrak{p}).$$

An L-homomorphism $\mathfrak{h}$ is called an *L-isomorphism* if $\mathfrak{h}$ is bijective.

Let Ξ be an equivalence relation on a set $\mathbb{L}$ and $\mathcal{P}(\mathbb{L})$ denote the power set of $\mathbb{L}$. For all $\mathfrak{b} \in \mathbb{L}$, let $[\mathfrak{b}]_{\Xi}$ denote the equivalence class of $\mathfrak{b}$ with respect to Ξ. Let Ξ_* and Ξ^* be mappings from $\mathcal{P}(\mathbb{L})$ to $\mathcal{P}(\mathbb{L})$ defined by

$$\Xi_* : \mathcal{P}(\mathbb{L}) \to \mathcal{P}(\mathbb{L}), \ A \mapsto \{\mathfrak{b} \in \mathbb{L} \mid [\mathfrak{b}]_{\Xi} \subseteq A\},$$

and

$$\Xi^* : \mathcal{P}(\mathbb{L}) \to \mathcal{P}(\mathbb{L}), \ A \mapsto \{\mathfrak{b} \in \mathbb{L} \mid [\mathfrak{b}]_{\Xi} \cap A \neq \emptyset\},$$

respectively. The pair $(\mathbb{L}, \Xi)$ is called an *approximation space* based on Ξ. A subset A of $\mathbb{L}$ is called *definable* if $\Xi_*(A) = \Xi^*(A)$, and *rough* otherwise. The set $\Xi_*(A)$ (resp., $\Xi^*(A)$) is called the *lower* (resp. *upper*) *approximation.*

Notation. In the following, we will assume that $(\mathbb{L}, \curvearrowright, 1)$ is an L-algebra (or simply denoted by $\mathbb{L}$) and $\mathbb{I}$ is an ideal of $\mathbb{L}$, unless stated otherwise.

3 Roughness of ideal of $\mathbb{L}$

In this section, we introduce the concept of lower and upper approximations of L-algebras and examine some of their properties. Additionally, we demonstrate that the lower and upper approximations constitute an interior operator and a closure operator, respectively.

Let $\cong_A$ denote a relation on $\mathbb{L}$ defined as follows:

$$\mathfrak{b} \cong_A \mathfrak{p} \text{ if and only if } \mathfrak{b} \curvearrowright \mathfrak{p} \in A \text{ and } \mathfrak{p} \curvearrowright \mathfrak{b} \in A.$$

It is evident that $\cong_A$ is an equivalence relation on $\mathbb{L}$ with respect to A, where $A \in \mathcal{I}(\mathbb{L})$. Moreover, $\cong_A$ satisfies the following condition:

$$\text{if } \mathfrak{v} \cong_A \mathfrak{a} \text{ and } \mathfrak{b} \cong_A \mathfrak{p}, \text{ then } (\mathfrak{v} \curvearrowright \mathfrak{b}) \cong_A (\mathfrak{a} \curvearrowright \mathfrak{p}).$$

Therefore, $\cong_A$ can be considered a congruence relation on $\mathbb{L}$, and we refer to it as the A-congruence relation on $\mathbb{L}$. Let $\frac{\mathbb{L}}{A}$ denote the collection of all equivalence classes, i.e., $\frac{\mathbb{L}}{A} = \{A[\mathfrak{b}] \mid \mathfrak{b} \in \mathbb{L}\}$. It is worth noting that $A[1] = A$. For any $A[\mathfrak{b}], A[\mathfrak{p}] \in \frac{\mathbb{L}}{A}$, we can define a binary operation denoted by "$\blacktriangleright$" on $\frac{\mathbb{L}}{A}$ as follows:

$$A[\mathfrak{b}] \blacktriangleright A[\mathfrak{p}] = A[\mathfrak{b} \curvearrowright \mathfrak{p}].$$

It can be easily verified that $(\frac{\mathbb{L}}{A}, \blacktriangleright, A[1])$ forms an L-algebra.

Let's consider the mappings for the A-congruence relation $\cong_A$ on $\mathbb{L}$:

$$\underline{\mathsf{appr}}_A : \mathcal{P}(\mathbb{L}) \to \mathcal{P}(\mathbb{L}), \ L \mapsto \{\mathfrak{b} \in \mathbb{L} \mid A[\mathfrak{b}] \subseteq L\},$$
$$\overline{\mathsf{appr}}_A : \mathcal{P}(\mathbb{L}) \to \mathcal{P}(\mathbb{L}), \ L \mapsto \{\mathfrak{b} \in \mathbb{L} \mid A[\mathfrak{b}] \cap L \neq \emptyset\}.$$

These mappings, known as the A-lower approximation and the A-upper approximation of L, respectively, define an approximation space on $(\mathbb{L}, \cong_A)$ based on the ideal A of $\mathbb{L}$. This approximation space is $(\mathbb{L}, A)$. A subset L of $\mathbb{L}$ is considered "definable" to A if the A-lower approximation and A-upper approximation of L are equal, i.e., $\underline{\mathsf{appr}}_A(L) = \overline{\mathsf{appr}}_A(L)$. If the A-lower approximation and A-upper approximation of L are not equal, then L is considered "rough".

The following proposition bears similarity to Proposition 3.3 in [16].

Proposition 3.1. [16] *Consider an A-approximation space $(\mathbb{L}, A)$. For any subsets L and M in the power set of $\mathbb{L}$, we have the following:*

(i) $\underline{\mathsf{appr}}_A(L) \subseteq L \subseteq \overline{\mathsf{appr}}_A(L)$,

(ii) $\underline{\mathsf{appr}}_A(L \cap M) = \underline{\mathsf{appr}}_A(L) \cap \underline{\mathsf{appr}}_A(M)$,

(iii) $\underline{\mathsf{appr}}_A(L) \cup \underline{\mathsf{appr}}_A(M) \subseteq \underline{\mathsf{appr}}_A(L \cup M)$,

(iv) $\overline{\mathsf{appr}}_A(L \cap M) \subseteq \overline{\mathsf{appr}}_A(L) \cap \overline{\mathsf{appr}}_A(M)$,

(v) $\overline{\mathsf{appr}}_A(L) \cup \overline{\mathsf{appr}}_A(M) = \overline{\mathsf{appr}}_A(L \cup M)$.

(vi) $\underline{\mathsf{appr}}_A(\overline{\mathsf{appr}}_A(L)) \subseteq \overline{\mathsf{appr}}_A(\overline{\mathsf{appr}}_A(L))$,

(vii) $\underline{\mathsf{appr}}_A(\underline{\mathsf{appr}}_A(L)) \subseteq \overline{\mathsf{appr}}_A(\underline{\mathsf{appr}}_A(L))$,

(viii) $\underline{\mathsf{appr}}_A(L^c) = (\overline{\mathsf{appr}}_A(L))^c$,

(ix) $\overline{\mathsf{appr}}_A(L^c) = \left(\underline{\mathsf{appr}}_A(L)\right)^c$,

(x) $\underline{\mathsf{appr}}_A(L) = \emptyset$ *for* $L \neq \mathbb{L}$,

(xi) $\overline{\mathsf{appr}}_A(L) = L$ *for* $L \neq \emptyset$.

(xii) $\underline{\mathsf{appr}}_A(L) = L$ *if and only if* $\overline{\mathsf{appr}}_A(L^c) = L^c$.

Definition 3.2. [23] *Let S be a set. A function $C : \mathcal{P}(S) \to \mathcal{P}(S)$ is referred to as a closure operator on the set S if it satisfies the following conditions for all subsets X and Y of S:*

(C_1) $X \subseteq C(X)$,

(C_2) *if* $X \subseteq Y$, *then* $C(X) \subseteq C(Y)$,

(C_3) $C(C(X)) = C(X)$.

