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On L-fuzzy approximation operators and L-fuzzy relations on residuated lattices

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Abstract

We consider properties of L-fuzzy relations and L-normal operators for a residuated lattice L in detail and show that the class $\mathcal{R}_L(U)$ of all L-fuzzy relations on U and the class $\mathcal{N}_L(U)$ of all L-normal operators are residuated lattices and they are isomorphic as lattices. Moreover, we prove that for any L-normal operator $\mathcal{F}$, it is reflexive (or transitive) if and only if the L-fuzzy relation $R_{\mathcal{F}}$ induced by $\mathcal{F}$ is reflexive (or transitive), respectively.

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1 Introduction

The rough set theory by Pawlak [8] has been actively researched as a valuable method for finding rules and features from incomplete data sets. The central concept of this theory is the notion of approximation space (U, E), where U is a non-empty finite set, and E is an equivalence relation on U. A subset of U divided by the equivalence relation E can be considered as representing some knowledge relative to U. When we extend this theory to a generalized approximation space (U, R), where U is (not necessarily finite) a set and R is (not necessarily equivalence) a binary relation on U, we face an essential problem that how we specify a subset representing rules or knowledge. As one of the methods to solve the problem, we use the approximation operator $\overline{\mathrm{R}}$ ($\underline{\mathrm{R}}$), called *the upper (lower) approximation operator* induced by the binary relation R to determine the subset representing knowledge about U. Since $\overline{\mathrm{R}}(A)$ and $\underline{\mathrm{R}}(A)$ for a subset $A \subseteq U$ is defined as follows:

$$R(x) = \{y \in U \mid (x, y) \in R\},$$
$$\overline{\mathrm{R}}(A) = \{x \in U \mid R(x) \cap A \neq \emptyset\},$$
$$\underline{\mathrm{R}}(A) = \{x \in U \mid R(x) \subseteq A\}.$$

Moreover, these sets are considered as the data similar to the elements of A; it is possible to express pieces of knowledge, and rules in an incomplete data set. As to the generalized approximation spaces (U, R), research on their topological properties [1, 5, 6, 12] and algebraic properties [4, 9, 13] are going on. Recently, for a mathematical structure, say a lattice L, research about L-fuzzy approximation spaces is also progressing. A map $A : U \to L$ is called an L-fuzzy set on U, and $R : U \times U \to L$ is said to be an L-fuzzy relation on U. For a lattice L, by an *L-fuzzy approximation space*, we mean the mathematical structure (U, R), where U is a non-empty set and R is an L-fuzzy relation on U. The mathematical object L can be selected such as distributive lattices, Boolean algebras and residuated lattices to the purposes. Research on L-fuzzy approximation spaces is one of the hot fields of rough set theory, and many papers have been published so far [3, 7, 9, 14]. Since most of the proofs in such papers are element-based, it is a tedious work to check the conditions of the definitions one by one, and they are relatively long proofs. It is not easy to apply the results to other mathematical structures L.
In this paper, we consider L-fuzzy approximation spaces (U, R) as operators on L^U and $L^{U\times U}$ for a residuated lattice L, and provide operator-based proofs for their properties. Since the proofs are operator-based, they are relatively short and a good outlook. Therefore, the results can be easily applied to other cases. In order to give operator-based proofs of properties of L-fuzzy approximation spaces, we prepare the following definitions and basic properties.

Let L be a residuated lattice, which definition is given later. For any L-fuzzy approximation space (U, R), we define operators called *upper (lower) approximation operator* $\overline{\mathrm{R}}(\underline{\mathrm{R}}) : L^U \to L^U$ as follows. For any $A \in \mathrm{L}^U$,

$$\overline{\mathrm{R}}(A)(x) = \bigvee_{y\in U} \left(R(x, y) \odot A(y)\right),$$
$$\underline{\mathrm{R}}(A)(x) = \bigwedge_{y\in U} \left(R(x, y) \to A(y)\right).$$

From these operators, we consider an upper approximation $\overline{\mathrm{R}}(A)$ (lower approximation $\underline{\mathrm{R}}(A)$) of an L-fuzzy set A, respectively. This means, we can get operators $\overline{\mathrm{R}}, \underline{\mathrm{R}}$ on L^U from the L-fuzzy relation R.

Conversely, the question of whether we can construct an L-fuzzy relation by an operator on L^U arises naturally. However, this is trivially No!. Because if we consider the cardinalities of the set of L-fuzzy relations $\mathcal{R}_L(U) = L^{U\times U}$ and that of the class of operators $(L^U)^{L^U}$ on L^U, then those classes do not have the same cardinality. So, there is no one-to-one correspondence between L-fuzzy relations and operators on L^U.

On the other hand, taking into account the properties of L-fuzzy relations on U, we have another interesting problem:

1. Under what conditions of operators on L^U do we have an L-fuzzy relations from the operator?
2. If so, is the correspondence between such operators and the L-fuzzy relations one-to-one?

We give an affirmative answer to the problem in this paper. We also provide operator-based algebraic proofs to the results in [7] instead of original element-based proofs. This allows the above question to be treated more generally.

In addition, by considering L-fuzzy approximation spaces as a fuzzy-version of Kripke semantics in modal logic, the properties of L-fuzzy relation, such as reflexivity, symmetry, and transitivity, are easy to understand for using operators. This means, we have new knowledge about the relationship between L-fuzzy approximation spaces and modal logic.

2 Residuated lattices and fuzzy approximation spaces

Let U be a non-empty set and $\mathcal{L} =< L, \wedge, \vee, \odot, 0, 1 >$ be a complete residuated lattice, that is,

(i) $< L, \wedge, \vee, 0, 1 >$ is a complete bounded lattice;

(ii) $< L, \odot, 1 >$ is a commutative monoid;

(iii) For all $x, y, z \in L$,

$$x \odot y \leq z \Leftrightarrow x \leq y \to z.$$

For any element $x \in L$, we define $x' = x \to 0$. We denote a residuated lattice by its support set L of $\mathcal{L}$ for the sake of simplicity. We have the following basic properties of residuated lattices [2].

Proposition 2.1. *For all $x, y, z, x_i, y_i \in L$, we have*

(1) $0' = 1, 1' = 0$;

(2) $x \odot x' = 0$;

(3) $x \leq y \iff x \to y = 1$;

(4) $x \odot (x \to y) \leq y$;

(5) $x \leq y \implies x \odot z \leq y \odot z, z \to x \leq z \to y, y \to z \leq x \to z$;

(6) $1 \to x = x$;

(7) $x \vee (y \to z) \leq y \to (x \vee z)$;

(8) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$;

(9) $(\bigvee_i x_i)' = \bigwedge_i x_i'$;

(10) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$, $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$.

