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The investigate of ΓUP-algebras

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"This paper is dedicated to Professor Young Bae Jun on the occasion of his 70th birthday."

Abstract

We first define a new concept, namely, the ΓUP-algebra. Then, we study and investigate the properties of its ΓUP-ideals and ΓUP-subalgebras. As a consequence, we construct a covariant functor between the ΓUP-category and the UP algebra-category. Some possible connections between these categories are also considered.

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1 Introduction and preliminaries

The algebras of logic form important class of algebras such as BCK-algebras, BCI-algebras, BCH-algebras, KU-algebras, SU-algebras and others that they are strongly connected with logic [2, 5, 6, 7]. The concept of BCI-algebras introduced by Iseki in 1966 such that have connections with BCI-logic which has application in the language of functional programming. Also, the class of BCK-algebras is a proper subclass of the class of BCI-algebras. The concept of a KU-algebra was first introduced by Prabpayak et. al. and they gave the concept of homomorphisms of KU-algebras and investigated some related properties. In continue, some researcher investigate the characterizations of KU-algebras. For example, Mostafa et.al. introduced the notion of intuitionistic fuzzy KU-ideals in KU-algebras and fuzzy intuitionistic image (preimage) of KU-ideals in KU-algebras. Also, they introduced the Cartesian product of two intuitionistic fuzzy KU-ideals in KU-algebras and investigated some results [8, 9] and Yaqoob et.al. introduced the notion of cubic

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KU-ideals of KU-algebras and several results are presented in this regard [14]. In 2013, Sithar Selvam et.al. introduced the concept of anti Q-fuzzy KU-subalgebras of KU-algebras and discussed few results of KU-ideals of KU-algebras under homomorphisms and anti homomorphisms and some of its properties [13]. The concept of UP-algebra introduced by Iampan as a new algebraic structure and he investigated fully UP-ideals, UP-subalgebras, congruences and UP-homomorphisms in UP-algebras, and considered some related properties of them [3]. Also, he introduced some new classes of algebras related to UP-algebras and semigroups, called a left UP-semigroup, a right UP-semigroup, a fully UP-semigroup, a left-left UP-semigroup, a right-left UP-semigroup, a left-right UP-semigroup, a right-right UP-semigroup, a fully-left UP-semigroup, a fully-right UP-semigroup, a left-fully UP-semigroup, a right-fully UP-semigroup, a fully-fully UP-semigroup [4]. In continue the concept of cubic sets in UP-subalgebras introduced by Senapati and relationships between the cubic UP-subalgebras and the cubic UP-ideals of a UP-algebra are investigated [12]. Also, Senapati et.al. introduced the concept of picture fuzzy sets in UP-algebras to the eight new concepts of picture fuzzy sets by means of a special type: Special picture fuzzy UP-subalgebras, special picture fuzzy near UP-filters, special picture fuzzy UP-filters, special picture fuzzy implicative UP-filters, special picture fuzzy comparative UP-filters, special picture fuzzy shift UP-filters, special picture fuzzy UP-ideals, and special picture fuzzy strong UP-ideals. Also, we discuss the relationship between the eight new concepts of picture fuzzy sets in UP-algebras [11].

In this paper, we first generalize the well known UP-algebras to the so-called ΓUP-algebra. Then, we consider and investigate ΓUP-ideals and the ΓUP-homomorphisms. Also, we define congruence relations on ΓUP-algebras and we construct quotient ΓUP-algebras. A covariant functor between the categories of ΓUP-algebras and UP-algebras is constructed and their related properties are therefore investigate. The fundamental theorems of homomorphisms of ΓUP-algebras are established.

Before we begin we will introduce the definition of a UP-algebras and some related topics:

Definition 1.1. [3] *An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called an UP-algebra if it satisfies the following axiom:*
(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0,$
(UP-2) $0 \cdot x = x,$
(UP-3) $x \cdot 0 = 0,$
(UP-4) $x \cdot y = y \cdot x = 0,$ *implies* $x = y,$
for every $x, y, z \in A$.

Example 1.2. *We define binary operation $\circ$ on $A = \{a, b, c, d\}$ by following table:*

$\circ$	a	b	c	d
a	a	b	c	d
b	a	a	a	a
c	a	b	a	d
d	a	b	c	a

Then, $(A, \circ, a)$ *is a* ΓUP*-algebra.*

Proposition 1.3. [3] *In an UP-algebra A, the following properties hold:*

(1) $x \cdot x = 0,$

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0,$

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0,$

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$,

(7) $x \cdot (y \cdot y) = 0$.

Definition 1.4. [3] *Let* A *be an* UP*-algebra and* B *be a non-empty subset of* A*. Then,* A *is called an* UP*-ideal if it satisfies the following properties:*

(1) *the constant* 0 *of* A *in* B,

(2) *for every* $x, y, z \in A$, $x \cdot (y \cdot z) \in B$ *and* $y \in B$ *imply* $x \cdot z \in B$.

Clearly, A and $\{0\}$ are UP- ideals of A.

Definition 1.5. [3] *Let* $(A, \cdot, 0)$ *be an* UP*-algebra and* S *be a non-empty subset of* A*. Then,* S *is called an* UP*-subalgebra of* A *if* $0 \in S$ *and* $(S, \cdot, 0)$ *itself forms an* UP*-algebra.*

Proposition 1.6. *A non-empty subset* S *of an* UP*-algebra* $(A, \cdot, 0)$ *is an* UP*-subalgebra of* A *if and only if* S *is closed under binary multiplication* $\cdot$.

