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Quotient bipolar fuzzy soft sets of hypervector spaces and bipolar fuzzy soft sets of quotient hypervector spaces

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Abstract

In this paper, two related quotient structures are investigated utilizing the concept of coset. At first, a new hypervector space $\frac{\mathcal{F}}{\mathcal{V}} = (\frac{\mathcal{F}}{\mathcal{V}}, \oplus, \circledcirc, \mathcal{K})$ is created, which is composed of all cosets of a bipolar fuzzy soft set $(\mathcal{F}, \mathcal{A})$ over a hypervector space $\mathcal{V}$. Then it will be shown that $\dim \frac{\mathcal{F}}{\mathcal{V}} = \dim \frac{\mathcal{V}}{\mathcal{W}}$, where the quotient hypervector space $\frac{\mathcal{V}}{\mathcal{W}}$ includes all cosets of an especial subhyperspace $\mathcal{W}$ of $\mathcal{V}$. Also, three bipolar fuzzy soft sets over the quotient hypervector space $\frac{\mathcal{V}}{\mathcal{W}}$ are presented, and in this way, some new bipolar fuzzy soft hypervector spaces are defined.

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1 Introduction

Zadeh [35] introduced fuzzy sets in 1965 as a way to represent objects with vague boundaries. Fuzzy sets assign a membership value $\mu(x)$ in the range $[0, 1]$ to each element x in set $\mathcal{X}$, indicating the degree of membership of x in $\mathcal{X}$. Over time, variations of fuzzy sets have emerged, such as fuzzy sets of type 2, L-fuzzy sets, interval-valued fuzzy sets, and intuitionistic fuzzy sets. While these variations share some similarities, they also have distinct characteristics.

In 1994, Zhang [36] introduced bipolar fuzzy sets as a new extension of fuzzy sets and applied them in decision analysis. Bipolar fuzzy sets have two membership degrees that represent the satisfaction level for a property and its counter-property. The membership degrees of bipolar fuzzy sets range from $[-1, 1]$, with a membership degree of 0 indicating irrelevance with the corresponding property. Membership degrees in the range $(0, 1]$ indicate some degree of satisfaction with the property, while membership degrees in the range $[-1, 0)$ indicate some degree of satisfaction with the implicit counter-property.

Molodtsov [25] introduced the concept of soft sets in 1999, as another mathematical tool for modelling uncertainty. This concept gained traction in various fields, including algebraic structures.

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For instance, Maji [23] explored its application in decision-making problems, while Aktas [5], Acar [3], and Sezgin [33] studied soft groups, soft rings, and soft vector spaces, respectively.

In 2011, Cogman [10] introduced a fuzzy soft set as a more precise modelling tool. In 2014, Abdollah [1] combined the concepts of bipolar fuzzy sets and soft sets, giving rise to the concept of bipolar fuzzy soft sets as a new generalization. Since then, this idea has been widely researched, resulting in numerous published papers. For example, Akram [4] examined the applications of bipolar fuzzy soft sets in K-algebras, Ali [6] utilized parameter reductions of bipolar fuzzy soft sets to solve decision-making problems, Abughazalah [2] applied bipolar fuzzy sets in BCI-algebras, and Riaz [29] discussed bipolar fuzzy soft topology in decision-making. Mahmood [22] introduced the notion of bipolar complex fuzzy soft sets as a generalization of bipolar complex fuzzy sets and soft sets. More recently, Khan [21] applied bipolar fuzzy soft matrices to solve decision-making problems.

On the other hand, in 1934, Marty [24] introduced the theory of algebraic hyperstructures by generalizing the notion of operation into hyperoperation. While an operation assigns a unique element of a set to any two elements of the context set, a hyperoperation assigns a unique subset of the set to any two elements. This theory has been extensively studied by researchers in various branches, with references to books such as [11], [13], and [34]. The concept of hypervector space was introduced by Scafati-Tallini [31] in 1990, and has since been investigated by Ameri [8], Sedghi [32], and author [15].

These concepts have had an impact on algebraic hyperstructures (as detailed in the book [12]). For example, Ameri [7] presented fuzzy hypervector spaces over valued fields in 2005, and studied their properties in [9]. The author expanded on this work and explored additional properties of fuzzy hypervector spaces in [16, 17]. Ranjbar [28] examined the properties of soft hypervector spaces and fuzzy soft hypervector spaces, while the author [20] investigated results in soft hypervector spaces. Norouzi [27] introduced new directions concerning soft hypermodules and soft fuzzy hypermodules. Sarwar [30] applied bipolar fuzzy soft sets to hypergraphs, and Muhiuddin [26] delved into the concept of bipolar-valued fuzzy soft hyper BCK-ideals.

Recently, the author [18, 19] applied the notion of bipolar fuzzy soft sets in hypervector spaces and explored results pertaining to bipolar fuzzy soft hypervector spaces. In this paper, we build upon the aforementioned studies and present new applications of bipolar fuzzy soft sets in hypervector spaces. Section 2 reviews some fundamental concepts from references [18] and [19]. In Section 3, we introduce the notion of bipolar fuzzy soft coset related to a bipolar fuzzy soft hypervector space, resulting in a hypervector space $\frac{\mathcal{F}}{\mathcal{V}} = (\frac{\mathcal{F}}{\mathcal{V}}, \oplus, \circledcirc, \mathcal{K})$. We also establish a relationship between the dimensions of $\frac{\mathcal{F}}{\mathcal{V}}$ and $\frac{\mathcal{V}}{\mathcal{W}}$, where $\mathcal{W}$ is a specific subhyperspace of $\mathcal{V}$. Additionally, we examine the correspondence between bipolar fuzzy soft hypervector spaces of $\frac{\mathcal{F}}{\mathcal{V}}$ and bipolar fuzzy soft hypervector spaces of $\mathcal{V}$. Section 4 focuses on the structure of the quotient hypervector space $\frac{\mathcal{V}}{\mathcal{W}}$ and defines bipolar fuzzy soft sets over $\frac{\mathcal{V}}{\mathcal{W}}$. Finally, in Section 5, we present ideas for future research directions.

2 Preliminaries

In this section, we will provide definitions, theorems, and examples that are necessary for our subsequent discussion. Most of the contents are from references [18] and [19].

Definition 2.1. [36] *A bipolar fuzzy set on $\mathcal{X}$ can be defined as a mapping that assigns each element x in $\mathcal{X}$ a pair of values $(\mu^+_{\mathcal{A}}(x), \mu^-_{\mathcal{A}}(x))$, where $\mu^+_{\mathcal{A}}(x)$ represents the degree to which x*

satisfies the property and $\mu_{\mathcal{A}}^{-}(x)$ represents the degree to which x satisfies the counter-property, i.e.

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}^{+}(x), \mu_{\mathcal{A}}^{-}(x)),\ x \in \mathcal{X}\}.$$

If the degree of positive satisfaction for $\mathcal{A}$, denoted by $\mu_{\mathcal{A}}^{+}(x)$, is non-zero ($\mu_{\mathcal{A}}^{+}(x) \neq 0$) and the degree of negative satisfaction for $\mathcal{A}$, denoted by $\mu_{\mathcal{A}}^{-}(x)$, is zero ($\mu_{\mathcal{A}}^{-}(x) = 0$), then it indicates that x has only positive satisfaction for $\mathcal{A}$. This means that x satisfies the property of $\mathcal{A}$ to some degree but does not satisfy the counter-property of $\mathcal{A}$.

On the other hand, if $\mu_{\mathcal{A}}^{+}(x) = 0$ and $\mu_{\mathcal{A}}^{-}(x) \neq 0$, it means that x does not satisfy the property of $\mathcal{A}$, but it satisfies the counter-property of $\mathcal{A}$. In this case, x does not exhibit positive satisfaction for $\mathcal{A}$, but it does exhibit negative satisfaction, indicating its compliance with the counter-property.

These scenarios illustrate the ability of bipolar fuzzy sets to capture degrees of satisfaction and counter-satisfaction separately, allowing for a more nuanced representation of fuzzy and uncertain information.

To simplify the representation, we can use the symbol $\mathcal{A} = (\mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-})$ or $\mathcal{A} = (\mathcal{A}^{+}, \mathcal{A}^{-})$ to denote a bipolar fuzzy set $\mathcal{A}$.

For example, $\mathcal{A} = \{(dragonfly, 0.3, 0), (mosquito, 1, 0), (turtle, 0, 0), (snake, 0, -1)\}$ is a bipolar fuzzy set which represents the fuzzy concept frog's prey.

Definition 2.2. [25] *Let $\mathcal{U}$ be a universe set, E be a set of parameters, $P(\mathcal{U})$ be the power set of $\mathcal{U}$, and $\mathcal{A}$ be a subset of E. A soft set over $\mathcal{U}$ is defined as a pair $(\mathcal{F}, \mathcal{A})$ where $\mathcal{F} : \mathcal{A} \to P(\mathcal{U})$ is a mapping that assigns subsets of $\mathcal{U}$ to elements in $\mathcal{A}$.*

Definition 2.3. [1] *Let $\mathcal{U}$ represent a set of elements, E represent a set of parameters, and $\mathcal{A}$ be a subset of E. Then, the term "bipolar fuzzy soft set" refers to a pair $(\mathcal{F}, \mathcal{A})$, where $\mathcal{F}$ is a function mapping $\mathcal{A}$ to $BF^{\mathcal{U}}$ ($BF^{\mathcal{U}}$ represents the collection of all bipolar fuzzy sets over $\mathcal{U}$), i.e.*

$$\forall e \in \mathcal{A};\ \mathcal{F}(e) = \{(x, \mu_{\mathcal{F}(e)}^{+}(x), \mu_{\mathcal{F}(e)}^{-}(x)),\ x \in \mathcal{U}\}.$$

For any element e belonging to set $\mathcal{A}$, $\mathcal{F}(e)$ refers to the collection of elements in the bipolar fuzzy soft set $(\mathcal{F}, \mathcal{A})$ that are approximately e. The degree to which an element x maintains the parameter e is denoted as $\mu_{\mathcal{F}(e)}^{+}(x)$, while the degree to which it deviates from e is denoted as $\mu_{\mathcal{F}(e)}^{-}(x)$. To simplify, we can represent $\mathcal{F}_e^{+}(x)$ as $\mu_{\mathcal{F}(e)}^{+}(x)$ and $\mathcal{F}_e^{-}(x)$ as $\mu_{\mathcal{F}(e)}^{-}(x)$. In simpler terms, for any e in $\mathcal{A}$, $\mathcal{F}(e)$ can be represented as $\mathcal{F}_e$ and is defined as the set of all $(x, \mathcal{F}_e^{+}(x), \mathcal{F}_e^{-}(x))$, where x belongs to the universal set.

If $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are bipolar fuzzy soft sets over $\mathcal{U}$, we can say that $(\mathcal{F}, \mathcal{A})$ is considered a bipolar fuzzy soft subset of $(\mathcal{G}, \mathcal{B})$ and it is denoted as $(\mathcal{F}, \mathcal{A}) \sqsubseteq (\mathcal{G}, \mathcal{B})$ if $\mathcal{A} \subseteq \mathcal{B}$ and for all $e \in \mathcal{A}$,

$$\mathcal{F}_e^{+}(x) \leq \mathcal{G}_e^{+}(x),\quad \mathcal{F}_e^{-}(x) \geq \mathcal{G}_e^{-}(x),\quad \forall x \in \mathcal{A}.$$

Definition 2.4. [31] *The hypervector space is represented by the quadruplet $(\mathcal{V}, +, \circ, \mathcal{K})$, where $\mathcal{V}$ is an Abelian group under addition "$+$", $\mathcal{K}$ is a field, "$\circ : \mathcal{K} \times \mathcal{V} \longrightarrow P_*(\mathcal{V})$" is an external hyperoperation that maps an element from $\mathcal{K}$ and an element from $\mathcal{V}$ to a non-empty subset of $\mathcal{V}$, and the following conditions must hold for all $a, b \in \mathcal{K}$ and $x, y \in \mathcal{V}$:*

(H_1) *Right Distributive Law: $a \circ (x + y) \subseteq a \circ x + a \circ y$. This means that the external hyperoperation distributes over addition from the right.*

(H_2) *Left Distributive Law: $(a + b) \circ x \subseteq a \circ x + b \circ x$. This means that the external hyperoperation distributes over addition from the left.*

(H_3) *Associativity:* $a \circ (b \circ x) = (ab) \circ x$.

(H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$. *This property presents the relation between additive inverses of scalars and vectors.*

(H_5) *Identity Element:* $x \in 1 \circ x$, *where* 1 *is the identity element of the field* $\mathcal{K}$.

Note that in (H_1), $a \circ x + a \circ y$ *is the set of all elements* $p + q$, *where* p *belongs to the set* $a \circ x$ *and* q *belongs to the set* $a \circ y$. *Similarly we have in (H_2). Moreover,* $a \circ (b \circ x)$ *equals to the union of all sets* $a \circ t$, *for all* $t \in b \circ x$.
Furthermore, $\mathcal{V}$ *is called strongly right distributive if equality holds in the right distributive law (H_1), and similarly, a hypervector space is strongly left distributive if equality holds in the left distributive law (H_2).*

A subhyperspace of a hypervector space $\mathcal{V}$ is a non-empty subset $\mathcal{W}$ of $\mathcal{V}$ that behaves like a separate hypervector space within $\mathcal{V}$. This means that $\mathcal{W}$ satisfies the properties of closure under subtraction and scalar multiplication from $\mathcal{V}$, where for any vectors x and y in $\mathcal{W}$ and any scalar a in the field $\mathcal{K}$, the subtraction of x and y, $x - y$ is still in $\mathcal{W}$, and the scalar multiplication of a and x, $a \circ x$ is also in $\mathcal{W}$.

In the rest of this paper, unless stated otherwise, the symbol $\mathcal{V}$ represents a hypervector space over the field $\mathcal{K}$.

Lemma 2.5. [14] *If* $\mathcal{W}$ *is a subhyperspace of* $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$ *and* $v, v_1, v_2 \in \mathcal{V}$, *then*

1) $v = v_1 + v_2 + x$, *for some* $x \in \mathcal{W}$, *if and only if* $v = \acute{v_1} + x_1 + \acute{v_2} + x_2$, *where* $x_1, x_2 \in \mathcal{W}$, $\acute{v_1} \in v_1 + \mathcal{W}$ *and* $\acute{v_2} \in v_2 + \mathcal{W}$.

