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Integral and obstinate prefilters of hyper EQ-algebras

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Abstract

The main goal of this paper is to introduce integral hyper EQ-algebras, integral (pre)filters and obstinate (pre)filters of hyper EQ-algebras. In the following, some characterizations of these (pre)filters in hyper EQ-algebras are investigated, and it is proved that the quotient hyper EQ-algebras induced by a filter F is an integral hyper EQ-algebra if and only if F is an integral filter. Moreover, the concept of obstinate (pre)filter in hyper EQ-algebras is introduced, and some related properties are provided. Finally, the relationship among obstinate (pre)filters and some type of other (pre)filters such integral, maximal, (positive) implicative and fantastic (pre)filters in hyper EQ-algebras are studied.

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1 Introduction

EQ-algebras were proposed by Novák and De Baets [12]. The first motivation for studying this algebra comes from fuzzy type theory (FTT) [11] that generalizes the system of classical type theory [1] in which the sole basic connective is equality. Another motivation is from the equational style of proof in logic. EQ-algebras are exciting and essential for studying and researching. In fact, EQ-algebras generalize non-commutative residuated lattices, [6]. From the point of view of logic, the main difference between residuated lattices and EQ-algebras lies in the way the implication operation is obtained. While in residuated lattices, it is obtained from (strong) conjunction, in EQ-algebras it is obtained from equivalence. The notion of the integral prefilter and integral EQ-algebras was introduced by Paad [13] as a special case of EQ-algebra. Algebraic hyperstructures represent a natural extension of classical algebraic structures. They were introduced in the 8th Congress of Scandinavian Mathematicians in 1934 by the French mathematician Marty [10]. In an algebraic hyperstructures, the composition of two elements is not an element but a set. Till now, the hyper structures have been studied from the theoretical point of view, for their applications to many subjects of pure and applied mathematics. For example hyper structure theory has been

intensively researched in [2, 4, 8, 9]. Recently, Borzooei has applied hyper theory to EQ-algebras to introduce hyper EQ-algebras [3] which are a generalization of EQ-algebras. Hyper filters (or hyper deductive systems) are important tools in studying hyperstructures [2, 5].
This paper is organized as follows: in Section 2, the basic definitions, properties and theorems of hyper EQ-algebras are reviewed. In Sections 3 and 4, we introduce the notion of integral of hyper EQ-algebras, integral (pre)filters and obstinate (pre)filters of hyper EQ-algebras. We characterize several essential properties of them, and we provide the quotient hyper EQ-algebras via hyper integral, and obstinate filters and we give some related results. Also, we study the relationship among obstinate (pre)filters and some type of other (pre)filters such integral, maximal, (positive)implicative and fantastic (pre)filters in hyper EQ-algebras.

2 Preliminaries

In this section, we give some fundamental definitions, propositions and theorems which provide some properties of hyper EQ-algebras. For more details, refer to the reference [3].

Definition 2.1. [12] *An EQ-algebra is an algebra* $(L, \wedge, \odot, \sim, 1)$ *of type* $(2, 2, 2, 0)$ *satisfying the following axioms:*
$(E1)$ $(L, \wedge, 1)$ *is a* $\wedge$*-semilattice with top element 1. We set* $x \leq y$ *if and only if* $x \wedge y = x$*,*
$(E2)$ $(L, \odot, 1)$ *is a commutative monoid and* $\odot$ *is isotone with respect to* $\leq$*,*
$(E3)$ $x \sim x = 1$ *(reflexivity axiom),*
$(E4)$ $((x \wedge y) \sim z) \odot (t \sim x) \leq z \sim (t \wedge y)$ *(substitution axiom),*
$(E5)$ $(x \sim y) \odot (z \sim t) \leq (x \sim z) \sim (y \sim t)$ *(congruence axiom),*
$(E6)$ $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$ *(monotonicity axiom),*
$(E7)$ $x \odot y \leq x \sim y$ *(boundedness axiom),*
for all $t, x, y, z \in L$*.*

A function $\circ : A \times A \to P^*(A)$, of the set $A \times A$ into the set of all nonempty subsets of A, is called a binary hyperoperation, and the pair $(A; \circ)$ is called a hypergroupoid. If $\circ$ is associative, then A is called a semihypergroup, which is said to be commutative if $\circ$ is commutative. Also, an element $1 \in A$ is called an identity element if $x \in 1 \circ x$, for all $x \in A$. Here after, we denote $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$ or $\{x\} \circ \{y\}$.

Definition 2.2. [3] *A hyper EQ-algebra* $(L, \wedge, \odot, \sim, 1)$ *is a nonempty set* L *with a binary operations* $\wedge$ *and two binary hyper operations* $\odot$, $\sim$ *and top element* 1 *satisfying the following conditions, for all* $x, y, z, t \in L$*:*
$(HEQ1)$ $(L, \wedge, 1)$ *is a commutative idempotent monoid with top element* 1*,*
$(HEQ2)$ $(L, \odot, 1)$ *is a commutative semihypergroup with* 1 *as an identity and* $\odot$ *is isotone w.r.t.* $\leq$*, i.e. if* $x \leq y$*, then* $x \odot z \ll y \odot z$ *(where* $x \leq y$ *if and only if* $x \wedge y = x$*),*
$(HEQ3)$ $((x \wedge y) \sim z) \odot (t \sim x) \ll z \sim (t \wedge y)$*,*
$(HEQ4)$ $(x \sim y) \odot (z \sim t) \ll (x \sim z) \sim (y \sim t)$*,*
$(HEQ5)$ $(x \wedge y \wedge z) \sim x \ll (x \wedge y) \sim x$*,*
$(HEQ6)$ $(x \wedge y) \sim x \ll (x \wedge y \wedge z) \sim (x \wedge z)$
$(HEQ7)$ $x \odot y \ll x \sim y$*,*
where $A \ll B$*, means that, for all* $a \in A$ *there exists* $b \in B$ *such that* $a \leq b$*. Moreover, for any nonempty subsets* $A, B \subseteq H$*, we write* $A \circ B = \bigcup_{a \in A, b \in B} (a \circ b)$*,* $\circ \in \{\odot, \sim\}$ *and* $A \wedge B = \{a \wedge b \mid a \in A, b \in B\}$*.*

In any hyper EQ-algebra $(L, \wedge, \odot, \sim, 1)$ the auxiliary hyperoperation $\rightarrow$ is defined as $x \rightarrow y := (x \wedge y) \sim x$ and $A \rightarrow B = (A \wedge B) \sim A = \bigcup_{x \in A \wedge B, y \in A} (x \rightarrow y)$ and since $a \rightarrow b = (a \wedge b) \sim a \subseteq \bigcup_{x \in A \wedge B, y \in A} (x \rightarrow y)$, for any $a \in A, b \in B$, we conclude that $\bigcup_{a \in A, b \in B} (a \rightarrow b) \subseteq A \rightarrow B$. Moreover, if it has a bottom element 0 (with respect to the order $\leq$), the set $x \rightarrow 0 = x \sim 0$ is denoted by $\neg x$ and hyper EQ-algebra L is called good, if $\tilde{x} = x \sim 1 = x$, for all $x \in L$.

Proposition 2.3. [3] *Let $(L, \wedge, \odot, \sim, 1)$ be a hyper EQ-algebra. Then the following conditions hold, for all $x, y, z \in L$ and $A, B, C \subseteq L$*
(i) $1 \in x \sim x$, $1 \ll x \rightarrow x$ and $1 \in A \rightarrow A$.
(ii) If $x \leq y$, then $1 \in x \rightarrow y$.
(iii) $x \odot (x \sim y) \ll \tilde{y}$ and $y \ll x \rightarrow y$.
(iv) If $x \leq y$, then $z \rightarrow x \ll z \rightarrow y$ and $y \rightarrow z \ll x \rightarrow z$.
(v) $x \sim y \ll y \sim x$ and $x \sim y \ll x \rightarrow y$.