Proof. The proof is straightforward and is hence omitted. □

Definition 1.7. *Let* ρ *be an equivalence relation on* UP*-algebra* $(A, \cdot, 0)$ *such that* $\rho(0) = 0$*. Then,* ρ *is called congruence if for any* $x, y, z \in A$,

$$x\rho y \Longrightarrow \rho(x \cdot z) = \rho(y \cdot z), \quad \rho(z \cdot x) = \rho(z \cdot y).$$

Proposition 1.8. *Let* $(A, \cdot, 0)$ *be an* UP*-algebra and* ρ *be a congruence. Then,* $A/\rho = \{\rho(x) : x \in A\}$ *is an* UP*-algebra by following binary operation:*

$$\rho(x) \odot \rho(y) = \rho(x \cdot y).$$

Proof. Suppose that $\rho(x_1) = \rho(x_2)$ and $\rho(y_1) = \rho(y_2)$. Hence $(x_1 \cdot y_1)\rho(x_2 \cdot y_1)$ and $(x_2 \cdot y_1)\rho(x_2 \cdot y_2)$. This implies $(x_1 \cdot y_1)\rho(x_2 \cdot y_2)$ and the binary operation $\odot$ is well-defined. Also, for every $\rho(x), \rho(y), \rho(z) \in A/\rho$,

$$(\rho(y) \odot \rho(z)) \odot ((\rho(x) \odot \rho(y)) \odot (\rho(x) \odot \rho(z)) = \rho(0).$$

Also,

$$\rho(0) \odot \rho(x) = \rho(0 \cdot x) = \rho(x), \quad \rho(x) \odot \rho(0) = \rho(0),$$

and

$$\rho(x) \odot \rho(y) = \rho(y) \odot \rho(x) = \rho(0)$$

implies that $\rho(x \cdot y) = \rho(y \cdot x) = \rho(0)$. Also, $x \cdot y \in \rho(0) = 0$ and $y \cdot x \in \rho(0) = 0$, implies that $x \cdot y = y \cdot x = 0$. By $(UP-4)$, $x = y$. Thus, $\rho(x) = \rho(y)$. Therefore, $(A/\rho, \odot, \rho(0))$ is an UP-algebra. □

Definition 1.9. [3] *Let* $(A, \cdot, 0_A)$ *and* $(B, \cdot, 0_B)$ *be two* UP*-algebras. A mapping* $\varphi : A \longrightarrow B$ *is called an* UP*-homomorphism, if for every* $x, y \in A$,

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \quad \varphi(0_A) = 0_B.$$

An UP*-homomorphism* $\varphi : A \longrightarrow B$ *is called an* UP*-epimorphism if* φ *is surjective, and is an* UP*-monomorphism if* φ *is injective, and is called an* UP*-isomorphism if* f *is bijective.*

Let A and B be two homomorphisms. Then, every homomorphism $\varphi : A \longrightarrow B$ induces the binary relation ρ_φ as follows:

$$\rho_\varphi = \{(x_1, x_2) \in A_1 \times A_1 : \varphi(x_1) = \varphi(x_2)\}.$$

Example 1.10. *Let A and B be two UP-algebras and $\varphi : A \longrightarrow B$ be a homomorphism. Then, the relation ρ_φ is a congruence relation.*

2 ΓUP-algebra

In this section, we concentrate on the ΓUP-algebras that is generalization of UP-algebras. Also, we now consider the covariant functor between the category of ΓUP-algebras and the category of UP-algebras.

Definition 2.1. *An algebra $A = (A, \Gamma, 0)$ of type $(2, 0)$ is called a ΓUP-algebra if every binary operation $\oplus_\alpha \in \Gamma$ satisfies the following axioms:*
(ΓUP-1): $(y \oplus_{\alpha_1} z) \oplus_{\alpha_2} ((x \oplus_{\alpha_3} y) \oplus_{\alpha_2} (x \oplus_{\alpha_1} z)) = 0$,
(ΓUP-2): $0 \oplus_\alpha x = x$,
(ΓUP-3): $x \oplus_\alpha 0 = 0$,
(ΓUP-4): $x \oplus_\alpha y = y \oplus_\alpha x = 0$ *implies* $x = y$,
for every $x, y, z \in A$ and $\oplus_\alpha, \oplus_{\alpha_i} \in \Gamma$, for $1 \leq i \leq 3$.

Example 2.2. *Let U be a universal set and $\oplus_\alpha$ defined on $P(U)$ as follows:*

$$A \oplus_\alpha B = A - B.$$

Then, $(P(U), \oplus_\alpha, \emptyset)$ is a ΓUP-algebra.

Proposition 2.3. *Let A be a ΓUP-algebra. Then, the following properties hold:*

(1) $x \oplus_\alpha x = 0$,

(2) $x \oplus_{\alpha_1} y = 0$ *and* $y \oplus_{\alpha_2} z = 0$ *imply* $x \oplus_{\alpha_3} z = 0$,

(3) *if* $x \oplus_{\alpha_1} y = 0$, *then* $(z \oplus_{\alpha_2} x) \oplus_{\alpha_3} (z \oplus_{\alpha_4} y) = 0$,

(4) $x \oplus_{\alpha_1} (y \oplus_{\alpha_2} x) = 0$,

(5) $x \oplus_{\alpha_1} (y \oplus_{\alpha_2} y) = 0$.