Definition 3.3. [24] *Let S be a set. A function $int : \mathcal{P}(S) \to \mathcal{P}(S)$ is considered an interior operator on the set S if it satisfies the following conditions for all subsets X and Y of S:*

(i) $int(X) \subseteq X$,

(ii) *if* $X \subseteq Y$, *then* $int(X) \subseteq int(Y)$,

(iii) $int(int(X)) = int(X)$.

Theorem 3.4. *Consider an A-approximation space $(\mathbb{L}, A)$. In this context, $\underline{\mathsf{appr}}_A$ and $\overline{\mathsf{appr}}_A$ function as an interior operator and a closure operator, respectively.*

Proof. For any $L \in \mathcal{P}(\mathbb{L})$, by Proposition 3.1(i), we have $L \subseteq \overline{\mathsf{appr}}_A(L)$ and $\overline{\mathsf{appr}}_A(L) \subseteq \overline{\mathsf{appr}}_A(\overline{\mathsf{appr}}_A(L))$. Let $\mathfrak{b} \in \overline{\mathsf{appr}}_A(\overline{\mathsf{appr}}_A(L))$. Then $A[\mathfrak{b}] \cap \overline{\mathsf{appr}}_A(L) \neq \emptyset$, and so $A[\mathfrak{b}] = A[\mathfrak{p}]$ for some $\mathfrak{p} \in \overline{\mathsf{appr}}_A(L)$. It follows that $A[\mathfrak{b}] \cap L = A[\mathfrak{p}] \cap L \neq \emptyset$. Thus $\mathfrak{b} \in \overline{\mathsf{appr}}_A(L)$, which shows that

$$\overline{\mathsf{appr}}_A(\overline{\mathsf{appr}}_A(L)) \subseteq \overline{\mathsf{appr}}_A(L)$$

Hence $\overline{\mathsf{appr}}_A$ is idempotent. Let $L, M \in \mathcal{P}(\mathbb{L})$ such that $L \subseteq M$. If $\mathfrak{b} \in \overline{\mathsf{appr}}_A(L)$, then $\emptyset \neq A[\mathfrak{b}] \cap L \subseteq A[\mathfrak{b}] \cap M$, and so $\mathfrak{b} \in \overline{\mathsf{appr}}_A(M)$. Hence, $\overline{\mathsf{appr}}_A(L) \subseteq \overline{\mathsf{appr}}_A(M)$. Therefore, $\overline{\mathsf{appr}}_A$ is a closure operator on $\mathbb{L}$.

Moreover, by Proposition 3.1(i), we get $\underline{\mathsf{appr}}_A(L) \subseteq L$ and $\underline{\mathsf{appr}}_A(\underline{\mathsf{appr}}_A(L)) \subseteq \underline{\mathsf{appr}}_A(L)$. Let $\mathfrak{b} \in \underline{\mathsf{appr}}_A(L)$. Then $A[\mathfrak{b}] \subseteq L$. Let $\mathfrak{w} \in A[\mathfrak{b}]$. Then $A[\mathfrak{w}] = A[\mathfrak{b}] \subseteq L$, and so $\mathfrak{w} \in \underline{\mathsf{appr}}_A(L)$. Thus $A[\mathfrak{b}] \subseteq \underline{\mathsf{appr}}_A(L)$ which implies that $\mathfrak{b} \in \underline{\mathsf{appr}}_A(\underline{\mathsf{appr}}_A(L))$. Therefore,

$$\underline{\mathsf{appr}}_A(\underline{\mathsf{appr}}_A(L)) = \underline{\mathsf{appr}}_A(L)$$

Now, let $L, M \in \mathcal{P}(\mathbb{L})$ such that $L \subseteq M$. If $\mathfrak{b} \in \underline{\mathsf{appr}}_A(L)$, then $A[\mathfrak{b}] \subseteq L \subseteq M$, and so $\mathfrak{b} \in \underline{\mathsf{appr}}_A(M)$. Hence, $\underline{\mathsf{appr}}_A(L) \subseteq \underline{\mathsf{appr}}_A(M)$. Therefore, $\underline{\mathsf{appr}}_A$ is an interior operator on $\mathbb{L}$. □

Proposition 3.5. *Consider an A-approximation space $(\mathbb{L}, A)$. In this context, for all $\mathfrak{b} \in \mathbb{L}$, $A[\mathfrak{b}]$ is definable with respect to A.*

Proof. By Proposition 3.1(i), clearly $\underline{\mathsf{appr}}_A(A[\mathfrak{b}]) \subseteq \overline{\mathsf{appr}}_A(A[\mathfrak{b}])$, for all $\mathfrak{b} \in \mathbb{L}$. Let $\mathfrak{p} \in \overline{\mathsf{appr}}_A(A[\mathfrak{b}])$. Then $A[\mathfrak{p}] \cap A[\mathfrak{b}] \neq \emptyset$, and so $A[\mathfrak{b}] = A[\mathfrak{p}]$. Thus $\mathfrak{p} \in \underline{\mathsf{appr}}_A(A[\mathfrak{b}])$. Hence $\overline{\mathsf{appr}}_A(A[\mathfrak{b}]) \subseteq \underline{\mathsf{appr}}_A(A[\mathfrak{b}])$. Therefore, $A[\mathfrak{b}]$ is definable with respective to A, for all $\mathfrak{b} \in \mathbb{L}$. □

Proposition 3.6. *Consider an A-approximation space $(\mathbb{L}, A)$ with $A = \{1\}$. In this case, every subset of $\mathbb{L}$ is definable.*

Proof. Let L be a subset of $\mathbb{L}$. Then by (L3), for all $\mathfrak{b} \in \mathbb{L}$ we have

$$A[\mathfrak{b}] = \{\mathfrak{p} \in \mathbb{L} \mid \mathfrak{b} \cong_A \mathfrak{p}\} = \{\mathfrak{p} \in \mathbb{L} \mid \mathfrak{b} \curvearrowright \mathfrak{p} = \mathfrak{p} \curvearrowright \mathfrak{b} = 1\} = \{\mathfrak{p} \in \mathbb{L} \mid \mathfrak{b} = \mathfrak{p}\} = \{\mathfrak{b}\}.$$

Thus

$$\underline{\mathsf{appr}}_A(L) = \{\mathfrak{b} \in \mathbb{L} \mid A[\mathfrak{b}] \subseteq L\} = \{\mathfrak{b} \in \mathbb{L} \mid \{\mathfrak{b}\} \subseteq L\} = \{\mathfrak{b} \in \mathbb{L} \mid \mathfrak{b} \in L\} = L,$$

and

$$\overline{\mathsf{appr}}_A(L) = \{\mathfrak{b} \in \mathbb{L} \mid A[\mathfrak{b}] \cap L \neq \emptyset\} = \{\mathfrak{b} \in \mathbb{L} \mid \{\mathfrak{b}\} \cap L \neq \emptyset\} = \{\mathfrak{b} \in \mathbb{L} \mid \mathfrak{b} \in L\} = L.$$

Therefore, L is definable. □

Corollary 3.7. *Every L-algebra is definable with respect to any ideal.*

Proposition 3.8. *Consider the equivalence relations $\cong_A$ and $\cong_B$ on $\mathbb{L}$, which are associated with the ideals A and B, respectively. If $A \subseteq B$, then $\cong_A \subseteq \cong_B$.*

Proof. Consider $\mathfrak{b}, \mathfrak{p} \in \mathbb{L}$ such that $\mathfrak{b} \cong_A \mathfrak{p}$. In this case, we have $\mathfrak{b} \curvearrowright \mathfrak{p}, \mathfrak{p} \curvearrowright \mathfrak{b} \in A \subseteq B$, which implies that $\mathfrak{b} \cong_B \mathfrak{p}$. Therefore, we can conclude that $\cong_A \subseteq \cong_B$. □

Let A and B be subsets of $\mathbb{L}$. We define the operation $\curvearrowright$ as follows:

$$A \curvearrowright B = \{\mathfrak{v} \curvearrowright \mathfrak{a} \mid \mathfrak{v} \in A, \mathfrak{a} \in B\},$$

If either A or B is empty, then we define $A \curvearrowright B = \emptyset$.