A mapping $A : U \to L$ ($i.e, A \in L^U$) is simply called an L-*fuzzy set* on U. For any element $a \in L$, we define L-fuzzy sets $\boldsymbol{a}$ and $\boldsymbol{a}_x$ for $x \in U$ as follows:

$$\boldsymbol{a}(x) = a \quad (\forall x \in U);$$
$$\boldsymbol{a}_x(y) = \begin{cases} a & (y = x) \\ 0 & (y \neq x) \end{cases} \quad (\forall x, y \in U).$$

Let $\mathbf{1}_S$ be the characteristic function of $S \subseteq U$. Thus, we have

$$\mathbf{1}_{U-\{x\}}(y) = \begin{cases} 1 & (y \neq x) \\ 0 & (y = x) \end{cases} \quad (\forall y \in U).$$

We define an order $\leq$ on L^U by the pointwise order, that is, for all $A, B \in L^U$,

$$A \leq B \text{ if and only if } A(x) \leq B(x) \text{ for all } x \in U.$$

Then $\mathbf{0}$ and $\mathbf{1}$ defined by

$$\mathbf{0}(x) = 0 \quad (\forall x \in U),$$
$$\mathbf{1}(x) = 1 \quad (\forall x \in U),$$

are the smallest and largest elements in L^U, respectively. Mathematical structures of L^U inherit from those of L as follows: For all $A, B, A_i \in L^U$, if we define

$$\begin{aligned}
A'(x) &= (A(x))', \quad \forall x \in U \\
(A \wedge B)(x) &= A(x) \wedge B(x), \quad \forall x \in U \\
(A \vee B)(x) &= A(x) \vee B(x), \quad \forall x \in U \\
(A \odot B)(x) &= A(x) \odot B(x), \quad \forall x \in U \\
(A \to B)(x) &= A(x) \to B(x), \quad \forall x \in U \\
(\bigwedge_i A_i)(x) &= \bigwedge_i A_i(x), \quad \forall x \in U \\
(\bigvee_i A_i)(x) &= \bigvee_i A_i(x), \quad \forall x \in U,
\end{aligned}$$

then $(L^U, \wedge, \vee, \odot, \mathbf{0}, \mathbf{1})$ is also a complete residuated lattice. We note $\mathbf{1}_{U-\{x\}} = (\mathbf{1}_x)'$.

We recall some definitions about L-fuzzy relations. Let U and V be non-empty sets. In general, a map $R : U \times V \to L$ is called an *L-fuzzy relation* from U to V. For the case of $U = V$, a map $R : U \times U \to L$ is simply called an L-fuzzy relation on U. An L-fuzzy relation R on U is also called

(1) *reflexive* if $R(x, x) = 1 \quad (\forall\, x \in U)$;

(2) *symmetric* if $R(x, y) = R(y, x) \quad (\forall\, x, y \in U)$;

(3) *transitive* if $R(x, y) \odot R(y, z) \leq R(x, z) \quad (\forall\, x, y, z \in U)$;

(4) *serial* if $\bigvee_{y \in U} R(x, y) = 1 \quad (\forall\, x \in U)$;

(5) *Euclidean* if $R(x, y) \odot R(x, z) \leq R(y, z) \quad (\forall\, x, y, z \in U)$.

For a non-empty set U and an L-fuzzy relation on U, a structure (U, R) is called an *L-fuzzy approximation space*. According to [7, 10, 11], we define an *upper (lower) L-fuzzy approximation operators* $\overline{\mathrm{R}}$ ($\underline{\mathrm{R}}$) $: L^U \to L^U$ as follows:

$$\overline{\mathrm{R}}(A)(x) = \bigvee_{y \in U} \left(R(x, y) \odot A(y)\right),$$
$$\underline{\mathrm{R}}(A)(x) = \bigwedge_{y \in U} \left(R(x, y) \to A(y)\right).$$

We mainly treat upper L-fuzzy approximation operators $\overline{\mathrm{R}}$ in this paper.

By $\mathcal{R}_L(U)$, we mean the class of all L-fuzzy relations on U. We note that $\mathcal{R}_L(U)$ is the complete residuated lattice, because, since $\mathcal{R}_L(U) = L^{U \times U}$, it inherits properties from those of L.

Proposition 2.2. *Let R be an L-fuzzy relation on U. Then for all $a \in L$ and $A, B, A_i \in L^U$, we have*

(1) $\overline{\mathrm{R}}(\boldsymbol{a} \odot (\bigvee_i A_i)) = \boldsymbol{a} \odot (\bigvee_i \overline{\mathrm{R}}(A_i))$;

(2) $\overline{\mathrm{R}}(\boldsymbol{a} \odot A) = \boldsymbol{a} \odot \overline{\mathrm{R}}(A)$;

(3) $\overline{\mathrm{R}}(\bigvee_i A_i) = \bigvee_i \overline{\mathrm{R}}(A_i)$;

(4) $\overline{\mathrm{R}}(\boldsymbol{a}) \leq \boldsymbol{a} \leq \underline{\mathrm{R}}(\boldsymbol{a})$;

(5) $A \leq B \Rightarrow \overline{\mathrm{R}}(A) \leq \overline{\mathrm{R}}(B),\ \underline{\mathrm{R}}(A) \leq \underline{\mathrm{R}}(B)$;

(6) $\underline{\mathrm{R}}(\bigwedge_i A_i) = \bigwedge_i \underline{\mathrm{R}}(A_i)$;

(7) $\underline{\mathrm{R}}(\boldsymbol{a} \vee A) \geq \boldsymbol{a} \vee \underline{\mathrm{R}}(A)$, *hence* $\underline{\mathrm{R}}(\boldsymbol{a} \vee (\bigwedge_i A_i)) \geq \boldsymbol{a} \vee (\bigwedge_i \underline{\mathrm{R}}(A_i))$;

(8) $\underline{\mathrm{R}}(\boldsymbol{a} \to A) = \boldsymbol{a} \to \underline{\mathrm{R}}(A)$;

(9) $\boldsymbol{a} \odot \underline{\mathrm{R}}(A) \leq \underline{\mathrm{R}}(\boldsymbol{a} \odot A)$.

Proof. We only prove the cases of (1) and (7). The other cases can be proved easily.

