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Modal operators on BCK-algebras with condition (S)

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Abstract

In this paper, modal operators on BCK-algebras with condition (S) are introduced and several properties and characterizations of them are investigated. Also, it is investigated under what conditions these modal operators form a lattice. Furthermore, some particular modal operators are introduced, and their properties and characterizations of them are obtained, especially in some classes of BCK-algebras, such as positive implicative BCK-algebras, and implicative BCK-algebras.

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1 Introduction

BCK-algebra was introduced by Imai and Iséki [5] as a generalization of set-theoretic difference and implication connective in propositional logic. Since then, many researchers worked in this area. Bounded commutative BCK-algebras are categorically equivalent to MV-algebras [10], which are themselves a subclass of residuated lattices. BCK-algebras with condition (S) are BCK-algebras $(X; *, 0)$ in which for any $a, b \in X$, the element $a \circ b$, the greatest element $x \in X$ with $x * a \leq b$, exists and with concerning to this operation form a commutative ordered monoid. The pair $(*, \circ)$ has similar properties to the pair $(*, \rightarrow)$ in residuated lattices, i.e., for any elements x, y, z of the background BCK-algebra, $x * y \leq z$ if and only if $x \leq y \circ z$. So, in characterizing modal operators on BCK-algebras with condition (S), the operation $\circ$ plays an essential role.

The study of modal operators on algebras of logic was started by Macnab's work [8] on Heyting algebras; he introduced the notion of a modal operator on Heyting algebras as a unary operation f satisfying $f^2(x) = f(x) \geq x$, which preserves the meet operation. He gave some basic properties and characterizations of modal operators and introduced some particular types of them and a complete characterization of them. Rachůnek [11] introduced the notion of a modal operator on ordered sets. After that, many authors have applied it to some other algebras of logic, such as MV-algebras [4], bounded commutative residuated ℓ-monoids [12], and residuated lattices [7] and investigated algebraic properties of them. Ciungu et al. [2] applied modal operators on pseudo BE-algebras (which are a generalization of reversed left pseudo BCK-algebras [6]) and obtained similar results. Since, BCK-algebras, with condition (S), are an essential class of algebras of logic and are close to the other algebras of logic, it motivates us to study the algebraic properties of modal operators on these structures and investigate the behavior of BCK-algebras with concerning to these operators.

This paper is organized as follows. In Section 2, some definitions and results from the literature are given. In Section 3, the notion of a modal operator on BCK-algebras is introduced, and their basic properties and equivalent conditions are investigated. We also focus on the image of a modal operator and investigate its properties, especially in some subclasses of BCK-algebras such as positive implicative BCK-algebras and commutative BCK-algebras. Furthermore, some conditions under which the set of modal operators form a lattice are investigated. In this respect, it is proved modal operators on a bounded implicative BCK-algebra form a lattice. In the sequel, some particular types of modal operators on a BCK-algebra are introduced, and their properties and exciting results are given.

2 Preliminaries

This section is devoted to introduce some notions, and results from the literature. For more details, we refer to the references [3, 9].

A BCK-algebra is an algebra $(X;*,0)$ of type (2,0) satisfying the following conditions:
(BCK1) $(x*y)*(x*z) \leq z*y$,
(BCK2) $x*(x*y) \leq y$,
(BCK3) $x \leq x$,
(BCK4) $x \leq y$ and $y \leq x$ imply $x = y$,
(BCK5) $0 \leq x$,
where the binary relation $\leq$ is defined as $x \leq y \Leftrightarrow x*y = 0$.

Proposition 2.1. *In any* BCK*-algebra* $(X;*,0)$*, the following hold:*

(1) $x \leq y$ *implies* $x*z \leq y*z$ *and* $z*y \leq z*x$,

(2) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$,

(3) $x*y \leq z$ *if and only if* $x*z \leq y$,

(4) $(x*z)*(y*z) \leq x*y$,

(5) $x*y \leq x$ *and* $x*0 = x$,

for all $x, y, z \in X$.

From (BCK3), (BCK4), (BCK5), and Proposition 2.1(2) it follows that $(X;\leq)$ is a partially ordered set with 0 as the least element. If X has the greatest element 1 (with respect to the ordering $\leq$), X is said to be bounded.

A BCK-algebra $(X;*,0)$ is said to be commutative if $x*(x*y) = y*(y*x)$. Any bounded commutative BCK-algebra is a lattice in which the meet and the join operations are given by $x \wedge y = y*(y*x)$ and $x \vee y = N(Nx \wedge Ny)$, where $Nx = 1*x$. X is said to be positive implicative if $(x*y)*z = (x*z)*(y*z)$. X is said to be implicative if $x = x*(y*x)$. It is well known that a BCK-algebra is implicative if and only if it is both commutative and positive implicative.