2) *if* $|1 \circ t| = 1$, *for all* $y \in \mathcal{V}$, *then for every nonzero* $b \in \mathcal{K}$ *and every* $y \in \mathcal{V}$, $t \in b \circ y + \mathcal{W}$ *if and only if* $t \in b \circ \acute{y}$ *for some* $\acute{y} \in y + \mathcal{W}$.

Definition 2.6. [18] *If* $(\mathcal{F}, \mathcal{A})$ *and* $(\mathcal{G}, \mathcal{B})$ *are bipolar fuzzy soft sets over a hypervector space* $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$ *and* $a \in \mathcal{K}$, *then the sum of* $(\mathcal{F}, \mathcal{A})$ *and* $(\mathcal{G}, \mathcal{B})$ *is represented as* $(\mathcal{F}, \mathcal{A}) + (\mathcal{G}, \mathcal{B})$ *and is defined as the bipolar fuzzy soft set* $(\mathcal{F} + \mathcal{G}, \mathcal{A} \cap \mathcal{B})$, *where*

$$(\mathcal{F} + \mathcal{G})_e^+(x) = \sup_{x=y+z} (\mathcal{F}_e^+(y) \wedge \mathcal{G}_e^+(z)),\ (\mathcal{F} + \mathcal{G})_e^-(x) = \inf_{x=y+z} (\mathcal{F}_e^-(y) \vee \mathcal{G}_e^-(z)).$$

Also, the scalar multiplication of a scalar value a *and a bipolar fuzzy soft set* $(\mathcal{F}, \mathcal{A})$ *is denoted as* $a \circ (\mathcal{F}, \mathcal{A})$ *and results in the bipolar fuzzy soft set* $(a \circ \mathcal{F}, \mathcal{A})$. *In this case,*

$$(a \circ \mathcal{F})_e^+(x) = \begin{cases} \sup\limits_{x \in a \circ t} \mathcal{F}_e^+(t) & \exists t \in \mathcal{V}, x \in a \circ t, \\ 0 & \textit{otherwise}, \end{cases}$$

$$(a \circ \mathcal{F})_e^-(x) = \begin{cases} \inf\limits_{x \in a \circ t} \mathcal{F}_e^-(t) & \exists t \in \mathcal{V}, x \in a \circ t, \\ 0 & \textit{otherwise}. \end{cases}$$

Lemma 2.7. [18] *Let* $(\mathcal{F}, \mathcal{A})$ *and* $(\mathcal{G}, \mathcal{B})$ *be bipolar fuzzy soft sets of hypervector space* $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$. *Then for all* $y \in \mathcal{V}$ *and* $e \in \mathcal{A}$, $(\mathcal{F}, \mathcal{A}) \sqsubseteq 1 \circ (\mathcal{F}, \mathcal{A})$ *and* $-(\mathcal{F}, \mathcal{A}) \sqsubseteq (-1) \circ (\mathcal{F}, \mathcal{A})$, *where* $(-\mathcal{F})_e^+(y) = \mathcal{F}_e^+(-y)$ *and* $(-\mathcal{F})_e^-(y) = \mathcal{F}_e^-(-y)$.

Definition 2.8. [18] *Consider a bipolar fuzzy soft set* $(\mathcal{F}, \mathcal{A})$ *in a hypervector space* $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$. *If for every element* e *in* $\mathcal{A}$, *the following conditions are satisfied, then we can say that* $(\mathcal{F}, \mathcal{A})$ *is a bipolar fuzzy soft hypervector space of* V*:*

1. $\mathcal{F}_e^+(x-y) \geq \mathcal{F}_e^+(x) \wedge \mathcal{F}_e^+(y)$, $\mathcal{F}_e^-(x-y) \leq \mathcal{F}_e^-(x) \vee \mathcal{F}_e^-(y)$,
2. $\inf\limits_{t\in a\circ x} \mathcal{F}_e^+(t) \geq \mathcal{F}_e^+(x)$, $\sup\limits_{t\in a\circ x} \mathcal{F}_e^-(t) \leq \mathcal{F}_e^-(x)$.

Example 2.9. [18] *In a classical vector space $(\mathbb{R}^3,+,.,\mathbb{R})$, we can define the external hyperoperation $\circ : \mathbb{R}\times\mathbb{R}^3 \to P_*(\mathbb{R}^3)$ as follows: For a given point (x_0,y_0,z_0) and a scalar a, $a\circ(x_0,y_0,z_0) = l$ represents a line "l" with the parametric equations: $x = ax_0$, $y = ay_0$ and $z = t$. By considering $\mathcal{V} = (\mathbb{R}^3,+,\circ,\mathbb{R})$, we can define $\mathcal{V}$ as a hypervector space over the field $\mathbb{R}$. Assuming $\mathcal{A} = \{a,b\}$ is a set of parameters, we can say that $(\mathcal{F},\mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\mathcal{V}$. Here, the functions $\mathcal{F}_a^+$ and $\mathcal{F}_b^+$ map $\mathbb{R}^3$ to the interval $[0,1]$, while the functions $\mathcal{F}_a^-$ and $\mathcal{F}_b^-$ map $\mathbb{R}^3$ to the interval $[-1,0]$, by the following rules ($\mathcal{X} = \{0\}\times\{0\}\times\mathbb{R}$, $\mathcal{Y} = \mathbb{R}\times\{0\}\times\mathbb{R}$):*

	$(x,y,z)\in\mathcal{X}$	$(x,y,z)\in\mathcal{Y}\setminus\mathcal{X}$	*otherwise*
$\mathcal{F}_a^+(x,y,z)$	0.7	0.3	0
$\mathcal{F}_a^-(x,y,z)$	−0.8	−0.4	−0.2
$\mathcal{F}_b^+(x,y,z)$	0.9	0.4	0.1
$\mathcal{F}_b^-(x,y,z)$	−0.6	−0.5	−0.1

Example 2.10. [18] *If $\mathcal{K} = \mathbb{Z}_2 = \{0,1\}$ is a field defined by the followings:*

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

then $(\mathbb{Z}_4,+,\circ,\mathbb{Z}_2)$ is a hypervector space, such that "$+ : \mathbb{Z}_4\times\mathbb{Z}_4 \to \mathbb{Z}_4$" and "$\circ : \mathbb{Z}_2\times\mathbb{Z}_4 \to P_*(\mathbb{Z}_4)$" are defined as follow:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

∘	0	1	2	3
0	$\{0,2\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0,2\}$	$\{1,2,3\}$	$\{0,2\}$	$\{1,2,3\}$

Suppose $\mathcal{A} = \{c,d,e\}$. Then $(\mathcal{F},\mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\mathcal{V}$, where "$F_c^+, F_d^+, F_e^+ : \mathbb{Z}_4 \to [0,1]$" and "$F_c^-, F_d^-, F_e^- : \mathbb{Z}_4 \to [-1,0]$" are given by the followings:

$$F_c^+(x) = \begin{cases} 0.5 & x\in\{0,2\} \\ 0.3 & x\in\{1,3\} \end{cases} \qquad F_c^-(x) = \begin{cases} -0.4 & x\in\{0,2\} \\ -0.2 & x\in\{1,3\} \end{cases}$$

$$F_d^+(x) = \begin{cases} 0.7 & x\in\{0,2\} \\ 0.2 & x\in\{1,3\} \end{cases} \qquad F_d^-(x) = \begin{cases} -0.6 & x\in\{0,2\} \\ -0.3 & x\in\{1,3\} \end{cases}$$

$$F_e^+(x) = \begin{cases} 0.8 & x\in\{0,2\} \\ 0.4 & x\in\{1,3\} \end{cases} \qquad F_e^-(x) = \begin{cases} -0.7 & x\in\{0,2\} \\ -0.5 & x\in\{1,3\} \end{cases}$$

Definition 2.11. [19] *Let $(\mathcal{F},\mathcal{A})$ denote a bipolar fuzzy soft set within a hypervector space $\mathcal{V} = (\mathcal{V},+,\circ,\mathcal{K})$. For any α in the range $(0,1]$ and β within the interval $[-1,0)$, we can represent the (α,β)-level soft subset of $\mathcal{V}$ as $(\mathcal{F},\mathcal{A})_{\alpha,\beta}$. This is defined as a soft set*

$$(\mathcal{F},\mathcal{A})_{\alpha,\beta} = \{(\mathcal{F}_e)_{\alpha,\beta};\ e\in\mathcal{A}\},$$

where for every e in $\mathcal{A}$, $(\mathcal{F}_e)_{\alpha,\beta}$ is an (α,β)-level subset of the bipolar fuzzy soft set $\mathcal{F}_e = (\mathcal{F}_e^+,\mathcal{F}_e^-)$ and can be expressed as:

$$(\mathcal{F}_e)_{\alpha,\beta} = \{x\in\mathcal{V};\ \mathcal{F}_e^+(x)\geq\alpha,\ \mathcal{F}_e^-(x)\leq\beta\}.$$

Theorem 2.12. [19] *Let $(\mathcal{F},\mathcal{A})$ represent a bipolar fuzzy soft set of $\mathcal{V}=(\mathcal{V},+,\circ,\mathcal{K})$. The necessary and sufficient condition for $(\mathcal{F},\mathcal{A})$ to be a bipolar fuzzy soft hypervector space of $\mathcal{V}$ is that for any values of α in the range $(0,1]$ and β within the interval $[-1,0)$, the soft subset $(\mathcal{F},\mathcal{A})_{\alpha,\beta}$ satisfies the properties of a soft hypervector space of $\mathcal{V}$. In other words, for any values of α in $(0,1]$, β in $[-1,0)$, and e in $\mathcal{A}$, the subset $(\mathcal{F}_e)_{\alpha,\beta}$ behaves like a subhyperspace of $\mathcal{V}$.*

Theorem 2.13. [19] *In a strongly left distributive hypervector space $\mathcal{V}=(\mathcal{V},+,\circ,\mathcal{K})$, if $(\mathcal{F},\mathcal{A})$ represents a bipolar fuzzy soft set of $\mathcal{V}$, then $(\mathcal{F},\mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\mathcal{V}$ if and only if for any a and b in $\mathcal{K}$, $a\circ(\mathcal{F},\mathcal{A})+b\circ(\mathcal{F},\mathcal{A})\sqsubseteq(\mathcal{F},\mathcal{A})$.*

Theorem 2.14. [19] *If $(\mathcal{F},\mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\mathcal{V}$, for any $\acute{e}\in A$, $\alpha\in(0,1]$ and $\beta\in[-1,0)$, the bipolar fuzzy soft set $(\mathcal{G},\mathcal{A})$ of $\mathcal{V}$ is defined as follows forms a bipolar fuzzy soft hypervector space of $\mathcal{V}$:*

$$\mathcal{G}_e^+(x)=\begin{cases}\mathcal{F}_e^+(x) & x\notin(\mathcal{F}_{\acute{e}})_{\alpha,\beta}\\ 1 & x\in(\mathcal{F}_{\acute{e}})_{\alpha,\beta}\end{cases}\qquad\qquad \mathcal{G}_e^-(x)=\begin{cases}\mathcal{F}_e^-(x) & x\notin(\mathcal{F}_{\acute{e}})_{\alpha,\beta}\\ -1 & x\in(\mathcal{F}_{\acute{e}})_{\alpha,\beta}\end{cases}$$

3 Quotient bipolar fuzzy soft sets relative to hypervector spaces

In this section, we define the notion of bipolar fuzzy soft coset related to a bipolar fuzzy soft hypervector space and obtain a hypervector space $\frac{\mathcal{F}}{\mathcal{V}}=(\frac{\mathcal{F}}{\mathcal{V}},\oplus,\circledcirc,\mathcal{K})$ by suitable operation and external hyperoperation over the set of all bipolar fuzzy soft cosets. Then in an important theorem, we show that the dimension of the mentioned hypervector space $\frac{\mathcal{F}}{\mathcal{V}}$ is equal to the dimension of a quotient hypervector space $\frac{\mathcal{V}}{\mathcal{W}}$, where $\mathcal{W}$ is a particular subhyperspace of $\mathcal{V}$. Next, we prove that every bipolar fuzzy soft hypervector space of $\frac{\mathcal{F}}{\mathcal{V}}$ corresponds to a bipolar fuzzy soft hypervector space of $\mathcal{V}$.

Lemma 3.1. *For any bipolar fuzzy soft hypervector space $(\mathcal{F},\mathcal{A})$ over strongly left distributive hypervector space $\mathcal{V}$, any parameter e in $\mathcal{A}$ and any vectors x and y in $\mathcal{V}$, the following conditions hold:*

1. $\mathcal{F}_e^+(x)\le\mathcal{F}_e^+(0)$ *and* $\mathcal{F}_e^-(x)\ge\mathcal{F}_e^-(0)$.
2. *if* $\mathcal{F}_e^+(x-y)\ge\mathcal{F}_e^+(0)$, *then* $\mathcal{F}_e^+(x)=\mathcal{F}_e^+(y)$, *and if* $\mathcal{F}_e^-(x-y)\le\mathcal{F}_e^-(0)$, *then* $\mathcal{F}_e^-(x)=\mathcal{F}_e^-(y)$.