Definition 2.4. [3] *Let D be a nonempty subset of a hyper EQ-algebra L. Then*
(i) D is called $S_{\sim}$-reflexive, if $(x \sim y) \cap D \neq \emptyset$, implies $x \sim y \subseteq D$, for any $x, y \in L$.
(ii) D is called $S_{\rightarrow}$-reflexive, if $(x \rightarrow y) \cap D \neq \emptyset$, implies $x \rightarrow y \subseteq D$, for any $x, y \in L$.
Every $S_{\sim}$-reflexive set is $S_{\rightarrow}$-reflexive.

Definition 2.5. [7] *Let L be a hyper EQ-algebra. Then we say that L is satisfies in*
(i) exchange principle condition or (EP) condition, if $A \rightarrow (B \rightarrow C) \ll B \rightarrow (A \rightarrow C)$, for all $A, B, C \subseteq L$.
(ii) residuated condition, when $A \odot B \ll C$ if and only if $A \ll B \rightarrow C$, for all $A, B, C \subseteq L$.

Definition 2.6. [3, 7] *Let F be a nonempty subset of a hyper EQ-algebra L such that $1 \in F$. Then F is called a(an)*
(i) Prefilter of L, if $x \rightarrow y \subseteq F$ and $x \in F$ imply $y \in F$ and $(x \odot y) \subseteq F$, for all $x, y \in F$.
(ii) Filter of L, if F is a prefilter and $x \rightarrow y \subseteq F$, implies $(x \odot z) \rightarrow (y \odot z) \subseteq F$ for all $x, y, z \in L$.
(iii) Positive implicative prefilter of L, if $z \rightarrow (y \rightarrow x) \subseteq F$ and $z \rightarrow y \subseteq F$, imply $z \rightarrow x \subseteq F$, for all $x, y, z \in L$ and $(x \odot y) \subseteq F$, for all $x, y \in F$.
(iv) Positive implicative filter of L, if F is a positive implicative prefilter and $x \rightarrow y \subseteq F$, implies $(x \odot z) \rightarrow (y \odot z) \subseteq F$ for all $x, y, z \in L$.
(v) Implicative prefilter of L, if $z \rightarrow ((x \rightarrow y) \rightarrow x) \subseteq F$ and $z \in F$, imply $x \in F$, and $(x \odot y) \subseteq F$, for all $x, y \in F$.
(vi) Implicative filter of L, if F is an implicative prefilter and $x \rightarrow y \subseteq F$, implies $(x \odot z) \rightarrow (y \odot z) \subseteq F$ for all $x, y, z \in L$.
(vii) Fantastic prefilter of L, if $z \rightarrow (y \rightarrow x) \subseteq F$ and $z \in F$, then $((x \rightarrow y) \rightarrow y) \rightarrow x \subseteq F$, and $(x \odot y) \subseteq F$, for all $x, y \in F$.
(viii) Fantastic filter of L, if F is a fantastic prefilter and $x \rightarrow y \subseteq F$, implies $(x \odot z) \rightarrow (y \odot z) \subseteq F$.

Proposition 2.7. [3] *Let F be an $S_{\sim}$-reflexive prefilter of a hyper EQ-algebra L. Then for any $x, y, z \in L$:*
(i) If $x \in F$ and $x \leq y$, then $y \in F$ and if $A \subseteq F$ and $A \ll B$, then $B \cap F \neq \emptyset$.
(ii) If $x \in F$ and $(x \sim y) \cap F \neq \emptyset$, then $y \in F$.
(iii) If $(x \sim y) \subseteq F$ and $(y \sim z) \subseteq F$, then $(x \sim z) \cap F \neq \emptyset$.
(iv) If $(x \rightarrow y) \subseteq F$ and $(y \rightarrow z) \subseteq F$, then $(x \rightarrow z) \cap F \neq \emptyset$.

Consider an $S_{\sim}$-reflexive filter F of a hyper EQ-algebra L and $x, y \in L$. Define a binary relation $\equiv_F$ on L as follows:

$$x \equiv_F y \text{ if and only if } (x \sim y) \cap F \neq \emptyset.$$

Then $\equiv_F$ is a congruence relation on L. We denote the equivalence class x w.r.t $\equiv_F$ by $[x]_{\equiv_F}$ ($[x]$ for short). Furthermore, we define quotient algebra $\frac{L}{F} := \{[x] \mid x \in L\}$ and the operations $\wedge$ and hyper operations $\overline{\sim}, \overline{\odot}$, for $[x], [y] \in \frac{L}{F}$ as follow:

$$[x] \wedge [y] := [x \wedge y], \ \ [x]\overline{\sim}[y] := \{[t] \mid t \in x \sim y\}, \ \ [x]\overline{\odot}[y] := \{[t] \mid t \in x \odot y\}.$$

The top element is $[1]$ and $[x] \leq [y]$ if and only if $[x] \wedge [y] = [x]$ if and only if $x \wedge y \equiv_F x$ if and only if $(x \wedge y \sim x) \cap F \neq \emptyset$ if and only if $(x \to y) \cap F \neq \emptyset$.

Theorem 2.8. [3] *Let F be an $S_\sim$-reflexive filter of a hyper EQ-algebra L. Then $(\frac{L}{F}, \wedge, \overline{\sim}, \overline{\odot}, [1])$ is a separated hyper EQ-algebra.*

Theorem 2.9. [7] *Let L be a good hyper EQ-algebra and satisfy the residuated condition. Then*
(i) Every $S_\sim$-reflexive implicative (pre)filter is a positive implicative (pre)filter.
(ii) If L is satisfied in (EP) condition and F is an $S_\sim$-reflexive filter of L, then F is positive implicative if and only if $(x \wedge y) \to (x \odot y) \subseteq F$, for any $x, y \in L$.

3 Integral hyper EQ-algebras and integral (pre)filters in hyper EQ-algebras

In this section, we introduce the concept of integral hyper EQ-algebras and integral (pre)filters, and we give some related results.

Definition 3.1. *Let L be a hyper EQ-algebra with bottom element 0 and F be a proper prefilter of L. Then F is called an integral prefilter, if for any $x, y \in L$:*

$$\neg(x \odot y) \subseteq F \text{ implies } \neg x \subseteq F \text{ or } \neg y \subseteq F.$$

An integral prefilter F is called integral filter, if for any $x, y, z \in L$, it satisfies:

$$x \to y \subseteq F \text{ implies } (x \odot z) \to (y \odot z) \subseteq F.$$

Example 3.2. [7] *Let $(L = \{0, a, b, 1\}, \leq)$ be a poset, such that $0 < a < b < 1$. Define $\wedge, \odot, \sim$ on L as follow:*

$$x \wedge y = x \odot y = \min\{x, y\},$$

and

$\sim$	0	a	b	1
0	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{1\}$	$\{b,1\}$	$\{a,1\}$
b	$\{0\}$	$\{b,1\}$	$\{1\}$	$\{b,1\}$
1	$\{0\}$	$\{a,1\}$	$\{b,1\}$	$\{1\}$

Then $(L, \wedge, \odot, \sim, 1)$ is a hyper EQ-algebra and $F = \{a, b, 1\}$ is an integral (pre)filter of L.

Definition 3.3. *A hyper EQ-algebra L with bottom element 0 is called an integral hyper EQ-algebra, if $0 \in x \odot y$, then $x = 0$ or $y = 0$, for any $x, y \in L$.*

Example 3.4. *Let L be a hyper EQ-algebra in Example 3.2. Then L is an integral hyper EQ-algebra.*

Example 3.5. [7] *Let $(L = \{0, a, b, 1\}, \leq)$ be a poset, such that $0 < a < b < 1$. Define $\wedge, \odot, \sim$ on L as follow:*

$$x \wedge y = \min\{x, y\},$$

and

$\sim$	0	a	b	1
0	$\{1\}$	$\{a,b,1\}$	$\{a,1\}$	$\{0,1\}$
a	$\{a,b,1\}$	$\{a,1\}$	$\{a,b,1\}$	$\{a,1\}$
b	$\{a,1\}$	$\{a,b,1\}$	$\{b,1\}$	$\{b,1\}$
1	$\{0,1\}$	$\{a,1\}$	$\{b,1\}$	$\{1\}$

$\odot$	0	a	b	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{0\}$	$\{0,a\}$	$\{b\}$	$\{b\}$
1	$\{0\}$	$\{0,a\}$	$\{b\}$	$\{1\}$

Then $(L, \wedge, \odot, \sim, 1)$ is a hyper EQ-algebra while L is not an integral hyper EQ-algebra, because $0 \in \{0, a\} = a \odot b$ but $a \neq 0$ and $b \neq 0$.