A BCK-algebra $(X;*,0)$ is said to satisfy the condition (S) if for all $a,b \in X$, the set $A(a,b) = \{x \in X : x*a \le b\}$ has the greatest element, denoted by $a \circ b$. $0, a, b \in A(a,b)$. By [9, Theorem I.7.16], every bounded commutative BCK-algebra satisfies the condition (S).

Proposition 2.2. *Any BCK-algebra X with condition (S) satisfies the following:*

(1) $0, x, y \le x \circ y$, $x \circ 0 = 0 \circ x = x$,

(2) $x * y \le z \Leftrightarrow x \le y \circ z$,

(3) $x \le y \circ (x * y)$ *and* $(x \circ y) * x \le y$,

(4) $(X, \circ, 0)$ *is a commutative ordered monoid; i.e.,* $(X, \circ, 0)$ *is a commutative monoid, and the order is monotone with concerning to the operation* $\circ$,

(5) $(x * y) * z = x * (y \circ z)$,

(6) $x * y \le (x * z) \circ (z * y)$,

(7) $(x \circ z) * (y \circ z) \le x * y \le x \circ y$.

(8) *If X is positive implicative, then*
   - (a) $x \circ x = x$ *and* $(x \circ y) * z = (x * z) \circ (y * z)$,
   - (b) $x \circ y = lub\{x, y\}$,
   - (c) $x \le y$ *implies* $x \circ y = y$,

(9) *If X is implicative with condition (S), then*
   - (a) $(X; \circ)$ *is an upper semilattice, i.e.,* $x \vee y = x \circ y$,
   - (b) $x * (y \circ z) = (x * y) \wedge (x * z)$,
   - (c) $(x \circ y) \wedge z = (x \wedge z) \circ (y \wedge z)$.

3 Modal operators on BCK-algebras

In the familiar algebras of logic such as MV-algebras, BL-algebras, and residuated lattices, a modal operator is defined based on the notion of a closure operator. We recall that a closure operator on a poset $(P;\le)$ is defined as a monotone mapping $f : P \longrightarrow P$ satisfying $x \le f(x) = f^2(x)$. So, the authors define a modal operator as a mapping satisfying the two first conditions of a closure operator together with an extra condition. In many cases, modal operators are monotone, but not all of them, see, say [7] and [8]. Also, in these algebras, the background structure corresponds to truth values close to the value 1, say, reversed left BCK-algebras, in [2]. Since, BCK-algebra introduced by Iséki is based on the truth values close to the value 0, in fact, we consider the notion of a dual closure operator (those monotone mappings f satisfies $f^2(x) = f(x) \le x$), but also under the name of "modal operator" due to the convenience and coincidence with the previous works.

In what follows, $X = (X;*,0)$ is a BCK-algebra, unless otherwise specified.

Definition 3.1. *A* modal operator *on BCK-algebra X is a mapping $g : X \longrightarrow X$ satisfying*

(DM1) $(\forall x \in X)$ $g(x) \le x$,
(DM2) $(\forall x \in X)$ $g(g(x)) = g(x)$,
(DM3) $(\forall x, y \in X)$ $g(x) * g(y) \le g(x * y)$.

If g is a modal operator on X, the image of g constitutes of those $y \in X$ with $g(x) = y$, for some $x \in X$. From (DM2) it follows that $g(y) = g(g(x)) = g(x) = y$. This means that $Im(g)$ constitutes those elements of X, which are fixed under g; i.e., $Im(g) = \{x \in X : g(x) = x\} := Fix(g)$.

Example 3.2. *Consider the BCK-algebra $(X; *, 0)$, where $X = \{0, a, b, c\}$ and the operation $*$ is given in Table 1 ([9]). Define a mapping $g : X \longrightarrow X$ by $g(0) = 0$, $g(a) = a$ and $g(b) = g(c) = b$. It is routine to check that g is a modal operator on X.*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
c	c	b	a	0

Table 1: Cayley's Table of $*$

Proposition 3.3. *If g is a modal operator on X, then*

(1) $g(0) = 0$,

(2) *g is monotone, and so it is a dual closure operator,*

(3) $g(g(a) * b) = g(a) * b = g(g(a) * g(b)) = g(a) * g(b) \leq g(a * b)$.

Proof. (1) is obvious.

(2) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so $g(x) * g(y) \leq g(x * y) = 0$, whence $g(x) * g(y) = 0$. This implies that $g(x) \leq g(y)$. The last part is obvious.

(3) Assume that $a, b \in X$. Then

$$\begin{aligned} g(g(a) * g(b)) &\leq g(a) * g(b) \quad \text{by (DM1)} \\ &= g(g(a)) * g(b) \quad \text{by (DM2)} \\ &\leq g(g(a) * b) \quad \text{by (DM3)} \\ &\leq g(a) * b \quad \text{by (DM1)} \\ &\leq g(a) * g(b) \quad \text{by (DM1) and Proposition 2.1(1)} \\ &= g(g(a)) * g(g(b)) \quad \text{by (DM2)} \\ &\leq g(g(a) * g(b)) \quad \text{by (DM3)} \end{aligned}$$

whence $g(g(a) * g(b)) = g(g(a) * b) = g(a) * b = g(a) * g(b)$. The last inequality, immediately follows from the definition. □

Proposition 3.4. *A mapping g on a BCK-algebra X with condition* (S) *is a modal operator if and only if it satisfies*

(1) $g(x) * g(y) = g(x) * y$,

(2) $g(x \circ y) \leq g(x) \circ g(y)$,

for all $x, y \in X$.