(1) This is proved as follows.

$$\begin{aligned}
\overline{\mathrm{R}}(\boldsymbol{a} \odot (\bigvee_i A_i))(x) &= \bigvee_{y \in U} \left(R(x,y) \odot (\boldsymbol{a} \odot (\bigvee_i A_i))(y) \right) \\
&= \bigvee_{y \in U} \left(R(x,y) \odot a \odot \bigvee_i A_i(y) \right) \\
&= a \odot \bigvee_{y \in U} \left(R(x,y) \odot \bigvee_i A_i(y) \right) \\
&= a \odot \bigvee_{y \in U} \left(\bigvee_i (R(x,y) \odot A_i(y)) \right) \\
&= a \odot \bigvee_{y \in U} \bigvee_i (R(x,y) \odot A_i(y)) \\
&= a \odot \bigvee_i \left(\bigvee_{y \in U} (R(x,y) \odot A_i(y)) \right) \\
&= a \odot \bigvee_i \overline{\mathrm{R}}(A_i)(x) \\
&= (\boldsymbol{a} \odot (\bigvee_i \overline{\mathrm{R}}(A_i)))(x).
\end{aligned}$$

(7) Since $\underline{\mathrm{R}}$ is an order preserving operator, it follows from (4) that

$$\boldsymbol{a} \vee \underline{\mathrm{R}}(A) \leq \underline{\mathrm{R}}(\boldsymbol{a}) \vee \underline{\mathrm{R}}(A) \leq \underline{\mathrm{R}}(\boldsymbol{a} \vee A).$$

□

Remark 2.3. *For (4), it follows that* $\overline{\mathrm{R}}(\boldsymbol{a}) = \boldsymbol{a}$ *if and only if* $\bigvee_{y \in U} R(x, y) = 1$ *for all* $x \in U$*, that is,* R *is serial.*

We note that a pair of two results (2) and (3) is equivalent to (1). Moreover, the results (5),(6),(8) and (9) are obtained by a general result that $\overline{\mathrm{R}}$ and $\underline{R^{-1}}$ (and also $\overline{R^{-1}}$ and $\underline{R}$) forms an adjoint pair (denoted by $\overline{\mathrm{R}} \dashv \underline{R^{-1}}$), that is,

Proposition 2.4. *For any* L*-fuzzy relation* R *on* U*, we have*

$$\overline{\mathrm{R}}(A) \leq B \;\Leftrightarrow\; A \leq \underline{\mathrm{R}^{-1}}(B) \quad (\forall A, B \in L^U);$$
$$\overline{\mathrm{R}^{-1}}(A) \leq B \;\Leftrightarrow\; A \leq \underline{R}(B) \quad (\forall A, B \in L^U).$$

Proof. Since

$$\begin{aligned}
\overline{\mathrm{R}}(A) \leq B &\Leftrightarrow (\overline{\mathrm{R}}(A))(x) \leq B(x) \quad (\forall x \in U) \\
&\Leftrightarrow \bigvee_{y \in U} (R(x, y) \odot A(y)) \leq B(x) \quad (\forall x \in U) \\
&\Leftrightarrow R(x, y) \odot A(y) \leq B(x) \quad (\forall x, y \in U) \\
&\Leftrightarrow A(y) \leq R(x, y) \to B(y) = \mathrm{R}^{-1}(y, x) \to B(x) \quad (\forall x, y \in U) \\
&\Leftrightarrow A(y) \leq \bigwedge_{x \in U} (\mathrm{R}^{-1}(y, x) \to B(x)) \quad (\forall y \in U) \\
&\Leftrightarrow A(y) \leq (\underline{\mathrm{R}^{-1}}(B))(y) \quad (\forall y \in U) \\
&\Leftrightarrow A \leq \underline{R^{-1}}(B),
\end{aligned}$$

a pair of two operators $\overline{\mathrm{R}}$ and $\underline{\mathrm{R}^{-1}}$ forms the adjoint pair. Another case can be proved similarly if we take R to be R^{-1}. □

For example, the result (9) $\boldsymbol{a} \odot \underline{\mathrm{R}}(A) \leq \underline{\mathrm{R}}(\boldsymbol{a} \odot A)$ can be proved as follows. Since $\overline{R^{-1}} \dashv \underline{\mathrm{R}}$, it is sufficient to show $\overline{R^{-1}}(\boldsymbol{a} \odot \underline{\mathrm{R}}(A)) \leq \boldsymbol{a} \odot A$. It is obvious that $\overline{R^{-1}}(\boldsymbol{a} \odot A) = \boldsymbol{a} \odot \overline{R^{-1}}(\underline{\mathrm{R}}(A))$ and $\overline{R^{-1}}(\underline{\mathrm{R}}(A)) \leq A$ by $\underline{\mathrm{R}}(A) \leq \underline{\mathrm{R}}(A)$. Therefore, we get

$$\overline{R^{-1}}(\boldsymbol{a} \odot \underline{\mathrm{R}}(A)) = \boldsymbol{a} \odot \overline{R^{-1}}(\underline{\mathrm{R}}(A)) \leq \boldsymbol{a} \odot A.$$

Corollary 2.5. *If* R *is symmetric, then* $\overline{\mathrm{R}} \dashv \underline{\mathrm{R}}$*.*

Let U and V be non-empty sets. An operator $\mathcal{F} : L^U \to L^V$ is called *normal* if it satisfies the condition:

$$\mathcal{F}(\boldsymbol{a} \odot \bigvee_i A_i) = \boldsymbol{a} \odot \bigvee_i \mathcal{F}(A_i) \qquad (\forall\, a \in L, \forall\, A_i \in L^U).$$

It is easy to prove that

Proposition 2.6. *For an operator* $\mathcal{F} : L^U \to L^V$*,* $\mathcal{F}$ *is normal if and only if it satisfies the conditions: For all* $a \in L, A_i \in L^U$*,*

(N1) $\mathcal{F}(\boldsymbol{a} \odot A) = \boldsymbol{a} \odot \mathcal{F}(A)$*;*

(N2) $\mathcal{F}(\bigvee_i A_i) = \bigvee_i \mathcal{F}(A_i)$*.*

It follows from the Proposition 2.2 that

Corollary 2.7. *For every L-fuzzy relation R on U, the operator $\overline{\mathrm{R}}$ is normal.*

It is clear to show the next result.

Proposition 2.8. *For all normal operators $\mathcal{F}, \mathcal{G} : L^U \to L^U$, the composition operator $\mathcal{F} \circ \mathcal{G}$ defined by $(\mathcal{F} \circ \mathcal{G})(A) = \mathcal{F}(\mathcal{G}(A))$ $(\forall\, A \in L^U)$ is a normal operator.*

For any map $\varphi : U \to V$, the *Zadeh's fuzzy backward operator* (simply backward operator) [7], $\varphi^{\leftarrow} : L^V \to L^U$ is defined by

$$\varphi^{\leftarrow}(B)(x) = B(\varphi(x)) \quad (\forall B \in L^V,\ \forall x \in U).$$

Proposition 2.9. *For every map $\varphi : U \to V$, the backward operator $\varphi^{\leftarrow} : L^V \to L^U$ is normal.*

Proof. We show that for any $x \in U, b \in L, B, B_i \in L^V$,

(1) $\varphi^{\leftarrow}(\boldsymbol{b} \odot B)(x) = \boldsymbol{b} \odot \varphi^{\leftarrow}(B)$ and

(2) $\varphi^{\leftarrow}(\bigvee_i B_i) = \bigvee_i \varphi^{\leftarrow}(B_i)$.