Theorem 3.6. *Let L be a hyper EQ-algebra with bottom element 0 and F be an $S_\sim$-reflexive filter of L. Then F is an integral filter of L if and only if $\dfrac{L}{F}$ is an integral hyper EQ-algebra.*

Proof. Let F be an integral filter of L and $[0] \in [x] \odot [y]$, for $[x], [y] \in \dfrac{L}{F}$. Then there exists $t \in x \odot y$, such that $[0] = [t]$ and so $t \equiv_F 0$. Hence, $(t \sim 0) \cap F \neq \emptyset$ and since $t \in x \odot y$, we get that

$$t \sim 0 \subseteq \bigcup_{t \in x \odot y} (t \sim 0) = (x \odot y) \sim 0 = \neg(x \odot y),$$

and so $\neg(x \odot y) \cap F \neq \emptyset$. Now, since F is $S_\sim$-reflexive, we conclude that $\neg(x \odot y) \subseteq F$ and since F is integral filter of L, we get that $\neg x \subseteq F$ or $\neg y \subseteq F$. Hence, $\neg x \cap F \neq \emptyset$ or $\neg y \cap F \neq \emptyset$ and so $x \equiv_F 0$ or $y \equiv_F 0$. Therefore, $[x] = [0]$ or $[y] = [0]$ and so $\dfrac{L}{F}$ is an integral hyper EQ-algebra.
Conversely, let $\frac{L}{F}$ be an integral hyper EQ-algebra and $\neg(x \odot y) \subseteq F$. Then $\neg(x \odot y) \cap F \neq \emptyset$ and so $x \odot y \equiv_F 0$. Hence, $[x \odot y] = [0]$ and so $[x] = [0]$ or $[y] = [0]$. Thus, $x \equiv_F 0$ or $y \equiv_F 0$ and so $\neg x \cap F \neq \emptyset$ or $\neg y \cap F \neq \emptyset$ and since F is $S_\sim$-reflexive, we conclude that $\neg x \subseteq F$ or $\neg y \subseteq F$. Therefore, F is an integral filter of L. □

Definition 3.7. *A hyper EQ-algebra L with bottom element 0 is called a spanned hyper EQ-algebra, if $\tilde{0} = 0 \sim 1 = 0$.*
Every good hyper EQ-algebra is a spanned hyper EQ-algebra.

Example 3.8. *Let L be a hyper EQ-algebra in Example 3.2. Then L is a spanned hyper EQ-algebra.*

Example 3.9. *Let L be a hyper EQ-algebra in Example 3.5. Then L is not a spanned hyper EQ-algebra, because, $\tilde{0} = \{0, 1\} \neq \{0\}$.*

Theorem 3.10. *(Extension Property) Let F and G be two proper prefilters of a spanned hyper EQ-algebra L such that $F \subseteq G$ and F be a $S_{\sim}$-reflexive integral prefilter of L. Then, G is an integral prefilter of L.*

Proof. Let $\neg(x \odot y) \subseteq G$ and $x, y \in L$. Then by Proposition 2.3(iii), we have $(x \odot y) \odot ((x \odot y) \sim 0) \ll \tilde{0} = 0$ and so $0 \sim 0 = 0 \to 0 \ll (x \odot y) \odot ((x \odot y) \sim 0) \to 0 = \neg((x \odot y) \odot \neg(x \odot y))$ and since $1 \in 0 \sim 0$, there exists $b \in \neg((x \odot y) \odot \neg(x \odot y))$ such that $1 \leq b$ and so $b = 1$. Hence, $\neg((x \odot y) \odot \neg(x \odot y)) \cap F \neq \emptyset$ and since F is $S_{\sim}$-reflexive, we get that $\neg((x \odot y) \odot \neg(x \odot y)) \subseteq F$. Now, since F is an integral prefilter of L, we conclude that $\neg(x \odot y) \subseteq F$ or $\neg\neg(x \odot y) \subseteq F$. If $\neg(x \odot y) \subseteq F$, then $\neg x \subseteq F \subseteq G$ or $\neg y \subseteq F \subseteq G$ and if $\neg\neg(x \odot y) \subseteq F$, then $\neg((x \odot y) \to 0) \subseteq F \subseteq G$ and so $\neg((x \odot y) \to 0) \cap G \neq \emptyset$ and so there exists $c \in \neg(x \odot y) \subseteq G$, such that $(c \to 0) \cap G \neq \emptyset$ and so $(c \sim 0) \cap G \neq \emptyset$ and since G is a prefilter of L and $c \in G$, by Proposition 2.7(ii), we conclude that $0 \in G$ and so by Proposition 2.7(i), $G = L$ which is impossible, because G is a proper prefilter of L. Therefore, G is an integral prefilter of L. □

4 Obstinate and maximal (pre)filter in hyper EQ-algebras

In this section, we introduce the concept of obstinate and maximal (per)filters of a hyper EQ-algebra and we study the relationship among obstinate (pre)filters, and some type of other (pre)filters such integral, maximal, (positive)implicative and fantastic (pre)filters in hyper EQ-algebras

Definition 4.1. *A proper (pre)filter F of a hyper EQ-algebra L is called an obstinate (pre)filter, if $x, y \notin F$, then $x \to y \subseteq F$ and $y \to x \subseteq F$, for any $x, y \in L$.*

Example 4.2. *Let $F = \{a, b, 1\}$ in Example 3.5. Then, F is an obstinate filter of L.*

Example 4.3. *Let $F = \{b, 1\}$ in Example 3.5. Then F is not an obstinate filter, because $0, a \notin F$ and $a \to 0 = (a \wedge 0) \sim a = 0 \sim a = \{a, b, 1\} \nsubseteq F$.*

Proposition 4.4. *Let F be an $S_{\sim}$-reflexive proper prefilter of a hyper EQ-algebra L. Then F is an obstinate prefilter of L if and only if $x, y \notin F$ implies $x \sim y \subseteq F$, for any $x, y \in L$.*

Proof. Let F be an $S_{\sim}$-reflexive proper prefilter of a hyper EQ-algebra L and $x, y \notin F$. Then $x \to y \subseteq F$ and $y \to x \subseteq F$ and so $(x \wedge y) \sim x \subseteq F$ and $(x \wedge y) \sim y \subseteq F$. Since by Proposition 2.3(v), $(x \wedge y) \sim x \ll x \sim (x \wedge y)$ and since $(x \wedge y) \sim x \subseteq F$, by Proposition 2.7($i$), we get that $(x \sim x \wedge y) \cap F \neq \emptyset$ and since F is $S_{\sim}$-reflexive we conclude that $x \sim (x \wedge y) \subseteq F$ and since $(x \wedge y) \sim y \subseteq F$, by Proposition 2.7($iii$), we get that $(x \sim y) \cap F \neq \emptyset$ and so $x \sim y \subseteq F$. Conversely, if $x \sim y \subseteq F$ and $x, y \notin F$, then by Proposition 2.3(v), $x \sim y \ll x \to y$ and so by Proposition 2.7(i), we have $(x \to y) \cap F \neq \emptyset$ and so $x \to y \subseteq F$. Moreover, since by Proposition 2.3(v), $x \sim y \ll y \sim x \ll y \to x$, we get that $x \sim y \ll y \to x$ and so by Proposition 2.7(i), $(y \to x) \cap F \neq \emptyset$. Hence, $y \to x \subseteq F$ and so F is an obstinate prefilter of L. □