Proof. Assume that g is a mapping on X with the given conditions. Let $x \in X$. Taking $y := x$ in (1), we get that $g(x) * x = g(x) * g(x) = 0$, whence $g(x) \leq x$, proving (DM1). Now, taking $y := g(x)$ in (1), we get that $g(x) * g(g(x)) = g(x) * g(x) = 0$ and so $g(x) \leq g(g(x))$, combining (DM1) it follows that $g(g(x)) = g(x)$, proving (DM2). Now, we prove that g is monotone. Let $x \leq y$. Then $x * y = 0$, and since $g(x) \leq x$, we get that $g(x) * g(y) = g(x) * y \leq x * y = 0$, whence $g(x) \leq g(y)$. Now, from $x \leq y \circ (x * y)$, by (2), it follows that $g(x) \leq g(y \circ (x * y)) \leq g(y) \circ g(x * y)$, whence by Proposition 2.2(2), $g(x) * g(y) \leq g(x * y)$, proving that g is a modal operator.

Conversely, assume that g is a modal operator on X. From $(x \circ y) * x \leq y$, we get $g(x \circ y) * g(x) \leq g((x \circ y) * x) \leq g(y)$, whence by Proposition 2.2(2), $g(x \circ y) \leq g(x) \circ g(y)$, proving (2). The proof of (1) follows from Proposition 3.3. □

Proposition 3.5. *Assume that X is a* BCK*-algebra with condition* (S) *and g is a modal operator on X. Then*

(1) $g(g(x) \circ y) = g(x \circ g(y)) = g(x \circ y)$.

(2) *If X is positive implicative, then* $g(x) \circ g(y) = g(g(x) \circ g(y)) = g(x \circ y)$.

Proof. (1) Let $x, y \in X$. From $g(x) \leq x$, by Proposition 2.2(4) it follows that $g(x) \circ g(y) \leq x \circ g(y)$ and so by Proposition 2.2(2) we get that $(g(x) \circ g(y)) * x \leq g(y)$. Now, from $g(x \circ y) \leq g(x) \circ g(y)$ it follows that $g(x \circ y) * x \leq g(y)$ and hence $g(x \circ y) \leq x \circ g(y)$, whence

$$g(x \circ y) = g(g(x \circ y)) \leq g(x \circ g(y)) \leq g(x \circ y),$$

because $g(y) \leq y$. Hence, $g(x \circ y) = g(x \circ g(y))$. Similarly, from $g(y) \leq y$ it follows that $g(x) \circ g(y) \leq g(x) \circ y$ and so $g(x \circ y) * g(x) \leq (g(x) \circ g(y)) * g(x) \leq y$. This implies that $g(x \circ y) \leq g(x) \circ y$. On the other hand, since $g(x) \leq x$, so $g(x) \circ y \leq x \circ y$ and so $g(x \circ y) \leq g(g(x) \circ y) \leq g(x \circ y)$. Thus, $g(x \circ y) = g(g(x) \circ y)$.

(2) Assume that X is positive implicative and let $x, y \in X$. From Proposition 2.2(8), we know that $x \circ y$ is the least upper bound for $\{x, y\}$. Now, since $x, y \leq x \circ y$, so $g(x), g(y) \leq g(x \circ y)$ and so by Proposition 2.2(8-c) we get that $g(x) \circ g(y) \leq g(y) \circ g(x \circ y) = g(x \circ y)$. Thus, $g(x \circ y) = g(x) \circ g(y)$. Hence, $g(x \circ y) = g(g(x \circ y)) = g(g(x) \circ g(y))$. □

We recall that a lower semilattice (or meet-semilattice) is a partially ordered set where every two elements have an infimum. If this is true for any (nonempty) family of elements, the semilattice is said to be complete (or $\wedge$-complete). A lattice is said to be complete if every (nonempty) family of the elements has infimum and supremum. Theorem 2.11 of [1] states that a $\wedge$-complete $\wedge$-semilattice, which has a top element, must be a complete lattice.