For the case (1), we have the following sequence of equations.

$$\begin{aligned}
\varphi^{\leftarrow}(\boldsymbol{b} \odot B)(x) &= (\boldsymbol{b} \odot B)(\varphi(x)) = \boldsymbol{b}(\varphi(x)) \odot B(\varphi(x)) \\
&= b \odot (\varphi^{\leftarrow}(B))(x) = \boldsymbol{b}(x) \odot (\varphi^{\leftarrow}(B))(x) \\
&= (\boldsymbol{b} \odot \varphi^{\leftarrow}(B))(x).
\end{aligned}$$

Therefore, we get $\varphi^{\leftarrow}(\boldsymbol{b} \odot B)(x) = \boldsymbol{b} \odot \varphi^{\leftarrow}(B)$.

As to the case (2), we also have

$$\varphi^{\leftarrow}(\bigvee_i B_i)(x) = (\bigvee_i B_i)(\varphi(x)) = \bigvee_i (B_i(\varphi(x)) = \bigvee_i (\varphi^{\leftarrow} B_i)(x) = (\bigvee_i \varphi^{\leftarrow} B_i)(x),$$

and thus $\varphi^{\leftarrow}(\bigvee_i B_i) = \bigvee_i B_i$.

Therefore, the backward operator $\varphi^{\leftarrow}$ is normal. □

In order to show the fundamental and essential property about normal operators, we need the following lemma [11].

Lemma 2.10. *For any L-fuzzy set $A \in L^U$, $A = \bigvee_{x \in U}(\boldsymbol{A}(x) \odot \mathbf{1}_x)$, where $\boldsymbol{A}(x)$ and $\mathbf{1}_x$ are defined respectively by*

$$\boldsymbol{A}(x)(t) = A(x) \in L \quad (\forall x \in U) \text{ and } \mathbf{1}_x(t) = \begin{cases} 1 & (t = x) \\ 0 & (\text{otherwise}). \end{cases}$$

For two operators $\mathcal{F}, \mathcal{G} : L^U \to L^V$, a partial order $\leq$ is defined as usual,

$$\mathcal{F} \leq \mathcal{G} \text{ if and only if } \mathcal{F}(A) \leq \mathcal{G}(A) \text{ for all } A \in L^U.$$

Now, we prove the fundamental and important property about normal operators. It says that the partial order $\leq$ on normal operators are determined only by the element $\mathbf{1}_x \in L^U$ for all $x \in U$.

Theorem 2.11. *For two normal operators $\mathcal{F}, \mathcal{G} : L^U \to L^V$, we have*

$$\mathcal{F} \leq \mathcal{G} \textit{ if and only if } \mathcal{F}(\mathbf{1}_x) \leq \mathcal{G}(\mathbf{1}_x) \textit{ for all } x \in U.$$

Proof. It is sufficient to show that $\mathcal{F} \leq \mathcal{G}$ if $\mathcal{F}(\mathbf{1}_x) \leq \mathcal{G}(\mathbf{1}_x)$ for all $x \in U$.

Let A be arbitrary element in L^U. Since $A = \bigvee_{x \in U}(\mathbf{A}(x) \odot \mathbf{1}_x)$, we have

$$\begin{aligned}
\mathcal{F}(A) = \mathcal{F}\left(\bigvee_{x\in U}(\mathbf{A}(x)\odot \mathbf{1}_x)\right) &= \bigvee_{x\in U}\mathcal{F}(\mathbf{A}(x)\odot \mathbf{1}_x) \quad (\because\ \mathcal{F} \text{ is normal})\\
&= \bigvee_{x\in U}(\mathbf{A}(x)\odot \mathcal{F}(\mathbf{1}_x)) \quad (\because\ \mathcal{F} \text{ is normal})\\
&\leq \bigvee_{x\in U}(\mathbf{A}(x)\odot \mathcal{G}(\mathbf{1}_x)) = \mathcal{G}\left(\bigvee_{x\in U}(\mathbf{A}(x)\odot \mathbf{1}_x)\right) \quad (\because\ \mathcal{G} \text{ is normal})\\
&= \mathcal{G}(A).
\end{aligned}$$

Therefore, we get $\mathcal{F}(A) \leq \mathcal{G}(A)$ for all $A \in L^U$. This means that $\mathcal{F} \leq \mathcal{G}$. □

Corollary 2.12. *For normal operators $\mathcal{F}, \mathcal{G}$,*

$$\mathcal{F} = \mathcal{G} \text{ if and only if } \mathcal{F}(\mathbf{1}_x) = \mathcal{G}(\mathbf{1}_x) \text{ for all } x \in U.$$

Now we consider properties of L-fuzzy relation from U to V. Let R and S be L-fuzzy relation from U to V. We define a partial order $\sqsubseteq$ on the set of all L-fuzzy relations from U to V as follows:

$$R \sqsubseteq S \ \Leftrightarrow\ R(x,y) \leq S(x,y) \quad (\forall\, x \in U, y \in V).$$

Proposition 2.13. *For any L-fuzzy relation R, S from U to V, we have*

$$R \sqsubseteq S \quad \text{if and only if} \quad \overline{\mathrm{R}} \leq \overline{\mathrm{S}}.$$

Proof. Suppose $R \sqsubseteq S$, that is, $R(x,y) \leq S(x,y)$ for all $x \in U, y \in V$. For all $B \in L^V$ and $x \in U$, since

$$\begin{aligned}
(\overline{\mathrm{R}}(B))(x) &= \bigvee_{y\in V}(R(x,y)\odot B(y))\\
&\leq \bigvee_{y\in V}(S(x,y)\odot B(y)) = (\overline{\mathrm{S}}(B))(x),
\end{aligned}$$

we have $\overline{\mathrm{R}}(B) \leq \overline{\mathrm{S}}(B)$ for all $B \in L^V$ and thus $\overline{\mathrm{R}} \leq \overline{\mathrm{S}}$.