Theorem 4.5. *Let F be an $S_{\sim}$-reflexive proper prefilter of a hyper EQ-algebra L with bottom element 0. Then F is an obstinate prefilter of L if and only if $x \notin F$ implies $\neg x \subseteq F$.*

Proof. Let F be an $S_{\sim}$-reflexive proper prefilter of a hyper EQ-algebra L and $x \notin F$. Since $0 \notin F$, by Proposition 4.4, we conclude that $\neg x = x \sim 0 \subseteq F$. Conversely, let $x, y \notin F$. Then $\neg x \subseteq F$ and $\neg y \subseteq F$ and so $x \sim 0 \subseteq F$ and $y \sim 0 \subseteq F$. Since by Proposition 2.3(v), $y \sim 0 \ll 0 \sim y$ and so by Proposition 2.7(i), $(0 \sim y) \cap F \neq \emptyset$ and since F is $S_{\sim}$ reflexive, we conclude that $0 \sim y \subseteq F$. Hence, by Proposition 2.7(iii), we have $(x \sim y) \cap F \neq \emptyset$ and so $x \sim y \subseteq F$. Therefore, by Proposition 4.4, F is an obstinate prefilter of L. □

Theorem 4.6. *(Extension Property) Let F and G be two proper prefilters of a hyper EQ-algebra L such that $F \subseteq G$ and F be an obstinate prefilter of L. Then G is an obstinate prefilter of L.*

Proof. Let $x, y \notin G$. Then by $F \subseteq G$ we get that $x, y \notin F$ and since F is an obstinate prefilters of L, we conclude that $x \to y \subseteq F$ and $y \to x \subseteq F$ and so $x \to y \subseteq G$ and $y \to x \subseteq G$. Therefore, G is an obstinate prefilter of L. □

Theorem 4.7. *Let F be an $S_{\sim}$-reflexive obstinate filter of a hyper EQ-algebra L. Then $\dfrac{L}{F}$ is a chain-separated hyper EQ-algebra.*

Proof. Let F be an $S_{\sim}$-reflexive filter of a hyper EQ-algebra L. Then, by Theorem 2.8, $\frac{L}{F}$ is a separated hyper EQ-algebra. Now, let $[x], [y] \in \dfrac{L}{F}$. If $x \in F$ or $y \in F$, then by Proposition 2.3(iii), $x \ll y \to x$ and so by Proposition 2.7(i), $(y \to x) \cap F \neq \emptyset$. Hence, $[y] \leq [x]$. Similarly, if $y \in F$, then $[x] \leq [y]$. Let $x, y \notin F$. Since F is an obstinate filter of L, by Proposition 4.4, we get that $x \sim y \subseteq F$ and so $(x \sim y) \cap F \neq \emptyset$. Hence, $x \equiv_F y$ and so $[x] = [y]$. Therefore, $\dfrac{L}{F}$ is a chain-separated hyper EQ-algebra. □

Lemma 4.8. *Let F be a prefilter of a hyper EQ-algebra L, $\emptyset \neq A, B \subseteq L$ and $b, x \in L$. Then*
(i) If $A \subseteq F$ and $A \to b \subseteq F$, then $b \in F$.
(ii) If $x \in F$ and $x \to B \subseteq F$, then $B \subseteq F$ and $B \cap F \neq \emptyset$.
(iii) If $A, B \subseteq F$, then $A \odot B \subseteq F$.

Proof. (i) Since $\emptyset \neq A \subseteq F$, there exists $a \in A$ such that $a \in F$ and since $a \to b \subseteq A \to b \subseteq F$, we get that $a \to b \subseteq F$ and so $(a \to b) \cap F \neq \emptyset$. Now, since F is a prefilter of L and $a \in F$, we get that $b \in F$.
(ii) Since $x \to B \subseteq F$, we get that for any $b \in B$, $\underset{b \in B}{\cup}(x \to b) \subseteq x \to B \subseteq F$ and since $x \to b \subseteq \underset{b \in B}{\cup}(x \to b)$, for any $b \in B$, we get that $x \to b \subseteq F$, for any $b \in B$ and by $x \in F$, we conclude that $b \in F$. Therefore, $B \subseteq F$ and since B is nonempty subset of L, we get that $B \cap F \neq \emptyset$.
(iii) Let $A, B \subseteq F$. Then for any $a \in A$ and $b \in B$, we have $a, b \in F$ and so $a \odot b \subseteq F$. Hence, $A \odot B = \underset{a \in A, b \in B}{\cup} a \odot b \subseteq F$. □

Theorem 4.9. *Let F be an $S_{\sim}$-reflexive obstinate prefilter of a hyper EQ-algebra L. Then F is an implicative prefilter of L.*

Proof. Let F be an $S_{\sim}$-reflexive obstinate prefilter of L and $x, y \in L$ such that $(x \to y) \to x \subseteq F$. If $y \in F$, then by Proposition 2.3(iii), $y \ll x \to y$ and so by Proposition 2.7(i), $(x \to y) \cap F \neq \emptyset$ and since F is $S_{\sim}$-reflexive, we get that $x \to y \subseteq F$. Now, since F is a prefilter of L and $(x \to y) \to x \subseteq F$, by Lemma 4.8($i$), $x \in F$. Moreover, if $x, y \notin F$, then $x \to y \subseteq F$ and since $(x \to y) \to x \subseteq F$, we conclude that $x \in F$, which is a contradiction. Therefore, F is an implicative prefilter of L. □

Corollary 4.10. *Let F be a $S_{\sim}$-reflexive obstinate prefilter of good hyper EQ-algebra L with residuated condition. Then F is a positive implicative prefilter of L.*

Proof. It follows from Theorem 2.9(i) and Theorem 4.9. □

Lemma 4.11. *Let L be a hyper EQ-algebra with residuated condition and $x, y, z \in L$. Then*
(i) $x \odot (x \to y) \ll y$.
(ii) $x \to (y \to z) \ll y \to (x \to z)$.
(iii) $(x \odot y) \to z \ll x \to (y \to z)$.
(iv) $x \to y \ll (x \odot z) \to (y \odot z)$.

Proof. (i) Since $x \to y \ll x \to y$, by residuated condition, we conclude that $x \odot (x \to y) \ll y$.
(ii) Let $x, y, z \in L$. Then by (i), $x \odot (x \to (y \to z)) \ll y \to z$, and so by residuated condition, we have $y \odot x \odot (x \to (y \to z)) \ll z$ and since $y \odot x \odot (x \to (y \to z)) = x \odot y \odot (x \to (y \to z))$, we conclude that $x \odot y \odot (x \to (y \to z)) \ll z$ and so $y \odot (x \to (y \to z)) \ll x \to z$. Hence, $x \to (y \to z) \ll y \to (x \to z)$.
(iii) Let $x, y, z \in L$. Then by (i), $(x \odot y) \odot ((x \odot y) \to z) \ll z$ and so by residuated condition, we have $y \odot ((x \odot y) \to z) \ll x \to z$ and so $(x \odot y) \to z \ll y \to (x \to z)$. Now, if $p \in (x \odot y) \to z$, then there exists $q \in y \to (x \to z)$ and since by (ii), $y \to (x \to z) \ll x \to (y \to z)$, there exists $r \in x \to (y \to z)$, such that $q \leq r$. Hence, $p \leq r$ and so $(x \odot y) \to z \ll x \to (y \to z)$.
(iv) Since by (i) $x \odot (x \to y) \ll y$, for $x, y, z \in L$, by $(HEQ2)$, we get that $x \odot z \odot (x \to y) \ll y \odot z$ and so by residuated condition, we have $x \to y \ll (x \odot z) \to (y \odot z)$. □

Theorem 4.12. *Let F be an $S_\sim$-reflexive obstinate filter of a good hyper EQ-algebra L with residuated condition. Then*
(i) $\frac{L}{F}$ *is an integral hyper EQ-algebra.*
(ii) F *is an integral filter of L.*