Proof. (1). By ΓUP-1 and ΓUP-2, we have

$$0 = (0 \oplus_\alpha x) \oplus_\alpha ((0 \oplus_\alpha 0) \oplus_\alpha (0 \oplus_\alpha x)) = (0 \oplus_\alpha x) \oplus_\alpha (0 \oplus_\alpha x) = x \oplus_\alpha x.$$

Hence, $x \oplus_\alpha x = 0$.
(2). Suppose that $x \oplus_{\alpha_1} y = 0$ and $y \oplus_{\alpha_2} z = 0$. Hence, by ΓUP-1 and ΓUP-2

$$x \oplus_{\alpha_2} z = 0 \oplus_{\alpha_3} (0 \oplus_{\alpha_3} (x \oplus_{\alpha_2} z)) = (y \oplus_{\alpha_2} z) \oplus_{\alpha_3} ((x \oplus_{\alpha_1} y) \oplus_{\{\alpha_3} (x \oplus_{\alpha_2} z)) = 0$$

(3). Assume that $x \oplus_{\alpha_1} y = 0$. Hence,

$$\begin{aligned}(z \oplus_{\alpha_2} x) \oplus_{\alpha_3} (z \oplus_{\alpha_4} y) &= 0 \oplus_{\alpha_3} ((z \oplus_{\alpha_2} x) \oplus_{\alpha_3} (z \oplus_{\alpha_1} y)) \\ &= (x \oplus_{\alpha_1} y) \oplus_{\alpha_3} ((z \oplus_{\alpha_2} x) \oplus_{\alpha_3} (z \oplus_{\alpha_1} y)) \\ &= 0.\end{aligned}$$

(4). By $(\Gamma UP-2)$ and $(\Gamma UP-3)$

$$\begin{aligned} x \oplus_{\alpha_1} (y \oplus_{\alpha_2} x) &= (0 \oplus_{\alpha_2} x) \oplus_{\alpha_1} (y \oplus_{\alpha_2} x) \\ &= (0 \oplus_{\alpha_2} x) \oplus_{\alpha_1} ((y \oplus_{\alpha_2} 0) \oplus_{\alpha_1} (y \oplus_{\alpha_2} x)) \\ &= 0. \end{aligned}$$

(5). $x \oplus_{\alpha_1} (y \oplus_{\alpha_1} y) = x \oplus_{\alpha_1} 0 = 0.$ □

Definition 2.4. *Let A be a ΓUP-algebra and B be a non-empty subset of A. Then, B is called a ΓUP-ideal if satisfies in the following properties:*

(1) *the constant 0 of A in B,*

(2) *for every $x, y, z \in A$, $x \oplus_{\alpha_1} (y \oplus_{\alpha_2} z) \in B$ and $y \in B$ implies that for every $\alpha \in \Gamma$, $x \oplus_\alpha z \in B$*

Clearly, A and $\{0\}$ are ΓUP- ideals of A.

Example 2.5. *Let $A = \{a, b, c, d\}$ be a set and $\oplus_\alpha$ defined on A as follows:*

$\oplus_\alpha$	a	b	c	d	e
a	a	b	c	d	e
b	a	b	c	d	e
c	a	a	a	d	e
d	a	a	c	a	e
e	a	a	a	a	a

Then, $(A, \oplus_\alpha, a)$ is a Γ UP-algebra and $\{a, b, d\}$ and $\{a, b, c\}$ are Γ UP-idelas of Γ UP-algebra $(A, \oplus_\alpha, a)$.

Proposition 2.6. *Let I be a Γ UP-ideal of a Γ UP-algebra such that $x_1, x_2 \in I$. Then, for every $\alpha \in \Gamma$, $x_1 \oplus_\alpha x_2 \in I$. Also, for every $z \in A$, $z \oplus_\alpha x_1 \in I$.*

Proof. Suppose that $x_1, x_2 \in I$ and $\alpha \in \Gamma$. Hence,

$$x_1 \oplus_\alpha 0 = x_1 \oplus_\alpha (x_2 \oplus_\alpha x_2) = 0 \in I.$$

Since, I is a ΓUP-ideal, $x_1 \oplus_\alpha x_2 \in I$. Also, $z \oplus_\alpha (x_1 \oplus_\alpha x_1) = z \oplus_\alpha 0 = 0 \in I$. Hence, $z \oplus_\alpha x_1 \in I$. □

Lemma 2.7. *Let A be a Γ UP-algebra and $\{I_i\}_{i \in J}$ be a family of Γ UP-ideals in A. Then, $\bigcap_{i \in J} I_i$ is a Γ UP-ideal.*

Proof. Suppose that for every $i \in J$, I_j are ΓUP-ideals of A. Then, for every $i \in J$, $0 \in I_i$. This implies that $0 \in \cap_{i \in J} I_i$. Let $x_1 \oplus_{\alpha_1} (x_2 \oplus_{\alpha_2} x_3) \in \cap_{i \in J} I_i$ and $x_2 \in \cap_{i \in} I_i$. Since, I_i is a ΓUP-algebra, for every $\alpha \in \Gamma$, $x_1 \oplus_\alpha x_3 \in I_i$. Therefore, $\bigcap_{i \in J} I_i$ is a ΓUP-algebra. □

Theorem 2.8. *Let A be a Γ UP-algebra and B be a Γ UP-ideal. Then, $A \oplus_\alpha B \subseteq B$ and B is a Γ UP-subalgebra of A.*

Proof. Suppose that $x \in A \oplus_\alpha B$. Then, for some $a \in A$, $b \in B$ and $\alpha \in \Gamma$, $x = a \oplus_\alpha b$. Hence, $a \oplus_\alpha (b \oplus_\alpha b) = a \oplus_\alpha 0 = 0 \in B$. Since, B is a ΓUP-ideal of A and B, we have $x = a \oplus_\alpha b \in B$. Hence, $A \oplus_\alpha B \subseteq B$. Also, $B \oplus_\alpha B \subseteq A \oplus_\alpha B \subseteq B$, implies that B is a ΓUP-subalgebra of A. □