Conversely, we assume $\overline{\mathrm{R}} \leq \overline{\mathrm{S}}$, namely, $\overline{\mathrm{R}}(B) \leq \overline{\mathrm{S}}(B)$ for all $B \in L^V$. Let $x \in U, y \in V$. If we take $\mathbf{1}_y \in L^V$ as $B \in L^V$, that is, $\overline{\mathrm{R}}(\mathbf{1}_y) \leq \overline{\mathrm{S}}(\mathbf{1}_y)$, then

$$(\overline{\mathrm{R}}(\mathbf{1}_y))(x) \leq (\overline{\mathrm{S}}(\mathbf{1}_y))(x).$$

By definition of $\overline{\mathrm{R}}$, we have

$$(\overline{\mathrm{R}}(\mathbf{1}_y))(x) = \bigvee_{t\in V}(R(x,t)\odot \mathbf{1}_y(t)) = R(x,y)\odot \mathbf{1}_y(y) = R(x,y).$$

Similarly, $(\overline{\mathrm{S}}(\mathbf{1}_y))(x) = S(x,y)$. It follows that

$$R(x,y) \leq S(x,y) \quad (\forall\, x \in U, y \in V).$$

This means that $R \sqsubseteq S$. □

Corollary 2.14. *For any L-fuzzy relations R, S on U, i.e. $R, S; U \times U \to L$, $R = S$ if and only if $\overline{\mathrm{R}} = \overline{\mathrm{S}}$.*

3 Fuzzy natural transformation

Let (X, R) and (Y, S) be two L-fuzzy approximation spaces. In [7], two important notions about L-fuzzy approximation spaces are defined and studied. A one-to-one map $\varphi : X \to Y$ is called an *upper fuzzy backward natural transformation* from (X, R) to (Y, S) if

$$\overline{\mathrm{R}}(\varphi^{\leftarrow}(B)) \leq \varphi^{\leftarrow}(\overline{\mathrm{S}}(B)), \quad (\forall B \in L^Y).$$

It is also represented by $\overline{\mathrm{R}} \circ \varphi^{\leftarrow} \leq \varphi^{\leftarrow} \circ \overline{\mathrm{S}}$ in operator-based notation.

$$\begin{array}{ccc} L^V & \xrightarrow{\varphi^{\leftarrow}} & L^U \\ {\scriptstyle \overline{\mathrm{S}}}\downarrow & & \downarrow{\scriptstyle \overline{\mathrm{R}}} \\ L^V & \xrightarrow[\varphi^{\leftarrow}]{} & L^U \end{array}$$

A map $\varphi : X \to Y$ is called *relation preserving* if

$$R(x, y) \leq S(\varphi(x), \varphi(y)), \quad (\forall x, y \in U).$$

The following result is proved in [7]:

> Proposition 4.1 Let (X, R) and (Y, S) be two fuzzy approximation spaces and $\varphi : X \to Y$ be a one-to-one map. Then φ is an upper fuzzy backward natural transformation if and only if φ is relation preserving.

The result is proved by an element-based method, so it has a long proof and also has few generalizations. We here provide operator-based proof about it. Our proof makes the result to apply to more wide cases. We prepare some results to do so.

At first, we note that

$$R(x, y) = \overline{\mathrm{R}}(\mathbf{1}_y)(x) \text{ and } S(\varphi(x), \varphi(y)) = \varphi^{\leftarrow}(\overline{\mathrm{S}}(\mathbf{1}_{\varphi(y)}))(x), \quad (\forall x, y \in U).$$

So, a relation preserving map $\varphi : U \to V$ from (U, R) to (V, S) can be represented by

$$\overline{\mathrm{R}}(\mathbf{1}_u) \leq \varphi^{\leftarrow}(\overline{\mathrm{S}}(1_{\varphi(u)})), \quad (\forall u \in U).$$

Lemma 3.1. *For any map $\varphi : U \to V$, we have*

$$\mathbf{1}_u \leq \varphi^{\leftarrow}(\mathbf{1}_{\varphi(u)}), \quad (\forall u \in U).$$

Moreover, if φ is injective (one-to-one), then

$$\mathbf{1}_u = \varphi^{\leftarrow}(\mathbf{1}_{\varphi(u)}), \quad (\forall u \in U).$$

Proof. It follows from $\{t \in U \mid u = t\} \subseteq \{t \in U \mid \varphi(u) = \varphi(t)\}$ that $\mathbf{1}_u(t) \leq \varphi^{\leftarrow}(\mathbf{1}_{\varphi(u)})(t)$ for all $t \in U$ and thus

$$\mathbf{1}_u \leq \varphi^{\leftarrow}(\mathbf{1}_{\varphi(u)}), \quad (\forall u \in U).$$

Moreover, if φ is injective, since $\{t \in U \mid t = u\} = \{t \in U \mid \varphi(u) = \varphi(t)\}$, then we get $\mathbf{1}_u(t) = \varphi^{\leftarrow}(\mathbf{1}_{\varphi(u)})(t)$ for all $t \in U$, thus

$$\mathbf{1}_u = \varphi^{\leftarrow}(\mathbf{1}_{\varphi(u)}), \quad (\forall u \in U).$$

□

Lemma 3.2. *Let $\varphi : U \to V$ be a map and R, S are L-fuzzy relations on U and on V, respectively. Then we have*

(1) $(\overline{\mathrm{R}} \circ \varphi^{\leftarrow})(B) = (\overline{\mathrm{R}} \circ \varphi^{\leftarrow})(B^*)$ $(\forall B \in L^V)$, *where B^* is defined by*

$$B^*(v) = \begin{cases} B(v) & (v \in \varphi(U)) \\ 0 & (v \notin \varphi(U)) \end{cases},$$

(2) $(\varphi^{\leftarrow} \circ \overline{\mathrm{S}})(B^*) \le (\varphi^{\leftarrow} \circ \overline{\mathrm{S}})(B)$, $(\forall B \in L^V)$.

Proof. For the case (1), let $B \in L^V$ and $x \in U$. Since

$$\begin{aligned}(\overline{\mathrm{R}} \circ \varphi^{\leftarrow})(B)(x) &= \overline{\mathrm{R}}(\varphi^{\leftarrow} B)(x) \\ &= \bigvee_{y \in U} (R(x,y) \odot (\varphi^{\leftarrow} B)(y)) \\ &= \bigvee_{y \in U} (R(x,y) \odot B(\varphi(y))) \\ &= \bigvee_{y \in U} (R(x,y) \odot B^*(\varphi(y))) \\ &= (\overline{\mathrm{R}} \circ \varphi^{\leftarrow})(B^*)(x),\end{aligned}$$

we get $(\overline{\mathrm{R}} \circ \varphi^{\leftarrow})(B) = (\overline{\mathrm{R}} \circ \varphi^{\leftarrow})(B^*)$ for all $B \in L^V$.

The case (2) can be proved similarly. □

Now, we provide an operator-based proof to the result in [7] above.