Proof. (i) Let F be an $S_\sim$-reflexive obstinate filter of good hyper EQ-algebra L with residuated condition and $[0] \in [x]\overline{\odot}[y]$, for $[x], [y] \in \frac{L}{F}$. Then $0 \in x \odot y$ and so

$$0 \sim 0 \subseteq \bigcup_{t \in x \odot y} (t \sim 0) = (x \odot y) \sim 0,$$

and since by Proposition 2.3(i), $1 \in 0 \sim 0$, we get that $1 \in (x \odot y) \sim 0$ and so $((x \odot y) \sim 0) \cap F \neq \emptyset$. Hence, $((x \odot y) \to 0) \cap F \neq \emptyset$ and since F is $S_\sim$-reflexive, we get that $((x \odot y) \to 0) \subseteq F$ and since by Lemma 4.11(iii), $(x \odot y) \to 0 \ll x \to (y \to 0)$, by Proposition 2.7$(i)$, we get that $(x \to (y \to 0)) \cap F \neq \emptyset$. Now, if $x, y \in F$, then by Lemma 4.8(ii), $(y \to 0) \subseteq F$ and since $y \in F$, we conclude that $0 \in F$, which is impossible, because F is a proper filter of L. Moreover, if $x \in F$ and $y \notin F$, then by $(x \to (y \to 0)) \cap F \neq \emptyset$ and Lemma 4.8(ii), we have $(y \to 0) \subseteq F$ and so $(y \sim 0) \subseteq F$. Hence, $(y \sim 0) \cap F \neq \emptyset$ and so $y \equiv_F 0$. Therefore, $[y] = [0]$ and by similar way, if $y \in F$ and $x \notin F$, by $(x \to (y \to 0)) \cap F \neq \emptyset$ and Lemma 4.11(ii), we conclude that $[x] = [0]$. Also, if $x, y \notin F$, since F is an obstinate filter of L we have $(x \to y) \subseteq F$ and $(y \to x) \subseteq F$ and by Corollary 4.10, F is a positive implicative . Now, since $(x \to y) \subseteq F$ and $x \to (y \to 0) \subseteq F$, we conclude that $(x \to 0) \subseteq F$ and so $(x \sim 0) \subseteq F$. Hence, $(x \sim 0) \cap F \neq \emptyset$ and so $x \equiv_F 0$. Thus, $[x] = [0]$. Moreover, since by Lemma 4.11(ii), $(x \to (y \to 0)) \ll (y \to (x \to 0))$ and $(x \to (y \to 0)) \cap F \neq \emptyset$, by Proposition 2.7$(i)$, we conclude that $(y \to (x \to 0)) \cap F \neq \emptyset$ and so $(y \to (x \to 0)) \subseteq F$. Now, by $(y \to x) \subseteq F$, we have $(y \sim 0) \subseteq F$. Hence, $(y \to 0) \cap F \neq \emptyset$ and so $y \equiv_F 0$. Therefore, $[y] = [0]$ and so $\frac{L}{F}$ is an integral hyper EQ-algebra.
(ii) It follows from Theorem 3.6. □

Theorem 4.13. *Let F be an $S_\sim$-reflexive fantastic prefilter of a hyper EQ-algebra L with bottom element 0. Then $\neg\neg x \to x \subseteq F$, for any $x \in L$.*

Proof. Since by Proposition 2.3(ii), $1 \in 0 \to x$, for any $x \in L$ and since $1 \in F$, we get that $(0 \to x) \cap F \neq \emptyset$ and so $(0 \to x) \subseteq F$ and since by Proposition 2.3(iii), $(0 \to x) \ll 1 \to (0 \to x)$, we have $(1 \to (0 \to x)) \cap F \neq \emptyset$, by Proposition 2.7($i$), and so $(1 \to (0 \to x)) \subseteq F$. Now, since F is a fantastic prefilter of L and $1 \in F$, we conclude that $((x \to 0) \to 0) \to x \subseteq F$. Therefore, $\neg\neg x \to x \subseteq F$, for any $x \in L$. □

Theorem 4.14. *Let F be an $S_\sim$-reflexive integral and fantastic prefilter of a spanned hyper EQ-algebra L with bottom element 0. Then, F is an obstinate prefilter of L.*

Proof. Let $x, y \in L$ such that $x, y \notin F$. Since F is a fantastic prefilter of L, by Theorem 4.13, we get that $\neg\neg x \to x \subseteq F$ and since by Proposition 2.3(iii), $x \odot \neg x = x \odot (x \sim 0) \ll \tilde{0} = 0$. Hence, by Proposition 2.3(iv), $0 \sim 0 = 0 \to 0 \ll (x \odot \neg x) \to 0$ and since $1 \in 0 \sim 0$, there exists $b \in (x \odot \neg x) \to 0$, such that $1 \leq b$ and since 1 is top element of L, we conclude that $b = 1$ and so $1 \in \neg(x \odot \neg x)$. Hence, $\neg(x \odot \neg x) \cap F \neq \emptyset$ and since F is $S_\sim$-reflexive, we get that $\neg(x \odot \neg x) \subseteq F$. Now, since F is an integral prefiletr of L, we conclude that $\neg x \subseteq F$ or $\neg\neg x \subseteq F$. If $\neg\neg x \subseteq F$, then by $\neg\neg x \to x \subseteq F$, we have $x \in F$ which impossible because $x \notin F$. Hence, $\neg x \subseteq F$ and by similarly, since $y \notin F$, we get that $\neg y \subseteq F$. Now, by Proposition 2.3(iv), $x \to 0 \ll x \to y$ and since $\neg x \subseteq F$, by Proposition 2.7(i), we conclude that $(x \to y) \cap F \neq \emptyset$ and so $x \to y \subseteq F$. Similarly, since $\neg y \subseteq F$, we get that $y \to x \subseteq F$. Therefore, F is an obstinate prefilter of L. □

Theorem 4.15. *Let F be an $S_\sim$-reflexive obstinate prefilter of a hyper EQ-algebra L. Then, F is a fantastic prefilter of L.*

Proof. Let F be an $S_\sim$-reflexive obstinate prefilter of L and $x, y, z \in L$ such that $z \to (y \to x) \subseteq F$ and $z \in F$. Then by Lemma 4.8(ii), $y \to x \subseteq F$. If $x \in F$, then by Proposition 2.3(ii), $x \ll ((x \to y) \to y) \to x$ and so by Proposition 2.7(i), $(((x \to y) \to y) \to x) \cap F \neq \emptyset$ and since F is $S_\sim$-reflexive, we get that $(((x \to y) \to y) \to x) \subseteq F$. If $x \notin F$ and $((x \to y) \to y) \subseteq F$, since $y \to x \subseteq F$ by Proposition 2.7(iv), we get that $((x \to y) \to x) \cap F \neq \emptyset$ and since F is $S_\sim$-reflexive, we conclude that $((x \to y) \to x) \subseteq F$. Moreover, since by Proposition 2.3(iii), $((x \to y) \to x) \ll 1 \to ((x \to y) \to x)$, we get that $(1 \to ((x \to y) \to x)) \cap F \neq \emptyset$ and since by Theorem 4.9, F is an implicative prefilter of L and $1 \in F$, we conclude that $x \in F$ which is impossible. Now, if $x \notin F$ and $((x \to y) \to y) \nsubseteq F$, then $((x \to y) \to y) \cap F = \emptyset$, indeed if $((x \to y) \to y) \cap F \neq \emptyset$, since F is $S_\sim$-reflexive, we have $((x \to y) \to y) \subseteq F$ which impossible. Hence, for any $k \in ((x \to y) \to y)$, we have $k \notin F$. Now, since F is an obstinate perfilter of L, we conclude that $k \to x \subseteq F$ and since $(k \to x) \subseteq \bigcup_{k \in ((x \to y) \to y)} (k \to x) \subseteq ((x \to y) \to y) \to x$, we get that $((x \to y) \to y) \to x \cap F \neq \emptyset$ and so $((x \to y) \to y) \to x \subseteq F$. Therefore, F is a fantastic prefilter of L. □