Proof. 1) By Theorem 2.13, putting $a=1$ and $b=-1$, it follows that $1\circ(\mathcal{F},\mathcal{A})+(-1)\circ(\mathcal{F},\mathcal{A})\sqsubseteq(\mathcal{F},\mathcal{A})$. Then by Lemma 2.7,

$$\begin{aligned}\mathcal{F}_e^+(0)&\ge (1\circ(\mathcal{F},\mathcal{A})+(-1)\circ(\mathcal{F},\mathcal{A}))_e^+(0)\\ &=\sup_{0=y+z}\big((1\circ(\mathcal{F},\mathcal{A}))_e^+(y)\wedge((-1)\circ(\mathcal{F},\mathcal{A}))_e^+(z)\big)\\ &\ge (1\circ(\mathcal{F},\mathcal{A}))_e^+(x)\wedge((-1)\circ(\mathcal{F},\mathcal{A}))_e^+(-x)\\ &\ge \mathcal{F}_e^+(x)\wedge(-\mathcal{F})_e^+(-x)\\ &=\mathcal{F}_e^+(x)\wedge\mathcal{F}_e^+(x)\\ &=\mathcal{F}_e^+(x),\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_e^-(0) &\leq (1\circ(\mathcal{F},\mathcal{A})+(-1)\circ(\mathcal{F},\mathcal{A}))_e^-(0)\\
&= \inf_{0=y+z}\left((1\circ(\mathcal{F},\mathcal{A}))_e^-(y)\vee((-1)\circ(\mathcal{F},\mathcal{A}))_e^-(z)\right)\\
&\leq (1\circ(\mathcal{F},\mathcal{A}))_e^-(x)\vee((-1)\circ(\mathcal{F},\mathcal{A}))_e^-(-x)\\
&\leq \mathcal{F}_e^-(x)\vee(-\mathcal{F})_e^-(-x)\\
&= \mathcal{F}_e^-(x)\vee\mathcal{F}_e^-(x)\\
&= \mathcal{F}_e^-(x).
\end{aligned}$$

2) $\mathcal{F}_e^+(x) = \mathcal{F}_e^+(x-y+y) \geq \mathcal{F}_e^+(x-y)\wedge\mathcal{F}_e^+(y) \geq \mathcal{F}_e^+(0)\wedge\mathcal{F}_e^+(y) = \mathcal{F}_e^+(y)$. Similarly, $\mathcal{F}_e^+(y) \geq \mathcal{F}_e^+(x)$ and so $\mathcal{F}_e^+(x) = \mathcal{F}_e^+(y)$. Also,

$$\mathcal{F}_e^-(x) = \mathcal{F}_e^-(x-y+y) \leq \mathcal{F}_e^-(x-y)\vee\mathcal{F}_e^-(y) \leq \mathcal{F}_e^-(0)\vee\mathcal{F}_e^-(y) = \mathcal{F}_e^-(y).$$

Similarly, $\mathcal{F}_e^-(y) \leq \mathcal{F}_e^-(x)$ and so $\mathcal{F}_e^-(x) = \mathcal{F}_e^-(y)$. □

If $\mathcal{W}$ is a subhyperspace of $(\mathcal{V},+,\circ,\mathcal{K})$, then $(\mathcal{V}/\mathcal{W},+,*,\mathcal{K})$ is a hypervector space over $\mathcal{K}$. In this context, the external hyperoperation $*:\mathcal{K}\times\mathcal{V}/\mathcal{W}\to P_*(\mathcal{V}/\mathcal{W})$ is defined by $a*(v+\mathcal{W}) = a\circ v+\mathcal{W}$.
The next theorem is a valuable resource when examining fuzzy cosets within a bipolar fuzzy soft hypervector space.

Theorem 3.2. *Suppose $\mathcal{V}=(\mathcal{V},+,\circ,\mathcal{K})$ is a strongly left distributive hypervector space. Then*
1) If $(\mathcal{F},\mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\mathcal{V}$, where $\mathcal{W}$ is defined as

$$\mathcal{W} = \{y\in\mathcal{V};\ \mathcal{F}_e^+(y)\geq\mathcal{F}_e^+(0),\ \mathcal{F}_e^-(y)\leq\mathcal{F}_e^-(0),\ \forall e\in\mathcal{A}\},$$

then $(\hat{\mathcal{F}},A)$ represents a bipolar fuzzy soft hypervector space of $\frac{\mathcal{V}}{\mathcal{W}}$. In this case, $\hat{\mathcal{F}}_e^+(x+\mathcal{W}) = \mathcal{F}_e^+(x)$ and $\hat{\mathcal{F}}_e^-(x+\mathcal{W}) = \mathcal{F}_e^-(x)$, for all $e\in\mathcal{A}$ and $x\in\mathcal{V}$.
2) Consider $\mathcal{U}$ is a subhyperspace of $\mathcal{V}$ and $(\mathcal{G},\mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\frac{\mathcal{V}}{\mathcal{U}}$, satisfying the conditions $\mathcal{G}_e^+(x+\mathcal{U}) = \mathcal{G}_e^+(\mathcal{U})$ and $\mathcal{G}_e^-(x+\mathcal{U}) = \mathcal{G}_e^-(\mathcal{U})$, only when $x\in\mathcal{U}$. In such a scenario, there exists a bipolar fuzzy soft hypervector space $(\mathcal{F},\mathcal{A})$ of $\mathcal{V}$, such that $\hat{\mathcal{F}}=\mathcal{G}$ and

$$\left\{z\in\mathcal{V};\ \mathcal{F}_e^+(z)\geq\mathcal{F}_e^+(0),\ \mathcal{F}_e^-(z)\leq\mathcal{F}_e^-(0),\ \forall e\in\mathcal{A}\right\} = \mathcal{U}.$$

Proof. 1) If $x,y\in\mathcal{W}$, $a\in\mathcal{K}$, then for all $e\in\mathcal{A}$,

$$\mathcal{F}_e^+(x-y)\geq\mathcal{F}_e^+(x)\wedge\mathcal{F}_e^+(y)\geq\mathcal{F}_e^+(0)\wedge\mathcal{F}_e^+(0)=\mathcal{F}_e^+(0),$$

$$\mathcal{F}_e^-(x-y)\leq\mathcal{F}_e^-(x)\vee\mathcal{F}_e^-(y)\leq\mathcal{F}_e^-(0)\wedge\mathcal{F}_e^-(0)=\mathcal{F}_e^-(0),$$

so $x-y\in\mathcal{W}$. Also, by Definition 2.8, for all $s\in a\circ x$, $\mathcal{F}_e^+(s)\geq\inf_{t\in a\circ x}\mathcal{F}_e^+(t)\geq\mathcal{F}_e^+(x)\geq\mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(s)\leq\sup_{t\in a\circ x}\mathcal{F}_e^-(t)\leq\mathcal{F}_e^-(x)\leq\mathcal{F}_e^-(0)$. Thus $a\circ x\subseteq\mathcal{W}$ and $\mathcal{W}$ is a subhyperspace of $\mathcal{V}$.
Now if $e\in\mathcal{A}$, $x,y\in\mathcal{V}$ and $x+\mathcal{W}=y+\mathcal{W}$, then $x,y\in\mathcal{W}$, so $\mathcal{F}_e^+(x-y)\geq\mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(x-y)\leq\mathcal{F}_e^-(0)$, for all $e\in\mathcal{A}$. Thus by Lemma 3.1, $\mathcal{F}_e^+(x)=\mathcal{F}_e^+(y)$ and $\mathcal{F}_e^-(x)=\mathcal{F}_e^-(y)$, for all $e\in\mathcal{A}$. Hence $\hat{\mathcal{F}}_e^+(x+\mathcal{W})=\hat{\mathcal{F}}_e^+(y+\mathcal{W})$ and $\hat{\mathcal{F}}_e^-(x+\mathcal{W})=\hat{\mathcal{F}}_e^-(y+\mathcal{W})$. Therefore, $\hat{\mathcal{F}}$ is well-defined.

Next, if $x, y \in \mathcal{V}$, $a \in \mathcal{K}$ and $e \in \mathcal{A}$, then

$$\begin{aligned} \hat{\mathcal{F}}_e^+((x+\mathcal{W})-(y+\mathcal{W})) &= \hat{\mathcal{F}}_e^+((x-y)+\mathcal{W}) \\ &= \mathcal{F}_e^+(x-y) \\ &\geq \mathcal{F}_e^+(x) \wedge \mathcal{F}_e^+(y) \\ &= \hat{\mathcal{F}}_e^+(x+\mathcal{W}) \wedge \hat{\mathcal{F}}_e^+(y+\mathcal{W}), \end{aligned}$$

and similarly, $\hat{\mathcal{F}}_e^-((x+\mathcal{W})-(y+\mathcal{W})) \leq \hat{\mathcal{F}}_e^-(x+\mathcal{W}) \vee \hat{\mathcal{F}}_e^-(y+\mathcal{W})$.
Also, if $s \in a \circ x$, then by Definition 2.8,

$$\hat{\mathcal{F}}_e^+(s+\mathcal{W}) = \mathcal{F}_e^+(s) \geq \inf_{t \in a \circ x} \mathcal{F}_e^+(t) \geq \mathcal{F}_e^+(x) = \hat{\mathcal{F}}_e^+(x+\mathcal{W}),$$

and

$$\hat{\mathcal{F}}_e^-(s+\mathcal{W}) = F_e^-(s) \leq \sup_{t \in a \circ x} \mathcal{F}_e^-(t) \leq \mathcal{F}_e^-(x) = \hat{\mathcal{F}}_e^-(x+\mathcal{W}).$$

Thus

$$\inf_{s+\mathcal{W} \in a*(x+\mathcal{W}) = a\circ x+\mathcal{W}} \hat{\mathcal{F}}_e^+(s+\mathcal{W}) \geq \hat{\mathcal{F}}_e^+(x+\mathcal{W}),$$

and

$$\sup_{s+\mathcal{W} \in a*(x+\mathcal{W}) = a\circ x+\mathcal{W}} \hat{\mathcal{F}}_e^-(s+\mathcal{W}) \leq \hat{\mathcal{F}}_e^-(x+\mathcal{W}).$$

Hence, by Definition 2.8, $(\hat{\mathcal{F}}, \mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\frac{\mathcal{V}}{\mathcal{W}}$.
2) Define a bipolar fuzzy soft set $(\mathcal{F}, \mathcal{A})$ of $\mathcal{V}$, such that for every $e \in \mathcal{A}$, $\mathcal{F}(e) = \mathcal{F}_e : \mathcal{V} \to [0,1] \times [-1,0]$, given by $\mathcal{F}_e^+(x) = \mathcal{G}_e^+(x+\mathcal{U})$ and $\mathcal{F}_e^-(x) = \mathcal{G}_e^-(x+\mathcal{U})$, where x belongs to $\mathcal{V}$. It can be easily verified that $(\mathcal{F}, \mathcal{A})$ qualifies as a bipolar fuzzy soft hypervector space of $\mathcal{V}$.
Moreover, $x \in \mathcal{V}$ such that for every $e \in \mathcal{A}$, $\mathcal{F}_e^+(x) \geq \mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(x) \leq \mathcal{F}_e^+(0)$, if and only if for all $e \in \mathcal{A}$, $\mathcal{G}_e^+(x+\mathcal{U}) = \mathcal{G}_e^+(\mathcal{U})$ and $\mathcal{G}_e^-(x+\mathcal{U}) = \mathcal{G}_e^-(\mathcal{U})$, if and only if $x \in \mathcal{U}$. Thus

$$\hat{\mathcal{F}}_e^+(x+\mathcal{U}) = \hat{\mathcal{F}}_e^+(x+\mathcal{W}) = \mathcal{F}_e^+(x) = \mathcal{G}_e^+(x+\mathcal{U}),$$

and

$$\hat{\mathcal{F}}_e^-(x+\mathcal{U}) = \hat{\mathcal{F}}_e^-(x+\mathcal{W}) = \mathcal{F}_e^-(x) = \mathcal{G}_e^-(x+\mathcal{U}).$$

Hence, $\hat{\mathcal{F}} = \mathcal{G}$ and the proof is completed. □

Example 3.3. *If $(\mathcal{F}, \mathcal{A})$ represents the bipolar fuzzy soft hypervector space of $(\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 2.9, then*

$$\begin{aligned} \mathcal{W} &= \left\{ x \in \mathbb{R}^3,\ \mathcal{F}_a^+(x) \geq 0.7, \mathcal{F}_b^+(x) \geq 0.9, \mathcal{F}_a^-(x) \leq -0.8, \mathcal{F}_b^-(x) \leq -0.6 \right\} \\ &= \{0\} \times \{0\} \times \mathbb{R}. \end{aligned}$$

Thus $(\hat{\mathcal{F}}, \mathcal{A})$ defined in Theorem 3.2, is a bipolar fuzzy soft hypervector space of $\frac{\mathbb{R}^3}{\{0\}\times\{0\}\times\mathbb{R}}$.

Example 3.4. *If $(\mathcal{F}, \mathcal{A})$ represents the bipolar fuzzy soft hypervector space of $(\mathbb{Z}_4, +, \circ, \mathbb{Z}_2)$ in Example 2.10, then*

$$\begin{aligned} \mathcal{W} &= \left\{ \begin{array}{c} x \in \mathbb{Z}_4,\ \mathcal{F}_c^+(x) \geq 0.5, \mathcal{F}_d^+(x) \geq 0.7, \mathcal{F}_e^+(x) \geq 0.8, \\ \mathcal{F}_c^-(x) \leq -0.4, \mathcal{F}_d^-(x) \leq -0.6, \mathcal{F}_e^-(x) \leq -0.7 \end{array} \right\} \\ &= \{0, 2\}. \end{aligned}$$

Thus $(\hat{\mathcal{F}}, \mathcal{A})$ defined in Theorem 3.2, is a bipolar fuzzy soft hypervector space of $\frac{\mathbb{Z}_4}{\{0,2\}}$.

Definition 3.5. *Let $(\mathcal{F}, \mathcal{A})$ be a bipolar fuzzy soft hypervector space of $\mathcal{V}$ with $v \in \mathcal{V}$. Then the bipolar fuzzy soft set $v\mathcal{F}$ of $\mathcal{V}$ ($v\mathcal{F} : \mathcal{A} \to BF^{\mathcal{V}}$), is defined as follows:*

$$\forall e \in \mathcal{A}, x \in \mathcal{V},\ (v\mathcal{F})^+_e(x) = \mathcal{F}F^+_e(v - x), (v\mathcal{F})^-_e(x) = \mathcal{F}^-_e(v - x),$$

$v\mathcal{F}$ is called the bipolar fuzzy soft coset related to $\mathcal{F}$ and v. The collection of all bipolar fuzzy soft cosets related to $\mathcal{F}$ is denoted by $\frac{\mathcal{F}}{\mathcal{V}}$, i.e.

$$\frac{\mathcal{F}}{\mathcal{V}} = \{v\mathcal{F};\ v \in \mathcal{V}\}.$$

Theorem 3.6. *In a bipolar fuzzy soft hypervector space $(\mathcal{F}, \mathcal{A})$ over $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$, the set $\left(\frac{\mathcal{F}}{\mathcal{V}}, \oplus, \circledcirc, \mathcal{K}\right)$ is a hypervector space, where the operation $\oplus : \frac{\mathcal{F}}{\mathcal{V}} \times \frac{\mathcal{F}}{\mathcal{V}} \to \frac{\mathcal{F}}{\mathcal{V}}$ and the external hyperoperation $\circledcirc : \mathcal{K} \times \frac{\mathcal{F}}{\mathcal{V}} \to P_*(\frac{\mathcal{F}}{\mathcal{V}})$ are defined as follows:*

$$x\mathcal{F} \oplus y\mathcal{F} = (x + y)\mathcal{F},$$

$$a \circledcirc (x\mathcal{F}) = \{t\mathcal{F};\ t \in a \circ x\},$$

for all $x\mathcal{F}, y\mathcal{F} \in \frac{\mathcal{F}}{\mathcal{V}}$, $a \in \mathcal{K}$.