Proposition 3.6. *Let g be a modal operator on X. Then*

(1) $(Fix(g); *, 0)$ *is a* BCK*-algebra.*

(2) *If X is commutative (positive implicative, implicative), then so is $Fix(g)$.*

(3) *If X is bounded with 1 as the greatest element, $Fix(g)$ is bounded with $g(1)$ as the greatest element. In this case, if the arbitrary infimums exist, $(Fix(g), \leq)$ is a complete lattice.*

(4) *If $(X; *, 0, 1)$ is a bounded commutative* BCK*-algebra, $(Fix(g); *, 0, g(1))$ is also a bounded commutative* BCK*-algebra and so is a lattice in which the meet operation is the same as in X and the join operation is given by $x \vee_g y = N_g(N_g x \wedge N_g y)$, where $N_g x = g(1) * x$.*

Proof. (1) Since $g(0) = 0$, so $0 \in Fix(g)$. Now, for $x, y \in Fix(g)$, we have $x * y = g(x) * g(y) \leq g(x * y) \leq x * y$, whence $g(x * y) = x * y$. Hence, $x * y \in Fix(g)$. Thus, $Fix(g)$ is closed with respect to the operation $*$, proving that $Fix(g)$ is a BCK-algebra.

(2) Assume that X is commutative and let $x, y \in Fix(g)$. By Proposition 3.3(3) we get that $g(x \wedge y) = g(y * (y * x)) \geq g(y) * (y * x) = y * (y * x) = x \wedge y$. Combining (DM1), it follows that $g(x \wedge y) = x \wedge y$, i.e., $Fix(g)$ is closed with respect to the meet operation, and so is commutative. The argument for the cases of positive implicativity and implicativity be obvious.

(3) Now, let X is bounded with 1 as the greatest element. First of all, we observe that $g(g(1)) = g(1)$, which shows that $g(1) \in Fix(g)$. If $x \in Fix(g)$, from $x \leq 1$, it follows that $x = g(x) \leq g(1)$, and so $g(1)$ is the greatest element of $Fix(g)$. This means that $(Fix(g); *, 0, g(1))$ is a bounded BCK-algebra.

Now, assume that the infimums exist. Then for $\{a_i : i \in \Lambda\}$ a family of elements of $Fix(g)$, we have $g(a_i) = a_i$, for all $i \in \Lambda$. Now, $g(\inf a_i) \leq \inf a_i \leq a_i$, for all $i \in \Lambda$. If $c \leq a_i$ with $g(c) = c$, so $c \leq \inf a_i$ and hence $c = g(c) \leq g(\inf a_i)$; i.e., $g(\inf a_i) = \inf a_i$. Thus, $Fix(g)$ is closed with respect to the arbitrary infimums. Now, $Fix(g)$ is a complete lattice.

(4) We recall that if X is a bounded commutative BCK-algebra, it is a lattice with respect to the ordering $\leq$ in which the meet and the join operations are given by $x \wedge y = y * (y * x)$ and $x \vee y = N(Nx \wedge Ny)$, where $Nx = 1 * x$. We shall prove that $Fix(g)$ is closed with respect to the operations $\wedge$, $\vee_g$ and N_g. By (3), $Fix(g)$ is closed with respect to $\wedge$. Moreover, by Proposition 3.3(3) we get $g(N_g x) = g(g(1) * x) = g(g(1)) * g(x) = g(1) * x = N_g x$, for any $x \in Fix(g)$. Hence, for all $x, y \in Fix(g)$, $N_g x \wedge N_g y \in Fix(g)$ and so $N_g(N_g x \wedge N_g y) \in Fix(g)$. Hence, $x \vee_g y \in Fix(g)$. Now, we prove that $x \vee_g y = \sup\{x, y\}$ in $Fix(g)$. First of all, we observe that for all $x \in Fix(g)$, $N_g N_g x = g(1) * (g(1) * x) = x * (x * g(1)) = x * 0 = x$. Now, if $x, y \in Fix(g)$, from $N_g x \wedge N_g y \leq N_g x, N_g y$, it follows that

$$x = N_g N_g x \leq N_g(N_g x \wedge N_g y) = x \vee_g y.$$

Similarly, $y \leq x \vee_g y$, which shows that $x \vee_g y$ is an upper bound for $\{x, y\}$. If $z \in Fix(g)$ is such that $x, y \leq z$, so $N_g z = g(1) * z \leq g(1) * x = N_g x$ and similarly, $N_g z \leq N_g y$ and hence $N_g z \leq N_g x \wedge N_g y$. Now, $x \vee_g y = N_g(N_g x \wedge N_g y) \leq N_g N_g z = z$, which shows that $x \vee_g y$ is the supremum of $\{x, y\}$ in $Fix(g)$, completes the proof. □

For $a \in X$, we say that a is *idempotent* if $a \circ a = a$. In every BCK-algebra, the element 0 is idempotent. Moreover, in a positive implicative BCK-algebra with condition (S), every element is idempotent (Proposition 2.2(8)).