Theorem 2.9. *Let A be a Γ UP-algebra and B be a Γ UP-ideal. Then, the following statement hold:*

(1) *if* $b \oplus_\alpha x \in B$ *and* $b \in B$*, then* $x \in B$*. Also, if* $b \oplus_\alpha X$ *and* $b \in B$*, then* $X \subseteq B$*.*

(2) *if* $b \in B$*, then* $x \oplus_\alpha b \in B$*. Also,* $X \oplus_\alpha b \subseteq B$*,*

where $\alpha, \alpha_1, \alpha_2, \alpha_3 \in \Gamma$*.*

Proof. (1). Suppose that $b \oplus_\alpha x \in B$ and $b \in B$. Hence, $0 \oplus_\alpha (b \oplus_\alpha x) \in B$. Since, B is a ΓUP-ideal of A, $x = 0 \oplus_\alpha x \in B$. Let $b \oplus_\alpha X \subseteq B$ and $b \in B$. Then, for every $x \in X$, $b \oplus_\alpha x \in B$. By a similar argument, $x \in B$. Hence, $X \subseteq B$.
(2). Assume that $x \in A$ and $b \in B$. Hence, $x \oplus_\alpha (b \oplus b) = x \oplus_\alpha 0 = 0 \in B$. Since, B is a ΓUP-ideal of A and $b \in B$, we have $x \oplus_\alpha b \in B$. Obviously, $X \oplus_\alpha B \subseteq B$. □

Definition 2.10. *Let* $A = (A, \Gamma, 0)$ *be a* Γ*UP-algebra and* T *be a subset of* A*. Then,* T *is called a* Γ*UP-subalgebra of* A *if* $0 \in B$ *and* $(T, \Gamma, 0)$ *itself forms a* Γ*UP-algebra. Obviously,* A *and* $\{0\}$ *are* Γ*UP-subalgebras of* A*.*

Lemma 2.11. *Let* $A = (A, \Gamma, 0)$ *be a* Γ*UP-algebra and* B *a non-empty subset of* A*. Then,* B *is called a* Γ*UP-subalgebra of* A *if* $0 \in B$ *and for every* $\alpha \in \Gamma$ *and* $x_1, x_2 \in B$*,* $x_1 \oplus_\alpha x_2 \in B$*.*

Proof. The proof is straightforward and is hence omitted. □

Proposition 2.12. *Let* $(A, \Gamma, 0)$ *be a* Γ*UP-algebra and* I *be a* Γ*UP-ideal. Then,* I *is a* Γ*UP-subalgebra.*

Proof. By using Proposition 2.6 and Lemma 2.11 the proof is completed. □

Theorem 2.13. *Let* A *be a* Γ*UP-algebra and* $\{B_i\}_{i \in I}$ *be a family* Γ*UP-subalgebra of* A*. Then,* $\bigcap_{i \in I} B_i$ *is a* Γ*UP-subalgebra of* A*.*

Proof. Suppose that for every $i \in I$, B_i are ΓUP-subalgebra. Hence, $0 \in B_i$, for all $i \in I$. It follows that $0 \in \bigcap_{i \in I} B_i \neq \emptyset$. Let $x, y \in \bigcap_{i \in I} B_i$. Then, for every $i \in I$, $x, y \in B_i$. Since, B_i are ΓUP-subalgebras, $x \oplus_\alpha y \in B_i$, for every $\alpha \in \Gamma$. Therefore, $x \oplus_\alpha y \in \bigcap_{i \in I} B_i$ and $\bigcap_{i \in I} B_i$ is a ΓUP-subalgebra. □

Definition 2.14. *Let* Θ *be an equivalence relation on* Γ*UP-algebra* A*. Then,* Θ *is congruence when*

$$x_1 \Theta x_2 \Longleftrightarrow \forall \alpha \in \Gamma,\ z \in A,\ (x_1 \oplus_\alpha z) \Theta (x_2 \oplus_\alpha z).$$

Example 2.15. *Let* I *be a* Γ*UP-ideal of an associated* Γ*UP-algebra* A*. Then, we define the relation* $\equiv$ *on* A *as follows:*

$$x_1 \equiv x_2 \Longleftrightarrow \forall\ \alpha \in \Gamma,\ x_1 \in I \oplus_\alpha x_2.$$

For every $x \in A$ *and* $\alpha \in \Gamma$*,*

$$x = 0 \oplus_\alpha x \in I \oplus_\alpha x.$$

Hence the relation $\equiv$ *is reflexive. Also,* $x_1 \equiv x_2$*, implies that there exists* $a \in I$ *such that* $x_1 = a \oplus_\alpha x_2$*. Hence,* $a \oplus_\alpha x_2 = a \oplus_\alpha (a \oplus_\alpha x_2) = (a \oplus_\alpha a) \oplus_\alpha x_2$*. Hence,* $x_2 = a \oplus_\alpha x_1 \in I \oplus_\alpha x_1$ *and the relation* $\equiv$ *symmetric. Let* $x_1 \equiv x_2$ *and* $x_2 \equiv x_3$*. Then, for some* $a_1, a_2 \in I$*,* $x_1 = a_1 \oplus_\alpha x_2$ *and* $x_2 = a_2 \oplus_\alpha x_3$*. Then, the relation* $\equiv$ *is transitive. Therefore,* $\equiv$ *is equivalence. The equivalence relation* ρ_I *is congruence.*