Proof. One can easily see that " $\oplus$ " and " $\circledcirc$ " are well-defined and $(\frac{\mathcal{F}}{\mathcal{V}}, \oplus)$ is an Abelian group. Now let $x\mathcal{F}, x_1\mathcal{F}, x_2\mathcal{F}, y\mathcal{F} \in \frac{\mathcal{F}}{\mathcal{V}}$ and $a_1, a_2, b, c \in \mathcal{K}$. Then

(H_1)

$$\begin{aligned}
b \circledcirc (x_1\mathcal{F} + x_2\mathcal{F}) &= b \circledcirc ((x_1 + x_2)\mathcal{F}) \\
&= \{x\mathcal{F};\ x \in b \circ (x_1 + x_2)\} \\
&\subseteq \{x\mathcal{F};\ x \in b \circ x_1 + b \circ x_2\} \\
&= \{x\mathcal{F};\ x = s_1 + s_2, s_1 \in b \circ x_1, s_2 \in b \circ x_2\} \\
&= \{(s_1 + s_2)\mathcal{F};\ s_1 \in b \circ x_1, s_2 \in b \circ x_2\} \\
&= \{s_1\mathcal{F} \oplus s_2\mathcal{F};\ s_1 \in b \circ x_1, s_2 \in b \circ x_2\} \\
&= \{s_1\mathcal{F};\ s_1 \in b \circ x_1\} \oplus \{s_2\mathcal{F};\ s_2 \in b \circ x_2\} \\
&= (b \circledcirc (x_1\mathcal{F})) \oplus (b \circledcirc (x_2\mathcal{F}))\,.
\end{aligned}$$

(H_2)

$$\begin{aligned}
(a_1 + a_2) \circledcirc (y\mathcal{F}) &= \{s\mathcal{F};\ s \in (a_1 + a_2) \circ y\} \\
&\subseteq \{s\mathcal{F};\ s \in a_1 \circ y + a_2 \circ y\} \\
&= \{s\mathcal{F};\ s = s_1 + s_2, s_1 \in a_1 \circ y, s_2 \in a_2 \circ y\} \\
&= \{(s_1 + s_2)\mathcal{F};\ s_1 \in a_1 \circ y, s_2 \in a_2 \circ y\} \\
&= \{s_1\mathcal{F} \oplus s_2\mathcal{F};\ s_1 \in a_1 \circ y, s_2 \in a_2 \circ y\} \\
&= \{s_1\mathcal{F};\ s_1 \in a_1 \circ y\} \oplus \{s_2\mathcal{F};\ s_2 \in a_2 \circ y\} \\
&= (a_1 \circledcirc (y\mathcal{F})) \oplus (a_2 \circledcirc (y\mathcal{F}))\,.
\end{aligned}$$

(H$_3$)

$$\begin{aligned} b \circledcirc (c \circledcirc (x\mathcal{F})) &= \bigcup_{t\mathcal{F} \in c \circledcirc (x\mathcal{F})} b \circledcirc t\mathcal{F} \\ &= \bigcup_{t \in c \circ x} \{s\mathcal{F};\ s \in b \circ t\} \\ &= \{s\mathcal{F};\ s \in b \circ (c \circ x)\} \\ &= \{s\mathcal{F};\ s \in (bc) \circ x\} \\ &= (bc) \circledcirc (x\mathcal{F}). \end{aligned}$$

(H$_4$)

$$\begin{aligned} b \circledcirc (-(y\mathcal{F})) &= b \circledcirc ((-y)\mathcal{F}) \\ &= \{t\mathcal{F};\ t \in b \circ (-y)\} \\ &= \{t\mathcal{F};\ t \in (-b) \circ y\} \\ &= (-b) \circledcirc (y\mathcal{F}), \end{aligned}$$

similarly, $b \circledcirc (-(y\mathcal{F})) = -(b \circledcirc (b\mathcal{F}))$.

(H$_5$) $x\mathcal{F} \in \{t\mathcal{F};\ 1 \in a \circ x\} = 1 \circledcirc (x\mathcal{F})$.

Consequently, by Definition 2.4, the set $\left(\frac{\mathcal{F}}{\mathcal{V}}, \oplus, \circledcirc, \mathcal{K}\right)$ forms a hypervector space. $\square$

Example 3.7. *Suppose $(\mathcal{F}, \mathcal{A})$ is the bipolar fuzzy soft hypervector space of $(\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 2.9, and $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. Then the bipolar fuzzy soft coset $v\mathcal{F}$ related to $\mathcal{F}$ and v is defined by the followings:*

$$\begin{aligned} ((v_1, v_2, v_3)\mathcal{F})^+_a(x_1, x_2, x_3) &= \mathcal{F}^+_a(v_1 - x_1, v_2 - x_2, v_3 - x_3) \\ &= \begin{cases} 0.7 & (x_1, x_2, x_3) \in \{v_1\} \times \{v_2\} \times \mathbb{R}, \\ 0.3 & (x_1, x_2, x_3) \in (\mathbb{R} \times \{v_2\} \times \mathbb{R}) \setminus (\{v_1\} \times \{v_2\} \times \mathbb{R}), \\ 0 & \textit{otherwise}, \end{cases} \end{aligned}$$

$$\begin{aligned} ((v_1, v_2, v_3)\mathcal{F})^-_a(x_1, x_2, x_3) &= \mathcal{F}^-_a(v_1 - x_1, v_2 - x_2, v_3 - x_3) \\ &= \begin{cases} -0.8 & (x_1, x_2, x_3) \in \{v_1\} \times \{v_2\} \times \mathbb{R}, \\ -0.4 & (x_1, x_2, x_3) \in (\mathbb{R} \times \{v_2\} \times \mathbb{R}) \setminus (\{v_1\} \times \{v_2\} \times \mathbb{R}), \\ -0.2 & \textit{otherwise}, \end{cases} \end{aligned}$$

$$\begin{aligned} ((v_1, v_2, v_3)\mathcal{F})^+_b(x_1, x_2, x_3) &= \mathcal{F}^+_b(v_1 - x_1, v_2 - x_2, v_3 - x_3) \\ &= \begin{cases} 0.9 & (x_1, x_2, x_3) \in \{v_1\} \times \{v_2\} \times \mathbb{R}, \\ 0.4 & (x_1, x_2, x_3) \in (\mathbb{R} \times \{v_2\} \times \mathbb{R}) \setminus (\{v_1\} \times \{v_2\} \times \mathbb{R}), \\ 0.1 & \textit{otherwise}, \end{cases} \end{aligned}$$

$$\begin{aligned} ((v_1, v_2, v_3)\mathcal{F})^-_b(x_1, x_2, x_3) &= \mathcal{F}^-_b(v_1 - x_1, v_2 - x_2, v_3 - x_3) \\ &= \begin{cases} -0.6 & (x_1, x_2, x_3) \in \{v_1\} \times \{v_2\} \times \mathbb{R}, \\ -0.5 & (x_1, x_2, x_3) \in (\mathbb{R} \times \{v_2\} \times \mathbb{R}) \setminus (\{v_1\} \times \{v_2\} \times \mathbb{R}), \\ -0.1 & \textit{otherwise}. \end{cases} \end{aligned}$$

Now, for every $v\mathcal{F} = (v_1, v_2, v_3)\mathcal{F}, \acute{v}\mathcal{F} = (\acute{v}_1, \acute{v}_2, \acute{v}_3)\mathcal{F} \in \frac{\mathcal{F}}{\mathbb{R}^3}$, *it follows that:*

$$\begin{array}{rcl} (v\mathcal{F} \oplus \acute{v}\mathcal{F})^+_a(x_1, x_2, x_3) & = & ((v_1 + \acute{v}_1, v_2 + \acute{v}_2, v_3 + \acute{v}_3)\mathcal{F})^+_a(x_1, x_2, x_3) \\ & = & \left\{ \begin{array}{ll} 0.7 & (x_1, x_2, x_3) \in S = \{v_1 + \acute{v}_1\} \times \{v_2 + \acute{v}_2\} \times \mathbb{R}, \\ 0.3 & (x_1, x_2, x_3) \in (\mathbb{R} \times \{v_2 + \acute{v}_2\} \times \mathbb{R}) \setminus S, \\ 0 & \textit{otherwise}, \end{array} \right. \end{array}$$

$(v\mathcal{F} \oplus \acute{v}\mathcal{F})^-_a$, $(v\mathcal{F} \oplus \acute{v}\mathcal{F})^+_b$ *and* $(v\mathcal{F} \oplus \acute{v}\mathcal{F})^-_b$ *are similarly obtained. Also, for every* $r \in \mathbb{R}$, $v\mathcal{F} = (v_1, v_2, v_3)\mathcal{F} \in \frac{\mathcal{F}}{\mathbb{R}^3}$,

$$\begin{array}{rcl} r \circledcirc (v\mathcal{F}) & = & \{x\mathcal{F}; x \in r \circ (v_1, v_2, v_3)\} \\ & = & \{(rv_1, rv_2, t)\mathcal{F}; t \in \mathbb{R}\}, \end{array}$$

where for every $t \in \mathbb{R}$,

$$\begin{array}{rcl} ((rv_1, rv_2, t)\mathcal{F})^+_a(x_1, x_2, x_3) & = & \mathcal{F}^+_a(rv_1 - x, rv_2 - y, t - z) \\ & = & \left\{ \begin{array}{ll} 0.7 & (x_1, x_2, x_3) \in X = \{rv_1\} \times \{rv_2\} \times \mathbb{R}, \\ 0.3 & (x_1, x_2, x_3) \in (\mathbb{R} \times \{rv_2\} \times \mathbb{R}) \setminus X, \\ 0 & \textit{otherwise}, \end{array} \right. \end{array}$$

$(r \circledcirc ((v_1, v_2, v_3)\mathcal{F}))^-_a$, $(r \circledcirc ((v_1, v_2, v_3)\mathcal{F}))^+_b$ *and* $(r \circledcirc ((v_1, v_2, v_3)\mathcal{F}))^-_b$ *are similarly defined.*

Example 3.8. *Let's talk about the bipolar fuzzy soft hypervector space* $(\mathcal{F}, \mathcal{A})$ *which is defined on the algebraic structures* $(\mathbb{Z}_4, +, \circ, \mathbb{Z}_2)$ *in Example 2.10. Here* $\frac{\mathcal{F}}{\mathbb{Z}_4} = \{v\mathcal{F}, v \in \mathbb{Z}_4\} = \{0\mathcal{F}, 1\mathcal{F}, 2\mathcal{F}, 3\mathcal{F}\}$, *where for example,* $0\mathcal{F}$ *and* $1\mathcal{F}$ *are defined by the followings:*

$$(0\mathcal{F})^+_c(x) = \mathcal{F}^+_c(-x) = \left\{ \begin{array}{ll} 0.5 & x \in \{0, 2\} \\ 0.3 & x \in \{1, 3\} \end{array} \right. \qquad (0\mathcal{F})^-_c(x) = \mathcal{F}^-_c(-x) = \left\{ \begin{array}{ll} -0.4 & x \in \{0, 2\} \\ -0.2 & x \in \{1, 3\} \end{array} \right.$$

$$(0\mathcal{F})^+_d(x) = \mathcal{F}^+_d(-x) = \left\{ \begin{array}{ll} 0.7 & x \in \{0, 2\} \\ 0.2 & x \in \{1, 3\} \end{array} \right. \qquad (0\mathcal{F})^-_d(x) = \mathcal{F}^-_d(-x) = \left\{ \begin{array}{ll} -0.6 & x \in \{0, 2\} \\ -0.3 & x \in \{1, 3\} \end{array} \right.$$

$$(0\mathcal{F})^+_e(x) = \mathcal{F}^+_e(-x) = \left\{ \begin{array}{ll} 0.8 & x \in \{0, 2\} \\ 0.4 & x \in \{1, 3\} \end{array} \right. \qquad (0\mathcal{F})^-_e(x) = \mathcal{F}^+_e(-x) = \left\{ \begin{array}{ll} -0.7 & x \in \{0, 2\} \\ -0.5 & x \in \{1, 3\} \end{array} \right.$$

and

$$(1\mathcal{F})^+_c(x) = \mathcal{F}^+_c(1 - x) = \left\{ \begin{array}{ll} 0.5 & x \in \{1, 3\} \\ 0.3 & x \in \{0, 2\} \end{array} \right. \qquad (1\mathcal{F})^-_c(x) = \mathcal{F}^-_c(1 - x) = \left\{ \begin{array}{ll} -0.4 & x \in \{1, 3\} \\ -0.2 & x \in \{0, 2\} \end{array} \right.$$

$$(1\mathcal{F})^+_d(x) = \mathcal{F}^+_d(1 - x) = \left\{ \begin{array}{ll} 0.7 & x \in \{1, 3\} \\ 0.2 & x \in \{0, 2\} \end{array} \right. \qquad (1\mathcal{F})^-_d(x) = \mathcal{F}^-_d(1 - x) = \left\{ \begin{array}{ll} -0.6 & x \in \{1, 3\} \\ -0.3 & x \in \{0, 2\} \end{array} \right.$$

$$(1\mathcal{F})^+_e(x) = \mathcal{F}^+_e(1 - x) = \left\{ \begin{array}{ll} 0.8 & x \in \{1, 3\} \\ 0.4 & x \in \{0, 2\} \end{array} \right. \qquad (1\mathcal{F})^-_e(x) = \mathcal{F}^-_e(1 - x) = \left\{ \begin{array}{ll} -0.7 & x \in \{1, 3\} \\ -0.5 & x \in \{0, 2\} \end{array} \right.$$

$2\mathcal{F}$ *and* $3\mathcal{F}$ *are similarly obtained. Moreover,* $\oplus$ *and* $\circledcirc$ *are defined by the following tables:*

$\oplus$	$0\mathcal{F}$	$1\mathcal{F}$	$2\mathcal{F}$	$3\mathcal{F}$
$0\mathcal{F}$	$0\mathcal{F}$	$1\mathcal{F}$	$2\mathcal{F}$	$3\mathcal{F}$
$1\mathcal{F}$	$1\mathcal{F}$	$2\mathcal{F}$	$3\mathcal{F}\mathcal{F}$	$0\mathcal{F}$
$2\mathcal{F}$	$2\mathcal{F}$	$3\mathcal{F}$	$0\mathcal{F}$	$1\mathcal{F}$
$3\mathcal{F}$	$3\mathcal{F}$	$0\mathcal{F}$	$1\mathcal{F}$	$2\mathcal{F}$

$\circledcirc$	$0\mathcal{F}$	$1\mathcal{F}$	$2\mathcal{F}$	$3\mathcal{F}$
0	$\{0\mathcal{F}, 2\mathcal{F}\}$	$\{0\mathcal{F}\}$	$\{0\mathcal{F}\}$	$\{0\mathcal{F}\}$
1	$\{0\mathcal{F}, 2\mathcal{F}\}$	$\{1\mathcal{F}, 2\mathcal{F}, 3\mathcal{F}\}$	$\{0\mathcal{F}, 2\mathcal{F}\}$	$\{1\mathcal{F}, 2\mathcal{F}, 3\mathcal{F}\}$

Lemma 3.9. *If $(\mathcal{F},\mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\mathcal{V}$, then for every parameter $e\in\mathcal{A}$ and every vectors $x,y\in\mathcal{V}$, the followings hold:*

1. $\mathcal{F}_e^+(x)<\mathcal{F}_e^+(y)$ *implies that* $\mathcal{F}_e^+(x-y)=\mathcal{F}_e^+(x)=\mathcal{F}_e^+(y-x)$.