Example 3.7. *Consider the BCK-algebra $(X; *, 0)$, where $X = \{0, a, b, c, d\}$, and the operation $*$ is given in Table 2. It is seen that $d \circ d = d$, $a \circ a = d$, and $b \circ b$ does not exist. So, d is idempotent and a is not idempotent.*

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	b
c	c	b	a	0	b
d	d	a	a	a	0

Table 2: Cayley's Table of $*$

Proposition 3.8. *Assume that* $(X;*,0)$ *is a* BCK*-algebra with condition* (S) *and* $a \in X$ *is idempotent. Then,* $([0,a],\circ,0,a)$ *is a bounded commutative ordered monoid. Also, if* X *is bounded with 1 as the greatest element, then* $([a,1],\circ,a,1)$ *is a bounded commutative ordered semigroup.*

Proof. If $0 \leq x,y \leq a$, from Proposition 2.2(4) we get that $0 \leq x\circ y \leq a\circ a = a$, showing that $[0,a]$ is closed with respect to the operation $\circ$. A similar proof holds for the last part. □

By a modal operator on a (bounded) commutative ordered monoid $(X;\square,0)$ we mean a unary operation $\mathfrak{c}$ on X satisfying (DM1)-(DM3).

Proposition 3.9. *Assume that* $(X;*,0)$ *is an implicative* BCK*-algebra with condition* (S) *and* $a \in X$ *is idempotent. If* g *is a modal operator on* X*, then* g_a *with* $g_a(x) = g(x)\wedge a$ *is a modal operator on* $[0,a]$.

Proof. We first observe that $[0,a]$ is a bounded commutative ordered monoid. Now, for $x \in [0,a]$, from $g_a(x) \leq g(x) \leq x$ and $g_a(g_a(x)) = g(x)\wedge a\wedge a = g_a(x)$ it follows (DM1) and (DM2). Now, since any implicative BCK-algebra is positive implicative, by Propositions 2.2(9-c) and 3.5(2), we get that

$$g_a(x)\circ g_a(y) = (g(x)\wedge a)\circ(g(y)\wedge a) = (g(x)\circ g(y))\wedge a \leq g(x\circ y)\wedge a = g_a(x\circ y),$$

proving (DM3). □

The following example shows that the converse of Proposition 3.9 is not valid, in general.

Example 3.10. *Consider the implicative BCK-algebra* $(X;*,0)$*, where* $X = \{0,a,b,c,d\}$ *and the operation* $*$ *is given in Table 3* (see [9, Page 274])*. It is easy to verify that all elements are*

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Table 3: Cayley's Table of $*$

idempotent. Indeed, the Cayley's Table of $\circ$ *is given in Table 4. The interval* $[0,d]$ *contains only* 0 *and d; i.e.,* $[0,d] = \{0,d\}$*. Now, we define a mapping* $g : X \longrightarrow X$ *by*

$$g(0) = 0, g(a) = a, g(b) = c, g(c) = b, g(d) = d.$$

It is easily checked that g_d *is a modal operator on the interval* $[0,d]$*, while* g *is not a modal operator on* X *because* $g^2(b) = g(c) = b \neq g(b)$.

Lemma 3.11. *If* g_1 *and* g_2 *are monotone mappings on* X *satisfying* $g_i(x)*g_i(y) \leq g_i(x*y)$*, for all* $x,y \in X$ *and* $i = 1,2$*, then* g_1g_2 *and* g_2g_1*, too.*

Proof. Assume that $x,y \in X$. Then $g_1g_2(x)*g_1g_2(y) = g_1(g_2(x))*g_1(g_2(y)) \leq g_1(g_2(x)*g_2(y)) \leq g_1(g_2(x*y)) = g_1g_2(x*y)$. A similar proof holds for g_2g_1. □

Proposition 3.12. *For modal operators* g_1 *and* g_2*,* g_1g_2 *is a modal operator if and only if* $g_1g_2 = g_2g_1$.

$\circ$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	c	c	a
b	b	c	b	c	d
c	c	c	c	c	d
d	d	a	d	d	d

Table 4: Cayley's Table of $\circ$

Proof. Since every modal operator on a BCK-algebra is a dual closure operator, so by the dual of [11, Theorem 6], the properties (DM1) and (DM2) hold. We only need to investigate the correctness of (DM3). Let $x, y \in X$. Then $g_1g_2(x) * g_1g_2(y) \leq g_1(g_2(x) * g_2(y)) \leq g_1g_2(x * y)$, proving (DM3). Hence, g_1g_2 is a modal operator on X. □

Remark 3.13. *Let $\mathcal{DN}(X)$ denote the set of all modal operators on X, and define the binary relation $\preceq$ on $\mathcal{DN}(X)$ pointwise. We recall that when X is commutative, it is a lower-meet semilattice with $x \wedge y = y * (y * x)$* (see [9, Theorem I.5.5])*. In this case, if X is with condition* (S)*, then $(\mathcal{DN}(X), \preceq)$ is a lower meet-semilattice. Moreover, from* [9, Theorem I.7.17]*, we know that any bounded implicative BCK-algebra is with condition* (S) *and $x \circ y$ is exactly the join operation, with respect to the partial ordering $\leq$. Let us define $(g_1 \sqcup g_2)(x) = g_1(x) \circ g_2(x)$, for modal operators g_1 and g_2.*