Proposition 3.9. *Consider an A-approximation space $(\mathbb{L}, A)$. Let $\cong_A$ be an A-congruence relation on $\mathbb{L}$. For any subsets $L, M \in \mathcal{P}(\mathbb{L})$, we have the following:*

(i) $\overline{\mathsf{appr}}_A(L) \curvearrowright \overline{\mathsf{appr}}_A(M) \subseteq \overline{\mathsf{appr}}_A(L \curvearrowright M)$,

(ii) $\underline{\mathsf{appr}}_A(L) \curvearrowright \underline{\mathsf{appr}}_A(M) \subseteq \underline{\mathsf{appr}}_A(L \curvearrowright M)$,

(iii) *If* $\underline{\mathsf{appr}}_A(L \curvearrowright M) = \emptyset$, *then* $\underline{\mathsf{appr}}_A(L) = \emptyset$ *or* $\underline{\mathsf{appr}}_A(M) = \emptyset$.

Proof. (*i*) Let $\mathfrak{q} \in \overline{\mathsf{appr}}_A(L) \curvearrowright \overline{\mathsf{appr}}_A(M)$. Then $\mathfrak{q} = \mathfrak{v} \curvearrowright \mathfrak{a}$ for some $\mathfrak{v} \in \overline{\mathsf{appr}}_A(L)$ and $\mathfrak{a} \in \overline{\mathsf{appr}}_A(M)$, and so $A[\mathfrak{v}] \cap L \neq \emptyset$ and $A[\mathfrak{a}] \cap M \neq \emptyset$. It follows that there are $\mathfrak{b}, \mathfrak{p} \in \mathbb{L}$ such that $\mathfrak{b} \in A[\mathfrak{v}] \cap L$ and $\mathfrak{p} \in A[\mathfrak{a}] \cap M$. Since $\cong_A$ is an A-congruence relation on $\mathbb{L}$, we have

$$\mathfrak{b} \curvearrowright \mathfrak{p} \in A[\mathfrak{v}] \curvearrowright A[\mathfrak{a}] = A[\mathfrak{v} \curvearrowright \mathfrak{a}] = A[\mathfrak{q}].$$

From $\mathfrak{b} \curvearrowright \mathfrak{p} \in L \curvearrowright M$, we consequence $\mathfrak{b} \curvearrowright \mathfrak{p} \in A[\mathfrak{q}] \cap (L \curvearrowright M)$, and so $\mathfrak{q} \in \overline{\mathsf{appr}}_A(L \curvearrowright M)$.
(*ii*) Let $\mathfrak{q} \in \underline{\mathsf{appr}}_A(L) \curvearrowright \underline{\mathsf{appr}}_A(M)$. Then $\mathfrak{q} = \mathfrak{v} \curvearrowright \mathfrak{a}$ for some $\mathfrak{v} \in \underline{\mathsf{appr}}_A(L)$ and $\mathfrak{a} \in \underline{\mathsf{appr}}_A(M)$. Hence $A[\mathfrak{v}] \subseteq L$ and $A[\mathfrak{a}] \subseteq M$. It follows that

$$A[\mathfrak{v} \curvearrowright \mathfrak{a}] = A[\mathfrak{v}] \curvearrowright A[\mathfrak{a}] \subseteq L \curvearrowright M.$$

Then $\mathfrak{q} = \mathfrak{v} \curvearrowright \mathfrak{a} \in \underline{\mathsf{appr}}_A(L \curvearrowright M)$.
(iii) Let $L, M \in \mathcal{P}(\mathbb{L})$ such that $\underline{\mathsf{appr}}_A(L) \neq \emptyset$ and $\underline{\mathsf{appr}}_A(M) \neq \emptyset$. Then there exist $\mathfrak{v} \in \underline{\mathsf{appr}}_A(L)$ and $\mathfrak{a} \in \underline{\mathsf{appr}}_A(M)$, such that $A[\mathfrak{v}] \subseteq L$ and $A[\mathfrak{a}] \subseteq M$. Since $\mathfrak{v} \in A[\mathfrak{v}]$ and $\mathfrak{a} \in A[\mathfrak{a}]$, we have $\mathfrak{v} \in L$ and $\mathfrak{a} \in M$. Then $\mathfrak{v} \curvearrowright \mathfrak{a} \in L \curvearrowright M$, and so

$$\mathfrak{v} \curvearrowright \mathfrak{a} \in A[\mathfrak{v} \curvearrowright \mathfrak{a}] = A[\mathfrak{v}] \curvearrowright A[\mathfrak{a}] \subseteq L \curvearrowright M.$$

Hence $\underline{\mathsf{appr}}_A(L \curvearrowright M) \neq \emptyset$, which is a contradiction. Therefore, $\underline{\mathsf{appr}}_A(L) = \emptyset$ or $\underline{\mathsf{appr}}_A(M) = \emptyset$. □

Corollary 3.10. *Consider an A-approximation space $(\mathbb{L}, A)$. Let $\overline{APPR}(L)$ and $\underline{APPR}(L)$ be the sets defined as follows:*

$$\overline{APPR}(L) = \{\overline{\mathsf{appr}}_A(X) \mid X \subseteq L\}$$
$$\underline{APPR}(L) = \{\underline{\mathsf{appr}}_A(X) \mid X \subseteq L\}.$$

Then both $\overline{APPR}(L)$ and $\underline{APPR}(L)$ are closed under the operation $\curvearrowright$, where $X, Y \in \mathcal{I}d(\mathbb{L})$ and $A \subseteq X$.

Proof. To prove this proposition, we prove the following statements:

(i) $\overline{\mathsf{appr}}_A(X) \curvearrowright \overline{\mathsf{appr}}_A(Y) = \overline{\mathsf{appr}}_A(X \curvearrowright Y)$,

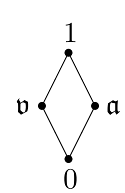
(ii) $\underline{\mathsf{appr}}_A(X) \curvearrowright \underline{\mathsf{appr}}_A(Y) = \underline{\mathsf{appr}}_A(X \curvearrowright Y)$,