Example 2.16. *Let* I *be a* Γ*UP-ideal of a* Γ*UP-algebra* A*. Then, we define the relation* ρ_I *on* A *as follows:*

$$x_1\rho_I x_2 \Longleftrightarrow \ \forall\alpha\in\Gamma,\ x_1\oplus_\alpha x_2\in I \text{ and } x_2\oplus_\alpha x_1\in I.$$

For every $x\in A$ *and* $\alpha\in\Gamma$*,* $x\oplus_\alpha x=0\in I$*. Hence,* ρ_I *is reflexive. Also, the relation* ρ_I *is symmetric. Let* $x,y,z\in A$ *such that* $x\rho_I y$ *and* $y\rho_I z$*. Then, for every* $\alpha\in\Gamma$*,* $x\oplus_\alpha y\in I$*,* $y\oplus_\alpha x\in I$ *and* $y\oplus_\alpha z\in I$*,* $z\oplus_\alpha y\in I$ *Hence,*

$$(y\oplus_\alpha z)\oplus_\alpha((x\oplus_\alpha y)\oplus_\alpha(x\oplus_\alpha z))=0.$$

Since, $y\oplus_\alpha x\in I$*,* $(x\oplus_\alpha y)\oplus_\alpha(x\oplus_\alpha z)\in I$*. Also,* $x\oplus_\alpha y\in I$*, implies that* $x\oplus_\alpha z\in I$ *and the relation* ρ_I *is equivalence.*

Theorem 2.17. *Let* A *be a* Γ*UP-algebra,* Θ *be a congruence relation on* A *and* T *be a* Γ*UP-ideal of* A/Θ*. Then,* $T=I/\Theta$*, where* I *is a* Γ*UP-ideal.*

Proof. Suppose that $\pi: A\longrightarrow A/\Theta$, is a natural homomorphism and T is a ΓUP-ideal of A/Θ. Hence, by Proposition 2.19, $\pi^{-1}(T)$ is a ΓUP-ideal of A. Also, $T=\pi^{-1}(T)/\Theta$. □

Definition 2.18. *Let* A_1 *and* A_2 *be* ΓUP*-algebra. Then, map* $\varphi: A_1\longrightarrow A_2$ *is called homomorphism if for every* $x_1,x_2\in A_1$ *and* $\alpha\in\Gamma$*,*

$$\varphi(x_1\oplus_\alpha x_2)=\varphi(x_1)\oplus_\alpha\varphi(x_2).$$

Also, the kernel of φ *defined as follows:*

$$ker\varphi=\{x\in A_1:\varphi(x)=0_{A_2}\}.$$

Proposition 2.19. *Let* $\varphi: A_1\longrightarrow A_2$ *be an epimorphism and* I *be a* Γ*UP-ideal of* A_2*. Then,* $\varphi^{-1}(I)$ *is a* Γ*UP- ideal of* A_1*.*

Proof. Obviously, $\varphi^{-1}(I)$ is a non-empty subset of A_1. Let $x_1\oplus_{\alpha_1}(x_2\oplus_{\alpha_2}x_3)\in\varphi^{-1}(I)$ and $x_2\in\varphi^{-1}(I)$. Then, $\varphi(x_1)\oplus_{\alpha_1}(\varphi(x_2)\oplus_{\alpha_2}\varphi(x_3))\in I$. Since I is a ΓUP-ideal, $\varphi(x_1)\oplus_{\alpha_1}\varphi(x_3)\in I$. Hence, $\varphi(x_1\oplus_{\alpha_1}x_3)\in I$ and $x_1\oplus_{\alpha_1}x_3\in\varphi^{-1}(I)$. Therefore, $\varphi^{-1}(I)$ is a ΓUP-ideal. □

Corollary 2.20. *Let* $\varphi: A_1\longrightarrow A_2$ *be a homomorphism. Then,* $ker\varphi$ *is a* Γ*UP-ideal of* A_1*.*

Proposition 2.21. *Let* $\varphi: A_1\longrightarrow A_2$*. Then,* $(a,b)\in\rho_\varphi$ *if and only if* $a\oplus_\alpha b\in ker\varphi$*.*

Proof. Suppose that $(a,b)\in\rho_\varphi$. Hence, $\varphi(a)=\varphi(b)$ and for every $\alpha\in\Gamma$,

$$0=\varphi(a)\oplus_\alpha\varphi(a)=\varphi(a)\oplus_\alpha\varphi(b)=\varphi(a\oplus_\alpha b).$$

Then, $a\oplus_\alpha b\in ker\varphi$. Similarly, we obtain the converse inclusion. □

Theorem 2.22. *Let* A *be a* Γ*UP-algebra and* ρ *be a congruence relation on* A*. Then,* A/ρ *is a* Γ*UP-algebra by following operation:*

$$\rho(x_1)\widehat{\oplus}_\alpha\rho(x_2)=\rho(x_1\oplus_\alpha x_2),$$

where $\alpha\in\Gamma$*,* $\rho(x_1),\rho(x_2)\in A/\rho$*.*

Proof. The proof is similar to the proof of Theorem 1.8. □

Theorem 2.23. *Let ρ_1 be a congruence relation on a ΓUP-algebra A_1 and $\varphi : A_1 \longrightarrow A_2$ be a homomorphism such that $\rho_1 \subseteq \rho_\varphi$. Then, there is a unique homomorphism $\beta : A_1/\rho_1 \longrightarrow A_2$, such that $Im\beta = Im\varphi$ such that $\beta \circ \pi = Im\varphi$, where $\pi : A_1 \longrightarrow A_1/\rho_1$ is a natural homomorphism.*