Theorem 3.3. *Let (X, R) and (Y, S) be two fuzzy approximation spaces and $\varphi : X \to Y$ a one-to-one map. Then φ is a (upper) fuzzy backward natural transformation if and only if φ is relation preserving, that is,*

$$\varphi \text{ is relation preserving } \Leftrightarrow\ \overline{\mathrm{R}} \circ \varphi^{\leftarrow} \le \varphi^{\leftarrow} \circ \overline{\mathrm{S}}.$$

Proof. $(\Rightarrow)$ We assume that a one-to-one map φ is relation preserving, that is, $\overline{\mathrm{R}}(\mathbf{1}_u) \le \varphi^{\leftarrow}(\overline{\mathrm{S}}(1_{\varphi(u)}))$ for all $u \in U$. Since the operators $\overline{\mathrm{R}} \circ \varphi^{\leftarrow}$ and $\varphi^{\leftarrow} \circ \overline{\mathrm{S}}$ are both normal, in order to show $\overline{\mathrm{R}}(\varphi^{\leftarrow}(B)) \le \varphi^{\leftarrow}(\overline{\mathrm{S}}(B))$ $(\forall B \in L^Y)$, it is sufficient to show by Lemma 3.2 that

$$(\overline{\mathrm{R}} \circ \varphi^{\leftarrow})(\mathbf{1}_{\varphi(u)}) \le (\varphi^{\leftarrow} \circ \overline{\mathrm{S}})(\mathbf{1}_{\varphi(u)}) \quad (\forall u \in U).$$

Since φ is the one-to-one map, we have $\overline{\mathrm{R}}(\mathbf{1}_u) = \overline{\mathrm{R}}(\varphi^{\leftarrow}(\mathbf{1}_{\varphi(u)}))$. It follows from assumption that

$$\begin{aligned}(\overline{\mathrm{R}} \circ \varphi^{\leftarrow})(\mathbf{1}_{\varphi(u)}) &= \overline{\mathrm{R}}(\varphi^{\leftarrow}(\mathbf{1}_{\varphi(u)})) = \overline{\mathrm{R}}(\mathbf{1}_u) \quad (\because\ \varphi \text{ is one-to-one}) \\ &\le (\varphi^{\leftarrow} \circ \overline{\mathrm{S}})(\mathbf{1}_{\varphi(u)}) \quad (\text{by assumption})\end{aligned}$$

$(\Leftarrow)$ Conversely, we suppose that $\overline{\mathrm{R}} \circ \varphi^{\leftarrow} \le \varphi^{\leftarrow} \circ \overline{\mathrm{S}}$. Since φ is one-to-one, we have $\mathbf{1}_u = \varphi^{\leftarrow}(\mathbf{1}_{\varphi(u)})$ and

$$\overline{\mathrm{R}}(\mathbf{1}_u) = \overline{\mathrm{R}}(\varphi^{\leftarrow}(\mathbf{1}_{\varphi(u)})) = (\overline{\mathrm{R}} \circ \varphi^{\leftarrow})(\mathbf{1}_{\varphi(u)}) \le (\varphi^{\leftarrow} \circ \overline{\mathrm{S}})(\mathbf{1}_{\varphi(u)}).$$

Therefore, φ is the relation preserving map. □

4 Normal operators and L-fuzzy relations

In the last section, we consider a relation between the class $\mathcal{N}_L(U)$ of all normal operators on U and the class $\mathcal{R}_L(U)$ of all L-fuzzy relations on U. We show that there is a one-to-one correspondence between them, and these classes are isomorphic as lattices.

Let U be a non-empty set and $\mathcal{F} : L^U \to L^U$ be an operator. We define a (upper) fuzzy transformation system according to [7]. A structure $(U, \mathcal{F})$ is called a *(upper) fuzzy transformation system* if

(1) $A(x) \leq \mathcal{F}(A)(x), \quad (\forall A \in L^U,\ x \in U);$

(2) $\mathcal{F}(\bigvee_i A_i) = \bigvee_i \mathcal{F}(A_i), \quad (\forall \{A_i \mid i \in I\} \subseteq L^U);$

(3) $\mathcal{F}(\boldsymbol{a} \odot A) = \boldsymbol{a} \odot \mathcal{F}(A), \quad (\forall a \in L, A \in L^U);$

Remark 4.1. *In our definition of a fuzzy transformation system is different from the original one in* [7]. *In* [7]*, the following condition*

(4) *core*$(\mathcal{F}(1_x)) \neq \emptyset$*, where core*$(A) = \{x \mid A(x) = 1, x \in U\}$*,*

is assumed for the definition of fuzzy transformation systems. However, the condition can be obtained from the condition (1) as follows.

Since $\mathbf{1}_x \leq \mathcal{F}(\mathbf{1}_x)$ by (1) and $\mathbf{1}_x(x) = 1$, we have $1 = \mathbf{1}_x(x) \leq \mathcal{F}(\mathbf{1}_x)(x)$ and thus $1 = \mathcal{F}(\mathbf{1}_x)(x)$. This means that $x \in \mathrm{core}(\mathcal{F}(\mathbf{1}_x))$ and $\mathrm{core}(\mathcal{F}(\mathbf{1}_x)) \neq \emptyset$. Therefore, the original condition (4) above is redundant.

Moreover, an operator $\mathcal{F} : L^U \to L^U$ is called *reflexive* if $A \leq \mathcal{F}(A)$ for all $A \in L^U$. Therefore, the notion of fuzzy transformation systems $\mathcal{F} : L^U \to L^U$ is precisely the same as that of reflexive normal operators.

We show that there is a one to one correspondence between the normal operators and the L-fuzzy relations on U. At first, we treat a general case.

Theorem 4.2. *Let U, V be non-empty sets. For any normal operator $\mathcal{F} : L^V \to L^U$, there exists a unique L-fuzzy relation $R : U \times V \to L$ from U to V such that $\mathcal{F} = \overline{\mathrm{R}}$.*

Proof. For any normal operator $\mathcal{F} : L^V \to L^U$, we define $R : U \times V \to L$ by

$$R(x, y) = \mathcal{F}(\mathbf{1}_y)(x), \quad (\forall\, x \in U,\ y \in V).$$

It is clear that $\overline{\mathrm{R}}(\mathbf{1}_y) = \mathcal{F}(\mathbf{1}_y)$ for all $y \in V$. Since $\overline{\mathrm{R}}$ ad $\mathcal{F}$ are normal, we get

$$\overline{\mathrm{R}} = \mathcal{F}.$$

The uniqueness is proved as follows. If $\overline{\mathrm{R}} = \mathcal{F} = \overline{\mathrm{S}}$ for two L-fuzzy relations R and S, then we have $\overline{\mathrm{R}} = \overline{\mathrm{S}}$ and thus $R = S$ by Corollary 2.14. □

In any fuzzy transformation system $(U, \mathcal{F})$, since $\mathcal{F}$ is the reflexive normal operator, the result holds immediately.