Proposition 3.14. *Assume that X is a positive implicative* BCK*-algebra with condition* (S)*. If g_1 and g_2 are modal operators on X, then so is $g_1 \sqcup g_2$.*

Proof. Assume that g_1 and g_2 are modal operators on X. For $x \in X$ we have $g_1(x) \leq x$ and $g_2(x) \leq x$ and so $(g_1 \sqcup g_2)(x) = g_1(x) \circ g_2(x) \leq x \circ x = x$. For $x, y \in X$ we have

$$
\begin{aligned}
g_1 \sqcup g_2(g_1 \sqcup g_2)(x) &= (g_1 \sqcup g_2)(g_1(x) \circ g_2(x)) = g_1(g_1(x) \circ g_2(x)) \circ g_2(g_1(x) \circ g_2(x)) \\
&= g_1g_1(x) \circ g_1g_2(x) \circ g_2g_1(x) \circ g_2g_2(x) \quad \text{by Proposition 3.5} \\
&= g_1(x) \circ g_1g_2(x) \circ g_2g_1(x) \circ g_2(x) \\
&= g_1(x) \circ g_2(x) \circ g_1g_2(x) \circ g_2g_1(x) \quad \text{by Proposition 2.2(4)} \\
&\geq g_1(x) \circ g_2(x) \quad \text{by Proposition 2.2(1)} \\
&= (g_1 \sqcup g_2)(x)
\end{aligned}
$$

proving (DM2). Now, we get that

$$
\begin{aligned}
& (g_1 \sqcup g_2)(x) * (g_1 \sqcup g_2)(y) \\
= \;& (g_1(x) \circ g_2(x)) * (g_1(y) \circ g_2(y)) \\
= \;& (g_1(x) * (g_1(y) \circ g_2(y))) \circ (g_2(x) * (g_1(y) \circ g_2(y))) \quad \text{by Proposition 2.2(8)} \\
= \;& ((g_1(x) * g_1(y)) * g_2(y)) \circ ((g_2(x) * g_2(y)) * g_1(y)) \quad \text{by Proposition 2.2(5)} \\
\leq \;& (g_1(x * y) * g_2(y)) \circ (g_2(x * y) * g_1(y)) \quad \text{by (DM3) and Proposition 2.1(1)} \\
\leq \;& g_1(x * y) \circ g_2(x * y) \quad \text{by Proposition 2.1(5)} \\
= \;& g_1 \sqcup g_2(x * y)
\end{aligned}
$$

proving (DM3). Hence $g_1 \sqcup g_2$ is a modal operator on X. □

Considering Remark 3.13 and Proposition 3.14 we get that

Corollary 3.15. *Let X be a* BCK*-algebra.*

(1) *If X is commutative with condition* (S)*, then $(\mathcal{DN}(X), \preceq)$ is a lower-meet semilattice.*

(2) *If X is a bounded implicative* BCK*-algebra, $(\mathcal{DN}(X), \preceq)$ is a lattice.*

Example 3.16. *Consider the commutative BCK-algebra $(X; *, 0)$, where $X = \{0, a, b, c\}$ and satisfy $0 \leq a \leq b \leq c$, and the operation $*$ is given in Table 5* (see [9, Page 244])*. Routine calculations*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
c	c	b	a	0

Table 5: Cayley's Table of $*$

show that X satisfies the condition (S)*; the Cayley's Table of $\circ$ is given in Table 6. Now, we*

$\circ$	0	a	b	c
0	0	a	b	c
a	a	b	c	c
b	b	c	c	c
c	c	c	c	c

Table 6: Cayley table of $\circ$

assume that $g : X \longrightarrow X$ is a mapping. If g is a modal operator, we must have $g(x) \leq x$, whence the only possible choices are

$$g(0) = 0, g(a) \in \{0, a\},\ g(b) \in \{0, a, b\},\ g(c) \in \{0, a, b, c\}$$

i.e., 24 cases. Nine of them do not satisfy (DM2), which are as follows:

$$g(0) = g(a) = g(b) \in \{0, b\}, g(c) = a$$
$$g(0) = 0, g(a) = a, g(b) \in \{0, a\}, g(c) = b$$
$$g(0) = g(a) = 0, g(b) = a, g(c) \in \{0, a, b, c\}$$

So it remains 15 of them, which 10 of them don't satisfy (DM3); some of them satisfy $x \leq y$, and $g(y) \geq g(x)$ and so $g(y) * g(x) \neq 0 = g(x * y)$. Finally, it remains five mappings satisfying all of the conditions (DM1), (DM2), and (DM3). These are as follows:

$$g_0(x) = 0,\ \forall x \in X$$
$$g_1(0) = 0,\ g_1(a) = g_1(b) = g_1(c) = a$$
$$g_2(0) = 0,\ g_2(a) = g_2(b) = a,\ g_2(c) = c$$
$$g_3(0) = 0,\ g_3(a) = a,\ g_3(b) = g_3(c) = b$$
$$g_4(x) = x,\ \forall x \in X$$