Proof. Suppose that $\beta : A_1/\rho_1 \longrightarrow A_2$ defined by $\beta(\rho_1(x)) = \varphi(x)$, for $\rho_1(x_1) \in A_1/\rho_1$. Then, β is well-defined, since, for all $x_1, x_2 \in A_1$,

$$\rho_1(x_1) = \rho_1(x_2) \Longrightarrow (x_1, x_2) \in \rho_1 \Longrightarrow (x_1, x_2) \in \rho_\varphi \Longrightarrow \varphi(x_1) = \varphi(x_2).$$

For every $\rho_1(x_1), \rho_1(x_2) \in A_1/\rho_1$ and $\alpha \in \Gamma$,

$$\begin{aligned} \beta(\rho_1(x_1)\widehat{\oplus}_\alpha\rho_1(x_2)) &= \beta(\rho_1(x_1 \oplus_\alpha x_2)) \\ &= \varphi(x_1 \oplus_\alpha x_2) = \varphi(x_1) \oplus_\alpha \varphi(x_2) \\ &= \beta(\rho_1(x_1)) \oplus_\alpha \beta(\rho_1(x_2)). \end{aligned}$$

Also, $Im\beta = Im\varphi$ and $\beta \circ \pi = Im\varphi$. □

Theorem 2.24. *Let A be a ΓUP-algebra and ρ, σ be congruence relation on A such that $\rho \subseteq \sigma$. Then,*

$$\sigma/\rho = \{(\rho(x_1), \rho(x_2)) \in A/\rho \times A/\rho : (x_1, x_2) \in \sigma\},$$

is a congruence relation on A/ρ, and $(A/\rho)/(\sigma/\rho) \cong A/\sigma$

Proof. Suppose that $\rho(x_1), \rho(x_2), \rho(x) \in A/\rho$, $\alpha \in \Gamma$ such that $(\rho(x_1), \rho(x_2)) \in \sigma/\rho$. Hence, $x_1\sigma x_2$ and for every $\alpha \in \Gamma$, $(x_1 \oplus_\alpha x)\sigma(x_2 \oplus_\alpha x)$. Then,

$$(\rho(x_1)\widehat{\oplus}_\alpha\rho(x))\sigma/\rho(\rho(x_2)\widehat{\oplus}_\alpha\rho(x)).$$

So, an equivalence relation σ/ρ is congruence. We define $\varphi : (A/\rho)/(\sigma/\rho) \longrightarrow A/\sigma$, by $\varphi((\sigma/\rho)(\rho(x)) = \sigma(x)$. By a routine argument φ is an isomorphism. □

3 The associated UP-algebras induced by ΓUP-algebra

In this section, we turn to study those associated UP-algebras induced by a given ΓUP-algebra.

Let $(A, \Gamma, 0)$ be a ΓUP-algebra such that for every $\alpha \in \Gamma$, $\oplus_\alpha$ is an associated binary operation and for every $x, z \in A$ and $\alpha \in \Gamma$, the equation $x \oplus_\alpha z_1 = z$, has the solution $z_1 \in A$.

Definition 3.1. *Let $(A, \Gamma, 0)$ be a ΓUP-algebra. Then, we define a relation ρ on $A \times \Gamma$ as follows:*

$$(x_1, \alpha_1)\Delta(x_2, \alpha_2) \Longleftrightarrow x_1 \oplus_{\alpha_1} x = x_2 \oplus_{\alpha_2} x,$$

for every $x \in A$.

Obviously, Δ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class contains (x, α) and $M = \{[x, \alpha] : x \in A, \alpha \in \Gamma\}$. We define the binary relation $\circ$ on M as follows:

$$[x_1, \alpha_1] \circ [x_2, \alpha_2] := [x_1 \oplus_{\alpha_1} x_2, \alpha_2].$$

Theorem 3.2. *Let $(A, \Gamma, 0)$ be a ΓUP-algebra. Then, M is an UP-algebra.*

Proof. Obviously, the binary operation defined on M is well-defined. Suppose that $[x,\alpha_1]$, $[y,\alpha_2]$, $[\alpha_3, z] \in M$. Hence,

$$\begin{aligned}[y,\alpha_2] \quad & \circ[z,\alpha_3]\circ(([x,\alpha_1]\circ[y,\alpha_2])\circ([x,\alpha_1]\circ[z,\alpha_3]))\\ &=[y\oplus_{\alpha_2} z,\alpha_3]\circ([(x\oplus_{\alpha_1} y),\alpha_2]\circ[(x\oplus_{\alpha_1} z,\alpha_3)])\\ &=[y\oplus_{\alpha_2} z,\alpha_3]\circ[(x\oplus_{\alpha_1} y)\oplus_{\alpha_2}(x\oplus_{\alpha_1} z),\alpha_3]\\ &=[(y\oplus_{\alpha_2} z)\oplus_{\alpha_3}(x\oplus_{\alpha_1} y)\oplus_{\alpha_2}(x\oplus_{\alpha_1} z),\alpha_3)]\\ &=[0,\alpha_3]\\ &=0_M.\end{aligned}$$

Also, for every $[x,\alpha]\in M$,

$$[0,\alpha]\circ[x,\alpha]=[0\oplus_\alpha x,\alpha]=[x,\alpha].$$

and

$$[x,\alpha]\circ[0,\alpha]=[x\oplus_\alpha 0,\alpha]=[0,\alpha]=0_M.$$

Let $[x_1,\alpha_1],[x_2,\alpha_2]\in M$ such that

$$[x_1,\alpha_1]\circ[x_2,\alpha_2]=[x_2,\alpha_2]\circ[x_1,\alpha_1]=[0,\alpha_1]=[0,\alpha_2].$$

(By ΓUP-2, for every $\alpha_1,\alpha_2\in\Gamma$, $[0,\alpha_1]=[0,\alpha_2]$.)