(i) By Proposition 3.9(i) one side is clear.
Conversely, assume $\mathfrak{v} \in \overline{\mathsf{appr}}_A(X \curvearrowright Y)$ that by definition $A[\mathfrak{v}] \cap (X \curvearrowright Y) \neq \emptyset$. Let $w \in A[\mathfrak{v}] \cap (X \curvearrowright Y)$. Then $w \in A[\mathfrak{v}]$ and $w \in X \curvearrowright Y$. Since $w \in X \curvearrowright Y$, there exist $x \in X$ and $y \in Y$ such that $w = x \curvearrowright y$. So, $A[\mathfrak{v}] = A[w] = A[x \curvearrowright y] = A[x] \curvearrowright A[y]$. So, since $w \in A[\mathfrak{v}]$, we get $w \in A[x] \curvearrowright A[y]$. Therefore, there is $x \in A[x] \cap X \neq \emptyset$ and $y \in A[y] \cap Y \neq \emptyset$. Thus $w = x \curvearrowright y$ such that $x \in \overline{app}_A(X)$ and $y \in \overline{\mathsf{appr}}_A(Y)$. Hence $w \in \overline{\mathsf{appr}}_A(X) \curvearrowright \overline{\mathsf{appr}}_A(Y)$.
(ii) By Proposition 3.9(ii) $\underline{\mathsf{appr}}_A(X) \curvearrowright \underline{\mathsf{appr}}_A(Y) \subseteq \underline{\mathsf{appr}}_A(X \curvearrowright Y)$ one side is clear.
Conversely, let $\mathfrak{v} \in \underline{\mathsf{appr}}_A(X \curvearrowright Y)$. Then $A[\mathfrak{v}] \subseteq X \curvearrowright Y$. Since $\mathfrak{v} \in A[\mathfrak{v}]$, we have $\mathfrak{v} \in X \curvearrowright Y$ which by definition there exist $x \in X$ and $y \in Y$ such that $\mathfrak{v} = x \curvearrowright y$. Now, let $\alpha \in A[x]$ and $\beta \in A[y]$, then $\alpha \curvearrowright x, x \curvearrowright \alpha \in A \subseteq X$ and $\beta \curvearrowright y, y \curvearrowright \beta \in A \subseteq Y$. So $x \curvearrowright \alpha \in X$ and $y \curvearrowright \beta \in Y$. Since $X, Y \in Id(L)$, so $\alpha \in X$ and $\beta \in Y$. Therefore $A[x] \subseteq X$ and $A[y] \subseteq Y$. So we have $x \in \underline{\mathsf{appr}}_A(X)$ and $y \in \underline{\mathsf{appr}}_A(Y)$. Hence $\mathfrak{v} = x \curvearrowright y \in \underline{\mathsf{appr}}_A(X) \curvearrowright \underline{\mathsf{appr}}_A(Y)$. □

Definition 3.11. *Consider an A-approximation space $(\mathbb{L}, A)$. We define the following terms:*

- *A subset L of $\mathbb{L}$ is called an A-lower rough ideal of $\mathbb{L}$ if $\underline{\mathsf{appr}}_A(L)$ is an ideal of $\mathbb{L}$.*
- *A subset L of $\mathbb{L}$ is called an A-upper rough ideal of $\mathbb{L}$ if $\overline{\mathsf{appr}}_A(L)$ is an ideal of $\mathbb{L}$.*
- *If a subset L of $\mathbb{L}$ is both an A-lower rough ideal and an A-upper rough ideal of $\mathbb{L}$, then we refer to L as an A-rough ideal of $\mathbb{L}$.*

Example 3.12. *Consider the poset $(\mathbb{L} = \{0, \mathfrak{v}, \mathfrak{a}, 1\}, \leq)$ with the following Hasse diagram, where* $0 < \mathfrak{v}, \mathfrak{a} < 1$.



Define the operation $\curvearrowright$ *on* $\mathbb{L}$ *as follows:*

$\curvearrowright$	0	$\mathfrak{v}$	$\mathfrak{a}$	1
0	1	1	1	1
$\mathfrak{v}$	$\mathfrak{a}$	1	$\mathfrak{a}$	1
$\mathfrak{a}$	$\mathfrak{v}$	$\mathfrak{v}$	1	1
1	0	$\mathfrak{v}$	$\mathfrak{a}$	1

Then $(\mathbb{L}, \curvearrowright, 0, 1)$ *is a bounded L-algebra. Consider an A-approximation space* $(\mathbb{L}, A)$ *where* $A = \{\mathfrak{v}, 1\}$ *is an ideal of* $\mathbb{L}$*. Then* $A[\mathfrak{v}] = A[1] = \{\mathfrak{v}, 1\}$ *and* $A[\mathfrak{a}] = A[0] = \{\mathfrak{a}, 0\}$*. For a subset* $L = \{0, \mathfrak{v}, 1\}$ *of* $\mathbb{L}$*, we have*

$$\underline{\mathsf{appr}}_A(L) = \{\mathfrak{b} \in \mathbb{L} \mid A[\mathfrak{b}] \subseteq \{0, \mathfrak{v}, 1\}\} = \{\mathfrak{v}, 1\},$$

and

$$\overline{\mathsf{appr}}_A(L) = \{\mathfrak{b} \in \mathbb{L} \mid A[\mathfrak{b}] \cap \{0, \mathfrak{v}, 1\} \neq \emptyset\} = \{0, \mathfrak{v}, \mathfrak{a}, 1\},$$

are ideals of $\mathbb{L}$*. Hence A is an A-rough ideal of* $\mathbb{L}$*. If we take a subset* $M = \{\mathfrak{a}\}$ *of* $\mathbb{L}$*, then* $\underline{\mathsf{appr}}_A(M) = \emptyset$ *and* $\overline{\mathsf{appr}}_A(M) = \{0, \mathfrak{a}\}$ *are not ideals of* $\mathbb{L}$*. Hence A is not an A-rough ideal of* $\mathbb{L}$*. Also, if we take a subset* $K = \{\mathfrak{v}\}$ *of* $\mathbb{L}$*, then* $\underline{\mathsf{appr}}_A(K) = \emptyset$ *that is not an ideal of* $\mathbb{L}$ *and* $\overline{\mathsf{appr}}_A(K) = \{\mathfrak{v}, 1\}$ *is an ideal of* $\mathbb{L}$*. Hence A is an A-upper rough ideal of* $\mathbb{L}$*.*

The extension theorem for A-upper rough ideals of $\mathbb{L}$ is derived from the following theorem.

Theorem 3.13. *Consider an A-approximation space* $(\mathbb{L}, A)$*, where* $\mathbb{L}$ *is a CKL-algebra. We have the following:*
(i) *Every ideal L of* $\mathbb{L}$ *that contains A is an A-upper rough ideal of* $\mathbb{L}$*.*
(ii) *Every ideal L of* $\mathbb{L}$ *that contains A is an A-lower rough ideal of* $\mathbb{L}$*.*