Corollary 4.3 (Theorem 5.1 in [7])**.** *For any (upper) fuzzy transformation system $(U, \mathcal{F})$, that is, $\mathcal{F} : L^U \to L^U$, there exists a unique L-fuzzy relation $\overline{\mathrm{R}}$ on U such that $F = \overline{\mathrm{R}}$.*

Now we consider the relation between normal operators and L-fuzzy relations. For an operator $\mathcal{F} : L^U \to L^U$, an L-fuzzy relation $R_{\mathcal{F}} : U \times U \to L$ is defined by

$$R_{\mathcal{F}}(x, y) = (\mathcal{F}(\mathbf{1}_y))(x), \quad (\forall x, y \in U).$$

Conversely, for an L-fuzzy relation $R : U \times U \to L$, we define an operator $\mathcal{F}_R : L^U \to L^U$ by $\mathcal{F}_R = \overline{\mathrm{R}}$, that is,

$$(\mathcal{F}_R(A))(x) = \bigvee_{y \in U} (R(x, y) \odot A(y)), \quad (\forall x \in U).$$

Theorem 4.4. *Let $\mathcal{F} : L^U \to L^U$ be a normal operator and $R : U \times U \to L$ an L-fuzzy relation.*

(1) $\mathcal{F} = \overline{R_{\mathcal{F}}}$*;*

(2) $R = R_{(\mathcal{F}_R)}$*;*

(3) $R \sqsubseteq S$ *iff* $\mathcal{F}_R \leq \mathcal{F}_S$ *(i.e. iff* $\overline{\mathrm{R}} \leq \overline{\mathrm{S}}$*).*

Therefore, the class $\mathcal{N}_L(U)$ of all normal operators on U and the class $\mathcal{R}_L(U)$ of all L-fuzzy relations on U both form lattices and they are isomorphic:

$$\mathcal{N}_L(U) \cong \mathcal{R}_L(U).$$

Proof. (1) Since $R_{\mathcal{F}} : U \times U \to L$ is an L-fuzzy relation, the operator $\overline{R_{\mathcal{F}}}$ is normal. For all $x, y \in U$, we have

$$(\overline{R_{\mathcal{F}}}(\mathbf{1}_y))(x) = \bigvee_{t \in U} (R_{\mathcal{F}}(x, t) \odot \mathbf{1}_y(t)) = R_{\mathcal{F}}(x, y) = (\mathcal{F}(\mathbf{1}_y))(x) \quad (\forall x \in U).$$

and $\overline{R_{\mathcal{F}}}(\mathbf{1}_y) = \mathcal{F}(\mathbf{1}_y)$ for all $y \in U$. Since $\overline{R_{\mathcal{F}}}, \mathcal{F}$ are normal, this means that $\mathcal{F} = \overline{R_{\mathcal{F}}}$.

(2) For all $x, y \in U$, we get

$$\begin{aligned}(\overline{R_{(\mathcal{F}_R)}}(\mathbf{1}_y))(x) &= \bigvee_{t \in U} (R_{(\mathcal{F}_R)}(x, t) \odot \mathbf{1}_y(t)) = R_{(\mathcal{F}_R)}(x, y) \\ &= (\mathcal{F}_R(\mathbf{1}_y))(x) = (\overline{\mathrm{R}}(\mathbf{1}_y))(x),\end{aligned}$$

and $\overline{R_{(\mathcal{F}_R)}}(\mathbf{1}_y) = \overline{\mathrm{R}}(\mathbf{1}_y)$. The fact that $\overline{\mathrm{R}}, \overline{R_{(\mathcal{F}_R)}}$ are both normal operators implies $\overline{R_{(\mathcal{F}_R)}} = \overline{\mathrm{R}}$ and $R = R_{(\mathcal{F}_R)}$.

(3) Proposition 2.13.

It follows from the above that a map $\xi : \mathcal{N}_L(U) \to \mathcal{R}_L(U)$ defined by

$$\xi(\mathcal{F}) = R_{\mathcal{F}}, \quad (\forall \mathcal{F} \in \mathcal{N}_L(U)),$$

gives a lattice isomorphism between $\mathcal{N}_L(U)$ and $\mathcal{R}_L(U)$. □

Let $\mathcal{F} : L^U \to L^U$ be a normal operator. For a normal operator $\mathcal{F}$, it is called *reflexive* if $A \leq \mathcal{F}(A)$ for all $A \in L^U$. The following results are proved in [11]. However, we provide the proofs using the normality property, that is, the operator $\overline{\mathrm{R}}$ or $R_{\mathcal{F}}$ is determined by only $\mathbf{1}_x$ for all $x \in U$.

Proposition 4.5. [11] *Let $\mathcal{F}$ be a normal operator $\mathcal{F} : L^U \to L^U$. Then*

$\mathcal{F}$ is reflexive if and only if $R_{\mathcal{F}}$ is reflexive.

Proof. Let $\mathcal{F}$ be reflexive. Since

$$R_{\mathcal{F}}(x,x) = \overline{R_{\mathcal{F}}}(\mathbf{1}_x)(x) = \mathcal{F}((\mathbf{1}_x))(x) \geq \mathbf{1}_x(x) = 1,$$

we have $R_{\mathcal{F}}(x,x) = 1$ for all $x \in U$, that is, $R_{\mathcal{F}}$ is reflexive.

Conversely, suppose that $R_{\mathcal{F}}$ is reflexive. It is sufficient to show that $\mathbf{1}_x \leq \mathcal{F}(\mathbf{1}_x)$ for all $x \in U$, because $\mathbf{1}$ and $\mathcal{F}$ are normal. Since $R_{\mathcal{F}}$ is reflexive, we have

$$\mathbf{1}_x(t) \leq R_{\mathcal{F}}(t,x) = \mathcal{F}(\mathbf{1}_x)(t), \quad (\forall\, t, x \in U),$$

and $\mathbf{1}_x \leq \mathcal{F}(\mathbf{1}_x)$ for all $x \in U$. This means that the operator $\mathcal{F}$ is reflexive. □

A normal operator $\mathcal{F} : L^U \to L^U$ is called *transitive* if $\mathcal{F}(\mathcal{F}(A)) \leq \mathcal{F}(A)$ for all $A \in L^U$

Theorem 4.6. [11] *Let $\mathcal{F} : L^U \to L^U$ be a normal operator. Then*

$\mathcal{F}$ is transitive if and only if $R_{\mathcal{F}}$ is transitive.