Hence,

$$[x_1\oplus_{\alpha_1} x_2,\alpha_2]=[0,\alpha_2],\ \ [x_2\oplus_{\alpha_2} x_1,\alpha_1]=[0,\alpha_1].$$

This implies that for every $z\in A$,

$$(x_1\oplus_{\alpha_1} x_2)\oplus_{\alpha_2} z=0\oplus_{\alpha_2} z=z,$$

and

$$(x_2\oplus_{\alpha_2} x_1)\oplus_{\alpha_1} z=0\oplus_{\alpha_1} z=z.$$

This implies that for every $z\in A$, $(x_1\oplus_{\alpha_1}x_2)\oplus_{\alpha_2}z=(x_2\oplus_{\alpha_2}x_1)\oplus_{\alpha_1}z$. Since the equations $x_1\oplus_{\alpha_1}z$ and $x_2\oplus_{\alpha_2} z$ have solutions, we have $[x_1,\alpha_1]=[x_2,\alpha_2]$. Therefore, M is an UP-algebra. □

We now give the following definition:

Definition 3.3. *Let A be a ΓUP-algebra and $B\subseteq A$, $C\subseteq M$. Then, we define $\widehat{B}\subseteq M$ and $C'\subseteq A$ as follows:*

$$C'=\{x\in A:\forall\alpha\in\Gamma,[x,\alpha]\in C\},\quad \widehat{B}=\{[x,\alpha]:x\in B,\alpha\in\Gamma\}.$$

Proposition 3.4. *Let A be a ΓUP-algebra and $C\subseteq M$ be an UP-ideal. Then, C' is a ΓUP-ideal of A.*

Proof. Suppose that $x_1\oplus_{\alpha_1}(x_2\oplus_{\alpha_2}x_3)\in C'$ and $x_2\in C'$. Hence, for every $\alpha\in\Gamma$, $[x_1\oplus_{\alpha_1}(x_2\oplus_{\alpha_2} x_3),\alpha]\in C$. This implies that $[x_1,\alpha_1]\circ([x_2,\alpha_2]\circ[x_3,\alpha])\in C$ and $[x_2,\alpha_2]\in C$. Since, C is a ΓUP-ideal of M, $[x_1,\alpha_1]\circ[x_3,\alpha]\in C$, for every $\alpha\in\Gamma$. Therefore, for every $\alpha\in\Gamma$, $[x_1\oplus_{\alpha_2}x_3,\alpha]\in C$ and $x_1\oplus_{\alpha_2}x_3\in C'$ and C' is a ΓUP-ideal of A. □

Proposition 3.5. *Let $\varphi:A_1\longrightarrow A_2$ be an onto homomorphism. Then, there is a homomorphism $\widehat{\varphi}:M_1\longrightarrow M_2$, where M_1 and M_2 are associated UP-algebra A_1 and A_2, respectively.*

Proof. Suppose that $\widehat{\varphi} : M_1 \longrightarrow M_2$ defined by $\widehat{\varphi}([x_1, \alpha]) = [\varphi(x_1), \alpha]$, where $[x_1, \alpha] \in M_1$. Let $[x_1, \alpha_1] = [x_2, \alpha_2]$. Then, for every $x \in A_1$,

$$x_1 \oplus_{\alpha_1} x = x_2 \oplus_{\alpha_2} x.$$

Hence, $\varphi(x_1) \oplus_{\alpha_1} \varphi(x) = \varphi(x_2) \oplus_{\alpha_2} \varphi(x)$. Since φ is onto, for every $z \in A_2$, there exists $x \in A_1$ such that $\varphi(x) = z$. Hence, $\varphi(x_1) \oplus_{\alpha_1} z = \varphi(x_2) \oplus_{\alpha_2} z$. Then, $\widehat{\varphi}$ is well-defined. Also, for every $[x_1, \alpha_1], [y_1, \beta_1] \in M_1$,

$$\begin{aligned} \widehat{\varphi}([x_1, \alpha_1] \circ [y_1, \beta_1]) = \widehat{\varphi}([x_1 \oplus_{\alpha_1} y_1, \beta_1]) &= [\varphi(x_1) \oplus_{\alpha_1} \varphi(y_1), \beta_1] \\ &= [\varphi(x_1), \alpha_1][\varphi(y_1), \beta_1] \\ &= \widehat{\varphi}([x_1, \alpha_1]\widehat{\varphi}([y_1, \beta_1]). \end{aligned}$$

Also, $\widehat{\varphi}([0_{A_1}, \alpha]) = [\varphi(0_{A_1}), \alpha] = [0_{A_2}, \alpha]$. Therefore, $\widehat{\varphi}$ is a homomorphism. □

Lemma 3.6. *Let $\varphi : A_1 \longrightarrow A_2$. Then, $\widehat{\ker \varphi} = ker\widehat{\varphi}$ and $\widehat{Im\varphi} \subseteq Im\widehat{\varphi}$.*