Proof. (i) Let $L \in \mathcal{I}d(\mathbb{L})$ such that $A \subseteq L$. Then $A[1] \cap L \neq \emptyset$, and so $1 \in \overline{\mathsf{appr}}_A(L)$. Suppose $\mathfrak{b}, \mathfrak{p} \in \mathbb{L}$ such that $\mathfrak{b} \in \overline{\mathsf{appr}}_A(L)$ and $\mathfrak{b} \curvearrowright \mathfrak{p} \in \overline{\mathsf{appr}}_A(L)$. Then $A[\mathfrak{b}] \cap L \neq \emptyset$ and $A[\mathfrak{b} \curvearrowright \mathfrak{p}] \cap L \neq \emptyset$, which imply that there exist $\mathfrak{v}, \mathfrak{a} \in L$ such that $\mathfrak{v} \in A[\mathfrak{b}]$ and $\mathfrak{a} \in A[\mathfrak{b} \curvearrowright \mathfrak{p}]$. Hence $\mathfrak{v} \cong_A \mathfrak{b}$ and $\mathfrak{a} \cong_A (\mathfrak{b} \curvearrowright \mathfrak{p})$. It follows that $\mathfrak{v} \curvearrowright \mathfrak{b} \in A \subseteq L$ and $\mathfrak{a} \curvearrowright (\mathfrak{b} \curvearrowright \mathfrak{p}) \in A \subseteq L$. Since $\mathfrak{v}, \mathfrak{a} \in L$ and $L \in \mathcal{I}d(\mathbb{L})$, we have $\mathfrak{b} \in L$ and $\mathfrak{b} \curvearrowright \mathfrak{p} \in L$, and so $\mathfrak{p} \in L$. Note that $\mathfrak{p} \in A[\mathfrak{p}]$, and thus $\mathfrak{p} \in A[\mathfrak{p}] \cap L$. Hence $\mathfrak{p} \in \overline{\mathsf{appr}}_A(L)$, and therefore $\overline{\mathsf{appr}}_A(L) \in \mathcal{I}d(\mathbb{L})$, that is, L is an A-upper rough ideal of $\mathbb{L}$.
(ii) Let $L \in \mathcal{I}d(\mathbb{L})$ such that $A \subseteq L$. Since $A = A[1]$, if $\mathfrak{b} \in A[1]$, then $\mathfrak{b} \in A \subseteq L$, and so $A[1] \subseteq L$. Hence $1 \in \underline{\mathsf{appr}}_A(L)$. Let $\mathfrak{b}, \mathfrak{p} \in \mathbb{L}$ such that $\mathfrak{b} \in \underline{\mathsf{appr}}_A(L)$ and $\mathfrak{b} \curvearrowright \mathfrak{p} \in \underline{\mathsf{appr}}_A(L)$. Then $A[\mathfrak{b}] \subseteq L$ and $A[\mathfrak{b}] \curvearrowright A[\mathfrak{p}] = A[\mathfrak{b} \curvearrowright \mathfrak{p}] \subseteq L$. Let $\mathfrak{v} \in A[\mathfrak{b}]$ and $\mathfrak{a} \in A[\mathfrak{p}]$. Then $\mathfrak{v} \cong_A \mathfrak{b}$ and $\mathfrak{a} \cong_A \mathfrak{p}$, which imply that $(\mathfrak{v} \curvearrowright \mathfrak{a}) \cong_A (\mathfrak{b} \curvearrowright \mathfrak{p})$, that is, $\mathfrak{v} \curvearrowright \mathfrak{a} \in A[\mathfrak{b} \curvearrowright \mathfrak{p}] \subseteq L$. Since $\mathfrak{v} \in L$ and $L \in \mathcal{I}d(\mathbb{L})$, we get $\mathfrak{a} \in L$ and $A[\mathfrak{p}] \subseteq L$. Thus $\mathfrak{p} \in \underline{\mathsf{appr}}_A(L)$, and therefore $\underline{\mathsf{appr}}_A(L) \in \mathcal{I}d(\mathbb{L})$. Consequently, L is an A-lower rough ideal of $\mathbb{L}$. □

Corollary 3.14. *Consider an A-approximation space* $(\mathbb{L}, A)$ *with* $A = \{1\}$*, where* $\mathbb{L}$ *is a CKL-algebra. We have the following:*
(i) *Every ideal L of* $\mathbb{L}$ *is an A-upper rough ideal of* $\mathbb{L}$*.*
(ii) *Every ideal L of* $\mathbb{L}$ *is an A-lower rough ideal of* $\mathbb{L}$*.*

In the following example, we demonstrate that the converse of Theorem 3.13 does not generally hold.

Example 3.15. *Consider the L-algebra $\mathbb{L}$ as described in Example 3.12, and let $(\mathbb{L}, A)$ be an A-approximation space of $\mathbb{L}$. It is evident that $A = \{\mathfrak{v}, 1\} \in \mathcal{I}d(\mathbb{L})$ and $\cong_A$ is an equivalence relation on $\mathbb{L}$ associated with A. Consequently, we have $A[0] = \{0, \mathfrak{a}\} = A[\mathfrak{a}]$ and $A[\mathfrak{v}] = A = A[1]$. Let $L = \{\mathfrak{a}, 1\}$ be a subset of $\mathbb{L}$. Then L does not contain A and*

$$\overline{\mathsf{appr}}_A(L) = \{\mathfrak{b} \in \mathbb{L} \mid A[\mathfrak{b}] \cap L \neq \emptyset\} = \mathbb{L}.$$

Thus L is an A-upper rough ideal of $\mathbb{L}$.

Proposition 3.16. *Consider an A-approximation space $(\mathbb{L}, A)$. For any subset L of $\mathbb{L}$, the following statements hold:*

(i) *$A \subseteq L$ if and only if $A \subseteq \underline{\mathsf{appr}}_A(L)$.*

(ii) *$L \subseteq A$ if and only if $\overline{\mathsf{appr}}_A(L) = A$.*

Proof. (i) Assume that $A \subseteq L$. If $\mathfrak{b} \in A$, then $A[\mathfrak{b}] = A \subseteq L$. Hence $\mathfrak{b} \in \underline{\mathsf{appr}}_A(L)$, and so $A \subseteq \underline{\mathsf{appr}}_A(L)$. By Proposition 3.1(i), the proof of converse is clear.

(ii) Suppose $L \subseteq A$ and $\mathfrak{b} \in \overline{\mathsf{appr}}_A(L)$. Then $A[\mathfrak{b}] \cap L \neq \emptyset$, and thus there exists $\mathfrak{p} \in A[\mathfrak{b}] \cap L$ which implies that $A[\mathfrak{b}] = A[\mathfrak{p}]$ and $\mathfrak{p} \in L$. Hence $A[\mathfrak{p}] = A$, and so $\mathfrak{b} \in A$. This shows that $\overline{\mathsf{appr}}_A(L) \subseteq A$. Let $\mathfrak{w} \in A$. Then $A[\mathfrak{w}] = A$ and so $A[\mathfrak{w}] \cap L = A \cap L \neq \emptyset$. Thus $\mathfrak{w} \in \overline{\mathsf{appr}}_A(L)$, that is, $A \subseteq \overline{\mathsf{appr}}_A(L)$. By Proposition 3.1(i), the proof of converse is clear. □

Corollary 3.17. *Consider an A-approximation space $(\mathbb{L}, A)$. If $L \in \mathcal{I}d(\mathbb{L})$ and $L \subseteq A$, then L is an A-upper rough ideal of $\mathbb{L}$.*

Theorem 3.18. *If L is an ideal in an A-approximation space $(\mathbb{L}, A)$, then*

(i) *$A \subseteq \overline{\mathsf{appr}}_A(L)$.*

(ii) *$A \subseteq L$ if and only if $\underline{\mathsf{appr}}_A(L) \subseteq L = \overline{\mathsf{appr}}_A(L)$.*

Proof. (i) Let $\mathfrak{b} \in A$. Since $\mathfrak{b} \in A[\mathfrak{b}]$, clearly $1 \in A[\mathfrak{b}]$. Moreover, since L is an ideal in an A-approximation space $(\mathbb{L}, A)$, we have $1 \in L$ and so $1 \in A[\mathfrak{b}] \cap L$. Hence $\mathfrak{b} \in \overline{\mathsf{appr}}_A(L)$, and therefore $A \subseteq \overline{\mathsf{appr}}_A(L)$.

(ii) Assume that $A \subseteq L$. Then by Proposition 3.1(i), $\underline{\mathsf{appr}}_A(L) \subseteq L \subseteq \overline{\mathsf{appr}}_A(L)$. Let $\mathfrak{b} \in \overline{\mathsf{appr}}_A(L)$. Then $A[\mathfrak{b}] \cap L \neq \emptyset$ and thus there exists $\mathfrak{v} \in L \cap A[\mathfrak{b}]$. Since $A \subseteq L$, it follows that $\mathfrak{v} \curvearrowright \mathfrak{b} \in A \subseteq L$. Hence $\mathfrak{b} \in L$ and so $\overline{\mathsf{appr}}_A(L) \subseteq L$.