2. $\mathcal{F}_e^-(x)>\mathcal{F}_e^+(y)$ *implies that* $\mathcal{F}_e^-(x-y)=\mathcal{F}_e^-(x)=\mathcal{F}_e^-(y-x)$.

Proof. 1) $\mathcal{F}_e^+(x-y)\geq\mathcal{F}_e^+(x)\wedge\mathcal{F}_e^+(y)=\mathcal{F}_e^+(x)$. Also, $\mathcal{F}_e^+(x)=\mathcal{F}_e^+(x-y+y)\geq\mathcal{F}_e^+(x-y)\wedge\mathcal{F}_e^+(y)$ and so $\mathcal{F}_e^+(x)\geq\mathcal{F}_e^+(x-y)$. Then $\mathcal{F}_e^+(x-y)=\mathcal{F}_e^+(x)$. Similarly, $\mathcal{F}_e^+(y-x)=\mathcal{F}_e^+(x)$.

2) $\mathcal{F}_e^-(x-y)\leq\mathcal{F}_e^-(x)\vee\mathcal{F}_e^-(y)=\mathcal{F}_e^-(x)$. Moreover, $\mathcal{F}_e^-(x)=\mathcal{F}_e^-(x-y+y)\leq\mathcal{F}_e^-(x-y)\vee\mathcal{F}_e^-(y)$ and so $\mathcal{F}_e^-(x)\leq\mathcal{F}_e^-(x-y)$. Thus $\mathcal{F}_e^-(x-y)=\mathcal{F}_e^-(x)$, and similarly, $\mathcal{F}_e^-(y-x)=\mathcal{F}_e^-(x)$. □

Next lemma will be used for proving Theorem 3.11, about the dimension of the hypervector space $\frac{\mathcal{F}}{\mathcal{V}}$.

Lemma 3.10. *Suppose $\mathcal{V}=(\mathcal{V},+,\circ,\mathcal{K})$ is a strongly left distributive hypervector space and $(\mathcal{F},\mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\mathcal{V}$. Then for every vector $x\in\mathcal{V}$ and every parameter $e\in\mathcal{A}$, it follows that $\mathcal{F}_e^+(x)=\mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(x)=\mathcal{F}_e^-(0)$ if and only if $x\mathcal{F}=0\mathcal{F}$.*

Proof. If $\mathcal{F}_e^+(x)=\mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(x)=\mathcal{F}_e^-(0)$, then for all $z\in\mathcal{V}$, by Lemma 3.1, $\mathcal{F}_e^+(z)\leq\mathcal{F}_e^+(0)=\mathcal{F}_e^+(x)$ and $\mathcal{F}_e^-(z)\geq\mathcal{F}_e^-(0)=\mathcal{F}_e^-(x)$. Now if $\mathcal{F}_e^+(z)<\mathcal{F}_e^+(x)$ and $\mathcal{F}_e^-(z)>\mathcal{F}_e^-(x)$, then by Lemma 3.9, $\mathcal{F}_e^+(z-x)=\mathcal{F}_e^+(z)$ and $\mathcal{F}_e^-(z-x)=\mathcal{F}_e^-(z)$. Thus $(x\mathcal{F})(z)=(0\mathcal{F})(z)$ and so $x\mathcal{F}=0\mathcal{F}$. Also, if $\mathcal{F}_e^+(z)=\mathcal{F}_e^+(x)=\mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(z)=\mathcal{F}_e^-(x)=\mathcal{F}_e^-(0)$, then

$$\mathcal{F}_e^+(z-x)\geq\mathcal{F}_e^+(z)\wedge\mathcal{F}_e^+(x)=\mathcal{F}_e^+(0)\wedge\mathcal{F}_e^+(0)=\mathcal{F}_e^+(0),$$

and

$$\mathcal{F}_e^-(z-x)\leq\mathcal{F}_e^-(z)\vee\mathcal{F}_e^-(x)=\mathcal{F}_e^-(0)\vee\mathcal{F}_e^-(0)=\mathcal{F}_e^-(0).$$

Hence, by Lemma 3.1, $\mathcal{F}_e^+(z-x)=\mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(z-x)=\mathcal{F}_e^-(0)$. So in this case, $x\mathcal{F}=0\mathcal{F}$, too.

Conversely, $(x\mathcal{F})(0)=(0\mathcal{F})(0)$, and so $\mathcal{F}_e^+(x)=\mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(x)=\mathcal{F}_e^-(0)$, for all $x\in\mathcal{V}$, $e\in\mathcal{A}$. □

The notion of dimension of a hypervector space was defined and studied by Ameri [8] as follows: A subset $\mathcal{B}$ of the hypervector space $\mathcal{V}$ is considered linearly independent, if for every collections of vectors $v_1,\ldots,v_n$ in $\mathcal{B}$, and every coefficients $c_1,\ldots,c_n$ in the field $\mathcal{K}$, the equation $0\in c_1\circ v_1+\cdots+c_n\circ v_n$, implies that $c_1=\cdots=c_n=0$. A basis for $\mathcal{V}$ is a linearly independent subset $\mathcal{B}$ of $\mathcal{V}$ such that every vector x in $\mathcal{V}$ can be expressed as an element of a linear combination of vectors $x_1,\ldots,x_n$ in $\mathcal{B}$ with coefficients $c_1,\ldots,c_n$ in $\mathcal{K}$, i.e. $x\in c_1\circ x_1+\cdots+c_n\circ x_n$. If a hypervector space $\mathcal{V}$ has a finite basis, it is referred to as finite dimensional. If $\mathcal{W}$ is a subhyperspace of $\mathcal{V}$, then $\dim\frac{\mathcal{V}}{\mathcal{W}}=\dim\mathcal{V}-\dim\mathcal{W}$.

Theorem 3.11. *Let $\mathcal{V}=(\mathcal{V},+,\circ,\mathcal{K})$ be a hypervector space with strongly left distributive property, and $(\mathcal{F},\mathcal{A})$ be a bipolar fuzzy soft hypervector space of $\mathcal{V}$. Consider*

$$\mathcal{W}=\{x\in\mathcal{V};\ \mathcal{F}_e^+(x)\geq\mathcal{F}_e^+(0),\ \mathcal{F}_e^-(x)\leq\mathcal{F}_e^-(0),\ \forall e\in\mathcal{A}\}.$$

Then

$$\dim\frac{\mathcal{F}}{\mathcal{V}}=\dim\frac{\mathcal{V}}{\mathcal{W}}.$$

Proof. If for all $e \in \mathcal{A}$, $\mathcal{F}_e^+$ and $\mathcal{F}_e^-$ are constant, i.e. $\mathcal{F}_e^+(x) = \mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(x) = \mathcal{F}_e^-(0)$, for every vector x in $\mathcal{V}$, then by Lemma 3.10, $\frac{\mathcal{F}}{\mathcal{V}} = \{0\mathcal{F}\}$, $\mathcal{W} = \mathcal{V}$ and in this case, $\dim \frac{\mathcal{F}}{\mathcal{V}} = 0 = \dim \frac{\mathcal{V}}{\mathcal{W}}$.

So assume that $\mathcal{F}_e^+$ and $\mathcal{F}_e^-$ are not constant. Let $\dim \mathcal{V} = n$, $\dim \mathcal{W} = m$. Suppose $\{w_1, \dots, w_m, v_1, \dots, v_r\}$, $r = n - m$, is a basis for $\mathcal{V}$ and $\{w_1, \dots, w_m\}$ is a basis for $\mathcal{W}$. Then $\{v_1 + \mathcal{W}, \dots, v_r + \mathcal{W}\}$ is a basis for $\frac{\mathcal{V}}{\mathcal{W}}$. It will be shown that $\mathcal{B} = \{v_1\mathcal{F}, \dots, v_r\mathcal{F}\}$ is a basis for $\frac{\mathcal{F}}{\mathcal{V}}$.

Note that elements of $\mathcal{B}$ are pairwise distinct, for if $v_i\mathcal{F} = v_j\mathcal{F}$, $i \neq j$, then $(v_i - v_j)\mathcal{F} = 0\mathcal{F}$ and so by Lemma 3.10, $\mathcal{F}_e^+(v_i - v_j) = \mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(v_i - v_j) = \mathcal{F}_e^-(0)$, $\forall e \in \mathcal{A}$. Thus $v_i - v_j \in \mathcal{W}$ and $v_i + \mathcal{W} = v_j + \mathcal{W}$, which is a contradiction.

Now we show that $\mathcal{B}$ generates $\frac{\mathcal{F}}{\mathcal{V}}$. Let $v\mathcal{F} \in \frac{\mathcal{F}}{\mathcal{V}}$, $v\mathcal{F} \neq 0\mathcal{F}$. Then by Lemma 3.10, $\mathcal{F}_e^+(v) \neq \mathcal{F}_e^+(0)$ or $\mathcal{F}_e^-(v) \neq \mathcal{F}_e^-(0)$, for some $e \in \mathcal{A}$. Thus $v + \mathcal{W}$ is a nonzero element of $\frac{\mathcal{V}}{\mathcal{W}}$. Hence there exist coefficients $a_1, \dots, a_r$ in the field $\mathcal{K}$ such that

$$\begin{aligned} v + \mathcal{W} &\in a_1 * (v_1 + \mathcal{W}) + \cdots + a_r * (v_r + \mathcal{W}) \\ &= (a_1 \circ v_1 + \mathcal{W}) + \cdots + (a_r \circ v_r + \mathcal{W}) \\ &= (a_1 \circ v_1 + \cdots + a_r \circ v_r) + \mathcal{W}. \end{aligned}$$

So there exist vectors $t_1, \dots, t_r$ in $\mathcal{V}$ such that $t_i \in a_i \circ v_i$, for all $1 \leq i \leq r$ and $v + \mathcal{W} = t_1 + \cdots + t_r + \mathcal{W}$. Then $v - (t_1 + \cdots + t_r) \in \mathcal{W}$, i.e. $\mathcal{F}_e^+(v - t_1 - \cdots - t_r) = \mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(v - t_1 - \cdots - t_r) = \mathcal{F}_e^-(0)$, for all $e \in \mathcal{A}$. By Lemma 3.10, $(v - t_1 - \cdots - t_r)\mathcal{F} = 0F$ and by definition of operations on $\frac{\mathcal{F}}{\mathcal{V}}$, it follows that:

$$\begin{aligned} v\mathcal{F} &= (t_1 + \cdots + t_r)\mathcal{F} \\ &= t_1\mathcal{F} \oplus \cdots \oplus t_r\mathcal{F} \\ &\in (a_1 \circledcirc v_1\mathcal{F}) \oplus \cdots \oplus (a_r \circledcirc v_r\mathcal{F}). \end{aligned}$$

Finally, we prove that $\mathcal{B}$ is linearly independent. Let $0\mathcal{F} \in (a_1 \circledcirc v_1\mathcal{F}) \oplus \cdots \oplus (a_r \circledcirc v_r\mathcal{F})$, for some $a_1, \dots, a_r \in \mathcal{K}$. Then

$$0\mathcal{F} = t_1\mathcal{F} \oplus \cdots \oplus t_r\mathcal{F} = (t_1 + \cdots + t_r)\mathcal{F}, \text{ for some } t_i\mathcal{F} \in a_i \circledcirc v_i\mathcal{F},\ 1 \leq i \leq r.$$

Thus by Lemma 3.10, $\mathcal{F}_e^+(t_1 + \cdots + t_r) = \mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(t_1 + \cdots + t_r) = \mathcal{F}_e^-(0)$, for all $e \in \mathcal{A}$. Hence $t_1 + \cdots + t_r \in \mathcal{W}$ and so there exist coefficients $b_1, \dots, b_m$ in the field $\mathcal{K}$ such that

$$\begin{aligned} & t_1 + \cdots + t_r \in b_1 \circ w_1 + \cdots + b_m \circ w_m, \\ \Longrightarrow\ & \exists l_i \in b_i \circ w_i,\ 1 \leq i \leq m,\ t_1 + \cdots + t_r = l_1 + \cdots + l_m, \\ \Longrightarrow\ & 0 = t_1 + \cdots + t_r - l_1 - \cdots - l_m \in a_1 \circ v_1 + \cdots + a_r \circ v_r - b_1 \circ w_1 - \cdots - b_m \circ w_m. \end{aligned}$$

Hence $a_1 = \cdots = a_r = b_1 = \cdots = b_m = 0$.
Therefore, $\dim \frac{\mathcal{F}}{\mathcal{V}} = r = n - m = \dim \mathcal{V} - \dim \mathcal{W} = \dim \frac{\mathcal{V}}{\mathcal{W}}$. □

Example 3.12. *In Example 3.4, we saw that*

$$\mathcal{W} = \{x \in \mathcal{V}; \mathcal{F}_e^+(x) \geq \mathcal{F}F_e^+(0), \mathcal{F}_e^-(x) \leq \mathcal{F}_e^-(0), \forall e \in \mathcal{A}\} = \{0, 2\},$$

where $(\mathcal{F}, \mathcal{A})$ *is the bipolar fuzzy soft hypervector space of* $\mathcal{V} = (\mathbb{Z}_4, +, \circ, \mathbb{Z}_2)$ *defined in Example 2.10. Here* $\beta = \{1 + \{0, 2\}\}$ *is a basis for* $\frac{\mathcal{V}}{\mathcal{W}} = \frac{\mathbb{Z}_4}{\{0,2\}} = \{\mathcal{W}, 1 + \mathcal{W}\} = \{\{0, 2\}, \{1, 3\}\}$ *and so* $\dim \frac{\mathbb{Z}_4}{\{0,2\}} = 1$*. Also,* $\acute{\beta} = \{1\mathcal{F}\}$ *is a basis for* $\frac{\mathcal{F}}{\mathbb{Z}_4} = \{0\mathcal{F}, 1\mathcal{F}, 2\mathcal{F}, 3\mathcal{F}\}$*, and so* $\dim \frac{\mathcal{F}}{\mathbb{Z}_4} = 1$*. Thus* $\dim \frac{\mathcal{F}}{\mathbb{Z}_4} = \dim \frac{\mathbb{Z}_4}{\{0,2\}}$.