Proof. We suppose that $\mathcal{F}$ satisfies $\mathcal{F}(\mathcal{F}(A)) \leq \mathcal{F}(A)$ for all $A \in L^U$. Since $\mathcal{F}(\mathcal{F}(\mathbf{1}_z)) \leq \mathcal{F}(\mathbf{1}_z)$, for all $z \in U$ and $\mathcal{F} = \overline{R_{\mathcal{F}}}$, we have

$$\overline{R_{\mathcal{F}}}(\overline{R_{\mathcal{F}}}(\mathbf{1}_z))(x) \leq \overline{R_{\mathcal{F}}}(\mathbf{1}_z)(x), \quad (\forall\, x \in U).$$

The facts that

$$\overline{R_{\mathcal{F}}}(\overline{R_{\mathcal{F}}}(\mathbf{1}_z))(x) = \bigvee_{y \in U} (R_{\mathcal{F}}(x,y) \odot (\overline{R_{\mathcal{F}}}(\mathbf{1}_z)(y))) = \bigvee_{y \in U} (R_{\mathcal{F}}(x,y) \odot R_{\mathcal{F}}(y,z)),$$

and

$$\overline{R_{\mathcal{F}}}(\mathbf{1}_z)(x) = R_{\mathcal{F}}(x,z),$$

imply

$$\bigvee_{y \in U} (R_{\mathcal{F}}(x,y) \odot R_{\mathcal{F}}(y,z)) \leq R_{\mathcal{F}}(x,z).$$

Therefore, $R_{\mathcal{F}}$ is transitive.

Conversely, let $R_{\mathcal{F}}$ be transitive, that is, for all $x, y, z \in U$

$$R_{\mathcal{F}}(x,y) \odot R_{\mathcal{F}}(y,z) \leq R_{\mathcal{F}}(x,z).$$

Taking into account of

$$(\mathcal{F}(\mathbf{1}_z))(x) = (\overline{R_{\mathcal{F}}}(\mathbf{1}_z))(x) = R_{\mathcal{F}}(x,z),$$

we have

$$\begin{aligned}
\overline{R_{\mathcal{F}}}(\overline{R_{\mathcal{F}}}(\mathbf{1}_z))(x) &= \bigvee_{y \in U} (R_{\mathcal{F}}(x,y) \odot (\overline{R_{\mathcal{F}}}(\mathbf{1}_z)(y))) \\
&= \bigvee_{y \in U} (R_{\mathcal{F}}(x,y) \odot R_{\mathcal{F}}(y,z)) \\
&\leq R_{\mathcal{F}}(x,z) = \overline{R_{\mathcal{F}}}(\mathbf{1}_z)(x),
\end{aligned}$$

and $\overline{R_{\mathcal{F}}}(\overline{R_{\mathcal{F}}}(\mathbf{1}_z)) \leq \overline{R_{\mathcal{F}}}(\mathbf{1}_z)$, in other words, $\mathcal{F}(\mathcal{F}(\mathbf{1}_z)) \leq \mathcal{F}(\mathbf{1}_z)$. It follows from normality that $\mathcal{F}$ is the transitive operator. □

We mainly consider properties of $\overline{\mathrm{R}}$ for L-fuzzy relation R so far. On the other hand, if we think about the properties of $\underline{\mathrm{R}}$ then the following result can be applicable that there is a one to one correspondence between $\underline{\mathrm{R}}$ and operators $\mathcal{H} : L^U \to L^U$ such that for all $a \in L$, $A, A_i \in L^U$

$$\mathcal{H}(\boldsymbol{a} \to A) = \boldsymbol{a} \to \mathcal{H}(A),$$
$$\mathcal{H}(\bigwedge_i A_i) = \bigwedge_i \mathcal{H}(A_i).$$

Therefore, it is similarly proved that for any operator $\mathcal{H} : L^U \to L^U$ satisfying the conditions above, there exists a unique L-fuzzy relation R on U such that $\mathcal{H} = \underline{\mathrm{R}}$.

Moreover, since our proof so far is mainly operator-based proof, not element-based one, other properties of L-fuzzy relations of R on U proved above can be represented by using $\overline{\mathrm{R}}$ and $\underline{\mathrm{R}}$ as follows. This comes from the idea of Kripke semantics for modal logic.

Theorem 4.7. [11] *Let R be an L-fuzzy relation on U. Then we have*

(1) *R is symmetric if and only if $A \le \underline{\mathrm{R}}(\overline{\mathrm{R}}(A))$, for all $A \in L^U$;*

(2) *R is serial if and only if $\overline{\mathrm{R}}(\mathbf{1}_x) = \mathbf{1}$, for all $x \in U$;*

(3) *R is Euclidean if and only if $\overline{\mathrm{R}}(\mathbf{1}_x) \le \underline{\mathrm{R}}(\overline{\mathrm{R}}(\mathbf{1}_x))$, for all $x \in U$.*

For a normal operator $\mathcal{F} : L^U \to L^U$, if it is reflexive and transitive, then it satisfies

(i) $\mathcal{F}(\mathbf{0}) = \mathbf{0}$;

(ii) $A \le \mathcal{F}(A), \quad (\forall A \in L^U)$;

(iii) $\mathcal{F}(\mathcal{F}(A)) = \mathcal{F}(A), \quad (\forall A \in L^U)$;

(iv) $\mathcal{F}(\bigvee_i A_i) = \bigvee_i \mathcal{F}(A_i), \quad (\forall A_i \in L^U)$.

This means that a reflexive and transitive normal operator is a closure operator on L^U, therefore,

$$\tau = \{\mathcal{F}(A) \mid A \in L^U\},$$

forms an L-fuzzy topology [11]. Moreover it is an *Alexandrov topology*, that is, $\bigvee_i A_i$ is also closed for every L-fuzzy closed set $A_i \in \tau$.

Theorem 4.8. *Every reflexive and transitive normal operator induces an Alexandrov topology on L^U*

5 Conclusions

In this paper, we consider properties of L-fuzzy relations and L-normal operators on a residuated lattice L by operator-based and prove that there is a one-to-one correspondence between the class of all L-fuzzy relations and the class of all L-normal operators. We also give a simple and general proof to the result (Theorem 3.3) that for L-fuzzy approximation spaces (X, R) and (Y, S) and a map $\varphi : X \to Y$, φ is a (upper) fuzzy backward natural transformation if and only if φ is relation preserving. We do not need the injectiveness of the map φ, which is needed in the original result in [7]. Moreover, we prove that for any L-normal operator $\mathcal{F}$, it is reflexive (or transitive) if and only if the L-fuzzy relation $R_{\mathcal{F}}$ induced by $\mathcal{F}$ is reflexive (or transitive), respectively.

Declarations

This article does not contain any studies with human participants or animals performed by any of the authors.
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