Conversely, suppose $\underline{\mathsf{appr}}_A(L) \subseteq L = \overline{\mathsf{appr}}_A(L)$ and $\mathfrak{b} \in A$. Since A and L are ideals, we get $1 \in A \cap L = A[\mathfrak{b}] \cap L$. Hence $\mathfrak{b} \in \overline{\mathsf{appr}}_A(L) = L$. Therefore, $A \subseteq L$. □

Corollary 3.19. *If L is an ideal of an A-approximation space $(\mathbb{L}, A)$, then*

$$\underline{\mathsf{appr}}_A(L) = L = \overline{\mathsf{appr}}_A(L),$$

and L is an A-rough ideal of $\mathbb{L}$.

For any nonempty subset L of $\mathbb{L}$, we let $L' = \{\mathfrak{b}' \mid \mathfrak{b} \in L\}$. Obviously, if L and M are nonempty subsets of $\mathbb{L}$, then $L \subseteq M$ satisfies $L' \subseteq M'$.

Proposition 3.20. *In an A-approximation space bounded CKL-algebra $(\mathbb{L}, A)$, for any $L \in \mathcal{P}(\mathbb{L}) \setminus \{\emptyset\}$, we have $(\overline{\mathsf{appr}}_A(L))' \subseteq \overline{\mathsf{appr}}_A(L')$.*

Proof. Let $\mathfrak{v} \in (\overline{\mathsf{appr}}_A(L))'$ for any nonempty subset L of $\mathbb{L}$. Then $\mathfrak{v} = \mathfrak{b}'$ for some $\mathfrak{b} \in \overline{\mathsf{appr}}_A(L)$ and so $A[\mathfrak{b}] \cap L \neq \emptyset$. It follows that there exists $\mathfrak{a} \in L$ such that $\mathfrak{a} \in A[\mathfrak{b}]$, which implies that $\mathfrak{a}' \in L'$ and $\mathfrak{a} \curvearrowright \mathfrak{b} \in A$. By Proposition 2.6(vii), we have

$$\mathfrak{a} \curvearrowright \mathfrak{b} \leq (\mathfrak{b} \curvearrowright 0) \curvearrowright (\mathfrak{a} \curvearrowright 0) = \mathfrak{b}' \curvearrowright \mathfrak{a}'.$$

Since $A \in \mathcal{I}d(\mathbb{L})$ and $\mathfrak{v} = \mathfrak{b}'$, it follows that $\mathfrak{v} \curvearrowright \mathfrak{a}' = \mathfrak{b}' \curvearrowright \mathfrak{a}' \in A$. Similarly $\mathfrak{a}' \curvearrowright \mathfrak{v} \in A$. Hence $\mathfrak{a}' \in A[\mathfrak{v}] \cap L'$, that is, $A[\mathfrak{v}] \cap L' \neq \emptyset$. Therefore, $\mathfrak{v} \in \overline{\mathsf{appr}}_A(L')$ which shows that $(\overline{\mathsf{appr}}_A(L))' \subseteq \overline{\mathsf{appr}}_A(L')$. □

The following example shows that the converse of Proposition 3.20 is not generally not true.

Example 3.21. *Consider the poset* $(\mathbb{L} = \{0, \mathfrak{v}, \mathfrak{a}, \mathfrak{q}, 1\}, \leq)$*, where* $0 < \mathfrak{q} < \mathfrak{v}, \mathfrak{a} < 1$*. We introduce a binary operation denoted by "*$\curvearrowright$*" on* $\mathbb{L}$ *using the following Table 1:*

Table 1: Table of the implication "$\curvearrowright$"

$\curvearrowright$	0	$\mathfrak{v}$	$\mathfrak{a}$	$\mathfrak{q}$	1
0	1	1	1	1	1
$\mathfrak{v}$	0	1	$\mathfrak{a}$	$\mathfrak{a}$	1
$\mathfrak{a}$	0	$\mathfrak{v}$	1	$\mathfrak{v}$	1
$\mathfrak{q}$	0	1	1	1	1
1	0	$\mathfrak{v}$	$\mathfrak{a}$	$\mathfrak{q}$	1

Then $(\mathbb{L}, \curvearrowright, 1)$ *is an* $\mathbb{L}$*-algebra. Let* $A = \{\mathfrak{v}, 1\}$*. Clearly,* A *is an ideal of* $\mathbb{L}$*. Let* $\cong_A$ *be an equivalence relation on* $\mathbb{L}$ *related to* A*. Then* $A[1] = A[\mathfrak{v}] = \{\mathfrak{v}, 1\}$*,* $A[\mathfrak{q}] = A[\mathfrak{a}] = \{\mathfrak{a}, \mathfrak{q}\}$ *and* $A[0] = \{0\}$*. If* $L = \{0, \mathfrak{v}\}$*, then* $L' = \{0, 1\}$*. Thus*

$$\overline{\mathsf{appr}}_A(L') = \{0, \mathfrak{v}, 1\} \quad , \quad \overline{\mathsf{appr}}_A(L) = \{0, \mathfrak{v}, 1\}.$$

But $(\overline{\mathsf{appr}}_A(L))' = (\{0, \mathfrak{v}, 1\})' = \{0, 1\}$*. Hence* $\overline{\mathsf{appr}}_A(L') \nsubseteq (\overline{\mathsf{appr}}_A(L))'$*.*

In the following example, we show that there exists a nonempty subset L of $\mathbb{L}$ such that $\underline{\mathsf{appr}}_A(L') \nsubseteq (\underline{\mathsf{appr}}_A(L))'$.

Example 3.22. *Consider an* A*-approximation space* $(\mathbb{L}, A)$*, where* $\mathbb{L}$ *is the* L*-algebra as described in Example 3.21, and* $A = \{\mathfrak{v}, 1\}$ *is an ideal of* $\mathbb{L}$*. If* $L = \{\mathfrak{v}, 0\}$*, then* $L' = \{0, 1\}$*. Thus* $\underline{\mathsf{appr}}_A(L') = \{0\}$ *and* $\underline{\mathsf{appr}}_A(L) = \{0\}$*, and so* $(\underline{\mathsf{appr}}_A(L))' = \{1\}$*. Hence* $\underline{\mathsf{appr}}_A(L') \nsubseteq (\underline{\mathsf{appr}}_A(L))'$*.*

Proposition 3.23. *Consider an* A*-approximation space* $(\mathbb{L}, A)$ *and let* L *be a nonempty subset of* $\mathbb{L}$*. Then,*
(i) $\mathcal{R}(\mathbb{L}) \cap \overline{\mathsf{appr}}_A(L') \subseteq (\overline{\mathsf{appr}}_A(L''))'$,
(ii) $\mathcal{R}(\mathbb{L}) \cap \overline{\mathsf{appr}}_A((L \cap \mathcal{R}(\mathbb{L}))') \subseteq (\overline{\mathsf{appr}}_A(L))'$,
where $\mathcal{R}(\mathbb{L}) := \{\mathfrak{b} \in \mathbb{L} \mid \mathfrak{b}'' = \mathfrak{b}\}$*.*