Note that $\mathcal{W} = \{0\} \times \{0\} \times \mathbb{R}$ in Example 3.3, is dimensionless, i.e. it does not have any linearly independent subset and so we can not check the satisfaction of Theorem 3.11, for the bipolar fuzzy soft hypervector space $(\mathcal{F}, \mathcal{A})$ of $\mathcal{V} = (\mathbb{R}^3, +, \circ, \mathbb{R})$, defined in Example 2.9.
A linear transformation was defined in [8] as a function $T : \mathcal{V} \to \mathcal{W}$ between hypervector spaces $\mathcal{V}$ and $\mathcal{W}$, such that $T(x+y) = T(x)+T(y)$ and $T(a \circ x) \subseteq a \circ T(x)$, for all vectors x, y in $\mathcal{V}$ and all scalars a in the field $\mathcal{K}$. If $T(a \circ x) = a \circ T(x)$, then T is said to be a good transformation. The kernel of a linear transformation $T : \mathcal{V} \to \mathcal{W}$ is the following set:

$$\ker T = \{x \in \mathcal{V}, T(x) \in 0 \circ 0_{\mathcal{W}}\}.$$

Theorem 3.13. *Let $(\mathcal{F}, \mathcal{A})$ represent a bipolar fuzzy soft hypervector space of $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$ and*

$$\mathcal{W} = \{x \in \mathcal{V};\ \mathcal{F}_e^+(x) = \mathcal{F}_e^+(0),\ \mathcal{F}_e^-(x) = \mathcal{F}_e^-(0),\ \forall e \in \mathcal{A}\}.$$

Then

1. *The bipolar fuzzy soft set $(\acute{\mathcal{F}}, \mathcal{A})$ of $\frac{\mathcal{F}}{\mathcal{V}} = (\frac{\mathcal{F}}{\mathcal{V}}, \oplus, \circledcirc, \mathcal{K})$ defined by*

$$\acute{\mathcal{F}}_e^+(x\mathcal{F}) = \mathcal{F}_e^+(x),\ \acute{\mathcal{F}}_e^-(x\mathcal{F}) = \mathcal{F}_e^-(x),\ \forall e \in \mathcal{A}, x \in \mathcal{V},$$

is a bipolar fuzzy soft hypervector space of $\frac{\mathcal{F}}{\mathcal{V}}$.

2. *The mapping $\phi : \mathcal{V} \to \frac{\mathcal{F}}{\mathcal{V}}$ defined by $\phi(x) = x\mathcal{F}$, is an onto good transformation with the kernel equal to $\mathcal{W}$.*

3. *Every bipolar fuzzy soft hypervector space of $\frac{\mathcal{F}}{\mathcal{V}}$ corresponds to a bipolar fuzzy soft hypervector space of $\mathcal{V}$.*

Proof. 1) For every parameter e in $\mathcal{A}$, every vectors x, y in $\mathcal{V}$ and every scalar a in the field K, it follows that:

$$\begin{aligned} \acute{\mathcal{F}}_e^+(x\mathcal{F} \ominus y\mathcal{F}) &= \acute{\mathcal{F}}_e^+((x-y)\mathcal{F}) \\ &= \mathcal{F}_e^+(x-y) \\ &\geq \mathcal{F}_e^+(x) \wedge \mathcal{F}_e^+(y) \\ &= \acute{\mathcal{F}}_e^+(x\mathcal{F}) \wedge \acute{\mathcal{F}}_e^+(y\mathcal{F}), \end{aligned}$$

and similarly, $\acute{\mathcal{F}}_e^-(x\mathcal{F} \ominus y\mathcal{F}) \leq \acute{\mathcal{F}}_e^-(x\mathcal{F}) \vee \acute{\mathcal{F}}_e^-(y\mathcal{F})$.
Also, for all $t\mathcal{F} \in a \circledcirc (x\mathcal{F})$, by Definition 2.8,

$$\acute{\mathcal{F}}_e^+(t\mathcal{F}) = \mathcal{F}_e^+(t) \geq \inf_{s \in a \circ x} \mathcal{F}_e^+(s) \geq \mathcal{F}_e^+(x) = \acute{\mathcal{F}}_e^+(x\mathcal{F}),$$

and so

$$\inf_{t\mathcal{F} \in a \circledcirc (x\mathcal{F})} \acute{\mathcal{F}}_e^+(t\mathcal{F}) \geq \acute{\mathcal{F}}_e^+(x\mathcal{F}).$$

Moreover, $\acute{\mathcal{F}}_e^-(t\mathcal{F}) = \mathcal{F}_e^-(t) \leq \bigvee_{s \in a \circ x} \mathcal{F}_e^-(s) \leq \mathcal{F}_e^-(x) = \acute{\mathcal{F}}_e^-(x\mathcal{F})$, and so $\sup_{t\mathcal{F} \in a \circledcirc (x\mathcal{F})} \acute{\mathcal{F}}_e^-(t\mathcal{F}) \leq \acute{\mathcal{F}}_e^-(x\mathcal{F})$. Therefore, $(\acute{\mathcal{F}}, \mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\frac{\mathcal{F}}{\mathcal{V}}$.
2) For all $x, y \in \mathcal{V}$, $a \in \mathcal{K}$, $\phi(x+y) = (x+y)\mathcal{F} = x\mathcal{F} \oplus y\mathcal{F} = \phi(x) \oplus \phi(y)$ and $\phi(a \circ x) = (a \circ x)\mathcal{F} = a \circledcirc (x\mathcal{F})$. Thus ϕ is a good transformation. Clearly, ϕ is onto. Also,

$$\begin{aligned} \ker \phi &= \{x \in \mathcal{V},\ \phi(x) \in 0 \circledcirc (0\mathcal{F})\} \\ &= \{x \in \mathcal{V},\ x\mathcal{F} \in \{t\mathcal{F},\ t \in 0 \circ 0\}\} \\ &= \{x \in \mathcal{V},\ x\mathcal{F} = t\mathcal{F},\ \text{for some } t \in 0 \circ 0\} \\ &= \{x \in \mathcal{V},\ \mathcal{F}_e^+(x) = \mathcal{F}_e^+(t), \mathcal{F}_e^-(x) = \mathcal{F}_e^-(t),\ \text{for some } t \in 0 \circ 0, \forall e \in \mathcal{A}\}. \end{aligned}$$

But for every $t \in 0 \circ 0$, by Definition 2.8,

$$\mathcal{F}_e^+(t) \geq \inf_{s \in 0 \circ 0} \mathcal{F}_e^+(s) \geq \mathcal{F}_e^+(0),$$

$$\mathcal{F}_e^-(t) \leq \sup_{s \in 0 \circ 0} \mathcal{F}_e^-(s) \leq \mathcal{F}_e^-(0),$$

so $\mathcal{F}_e^+(t) = \mathcal{F}_e^+(0)$ and $\mathcal{F}_e^-(t) = \mathcal{F}_e^-(0)$. Hence,

$$\ker \phi = \{x \in \mathcal{V},\ \mathcal{F}_e^+(x) = \mathcal{F}_e^+(0), \mathcal{F}_e^-(x) = \mathcal{F}_e^-(0), \forall e \in \mathcal{A}\} = \mathcal{W}.$$

3) If $(\mathcal{G}, \mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\frac{\mathcal{F}}{\mathcal{V}}$, then the bipolar fuzzy soft set $\mathcal{H} : \mathcal{A} \to BF^{\mathcal{V}}$ defined by $\mathcal{H}(e) = \mathcal{H}_e = (\mathcal{H}_e^+, \mathcal{H}_e^-)$, for all $e \in \mathcal{A}$, is indeed a bipolar fuzzy soft hypervector space of $\mathcal{V}$. In this case, $\mathcal{H}_e^+(x) = \mathcal{G}_e^+(x\mathcal{F})$ and $\mathcal{H}_e^-(x) = \mathcal{G}_e^-(x\mathcal{F})$, for all $x \in \mathcal{V}$. □

4 Some bipolar fuzzy soft sets of quotient hypervector spaces

In this part, we establish three bipolar fuzzy soft sets for quotient hypervector spaces while considering specific limitations. In fact, in this way, some new bipolar fuzzy soft hypervector spaces are given.

Theorem 4.1. *If $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$ is a strongly left distributive hypervector space and $\mathcal{W}$ is a subhyperspace of $\mathcal{V}$, where $0 \circ x \subseteq \mathcal{W}$, for every vector x in $\mathcal{V}$, then a bipolar fuzzy soft hypervector space $(\mathcal{G}, \mathcal{A})$ of $\frac{\mathcal{V}}{\mathcal{W}}$ can be constructed from the bipolar fuzzy soft hypervector space $(\mathcal{F}, \mathcal{A})$ of $\mathcal{V}$. In this construction, the bipolar fuzzy soft set $(\mathcal{G}, \mathcal{A})$ of $\frac{\mathcal{V}}{\mathcal{W}}$ is defined by*

$$\mathcal{G}_e^+(x + \mathcal{W}) = \sup_{w \in \mathcal{W}} \mathcal{F}_e^+(x + w),$$

$$\mathcal{G}_e^-(x + \mathcal{W}) = \inf_{w \in \mathcal{W}} \mathcal{F}_e^-(x + w),$$

where e is a parameter in $\mathcal{A}$ and x is a vector in $\mathcal{V}$.

Proof. We can demonstrate that $(\mathcal{G}, \mathcal{A})$ fulfills the requirements stated in Definition 2.8. Assume that e belongs to $\mathcal{A}$, and x and y are elements of $\mathcal{V}$, while a is an element of K. Then by Lemma 2.5, it follows that:

1)

$$\begin{aligned}
\mathcal{G}_e^+(x + \mathcal{W} + y + \mathcal{W}) &= \mathcal{G}_e^+(x + y + \mathcal{W}) \\
&= \sup_{w \in \mathcal{W}} \mathcal{F}_e^+(x + y + w) \\
&= \sup_{\substack{w_1, w_2 \in \mathcal{W} \\ \acute{x} \in x + \mathcal{W}, \acute{y} \in y + \mathcal{W}}} \mathcal{F}_e^+(\acute{x} + w_1 + \acute{y} + w_2) \\
&\geq \sup_{\substack{w_1, w_2 \in \mathcal{W} \\ \acute{x} \in x + \mathcal{W}, \acute{y} \in y + \mathcal{W}}} \left(\mathcal{F}_e^+(\acute{x} + w_1) \wedge \mathcal{F}_e^+(\acute{y} + w_2)\right) \\
&\geq \left(\sup_{\substack{w_1 \in \mathcal{W} \\ \acute{x} \in x + \mathcal{W}}} \mathcal{F}_e^+(\acute{x} + w_1)\right) \wedge \left(\sup_{\substack{w_2 \in \mathcal{W} \\ \acute{y} \in y + \mathcal{W}}} \mathcal{F}_e^+(\acute{y} + w_2)\right) \\
&= \left(\sup_{w_1 \in \mathcal{W}} \mathcal{F}_e^+(x + w_1)\right) \wedge \left(\sup_{w_2 \in \mathcal{W}} \mathcal{F}_e^+(y + w_2)\right) \\
&= \mathcal{G}_e^+(x + \mathcal{W}) \wedge \mathcal{G}_e^+(y + \mathcal{W}),
\end{aligned}$$

and similarly, $\mathcal{G}_e^-(x+\mathcal{W}+y+\mathcal{W}) \leq \mathcal{G}_e^-(x+\mathcal{W}) \vee \mathcal{G}_e^-(y+\mathcal{W})$. Also,

$$\begin{array}{rcl} \mathcal{G}_e^+(-(x+\mathcal{W})) & = & \mathcal{G}_e^+(-x+\mathcal{W}) \\ & = & \sup\limits_{w\in\mathcal{W}} \mathcal{F}_e^+(-x+w) \\ & = & \sup\limits_{w\in\mathcal{W}} \mathcal{F}_e^+(-(x-w)) \\ & \geq & \sup\limits_{w\in\mathcal{W}} \mathcal{F}_e^+(x-w) \\ & = & \sup\limits_{\acute{w}\in\mathcal{W}} \mathcal{F}_e^+(x+\acute{w}) \\ & = & \mathcal{G}_e^+(x+\mathcal{W}), \end{array}$$

and likewise, $\mathcal{G}_e^-(-(x+\mathcal{W})) \leq \mathcal{G}_e^-(x+\mathcal{W})$.