Lemma 4.16. *Let L be a spanned hyper EQ-algebra and $\emptyset \neq A \subseteq L$. Then $0 \in A \odot \neg A$.*

Proof. Let $\emptyset \neq A \subseteq L$ and L be a spanned hyper EQ-algebra. Then there exists $a \in A$ and since by Proposition 2.3(iii), $a \odot \neg a = a \odot (a \to 0) \ll \tilde{0} = 0$, we get that for any $x \in a \odot \neg a$, $x \leq 0$ and so $x = 0$. Hence, $a \odot \neg a = \{0\}$ and since $a \odot \neg a \subseteq A \odot \neg A = \bigcup_{a \in A, b \in \neg A} a \odot b$, we conclude that $0 \in A \odot \neg A$. □

Theorem 4.17. *Let F be an $S_\sim$-reflexive obstinate prefilter of spanned hyper EQ-algebra L. Then F is an integral prefilter of L.*

Proof. Let $\neg(x \odot y) \subseteq F$, $\neg x \nsubseteq F$ and $\neg y \nsubseteq F$, for some $x, y \in L$. Since F is a proper prefilter of L, we get that $0 \notin F$ and since F is an obstinate prefilter of L and $\neg x \nsubseteq F$, we conclude that $\neg\neg x = (\neg x \to 0) \subseteq F$, indeed, since $\neg x \nsubseteq F$, then exists $b \in \neg x$, such that $b \notin F$ and since F is an obstinate prefilter of L and $0 \notin F$, we conclude that $b \to 0 \subseteq F$ and since

$$b \to 0 \subseteq \bigcup_{t \in \neg x} (t \to 0) \subseteq \neg x \to 0,$$

we conclude that $(\neg x \to 0) \cap F \neq \emptyset$ and since F is $S_{\sim}$-reflexive, we get that $\neg\neg x \subseteq F$. Similarly, $\neg\neg y \subseteq F$ and since by Theorem 4.15, F is a fantastic prefilter of L, by Theorem 4.13, we have $\neg\neg x \to x \subseteq F$ and $\neg\neg y \to y \subseteq F$. Now, by Lemma 4.8($i$), we conclude that $x, y \in F$ and so $(x \odot y) \subseteq F$. Hence, by Lemma 4.8(iii), $(x \odot y) \odot (\neg(x \odot y)) \subseteq F$ and since by Lemma 4.16, $0 \in (x \odot y) \odot (\neg(x \odot y))$, we conclude that $0 \in F$, which is impossible. Therefore, $\neg x \subseteq F$ or $\neg y \subseteq F$ and so F is an integral prefilter of L. □

Corollary 4.18. *Let F be an $S_{\sim}$-reflexive integral prefilter of spanned hyper EQ-algebra L. Then $\neg x \subseteq F$ or $\neg\neg x \subseteq F$, for any $x \in L$.*

Proof. Since L is a spanned hyper EQ-algebra, by Lemma 4.16, we get that $x \odot \neg x = 0$ and so $1 \in 0 \sim 0 = \neg(x \odot \neg x)$. Hence, $\neg(x \odot \neg x) \cap F \neq \emptyset$ and since F is an $S_{\sim}$-reflexive integral prefilter of L, we get that $\neg(x \odot \neg x) \subseteq F$ and so $\neg x \subseteq F$ or $\neg\neg x \subseteq F$. □

Theorem 4.19. *Let F be an $S_{\sim}$-reflexive proper prefilter of a spanned hyper EQ-algebra L such that $x \in F$ or $\neg x \subseteq F$, for any $x \in L$. Then F is an integral filter of L.*

Proof. Let $\neg(x \odot y) \subseteq F$, $\neg x \nsubseteq F$ and $\neg y \nsubseteq F$, for some $x, y \in L$. Then $x \in F$ and $y \in F$ and so $(x \odot y) \subseteq F$. Hence, by Lemma 4.8(iii), $(x \odot y) \odot (\neg(x \odot y)) \subseteq F$ and since by Lemma 4.16, $0 \in (x \odot y) \odot (\neg(x \odot y))$, we conclude that $0 \in F$, which is impossible. Therefore, $\neg x \subseteq F$ or $\neg y \subseteq F$ and so F is an integral prefilter of L. □

Theorem 4.20. *Let F be an $S_{\sim}$-reflexive prefilter of a spanned hyper EQ-algebra L. Then the following conditions are equivalent:*
(i) *F is an obstinate prefilter of L.*
(ii) *F is an integral and fantastic prefilter of L.*

Proof. It follows from Theorem 4.14, Theorem 4.15 and Theorem 4.17. □

Proposition 4.21. *Let F be an $S_{\sim}$-reflexive prefilter of a hyper EQ-algebra L with residuated condition. Then F is a filter of L.*

Proof. Let F be an $S_{\sim}$-reflexive prefilter of L and $x, y, z \in L$ such that $x \to y \subseteq F$. Since by Lemma 4.11(iv), $x \to y \ll (x \odot z) \to (y \odot z)$, by Proposition 2.7($i$), $((x \odot z) \to (y \odot z)) \cap F \neq \emptyset$ and since F is $S_{\sim}$-reflexive, we get that $((x \odot z) \to (y \odot z)) \subseteq F$. Therefore, F is a filter of L. □

Definition 4.22. *A prefilter F of a hyper EQ-algebra L is called a maximal prefilter, if it is proper and no proper prefilter of L strictly contains F, that is, for each prefilter, $F \neq G$, if $F \subseteq G$, then $G = L$.*

Example 4.23. *Let $F = \{a, b, 1\}$ in Example 3.5. Then, F is a maximal filter of L.*

Proposition 4.24. *Let F be an $S_{\sim}$-reflexive obstinate prefilter of a hyper EQ-algebra L with bottom element 0. Then F is the maximal perfilter of L.*

Proof. Let F be an obstinate prefilter of L. Then F is a proper perfilter of L and if $F \subseteq G \subseteq L$ and $F \neq G$ such that G is a prefilter of L, then there exists $x \in G$ such that $x \notin F$ and since $0 \notin F$, we get that $x \to 0 \subseteq F \subseteq G$ and since $x \in G$ and G is a prefilter of L, we conclude that $0 \in G$ and so by Proposition 2.7(i), $G = L$. Therefore, F is maximal perfilter of L. □

Theorem 4.25. *Let F be an $S_\sim$-reflexive positive implicative prefilter of a hyper EQ-algebra L with the residuated condition. Then $F_a = \{x \in L \mid a \to x \subseteq F\}$ is the least filter of L contains F and $\{a\}$.*

Proof. Let F be an $S_\sim$-reflexive positive implicative prefilter of L and $a \in L$. Then by Proposition 2.3(ii), we get that $1 \in a \to 1$ and so $(a \to 1) \cap F \neq \emptyset$ and so $(a \to 1) \subseteq F$. Hence, $1 \in F_a$. Moreover, if $x \in F_a$ and $x \to y \subseteq F_a$, for $x, y \in L$, then $a \to x \subseteq F$ and $a \to (x \to y) \subseteq F$ and since F is a positive implicative prefilter of L, we conclude that $a \to y \subseteq F$ and so $y \in F_a$. Now, let $x, y \in F_a$, for $x, y, z \in L$. Then $a \to x \subseteq F$ and $a \to y \subseteq F$ and since by Proposition 4.21, F is a filter of L, we get that $(x \odot a) \to (x \odot y) \subseteq F$ and $(a \odot a) \to (x \odot a) \subseteq F$ and so by Proposition 2.7(iv), $((a \odot a) \to (x \odot y)) \cap F \neq \emptyset$ and since F is $S_\sim$-reflexive, we get that $(a \odot a) \to (x \odot y) \subseteq F$. Now, since by Theorem 2.9(ii), $a \to (a \odot a) = (a \wedge a) \to (a \odot a) \subseteq F$, by Proposition 2.7($iv$), we get that $(a \to (x \odot y)) \cap F \neq \emptyset$ and since F is $S_\sim$-reflexive, we get $a \to (x \odot y) \subseteq F$. Hence, $x \odot y \subseteq F_a$, indeed if $t \in x \odot y$, then $a \to t \subseteq a \to (x \odot y) \subseteq F$. Hence, $t \in F_a$ and so $x \odot y \subseteq F_a$. Therefore, F_a is a prefilter of L and so by Theorem 4.21, F is a filter of L. Moreover, since by Proposition 2.3(i), $1 \in a \to a$ and so $(a \to a) \cap F \neq \emptyset$ and so $(a \to a) \subseteq F$. Hence, $a \in F_a$ and since by Proposition 2.3(iii), $x \ll a \to x$, $x \in F$ by Proposition 2.7(i), for $x \in F$, we conclude that $(a \to x) \cap F \neq \emptyset$ and so $a \to x \subseteq F$ and so $x \in F_a$. Now, let B be a filter of L contains F and $\{a\}$ and $u \in F_a$. Then $a \to u \subseteq F \subseteq B$ and since $a \in B$, we get that $u \in B$. Therefore, F_a is the least filter of L contains F and $\{a\}$. □