Proof. (i) Let $\mathfrak{w} \in \mathcal{R}(\mathbb{L}) \cap \overline{\mathsf{appr}}_A(L')$. Then $\mathfrak{w}'' = \mathfrak{w}$ and $A[\mathfrak{w}] \cap L' \neq \emptyset$, which imply that there exists $\mathfrak{b} \in L$ such that $A[\mathfrak{b}'] = A[\mathfrak{w}]$. Hence

$$A[\mathfrak{w}'] \cap L'' = A[\mathfrak{b}''] \cap L'' \neq \emptyset,$$

i.e., $\mathfrak{w}' \in \overline{\mathsf{appr}}_A(L'')$. Therefore $\mathfrak{w} \in (\overline{\mathsf{appr}}_A(L''))'$.
(ii) Let $\mathfrak{v} \in \mathcal{R}(\mathbb{L}) \cap \overline{\mathsf{appr}}_A((L \cap \mathcal{R}(\mathbb{L}))')$. Then $\mathfrak{v}'' = \mathfrak{v}$ and $A[\mathfrak{v}] \cap (L \cap \mathcal{R}(\mathbb{L}))' \neq \emptyset$. It follows that there exists $\mathfrak{b} \in L \cap \mathcal{R}(\mathbb{L})$ such that $A[\mathfrak{v}] = A[\mathfrak{b}']$ and $\mathfrak{b}'' = \mathfrak{b}$. Hence

$$A[\mathfrak{v}'] \cap L = A[\mathfrak{b}''] \cap L = A[\mathfrak{b}] \cap L \neq \emptyset,$$

and so $\mathfrak{v}' \in \overline{\mathsf{appr}}_A(L)$, i.e., $\mathfrak{v} \in (\overline{\mathsf{appr}}_A(L))'$. Therefore

$$\mathcal{R}(\mathbb{L}) \cap \overline{\mathsf{appr}}_A((L \cap \mathcal{R}(\mathbb{L}))') \subseteq (\overline{\mathsf{appr}}_A(L))'.$$

□

Lemma 3.24. *If $\mathbb{L}$ is a bounded CKL-algebra, then the set*

$$\mathcal{E}(\mathbb{L}) := \{\mathfrak{b} \in \mathbb{L} \mid \mathfrak{b}' = 0\},$$

is an ideal of $\mathbb{L}$.

Proof. Obviously $1 \in \mathcal{E}(\mathbb{L})$. Let $\mathfrak{b}, \mathfrak{p} \in \mathbb{L}$ such that $\mathfrak{b} \in \mathcal{E}(\mathbb{L})$ and $\mathfrak{b} \curvearrowright \mathfrak{p} \in \mathcal{E}(\mathbb{L})$. Then $\mathfrak{b}' = 0$ and $(\mathfrak{b} \curvearrowright \mathfrak{p})' = 0$. Since by Proposition 2.2 and Proposition 2.6(x) $\mathfrak{p} \leq \mathfrak{p}''$, by Proposition 2.6(xiv), we get $\mathfrak{b} \curvearrowright \mathfrak{p} \leq \mathfrak{b} \curvearrowright \mathfrak{p}'' = \mathfrak{p}' \curvearrowright \mathfrak{b}'$. Hence by Proposition 2.6(x) we have

$$\mathfrak{p}' = \mathfrak{p}''' = (\mathfrak{p}' \curvearrowright 0)' = (\mathfrak{p}' \curvearrowright \mathfrak{b}')' \leq (\mathfrak{b} \curvearrowright \mathfrak{p})' = 0,$$

and so $\mathfrak{p}' = 0$, that is, $\mathfrak{p} \in \mathcal{E}(\mathbb{L})$. Therefore $\mathcal{E}(\mathbb{L}) \in \mathcal{I}d(\mathbb{L})$. □

Proposition 3.25. *Given an A-approximation space $(\mathbb{L}, A)$ and a nonempty subset L of CKL-algebra $\mathbb{L}$, we have the following:*

$$A \subseteq \overline{\mathsf{appr}}_A(\mathcal{E}(\mathbb{L})) \subseteq \{\mathfrak{p} \in \mathbb{L} \mid \mathfrak{p}'' \in A\}. \tag{3.1}$$

Proof. Using Lemma 3.24 and Theorem 3.18(i), we get $A \subseteq \overline{\mathsf{appr}}_A(\mathcal{E}(\mathbb{L}))$. Let $\mathfrak{b} \in \overline{\mathsf{appr}}_A(\mathcal{E}(\mathbb{L}))$. Then $A[\mathfrak{b}] \cap \mathcal{E}(\mathbb{L}) \neq \emptyset$ and so there exists $\mathfrak{v} \in A[\mathfrak{b}]$ such that $\mathfrak{v}' = 0$. Thus $\mathfrak{v} \curvearrowright \mathfrak{b} \in A$. By Proposition 2.6(x), $\mathfrak{v} \curvearrowright \mathfrak{b} \leq (\mathfrak{b} \curvearrowright 0) \curvearrowright (\mathfrak{v} \curvearrowright 0) = \mathfrak{b}' \curvearrowright \mathfrak{v}'$ and $A \in \mathcal{I}d(\mathbb{L})$, we have $\mathfrak{b}' \curvearrowright \mathfrak{v}' \in A$. Thus, $\mathfrak{b}'' = \mathfrak{b}' \curvearrowright 0 = \mathfrak{b}' \curvearrowright \mathfrak{v}' \in A$. Therefore $\overline{\mathsf{appr}}_A(\mathcal{E}(\mathbb{L})) \subseteq \{\mathfrak{p} \in \mathbb{L} \mid \mathfrak{p}'' \in A\}$. □

We provide conditions for a nonempty subset to be definable.

Theorem 3.26. *Consider an A-approximation space $(\mathbb{L}, A)$. A nonempty subset L of $\mathbb{L}$ is said to be definable concerning A if and only if either $\underline{\mathsf{appr}}_A(L) = L$ or $\overline{\mathsf{appr}}_A(L) = L$.*

Proof. Assume that L is definable with respect to A. Then $L \subseteq \overline{\mathsf{appr}}_A(L) = \underline{\mathsf{appr}}_A(L) \subseteq L$ and so

$$\overline{\mathsf{appr}}_A(L) = \underline{\mathsf{appr}}_A(L) = L.$$

Conversely, suppose that $\underline{\mathsf{appr}}_A(L) = L$ or $\overline{\mathsf{appr}}_A(L) = L$. For the case $\underline{\mathsf{appr}}_A(L) = L$, let $\mathfrak{b} \in \overline{\mathsf{appr}}_A(L)$. Then $A[\mathfrak{b}] \cap L \neq \emptyset$ which implies that $A[\mathfrak{b}] = A[\mathfrak{w}]$ for some $\mathfrak{w} \in L$. It follows from $\underline{\mathsf{appr}}_A(L) = L$ that $A[\mathfrak{b}] = A[\mathfrak{w}] \subseteq L$. Hence $\mathfrak{b} \in L$, and therefore $\overline{\mathsf{appr}}_A(L) \subseteq L$. Consequently, $\overline{\mathsf{appr}}_A(L) = L$. Suppose that $\overline{\mathsf{appr}}_A(L) = L$. For any $\mathfrak{b} \in L$ let $\mathfrak{w} \in A[\mathfrak{b}]$. Then $A[\mathfrak{w}] \cap L = A[\mathfrak{b}] \cap L \neq \emptyset$ and so $\mathfrak{w} \in \overline{\mathsf{appr}}_A(L) = L$. This shows that $A[\mathfrak{b}] \subseteq L$, that is, $\mathfrak{b} \in \underline{\mathsf{appr}}_A(L)$. Hence $L \subseteq \underline{\mathsf{appr}}_A(L)$, and so $\underline{\mathsf{appr}}_A(L) = L$. Therefore L is definable concerning A. □