The relation between g_i's is $g_0 \preceq g_1 \preceq g_2, g_3 \preceq g_4$, which shows that $(\mathcal{DN}, \preceq)$ is indeed a lattice.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Table 7: Cayley's Table of $*$

Example 3.17. *Consider the bounded implicative BCK-algebra $(X;*,0)$, where $X = \{0,a,b,c\}$ and satisfy $0 \leq a, b \leq c$, and the operation $*$ is given in Table 5* (see [9, Page 245]). *Similar to Example 3.16, we can calculate possible modal operators. Indeed, only two modal operators exist, which are zero mapping and identity mapping.*

In the sequel, we introduce two special modal operators on a BCK-algebra, which are essential to characterize the lattice of modal operators.

For $a \in X$, we define the mapping $\zeta_a : X \longrightarrow X$ by $\zeta_a(x) = x*a$. We mention that $a \notin Fix(\zeta_a)$, for all $a \in X \setminus \{0\}$. The following proposition gives a characterization of ζ_a.

Proposition 3.18. *Assume that X is a positive implicative BCK-algebra, and $a \in X$. Then ζ_a is a modal operator on X and if X is with condition* (S)*, then $Fix(\zeta_a) \subseteq \{0\} \cup a^{\uparrow} := \{0\} \cup \{x \in X : a \leq x\}$.*

Proof. Let $x \in X$. We know that $x * a \leq x$ and so $\zeta_a(x) \leq x$. Now, $\zeta_a(\zeta_a(x)) = (x*a)*a = (x*a)*(a*a) = x*a = \zeta_a(x)$. Let $x, y \in X$. Then $\zeta_a(x)*\zeta_a(y) = (x*a)*(y*a) = (x*y)*a = \zeta_a(x*y)$, proving (DM3). For the last part, we observe that $0 * a = 0$, whence $0 \in Fix(\zeta_a)$. Now, if $x \neq 0$ be such that $x * a = x$, from Proposition 2.2(2), it follows that $a \leq x \circ x = x$. Hence, $Fix(\zeta_a) \subseteq \{0\} \cup a^{\uparrow}$. □

Corollary 3.19. *If X is a positive implicative BCK-algebra with conditions* (S)*, for any $a \in X$ and natural number n, the unary operation $\zeta_{a^n}(x) = x * a^n := (\cdots((x*a)*a)*\cdots)*a$ is a modal operator with $Fix(\zeta_{a^n}) \subseteq \{0\} \cup a^{\uparrow}$.*

Example 3.20. *Consider the BCK-algebra $(X;*,0)$, where $X = \{0,a,b,c,d\}$ and satisfy $0 \leq a \leq b \leq c, d$, and the operation $*$ is given in Table 8* (see [9, Page 275]). *X is not positive implicative because $(c*b)*(a*b) = a*0 = a \neq 0 = a*b = (c*a)*b$. Moreover, $\zeta_a\zeta_a(b) = \zeta_a(b*a) = \zeta_a(a) = a*a = 0 \neq a = \zeta_a(b)$, whence ζ_a is not a modal operator on X. Hence, the condition 'positive implicative' is necessary for Proposition 3.18.*

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	a	a	0	a
d	d	d	d	d	0

Table 8: Cayley's Table of $*$

Example 3.21. *Consider the positive implicative BCK-algebra $(X;*,0)$, where $X = \{0,a,b,c\}$ and satisfy $0 \leq a \leq b \leq c$, and the operation $*$ is given in Table 9* (see [9, Page 246]). *It is routine to verify that X satisfies the condition* (S)*. Moreover, $Fix(\zeta_c) = \{0\} \neq c^{\downarrow}$. This shows that the equality may not hold in Proposition 3.18.*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	c	0

Table 9: Cayley's Table of $*$

For $a \in X$, we define the mapping $\phi_a : X \longrightarrow X$ by $\phi_a(x) = a * (a * x)$.

Proposition 3.22. *In any* BCK-*algebra X with condition* (S)*, the mapping ϕ_a, for all $a \in X$, is a modal operator on X with $Fix(\phi_a) \subseteq a^{\downarrow}$. Moreover, if X is also implicative, then $Fix(\phi_a) = a^{\downarrow}$.*