2) If $a \neq 0$ and $t \in a \circ x$, then it follows that $t + w_1 \in a \circ x + 1 \circ w_1 = a \circ x + a \circ \left(a^{-1} \circ w_1\right)$, and so $t + w_1 \in a \circ x + a \circ w_2$, for some $w_2 \in a^{-1} \circ w_1 \subseteq \mathcal{W}$. Thus $\mathcal{F}_e^+(t+w_1) \geq \inf\limits_{s\in a\circ x+a\circ w_2} \mathcal{F}_e^+(s)$ and $\mathcal{F}_e^-(t+w_1) \leq \inf\limits_{s\in a\circ x+a\circ w_2} \mathcal{F}_e^-(s)$. Hence

$$\begin{array}{rcl} \mathcal{G}_e^+(t+\mathcal{W}) & = & \sup\limits_{w_1\in\mathcal{W}} \mathcal{F}_e^+(t+w_1) \\ & \geq & \sup\limits_{w_2\in\mathcal{W}} \left(\inf\limits_{s\in a\circ x+a\circ w_2} \mathcal{F}_e^+(s) \right) \\ & = & \sup\limits_{w_2\in\mathcal{W}} \left(\inf\limits_{s\in a\circ(x+w_2)} \mathcal{F}_e^+(s) \right) \\ & \geq & \sup\limits_{w_2\in\mathcal{W}} \mathcal{F}_e^+(x+w_2) \\ & = & \mathcal{G}_e^+(x+\mathcal{W}), \end{array}$$

and likewise, $\mathcal{G}_e^-(t+\mathcal{W}) \leq \mathcal{G}_e^-(x+\mathcal{W})$. Therefore,

$$\begin{array}{rcl} \inf\limits_{t+\mathcal{W}\in a*(x+\mathcal{W})} \mathcal{G}_e^+(t+\mathcal{W}) & = & \inf\limits_{t+\mathcal{W}\in a\circ x+\mathcal{W}} \mathcal{G}_e^+(t+\mathcal{W}) \\ & = & \inf\limits_{t\in a\circ x} \mathcal{G}_e^+(t+\mathcal{W}) \\ & \geq & \mathcal{G}_e^+(x+\mathcal{W}), \end{array}$$

and

$$\begin{array}{rcl} \sup\limits_{t+\mathcal{W}\in a*(x+\mathcal{W})} \mathcal{G}_e^-(t+\mathcal{W}) & = & \sup\limits_{t+\mathcal{W}\in a\circ x+\mathcal{W}} \mathcal{G}_e^-(t+\mathcal{W}) \\ & = & \sup\limits_{t\in a\circ x} \mathcal{G}_e^-(t+\mathcal{W}) \\ & \leq & \mathcal{G}_e^-(x+\mathcal{W}). \end{array}$$

Moreover, if $a = 0$, then according to 3.1, it can be inferred that for every $t \in 0 \circ x$, the followings hold:

$$\begin{array}{rcl} \mathcal{G}_e^+(t+\mathcal{W}) & = & \sup\limits_{w\in\mathcal{W}} \mathcal{F}_e^+(t+w) \\ & = & \mathcal{F}_e^+(0) \qquad (t \in 0 \circ x \subseteq \mathcal{W}) \\ & \geq & \mathcal{G}_e^+(x+W), \qquad (Im\ \mathcal{F}_e^+ = Im\ \mathcal{G}_e^+) \end{array}$$

and

$$\begin{aligned}\mathcal{G}_e^-(t+\mathcal{W}) &= \inf_{w\in\mathcal{W}} \mathcal{F}_e^-(t+w)\\ &= \mathcal{F}_e^-(0) \qquad (t\in 0\circ x\subseteq \mathcal{W})\\ &\leq \mathcal{G}_e^-(x+\mathcal{W}). \quad (Im\ \mathcal{F}_e^- = Im\ \mathcal{G}_e^-)\end{aligned}$$

Hence, $\inf_{t+\mathcal{W}\in 0*(x+\mathcal{W})} \mathcal{G}_e^+(t+\mathcal{W}) \geq \mathcal{G}_e^+(x+\mathcal{W})$ and $\sup_{t+\mathcal{W}\in 0*(x+\mathcal{W})} \mathcal{G}_e^-(t+\mathcal{W}) \leq \mathcal{G}_e^-(x+\mathcal{W})$. Consequently, $(\mathcal{G},\mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\frac{\mathcal{V}}{\mathcal{W}}$. □

Example 4.2. *Consider the bipolar fuzzy soft hypervector space $(\mathcal{F},\mathcal{A})$ of $\mathcal{V} = (\mathbb{R}^3,+,\circ,\mathbb{R})$ in Example 2.9, and $\mathcal{W} = \mathbb{R}\times\{0\}\times\mathbb{R}$. Then $0\circ(x,y,z) \subseteq \mathcal{W}$, for every vector (x,y,z) in $\mathbb{R}^3$, and the bipolar fuzzy soft hypervector space $(\mathcal{G},\mathcal{A})$ of $\frac{\mathbb{R}^3}{\mathbb{R}\times\{0\}\times\mathbb{R}}$ in Theorem 4.1, is defined by the followings:*

$$\mathcal{G}_a^+(x,y,z) = \begin{cases} 0.7 & y=0,\\ 0 & y\neq 0,\end{cases} \qquad \mathcal{G}_a^-(x,y,z) = \begin{cases} -0.8 & y=0,\\ -0.2 & y\neq 0,\end{cases}$$

$$\mathcal{G}_b^+(x,y,z) = \begin{cases} 0.9 & y=0,\\ 0.1 & y\neq 0,\end{cases} \qquad \mathcal{G}_b^-(x,y,z) = \begin{cases} -0.6 & y=0,\\ -0.1 & y\neq 0.\end{cases}$$

Example 4.3. *Consider the bipolar fuzzy soft hypervector space $(\mathcal{F},\mathcal{A})$ of $\mathcal{V} = (\mathbb{Z}_4,+,\circ,\mathbb{Z}_2)$ in Example 2.10, and $\mathcal{W} = \{0,2\}$. Then $0\circ x\subseteq\mathcal{W}$, for all $x\in\mathbb{Z}_4$, and the bipolar fuzzy soft hypervector space $(\mathcal{G},\mathcal{A})$ of $\frac{\mathbb{Z}_4}{\{0,2\}}$ in Theorem 4.1, is defined by $\mathcal{G}_a^+(x+\mathcal{W}) = \mathcal{F}_a^+(x)$, $\mathcal{G}_a^-(x+\mathcal{W}) = \mathcal{F}_a^-(x)$, for all $a\in A=\{c,d,e\}$ and $x\in\mathbb{Z}_4$.*

Theorem 4.4. *In a strongly left distributive hypervector space $\mathcal{V} = (\mathcal{V},+,\circ,\mathcal{K})$ and given a subhyperspace $\mathcal{U}$ of $\mathcal{V}$ where $0\circ x\subseteq\mathcal{U}$ and $|1\circ \acute{x}| = 1$, for all vectors x in $\mathcal{V}$ and $\acute{x}\in\mathcal{U}$, if $(\mathcal{F},\mathcal{A})$ represents a bipolar fuzzy soft hypervector space of $\mathcal{V}$, then the bipolar fuzzy soft set $(\mathcal{H},\mathcal{A})$ of $\frac{\mathcal{V}}{\mathcal{U}}$ can be defined as follows, which also forms a bipolar fuzzy soft hypervector space of $\frac{\mathcal{V}}{\mathcal{U}}$:*

$$\mathcal{H}_e^+(x+\mathcal{U}) = \begin{cases} \inf_{u\in\mathcal{U}} \mathcal{F}_e^+(x+u) & x\notin\mathcal{U},\\ \mathcal{F}_e^+(0) & x\in\mathcal{U},\end{cases}$$

$$\mathcal{H}_e^-(x+\mathcal{U}) = \begin{cases} \sup_{u\in\mathcal{U}} \mathcal{F}_e^-(x+u) & x\notin\mathcal{U},\\ \mathcal{F}_e^-(0) & x\in\mathcal{U},\end{cases}$$

for all $e\in\mathcal{A}$, $x\in\mathcal{V}$.

Proof. We will demonstrate that the pair $(\mathcal{H},\mathcal{A})$ meets the requirements outlined in Definition 2.8. Given that e is an element of $\mathcal{A}$, and x and y belong to $\mathcal{V}$, while a belongs to $\mathcal{K}$, then
1) If x and y are vectors in $\mathcal{U}$, then

$$\begin{aligned}\mathcal{H}_e^+((x+\mathcal{U})+(y+\mathcal{U})) &= \mathcal{H}_e^+(\mathcal{U})\\ &= \mathcal{F}_e^+(0)\\ &= \mathcal{H}_e^+(x+\mathcal{U})\wedge\mathcal{H}_e^+(y+\mathcal{U}),\end{aligned}$$

and similarly, $\mathcal{H}_e^-((x+\mathcal{U})+(y+\mathcal{U})) = \mathcal{H}_e^-(x+\mathcal{U})\vee\mathcal{H}_e^-(y+\mathcal{U})$.

If $x \in \mathcal{U}$ and $y \notin \mathcal{U}$, then by Lemma 3.1,

$$\begin{aligned}\mathcal{H}_e^+\left((x+\mathcal{U})+(y+\mathcal{U})\right) &= \mathcal{H}_e^+(y+\mathcal{U})\\ &= \mathcal{F}_e^+(0)\wedge\mathcal{H}_e^+(y+\mathcal{U})\\ &= \mathcal{H}_e^+(x+\mathcal{U})\wedge\mathcal{H}_e^+(y+\mathcal{U}),\end{aligned}$$

and similarly, $\mathcal{H}_e^-\left((x+\mathcal{U})+(y+\mathcal{U})\right)=\mathcal{H}_e^-(x+\mathcal{U})\vee\mathcal{H}_e^-(y+\mathcal{U})$.
If $x \notin \mathcal{U}$ and $y \in \mathcal{U}$, the result is similarly obtained.
If $x, y \notin \mathcal{U}$, then

$$\begin{aligned}\mathcal{H}_e^+(x+y+\mathcal{U}) &= \inf_{u\in\mathcal{U}}\mathcal{F}_e^+(x+y+u)\\ &= \inf_{\acute{x}\in x+\mathcal{U},\ \acute{y}\in y+\mathcal{U}}\mathcal{F}_e^+(\acute{x}+\acute{y})\\ &\geq \inf_{\acute{x}\in x+\mathcal{U},\ \acute{y}\in y+\mathcal{U}}\left(\mathcal{F}_e^+(\acute{x})\wedge\mathcal{F}_e^+(\acute{y})\right)\\ &= \left(\inf_{\acute{x}\in x+\mathcal{U}}\mathcal{F}_e^+(\acute{x})\right)\wedge\left(\inf_{\acute{y}\in y+\mathcal{U}}\mathcal{F}_e^+(\acute{y})\right)\\ &= \left(\inf_{u_1\in\mathcal{U}}\mathcal{F}_e^+(x+u_1)\right)\wedge\left(\inf_{u_2\in\mathcal{U}}\mathcal{F}_e^+(y+u_2)\right)\\ &= \mathcal{H}_e^+(x+\mathcal{U})\wedge\mathcal{H}_e^+(y+\mathcal{U}),\end{aligned}$$

and similarly, $\mathcal{H}_e^-(x+y+\mathcal{U})\leq\mathcal{H}_e^-(x+\mathcal{U})\vee\mathcal{H}_e^-(y+\mathcal{U})$. Thus

$$\mathcal{H}_e^+\left((x+\mathcal{U})+(y+\mathcal{U})\right)\geq\mathcal{H}_e^+(x+\mathcal{U})\wedge\mathcal{H}_e^+(y+\mathcal{U}),$$

and

$$\mathcal{H}_e^-\left((x+\mathcal{U})+(y+\mathcal{U})\right)\leq\mathcal{H}_e^-(x+\mathcal{U})\vee\mathcal{H}_e^-(y+\mathcal{U}),$$

for all $x+\mathcal{U}, y+\mathcal{U}\in\frac{\mathcal{V}}{\mathcal{U}}$.
Also, if $x\in\mathcal{U}$, then $-x\in\mathcal{U}$ and so $\mathcal{H}_e^+\left(-(x+\mathcal{U})\right)=\mathcal{F}_e^+(0)=\mathcal{H}_e^+(x+\mathcal{U})$ and $\mathcal{H}_e^-\left(-(x+\mathcal{U})\right)=\mathcal{F}_e^-(0)=\mathcal{H}_e^-(x+\mathcal{U})$. If $x\notin\mathcal{U}$, then

$$\begin{aligned}\mathcal{H}_e^+\left(-(x+\mathcal{U})\right) &= \mathcal{H}_e^+(-x+\mathcal{U})\\ &= \inf_{u\in\mathcal{U}}\mathcal{F}_e^+(-x+u)\\ &= \inf_{u\in\mathcal{U}}\mathcal{F}_e^+\left(-(x-u)\right)\\ &\geq \inf_{u\in\mathcal{U}}\mathcal{F}_e^+(x-u)\\ &= \inf_{\acute{u}\in\mathcal{U}}\mathcal{F}_e^+(x+\acute{u})\\ &= \mathcal{H}_e^+(x+\mathcal{U}),\end{aligned}$$

and similarly, $\mathcal{H}_e^-\left(-(x+\mathcal{U})\right)=\mathcal{H}_e^-(x+\mathcal{U})$.