Theorem 4.26. *Let F be an $S_\sim$-reflexive maximal and positive implicative prefilter of a hyper EQ-algebra L with residuated and (EP)condition. Then F is an obstinate filter of L.*

Proof. Let F be an $S_\sim$-reflexive maximal and positive implicative prefilter of L and $x, y \in L$ such that $x, y \notin F$. Then by Proposition 4.21, F is a filter of L and by Theorem 4.25, $F \subsetneq F_x \subseteq L$ and since F is a maximal filter of L, we get that $F_x = L$ and so $y \in F_x$. Hence, $x \to y \subseteq F$ and by similar way, $y \to x \subseteq F$. Therefore, F is an obstinate filter of L. □

By Proposition 4.24 and Theorem 2.9(i) and Theorems 4.9, 4.20, 4.26, the following Theorem is provided.

Theorem 4.27. *Let F be an $S_\sim$-reflexive proper prefilter of a good hyper EQ-algebra L with residuated and (EP) condition. Then the following are equivalent:*
(i) *F is a maximal and implicative filter of L.*
(ii) *F is a maximal and positive implicative filter of L.*
(iii) *F is an obstinate filter of L.*
(iv) *F is an integral and fantastic filter of L.*

In the following, we introduce the concept of a good hyper EQ-homomorphism, and we give some related results.

Definition 4.28. *Let $(L_1, \wedge_1, \odot_1, \sim_1, 1)$ and $(L_2, \wedge_2, \odot_2, \sim_2, 1)$ be two hyper EQ-algebras and $\Phi : L_1 \longrightarrow L_2$ be a mapping. Then Φ is called a good hyper EQ-homomorphism, if $\Phi(1) = 1$ and for any $x, y \in L_1$ satisfies in the followings:*

(i) $\Phi(x \wedge_1 y) = \Phi(x) \wedge_2 \Phi(y)$,
(ii) $\Phi(x \odot_1 y) = \Phi(x) \odot_2 \Phi(y)$,
(iii) $\Phi(x \sim_1 y) = \Phi(x) \sim_2 \Phi(y)$.
Note that

$$\Phi(x \to_1 y) = \Phi((x \wedge_1 y) \sim_1 x) = \Phi(x \wedge_1 y) \sim_2 \Phi(x) = (\Phi(x) \wedge_2 \Phi(y)) \sim_2 \Phi(x) = \Phi(x) \to_2 \Phi(y).$$

Moreover, $\Phi(A) = \{\Phi(a) \mid a \in A\}$, *for any* $\emptyset \neq A \subseteq L_1$ *and* $\Phi^{-1}(B) = \{x \in L_1 \mid \Phi(x) \in B\}$, *for any* $\emptyset \neq B \subseteq L_2$.

Example 4.29. *Let* L *be a hyper EQ-algebra in Example 3.2 and* $\Phi(0) = 0, \Phi(a) = a, \Phi(b) = b, \Phi(1) = 1$. *Then, it is easy to check that* Φ *is a good hyper EQ-algebra homomorphism.*

Proposition 4.30. *Let* $\Phi : L_1 \longrightarrow L_2$ *be a good hyper EQ-homomorphism and* $\emptyset \neq A, B \subseteq L_1$. *Then*
(i) $\Phi(A \odot_1 B) = \Phi(A) \odot_2 \Phi(B)$,
(ii) $\Phi(A \sim_1 B) = \Phi(A) \sim_2 \Phi(B)$,
(iii) $\Phi(A \to_1 B) = \Phi(A) \to_2 \Phi(B)$.

Proof. (i) Let $\Phi : L_1 \longrightarrow L_2$ be a good hyper EQ-homomorphism, $\emptyset \neq A, B \subseteq L_1$ and $x \in \Phi(A \odot_1 B) = \phi(\bigcup_{a\in A, b\in B} a \odot b)$. Then there exists $a_0 \in A$ and $b_0 \in B$ such that $x \in \Phi(a_0 \odot_1 b_0)$ and by Definition 4.28 we have

$$\begin{aligned} \Phi(a_0 \odot_1 b_0) &= \Phi(a_0) \odot_2 \Phi(b_0) \\ &\subseteq \bigcup_{a\in A, b\in B} \Phi(a_0) \odot_2 \Phi(b_0) \\ &= \Phi(A) \odot_2 \Phi(B). \end{aligned}$$

Hence, $x \in \Phi(A) \odot_2 \Phi(B)$ and so $\Phi(A \odot_1 B) \subseteq \Phi(A) \odot_2 \Phi(B)$. If $x \in \Phi(A) \odot_2 \Phi(B)$, then there exists $a_0 \in A$ and $b_0 \in B$ such that $x \in \Phi(a_0) \odot_2 \Phi(b_0)$ and since by Definition 4.28

$$\begin{aligned} \Phi(a_0) \odot_2 \Phi(b_0) &= \Phi(a_0 \odot_1 b_0) \\ &\subseteq \Phi(\bigcup_{a\in A, b\in B} a_0 \odot_1 b_0) \\ &= \Phi(A \odot_1 B), \end{aligned}$$

we conclude that $x \in \Phi(A \odot_1 B)$ and so $\Phi(A) \odot_2 \Phi(B) \subseteq \Phi(A \odot_1 B)$.
(ii) The proof is similar (i).
(iii) It follows from Definition 4.28 and (ii). □

Proposition 4.31. *Let* $\Phi : L_1 \longrightarrow L_2$ *be a good hyper EQ-homomorphism and* $x, y \in L_1$. *Then*
(i) $1 \in \Phi(x \sim_1 x)$.
(ii) If L_2 *is separated and* $x \leq y$, *then* $\Phi(x) \leq \Phi(y)$.
(iii) If F *is a (pre)filter of* L_2, *then* $\Phi^{-1}(F)$ *is a (pre)filter of* L_1.
(iv) If F *is a (pre)filter of* L_2 *and* $\emptyset \neq A \subseteq L_2$, *then* $\Phi(A) \subseteq F$ *if and only if* $A \subseteq \Phi^{-1}(F)$.