Proposition 3.27. *Consider an A-approximation space $(\mathbb{L}, A)$. For any $L, M \in \mathcal{P}(\mathbb{L})$, we have the equality $\overline{\mathsf{appr}}^c_A(L) \cup \overline{\mathsf{appr}}_A(M) = \overline{\mathsf{appr}}_A(L^c \cup M)$.*

Proof. Let $x \in \overline{\mathsf{appr}}^c_A(L) \cup \overline{\mathsf{appr}}_A(M)$. Then $x \in \overline{\mathsf{appr}}^c_A(L)$ or $x \in \overline{\mathsf{appr}}_A(M)$ and we have $x \notin \overline{\mathsf{appr}}_A(L)$ or $x \in \overline{\mathsf{appr}}_A(M)$. So,

$$A[x] \cap L = \emptyset \Longrightarrow A[x] \cap L^c \neq \emptyset \Longrightarrow x \in \overline{\mathsf{appr}}_A(L^c),$$

or $x \in \overline{\mathsf{appr}}_A(M)$. Since, $A[x] \cap M \neq \emptyset$ so, $A[x] \cap (L^c \cup M) \neq \emptyset$. Thus, $x \in \overline{\mathsf{appr}}_A(L^c \cup M)$. Hence $\overline{\mathsf{appr}}^c_A(L) \cup \overline{\mathsf{appr}}_A(M) \subseteq \overline{\mathsf{appr}}_A(L^c \cup M)$. By similar way, we can prove that $\overline{\mathsf{appr}}_A(L^c \cup M) \subseteq \overline{\mathsf{appr}}^c_A(L) \cup \overline{\mathsf{appr}}_A(M)$. Finally, $\overline{\mathsf{appr}}^c_A(L) \cup \overline{\mathsf{appr}}_A(M) = \overline{\mathsf{appr}}_A(L^c \cup M)$. □

Theorem 3.28. *Let $\overline{APPR}(\mathbb{L}) = \{\overline{\mathsf{appr}}_A(L) | L \subseteq \mathbb{L}\}$. Define the operation $\curvearrowright$ on $\overline{APPR}(\mathbb{L})$ as follows:*

$$\overline{\mathsf{appr}}_A(L_1) \curvearrowright \overline{\mathsf{appr}}_A(L_2) = \overline{\mathsf{appr}}^c_A(L_1) \cup \overline{\mathsf{appr}}_A(L_2).$$

Then $(\overline{APPR}(\mathbb{L}), \curvearrowright, \overline{\mathsf{appr}}_A(\mathbb{L}))$ is an $\mathbb{L}$-algebra, where $\overline{\mathsf{appr}}_A(X) = \overline{\mathsf{appr}}_A(Y)$ implies $X = Y$.

Proof. Assume that $X \subseteq \mathbb{L}$. Then to verify the validity of (L_1), we can proceed as follows:

$$\overline{\mathsf{appr}}_A(X) \curvearrowright \overline{\mathsf{appr}}_A(X) = \overline{\mathsf{appr}}^c_A(X) \cup \overline{\mathsf{appr}}_A(X) \xlongequal{\text{by Proposition 3.27}} \overline{\mathsf{appr}}_A(X^c \cup X) = \overline{\mathsf{appr}}_A(\mathbb{L}),$$

$$\overline{\mathsf{appr}}_A(X) \curvearrowright \overline{\mathsf{appr}}_A(\mathbb{L}) = \overline{\mathsf{appr}}^c_A(X) \cup \overline{\mathsf{appr}}_A(\mathbb{L}) \xlongequal{\text{by Proposition 3.27}} \overline{\mathsf{appr}}_A(X^c \cup \mathbb{L}) = \overline{\mathsf{appr}}_A(\mathbb{L}),$$

$$\overline{\mathsf{appr}}_A(\mathbb{L}) \curvearrowright \overline{\mathsf{appr}}_A(X) = \overline{\mathsf{appr}}^c_A(\mathbb{L}) \cup \overline{\mathsf{appr}}_A(X) \xlongequal{\text{by Proposition 3.27}} \overline{\mathsf{appr}}_A(\mathbb{L}^c \cup X) = \overline{\mathsf{appr}}_A(X).$$

Thus (L_1) is holds. For proving (L_3), suppose $X, Y \subseteq \mathbb{L}$, then we get if $\overline{\mathsf{appr}}_A(X) \curvearrowright \overline{\mathsf{appr}}_A(Y) = \overline{\mathsf{appr}}_A(\mathbb{L})$ and $\overline{\mathsf{appr}}_A(Y) \curvearrowright \overline{\mathsf{appr}}_A(X) = \overline{\mathsf{appr}}_A(\mathbb{L})$, then

$$\overline{\mathsf{appr}}_A(X^c \cup Y) = \overline{\mathsf{appr}}_A(\mathbb{L}),$$

and

$$\overline{\mathsf{appr}}_A(Y^c \cup X) = \overline{\mathsf{appr}}_A(\mathbb{L}).$$

By assumption, we have $X^c \cup Y = \mathbb{L}$ and $Y^c \cup X = \mathbb{L}$. So, $X \subseteq Y$ and $Y \subseteq X$ that consequently $X = Y$. Thus (L_3) is holds. For proving (L_2), suppose $X, Y, Z \subseteq \mathbb{L}$, then we have:

$$\begin{aligned}
(\overline{\mathsf{appr}}_A(X) \curvearrowright \overline{\mathsf{appr}}_A(Y)) \curvearrowright (\overline{\mathsf{appr}}_A(X) \curvearrowright \overline{\mathsf{appr}}_A(Z))& \\
&= \overline{\mathsf{appr}}_A(X^c \cup Y) \curvearrowright \overline{\mathsf{appr}}_A(X^c \cup Z) \\
&= \overline{\mathsf{appr}}_A((X^c \cup Y)^c \cup (X^c \cup Z)) \\
&= \overline{\mathsf{appr}}_A((X \cap Y^c) \cup (X^c \cup Z)) \\
&= \overline{\mathsf{appr}}_A((X^c \cup Y^c) \cup Z) \\
&= \overline{\mathsf{appr}}_A((Y \cap X^c) \cup (Y^c \cup Z)) \\
&= \overline{\mathsf{appr}}_A((Y^c \cup X)^c \cup (Y^c \cup Z)) \\
&= \overline{\mathsf{appr}}_A((Y^c \cup X) \curvearrowright \overline{\mathsf{appr}}_A(Y^c \cup Z)) \\
&= \overline{\mathsf{appr}}_A(Y) \curvearrowright \overline{\mathsf{appr}}_A(X)) \curvearrowright (\overline{\mathsf{appr}}_A(Y) \curvearrowright \overline{\mathsf{appr}}_A(Z)).
\end{aligned}$$

Thus (L_2) is holds. Hence, $(\overline{APPR}(\mathbb{L}), \curvearrowright, \overline{\mathsf{appr}}_A(\mathbb{L}))$ is an $\mathbb{L}$-algebra. □

4 Conclusion

Considering the significance of this topic in the field of decision-making, we have chosen to introduce these concepts specifically on L-algebras. This will pave the way for future discussions on rough soft and soft rough L-algebras, as well as their fuzzification. In this paper, we introduce the concepts of lower and upper approximations on L-algebras and investigate their properties. We also explore the relationship between these approximations and an interior operator and a closure operator. Additionally, we offer criteria for a subset to be considered definable, and it must not be an empty set. Further research opportunities exist in exploring roughness with different ideals and ideals in L-algebras.

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Compliance with ethical standards

Conflict of interest: The authors declare that there is no conflict of interest.
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