2) If $a \neq 0$, then by Lemma 2.5,

$$\begin{aligned}
\inf_{t+\mathcal{U}\in a*(x+\mathcal{U})} \mathcal{H}_e^+(t+\mathcal{U}) &= \inf_{t+\mathcal{U}\in a\circ x+\mathcal{U}} \mathcal{H}_e^+(t+\mathcal{U}) \\
&= \inf_{t\in a\circ x} \mathcal{H}_e^+(t+\mathcal{U}) \\
&= \inf_{t\in a\circ x} \inf_{u\in\mathcal{U}} \mathcal{F}_e^+(t+u) \\
&= \inf_{\acute{t}\in a\circ x+\mathcal{U}} \mathcal{F}_e^+(\acute{t}) \\
&= \inf_{\acute{t}\in a\circ\acute{x},\acute{x}\in x+\mathcal{U}} \mathcal{F}_e^+(\acute{t}) \\
&= \inf_{\acute{x}\in x+\mathcal{U}} \inf_{\acute{t}\in a\circ\acute{x}} \mathcal{F}_e^+(\acute{t}) \\
&\geq \inf_{\acute{x}\in x+\mathcal{U}} \mathcal{F}_e^+(\acute{x}) \\
&= \inf_{\acute{u}\in\mathcal{U}} \mathcal{F}_e^+(x+\acute{u}) \\
&= \mathcal{H}_e^+(x+\mathcal{U}),
\end{aligned}$$

and similarly, $\sup\limits_{t+\mathcal{U}\in a*(x+\mathcal{U})} \mathcal{H}_e^-(t+\mathcal{U}) \leq \mathcal{H}_e^-(x+\mathcal{U})$.
If $a = 0$ and $t \in 0 \circ x$, then by Lemma 3.1, for all $e \in \mathcal{A}$, $\mathcal{H}_e^+(t+\mathcal{U}) = \mathcal{F}_e^+(0) \geq \mathcal{H}_e^+(x+\mathcal{U})$ and $\mathcal{H}_e^-(t+\mathcal{U}) = \mathcal{F}_e^-(0) \leq \mathcal{H}_e^-(x+\mathcal{U})$, since $t \in 0\circ x \subseteq \mathcal{U}$, $Im\ \mathcal{F}_e^+ = Im\ \mathcal{H}_e^+$ and $Im\ \mathcal{F}_e^- = Im\ \mathcal{H}_e^-$. Thus

$$\inf_{t+\mathcal{U}\in 0*(x+\mathcal{U})} \mathcal{H}_e^+(t+\mathcal{U}) = \inf_{t\in 0\circ x} \mathcal{H}_e^+(t+\mathcal{U}) \geq \mathcal{H}_e^+(x+\mathcal{U}),$$

and

$$\sup_{t+\mathcal{U}\in 0*(x+\mathcal{U})} \mathcal{H}_e^-(t+\mathcal{U}) = \sup_{t\in 0\circ x} \mathcal{H}_e^-(t+\mathcal{U}) \leq \mathcal{H}_e^-(x+\mathcal{U}).$$

Therefore, $(\mathcal{H}, \mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\frac{\mathcal{V}}{\mathcal{W}}$. □

Theorem 4.5. *Let $(\mathcal{F}, \mathcal{A})$ represent a bipolar fuzzy soft hypervector space of $\mathcal{V}$, where $\mathcal{U} = (\mathcal{F}_e)_{\alpha,\beta}$ is the (α, β)-level subset of $\mathcal{V}$, for some e belonging to $\mathcal{A}$, α belonging to $(0, 1]$ and β belonging to $[-1, 0)$. If $(\mathcal{G}, \mathcal{A})$ is the bipolar fuzzy soft hypervector space of $\mathcal{V}$ defined in Theorem 2.14, then the bipolar fuzzy soft set $(\tilde{\mathcal{G}}, \mathcal{A})$ of $\frac{\mathcal{V}}{\mathcal{U}}$ given by the following expressions is a bipolar fuzzy soft hypervector space of $\frac{\mathcal{V}}{\mathcal{U}}$:*

$$\tilde{\mathcal{G}}_e^+(x+\mathcal{U}) = \sup_{u\in\mathcal{U}} \mathcal{G}_e^+(x+u),$$

$$\tilde{\mathcal{G}}_e^-(x+\mathcal{U}) = \inf_{u\in\mathcal{U}} \mathcal{G}_e^-(x+u),$$

where e belongs to $\mathcal{A}$ and x belongs to $\mathcal{V}$.

Proof. We examine the criteria stated in Definition 2.8, for the case of $(\tilde{G}, A)$. Considering an element e belonging to $\mathcal{A}$, as well as elements x and y belonging to $\mathcal{V}$, and an element a belonging to K, similarly to the proof of Theorem 4.1, it follows that:

$$\begin{gathered}
\tilde{\mathcal{G}}_e^+((x+\mathcal{U}) + (y+\mathcal{U})) \geq \tilde{\mathcal{G}}_e^+(x+\mathcal{U}) \wedge \tilde{\mathcal{G}}_e^+(y+\mathcal{U}), \\
\tilde{\mathcal{G}}_e^-((x+\mathcal{U}) + (y+\mathcal{U})) \leq \tilde{\mathcal{G}}_e^-(x+\mathcal{U}) \vee \tilde{\mathcal{G}}_e^-(y+\mathcal{U}), \\
\tilde{\mathcal{G}}_e^+(-(x+\mathcal{U})) \geq \tilde{\mathcal{G}}_e^+(x+\mathcal{U}), \\
\tilde{\mathcal{G}}_e^-(-(x+\mathcal{U})) \leq \tilde{\mathcal{G}}_e^-(x+\mathcal{U}).
\end{gathered}$$

Moreover, for all $t+\mathcal{U} \in a*(x+\mathcal{U}) = a\circ x+\mathcal{U}$,

$$\begin{aligned}
\tilde{\mathcal{G}}_e^+(t+\mathcal{U}) &= \sup_{u\in\mathcal{U}} \mathcal{G}_e^+(t+u) \\
&\geq \sup_{u\in\mathcal{U}} \big(\mathcal{G}_e^+(t)\wedge \mathcal{G}_e^+(u)\big) \\
&= \mathcal{G}_e^+(t) \\
&\geq \inf_{s\in a\circ x} \mathcal{G}_e^+(s) \\
&\geq \mathcal{G}_e^+(x),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{G}}_e^-(t+\mathcal{U}) &= \inf_{u\in\mathcal{U}} \mathcal{G}_e^-(t+u) \\
&\leq \inf_{u\in\mathcal{U}} \big(\mathcal{G}_e^-(t)\vee \mathcal{G}_e^-(u)\big) \\
&= \mathcal{G}_e^-(t) \\
&\leq \sup_{s\in a\circ x} \mathcal{G}_e^-(s) \\
&\leq \mathcal{G}_e^-(x).
\end{aligned}$$

Then $\inf\limits_{t+\mathcal{U}\in a*(x+\mathcal{U})} \tilde{\mathcal{G}}_e^+(t+\mathcal{U}) \geq \mathcal{G}_e^+(x)$ and $\sup\limits_{t+\mathcal{U}\in a*(x+\mathcal{U})} \tilde{\mathcal{G}}_e^-(t+\mathcal{U}) \leq \mathcal{G}_e^-(x)$.
Likewise, if $u\in\mathcal{U}$, then given that $t+\mathcal{U}\in a*(x+u+\mathcal{U}) = a\circ(x+u)+\mathcal{U}$, it can be concluded that:

$$\begin{aligned}
\tilde{\mathcal{G}}_e^+(t+\mathcal{U}) &= \sup_{\acute{u}\in\mathcal{U}} \mathcal{G}_e^+(t+\acute{u}) \\
&\geq \sup_{\acute{u}\in\mathcal{U}} \big(\mathcal{G}_e^+(t)\wedge \mathcal{G}_e^+(\acute{u})\big) \\
&= \mathcal{G}_e^+(t) \\
&\geq \inf_{s\in a\circ(x+u)} \mathcal{U}_e^+(s) \\
&\geq \mathcal{G}_e^+(x+u),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{G}}_e^-(t+\mathcal{U}) &= \inf_{\acute{u}\in\mathcal{U}} \mathcal{G}_e^-(t+\acute{u}) \\
&\leq \inf_{\acute{u}\in\mathcal{U}} \big(\mathcal{G}_e^-(t)\vee \mathcal{G}_e^-(\acute{u})\big) \\
&= \mathcal{G}_e^-(t) \\
&\leq \sup_{s\in a\circ(x+u)} \mathcal{G}_e^-(s) \\
&\leq \mathcal{G}_e^-(x+u).
\end{aligned}$$

Hence,

$$\inf_{t+\mathcal{U}\in a*(x+u+\mathcal{U})} \tilde{\mathcal{G}}_e^+(t+\mathcal{U}) \geq \mathcal{U}_e^+(x+u),$$

$$\sup_{t+\mathcal{U}\in a*(x+u+\mathcal{U})} \tilde{\mathcal{G}}_e^-(t+\mathcal{U}) \leq \mathcal{G}_e^-(x+u),$$

for every vector u in $\mathcal{U}$.
But $x+\mathcal{U}=x+u+\mathcal{U}$ and $a*(x+\mathcal{U})=a*(x+u+\mathcal{U})$, for every u in $\mathcal{U}$, thus

$$\begin{aligned}\inf_{t+\mathcal{U}\in a*(x+\mathcal{U})}\tilde{\mathcal{G}}_e^+(t+\mathcal{U}) &= \inf_{t+\mathcal{U}\in a*(x+u+\mathcal{U})}\tilde{\mathcal{G}}_e^+(t+\mathcal{U})\\ &= \inf_{t+\mathcal{U}\in a\circ(x+u)+\mathcal{U}}\tilde{\mathcal{G}}_e^+(t+\mathcal{U})\\ &\geq \mathcal{G}_e^+(x+u),\end{aligned}$$

and

$$\begin{aligned}\sup_{t+\mathcal{U}\in a*(x+\mathcal{U})}\tilde{\mathcal{G}}_e^-(t+\mathcal{U}) &= \sup_{t+\mathcal{U}\in a*(x+u+\mathcal{U})}\tilde{\mathcal{G}}_e^-(t+\mathcal{U})\\ &= \sup_{t+\mathcal{U}\in a\circ(x+u)+\mathcal{U}}\tilde{\mathcal{G}}_e^-(t+\mathcal{U})\\ &\leq \mathcal{G}_e^-(x+u).\end{aligned}$$

Therefore,

$$\inf_{t+\mathcal{U}\in a*(x+\mathcal{U})}\tilde{\mathcal{G}}_e^+(t+\mathcal{U})\geq \sup_{u\in\mathcal{U}}\mathcal{G}_e^+(x+u)=\tilde{\mathcal{G}}_e^+(x+\mathcal{U}),$$

and

$$\sup_{t+\mathcal{U}\in a*(x+\mathcal{U})}\tilde{\mathcal{G}}_e^-(t+\mathcal{U})\leq \inf_{u\in\mathcal{U}}\mathcal{G}_e^-(x+u)=\tilde{\mathcal{G}}_e^-(x+\mathcal{U}).$$

Consequently, $(\tilde{\mathcal{G}},\mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\frac{\mathcal{V}}{\mathcal{W}}$. □

Example 4.6. *Consider the bipolar fuzzy soft hypervector space $(\mathcal{F},\mathcal{A})$ of $\mathcal{V}=(\mathbb{R}^3,+,\circ,\mathbb{R})$ in Example 2.9. Then for $e=a$, $\alpha=0.4$, $\beta=-0.3$ and $\mathcal{U}=(\mathcal{F}_e)_{\alpha,\beta}=(\mathcal{F}_a)_{0.4,-0.3}=\{0\}\times\{0\}\times\mathbb{R}$, the bipolar fuzzy soft hypervector space $(\tilde{\mathcal{G}},\mathcal{A})$ of $\frac{\mathbb{R}^3}{\{0\}\times\{0\}\times\mathbb{R}}$ in Theorem 4.5, is defined by the followings:*

$$\tilde{\mathcal{G}}_a^+(x+\mathcal{U})=\begin{cases}1 & x\in\mathcal{U},\\ 0.3 & x\in(\mathbb{R}\times\{0\}\times\mathbb{R})\setminus\mathcal{U},\\ 0 & \textit{otherwise},\end{cases}$$

$$\tilde{\mathcal{G}}_a^-(x+\mathcal{U})=\begin{cases}-1 & x\in\mathcal{U},\\ -0.4 & x\in(\mathbb{R}\times\{0\}\times\mathbb{R})\setminus\mathcal{U},\\ -0.2 & \textit{otherwise},\end{cases}$$

$$\tilde{\mathcal{G}}_b^+(x+\mathcal{U})=\begin{cases}1 & x\in\mathcal{U},\\ 0.4 & x\in(\mathbb{R}\times\{0\}\times\mathbb{R})\setminus\mathcal{U},\\ 0.1 & \textit{otherwise},\end{cases}$$

$$\tilde{\mathcal{G}}_b^-(x+\mathcal{U})=\begin{cases}-1 & x\in\mathcal{U},\\ -0.5 & x\in(\mathbb{R}\times\{0\}\times\mathbb{R})\setminus\mathcal{U},\\ -0.1 & \textit{otherwise}.\end{cases}$$

Example 4.7. *Consider the bipolar fuzzy soft hypervector space $(\mathcal{F},\mathcal{A})$ of $\mathcal{V}=(\mathbb{Z}_4,+,\circ,\mathbb{Z}_2)$ in Example 2.10. Then for $e=d$, $\alpha=0.5$, $\beta=-0.5$ and $\mathcal{U}=(\mathcal{F}_e)_{\alpha,\beta}=(\mathcal{F}_d)_{0.5,-0.5}=\{0,2\}$, the bipolar fuzzy soft hypervector space $(\tilde{\mathcal{G}},\mathcal{A})$ of $\frac{\mathbb{Z}_4}{\{0,2\}}$ in Theorem 4.5, is defined by the followings:*

$$\tilde{\mathcal{G}}_c^+(x+\mathcal{U})=\begin{cases}1 & x\in\{0,2\}\\ 0.3 & x\in\{1,3\}\end{cases}\qquad \tilde{\mathcal{G}}_c^-(x+\mathcal{U})=\begin{cases}-1 & x\in\{0,2\}\\ -0.2 & x\in\{1,3\}\end{cases}$$

$$\tilde{\mathcal{G}}_d^+(x+\mathcal{U}) = \begin{cases} 1 & x \in \{0,2\} \\ 0.2 & x \in \{1,3\} \end{cases} \qquad \tilde{\mathcal{G}}_d^-(x+\mathcal{U}) = \begin{cases} -1 & x \in \{0,2\} \\ -0.3 & x \in \{1,3\} \end{cases}$$

$$\tilde{\mathcal{G}}_e^+(x+\mathcal{U}) = \begin{cases} 1 & x \in \{0,2\} \\ 0.4 & x \in \{1,3\} \end{cases} \qquad \tilde{\mathcal{G}}_e^-(x+\mathcal{U}) = \begin{cases} -1 & x \in \{0,2\} \\ -0.5 & x \in \{1,3\} \end{cases}$$

5 Conclusions

The author, in references [18, 19], explored the properties of bipolar fuzzy soft hypervector spaces by incorporating the notion of bipolar fuzzy sets. This approach seems to integrate fuzzy logic and hypervector spaces to analyze and understand the characteristics of such spaces. It would be fascinating to delve deeper into their findings and understand the implications of these properties. In this paper, we followed these papers and investigated some new results in the mentioned algebraic structure. Next, we defined a hypervector space $\frac{F}{V} = (\frac{F}{V}, \oplus, \circledcirc, K)$ that consists of all cosets of a bipolar fuzzy soft hypervector space (F, A). Then we proved an interesting theorem concerning the fundamental concept of dimension in hypervector spaces. In fact, we showed that $\dim \frac{F}{V} = \dim \frac{V}{W}$, such that $\frac{V}{W}$ is the quotient hypervector space containing all cosets of W in V, and W is a particular subhyperspace of V. In addition, we have presented some bipolar fuzzy soft sets over the quotient hypervector space $\frac{V}{W}$. With these results, the following topics can be studied in the future:

- Investigation of isomorphism theorem in bipolar fuzzy soft hypervector spaces,
- Finding the applications of the introduced structure in decision making,
- Application of bipolar fuzzy soft sets over other algebraic structures/hyperstructures,
- Generalized bipolar fuzzy soft hypervector spaces based on bipolar fuzzy points,
- Dimension of bipolar fuzzy soft hypervector spaces.

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