Proof. (i) Since by Proposition 2.3(i), $1 \in \Phi(x) \sim_2 \Phi(x)$, for any $x \in L_1$ and since $\Phi(x \sim_1 x) = \Phi(x) \sim_2 \Phi(x)$, we get that $1 \in \Phi(x \sim_1 x)$.
(ii) Let L_2 be a separated hyper EQ-algebra and $x \leq y$. Then by Proposition 2.3(ii), $1 \in x \to y$

and so $1 = \Phi(1) \in \Phi(x \to_1 y) = \Phi(x) \to_2 \Phi(y)$ and so $1 \in (\Phi(x) \wedge_2 \Phi(y)) \sim_2 \Phi(x)$ and since L_2 is separated, we get that $\Phi(x) \wedge_2 \Phi(y) = \Phi(x)$. Hence, $\Phi(x) \leq \Phi(y)$.
(iii) Let F be a prefilter of L and $x, y \in L_1$, such that $x \in \Phi^{-1}(F)$ and $x \to_1 y \subseteq \Phi^{-1}(F)$. Then $\Phi(x) \in F$ and $\Phi(x \to_1 y) \subseteq F$ and since Φ is a good hyper EQ-homomorphism, we get that $\Phi(x) \to_2 \Phi(y) \subseteq F$. Hence, $\Phi(y) \in F$ and so $y \in \Phi^{-1}(F)$. Moreover, since $\Phi(1) = 1 \in F$, we conclude that $1 \in \Phi^{-1}(F)$. Also, if $a, b \in \Phi^{-1}(F)$, then $\Phi(a), \Phi(b) \in F$ and so $\Phi(a) \odot_2 \Phi(b) \subseteq F$ and since $\Phi(a) \odot_2 \Phi(b) = \Phi(a \odot_1 b)$, we get that $\Phi(a \odot_1 b) \subseteq F$. Hence, $a \odot_1 b \subseteq \Phi^{-1}(F)$ and so $\Phi^{-1}(F)$ is a prefilter of L_1. Now, let F be a filter of L_2 and $x, y, z \in L_1$ such that $x \to_1 y \subseteq \Phi^{-1}(F)$. Then $\Phi(x) \to_2 \Phi(y) = \Phi(x \to_1 y) \subseteq F$ and since F is a filter of L_2 and $\Phi(z) \in L_2$, we conclude that $(\Phi(x) \odot_2 \Phi(z)) \to_2 (\Phi(y) \odot_2 \Phi(z)) \subseteq F$ and since by Definition 4.28, $(\Phi(x) \odot_2 \Phi(z)) \to_2 (\Phi(y) \odot_2 \Phi(z)) = \Phi(x \odot_1 z \to_1 y \odot_1 z)$, we conclude that $\Phi(x \odot_1 z \to_1 y \odot_1 z) \subseteq F$ and so $(x \odot_1 z) \to_1 (y \odot_1 z) \subseteq \Phi^{-1}(F)$. Therefore, $\Phi^{-1}(F)$ is a filter of L_1.
(iv) Let $\Phi(A) \subseteq F$ and $x \in A$, then $\Phi(x) \in \Phi(A)$ and so $\Phi(x) \in F$. Hence, $x \in \Phi^{-1}(F)$ and so $A \subseteq \Phi^{-1}(F)$. Conversely, if $A \subseteq \Phi^{-1}(F)$ and $y \in \Phi(A)$, then there exists $a \in A$ such that $y = \Phi(a)$ and since $A \subseteq \Phi^{-1}(F)$, we get that $\Phi(a) \in \Phi(A) \subseteq \Phi\Phi^{-1}(F)$ and since $\Phi\Phi^{-1}(F) \subseteq F$, we conclude that $y = \Phi(a) \in F$ and so $\Phi(A) \subseteq F$. □

Proposition 4.32. *Let $\Phi : L_1 \longrightarrow L_2$ be a good hyper EQ-homomorphism and F be a (pre)filter of L_2. Then*
(i) If L_1, L_2 are hyper EQ-algebras with bottom element 0, $\Phi(0) = 0$ and F is an integral (pre)filter of L_2, then $\Phi^{-1}(F)$ is an integral (pre)filter of L_1.
(ii) If F is an obstinate (pre)filter of L_2, then $\Phi^{-1}(F)$ is an obstinate (pre)filter of L_1.
(iii) If F is a positive implicative (pre)filter of L_2, then $\Phi^{-1}(F)$ is a positive implicative (pre)filter of L_1.
(iv) If F is an implicative (pre)filter of L_2, then $\Phi^{-1}(F)$ is an implicative (pre)filter of L_1.

Proof. (i) Let F be an integral (pre)filter of L_2 and $\neg(x \odot_1 y) \subseteq \Phi^{-1}(F)$, for $x, y \in L_1$. Then $\Phi((x \odot_1 y) \sim_1 0) = \Phi(\neg(x \odot_1 y)) \subseteq F$ and by Proposition 4.31(iii), $\Phi^{-1}(F)$ is a (pre)filter of L_1. Now, since Φ is good hyper EQ-homomorphism, we get that $\neg(\Phi(x) \odot_2 \Phi(y)) = \Phi(x) \odot_2 \Phi(y) \sim_2 0 = \Phi(x \odot_1 y) \sim_2 \Phi(0) \subseteq F$ and since F is integral, we conclude that $\neg\Phi(x) \subseteq F$ or $\neg\Phi(y) \subseteq F$ and so $\Phi(\neg x) = \Phi(x) \sim_2 0 \subseteq F$ or $\Phi(\neg y) = \Phi(y) \sim_2 0 \subseteq F$. Hence, $\neg x \in \Phi^{-1}(F)$ or $\neg y \in \Phi^{-1}(F)$ and so $\Phi^{-1}(F)$ is an integral (pre)filter of L_1.
(ii) Let F be an obstinate (pre)filter of L_2 and $x, y \notin \Phi^{-1}(F)$, for $x, y \in L_1$. Then $\Phi(x) \notin F$ and $\Phi(y) \notin F$. Hence, $\Phi(x \to_1 y) = \Phi(x) \to_2 \Phi(y) \subseteq F$ and $\Phi(y \to_1 x) = \Phi(y) \to_2 \Phi(x) \subseteq F$ and so $x \to_1 y \subseteq \Phi^{-1}(F)$ and $y \to_1 x \subseteq \Phi^{-1}(F)$. Therefore, $\Phi^{-1}(F)$ is an obstinate (pre)filter of L_1.
(iii) Let F be a positive implicative (pre)filter of L_2 and $x \to_1 (y \to_1 z) \subseteq \Phi^{-1}(F)$ and $(x \to_1 y) \subseteq \Phi^{-1}(F)$, for $x, y, z \in L_1$. Then by Definition 4.28, $\Phi(x) \to_2 (\Phi(y) \to_2 \Phi(z)) = \Phi(x \to_1 (y \to_1 z)) \subseteq F$ and $\Phi(x) \to_2 \Phi(y) = \Phi(x \to_1 y) \subseteq F$ and since F is positive implicative, we conclude that $\Phi(x \to_1 z) = \Phi(x) \to_2 \Phi(z) \subseteq F$ and so $x \to_1 z \subseteq \Phi^{-1}(F)$. Therefore, $\Phi^{-1}(F)$ is a positive implicative (pre)filter of L_1.
(iv) Let F be an implicative (pre)filter of L_2 and $z \to_1 ((x \to_1 y) \to_1 x) \subseteq \Phi^{-1}(F)$ and $z \in \Phi^{-1}(F)$, for $x, y, z \in L_1$. Then by Definition 4.28, $\Phi(z) \to_2 ((\Phi(x) \to_2 \Phi(y)) \to_2 \Phi(x)) = \Phi(z \to_1 ((x \to_1 y) \to_1 x)) \subseteq F$ and $\Phi(z) \in F$ and since F is implicative, we conclude that $\Phi(x) \subseteq F$ and so $x \in \Phi^{-1}(F)$. Therefore, $\Phi^{-1}(F)$ is an implicative (pre)filter of L_1.

□

5 Conclusion

The results of this paper are devoted introducing integral hyper EQ-algebras, integral (pre)filters and obstinate(pre)filters of hyper EQ-algebras. We presented some characterizations of this (pre)filters in hyper EQ-algebras and we proved that the quotient hyper EQ-algebras induced by a hyper filter F is an integral hyper EQ-algebra if and only if F is an integral filter. Moreover, we introduced the concept of obstinate (pre)filter in hyper EQ-algebras and we studied relationship among obstinate (pre)filters and some types of other (pre)filters such integral, maximal, (positive)implicative and fantastic (pre)filters in hyper EQ-algebras. There are still some questions:
How do we define the notions of primary (pre)filters of hyper EQ-algebras, and what is the quotient algebra induced by these filters?
What is the relationship between these (pre)filters and other types (pre)filters of hyper EQ-algebras?
These could be a topic of further research.

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