Proof. From (BCK2), we know that $a*(a*x) \leq x$, for all $x \in X$, whence $\phi_a(x) \leq x$, for all $x \in X$. Now, we shall prove that ϕ_a satisfies the conditions of Proposition 3.4. So, for $x, y \in X$, by (BCK1), we have $\phi_a(x)*\phi_a(y) = (a*(a*x))*(a*(a*y)) \leq (a*y)*(a*x) = (a*(a*x))*y = \phi_a(x)*y$. On the other hand, from $\phi_a(y) \leq y$ and Proposition 2.1(1), it follows that $\phi_a(x) * y \leq \phi_a(x) * \phi_a(y)$ and hence $\phi_a(x) * \phi_a(y) = \phi_a(x) * y$. Now $\phi_a(x \circ y) \leq \phi_a(x) \circ \phi_a(y)$ if and only if $a * (a * (x \circ y)) \leq (a * (a * x)) \circ (a * (a * y))$ and this equivalent to

$$(a * (a * (x \circ y))) * (a * (a * x)) \leq a * (a * y). \tag{1}$$

Now, by Propositions 2.1(4) and 2.2(5) we have $(a * (a * (x \circ y))) * (a * (a * x)) = (a * ((a * x) * y)) * (a * (a * x)) \leq (a * x) * ((a * x) * y) = (a * x) * ((a * y) * x) \leq a * (a * y)$, proving (1). Therefore, ϕ_a is a modal operator on X.

For the last part, we observe that if $x \in Fix(\phi_a)$, then $a * (a * x) = x$ and so $x * a = (a * (a * x)) * a = (a * a) * (a * x) = 0$, whence $x \leq a$. Now, assume that X is also implicative and $x \leq a$. Then $x = x * (a * x) \leq a * (a * x) \leq x$, whence $\phi_a(x) = x$; i.e., $x \in Fix(\phi_a)$. □

Example 3.23. *Consider the BCK-algebra $(X; *, 0)$, where $X = \{0, a, b, c, d\}$ and satisfy $0 \leq a \leq b \leq c, d$, and the operation $*$ is given as in Table 10* (see [9, Page 274])*. X is not implicative because $b * (c * b) = b * a = a \neq b$. It is routine to verify that X satisfies the condition* (S)*. Now, $Fix(\phi_d) = \{0, d\} \neq d^{\downarrow} = X$. Hence, the condition 'implicative' is necessary in Proposition 3.22.*

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	a	a	0	a
d	d	d	d	d	0

Table 10: Cayley's Table of $*$

Proposition 3.24. *In any* BCK-*algebra X with condition* (S)*, for $a \in X$ and $g \in \mathcal{DN}(X)$, $\phi_a \leq g$ if and only if $a \in Fix(g)$.*

Proof. Let g be a modal operator on X and $a \in X$. If $\phi_a \leq g$, we have $a = \phi_a(a) \leq g(a) \leq a$, which means that $g(a) = a$; i.e., $a \in Fix(g)$. Conversely, if $g(a) = a$, from Proposition 3.5 we get that $\phi_a(x) = a * (a * x) = g(a) * (g(a) * x) = g(a) * (g(a) * g(x)) = a * (a * g(x)) \leq g(x)$. Hence, $\phi_a \leq g$. □

The next corollary follows immediately from Corollary 3.15 and Proposition 3.24.

Corollary 3.25. *If X is a commutative* BCK*-algebra with condition* (S), $(\mathcal{DN}(X), \preceq)$ *is a lower-semilattice with* ϕ_0 *as the least element.*

Example 3.26. *Consider the BCK-algebra X with condition* (S) *given in Example 3.10. It is easy to check that the mapping $f : X \longrightarrow X$ with*

$$f(0) = f(a) = 0,\ f(b) = f(c) = b,\ f(d) = d$$

is a modal operator on X, and $a \notin Fix(f)$. Also, we can see that $\phi_a \not\leq f$. On the other hand, the mapping $g : X \longrightarrow X$ with

$$g(0) = g(b) = g(d) = 0,\ g(a) = g(c) = a$$

is a modal operator on X with $a \in Fix(g)$. Moreover, $\phi_a \leq g$.

4 Conclusions

In this paper, the notion of a modal operator on BCK-algebras with condition (S) was introduced and several characterizations and properties were obtained. Also, in some classes of BCK-algebras, such as commutative and positive implicative BCK-algebras, because of being well-behaviour. We proved that modal operators on a commutative BCK-algebra with condition (S), under the point-wise ordering, form a lower semilattice with the least element. Also, in a bounded implicative BCK-algebra, modal operators form a lattice. Furthermore, it is proved that the image of any modal operator on a commutative/positive implicative/implicative BCK-algebra is a subalgebra of the background BCK-algebra. Significantly, the set of fixed elements of a modal operator on a bounded commutative BCK-algebra forms a lattice. It was proved that using some particular types of modal operators, such as $\phi_a(x) = a * (a * x)$, modal operators can be characterized. Particularly, that the mapping ϕ_0 is the least element of the induced lattice by modal operators on a BCK-algebra with condition (S).

There are still some open problems which will be helpful for future work.

1. What under conditions ϕ_a can be an upper bound for $\mathcal{DN}(X)$?
2. What other types of modal operators can be defined on a BCK-algebra?
3. What are the relationships among these modal operators?
4. How can we model the logical aspects of modal operators on BCK-algebras using BCK-logic?

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