Proof. Suppose that $\widehat{\varphi} : M_1 \longrightarrow M_2$ is a homomorphism between associated UP-algebras and $[x, \alpha] \in \widehat{ker\varphi}$. Then, $[x, \alpha] = [x_1, \alpha_1]$, for some $x_1 \in ker\varphi$ and $\alpha_1 \in \Gamma$. Then,

$$\widehat{\varphi}([x, \alpha]) = \widehat{\varphi}([x_1, \alpha_1] = [\varphi(x_1), \alpha_1] = [0_{A_2}, \alpha_1] = 0_{M_2}.$$

Hence, $\widehat{ker\varphi} \subseteq ker\widehat{\varphi}$. Also, $[x, \alpha] \in ker\widehat{\varphi}$, implies that

$$\widehat{\varphi}([x, \alpha]) = 0_{M_2} \Longleftrightarrow [\varphi(x), \alpha] = [0_{A_2}, \alpha] \Longleftrightarrow \ \forall z \in A_2, \varphi(x) \oplus_\alpha z = 0_{A_2} \oplus_\alpha z = z.$$

Since 0_{A_2} is unique, $\varphi(x) = 0_{A_2}$ and $[x, \alpha] \in \widehat{ker\varphi}$.
Let $[y, \alpha] \in \widehat{Im\varphi}$. Then, $[y, \alpha] = [y_1, \alpha_1]$, for some $y_1 \in Im\varphi$ and $\alpha \in \Gamma$. Hence, there exists $x_1 \in A_1$, $\varphi(x_1) = y_1$ and $\widehat{\varphi}([x_1, \alpha]) = [y, \alpha]$. Then, $\widehat{Im\varphi} \subseteq Im\widehat{\varphi}$. □

Definition 3.7. *A sequence $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3$ is called exact if $Im\varphi_1 = ker\varphi_2$.*

Proposition 3.8. *Let $\varphi_1 : A_1 \longrightarrow A_2$ and $\varphi_2 : A_2 \longrightarrow A_3$ be homomorphisms. Then, $\widehat{\varphi_2 \circ \varphi_1} = \widehat{\varphi}_2 \circ \widehat{\varphi}_1$.*

Proof. The proof is straightforward. □

Theorem 3.9. *Let $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3$ be an exact sequence of Γ UP-algebras and homomorphisms. Then, $M_1 \xrightarrow{\widehat{\varphi}_1} M_2 \xrightarrow{\widehat{\varphi}_2} M_3$ is an exact sequence of UP-algebras.*

Proof. Since $Im\varphi_1 = ker\varphi_2$ and Proposition 3.8, implies that $Im\widehat{\varphi}_1 \subseteq ker\widehat{\varphi}_2$. Let $[x_2, \alpha_2] \in ker\widehat{\varphi}_2$. Then, $\widehat{\varphi}([x_2, \alpha_2]) = 0_{M_3}$ and $[\varphi(x_2), \alpha_2] = [0_{A_3}, \alpha_2]$. Hence, for every $z \in A_3$, $\varphi(x_2) \oplus_{\alpha_2} z = 0_{A_3} \oplus_{\alpha_2} z = z$. Thus, $x_2 \in ker\varphi = Im\varphi_1$. Then, for some $x_1 \in A_1$, $\varphi_1(x_1) = x_2$ Therefore, $\widehat{\varphi}_1([x_1, \alpha_2])$ and $ker\widehat{\varphi}_2 \subseteq Im\widehat{\varphi}_1$. □

Corollary 3.10. *A homomorphism $\varphi : A_1 \longrightarrow A_2$ is an isomorphism if and only if $\widehat{\varphi} : M_1 \longrightarrow M_2$ is an isomorphism.*

Corollary 3.11. *Let $0 \longrightarrow A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \longrightarrow 0$ be a short exact sequence. Then, $0 \longrightarrow M_1 \xrightarrow{\widehat{\varphi}_1} M_2 \xrightarrow{\widehat{\varphi}_2} M_3 \longrightarrow 0$ is a short exact sequence.*

Proposition 3.12. *Let* $\varphi : A_1 \longrightarrow A_2$ *be a homomorphism. Then,* $\widehat{\varphi(A)} = \widehat{\varphi}(\widehat{A})$ *and* $\widehat{\varphi^{-1}(D)} \subseteq \widehat{\varphi}^{-1}(\widehat{D})$*, where* $A \subseteq A_1$ *and* $D \subseteq M_2$*.*

Proof. Suppose that $[y, \alpha] \in \widehat{\varphi(A)}$. Then, $[y, \alpha] = [y_1, \alpha_1]$, for some $\alpha \in \Gamma$ and $y_1 = \varphi(a)$. Hence, $[y, \alpha] = [\varphi(a), \alpha_1] = \widehat{\varphi}([\alpha_1, a,])$. Then, $\widehat{\varphi(A)} \subseteq \widehat{\varphi}(\widehat{A})$. Let $[y, \alpha] \in \widehat{\varphi}(\widehat{A})$. Then, for some $[x, \alpha_1] \in \widehat{A}$, $[y, \alpha] = \widehat{\varphi}([x, \alpha_1])$. Also, $[x, \alpha_1] = [a, \alpha_2]$, for some $a \in A$ and $\alpha_2 \in \Gamma$. This implies that $[y, \alpha] = \widehat{\varphi}([a, \alpha_2]) = [\varphi(a), \alpha_2]$. Hence, $[\varphi(a), \alpha_2] \in \widehat{\varphi(A)}$ and $\widehat{\varphi}(\widehat{A}) \subseteq \widehat{\varphi(A)}$.
Let $[x, \alpha] \in \widehat{\varphi^{-1}(D)}$. Then, $[x, \alpha] = [x_1, \alpha_1]$, for some $x_1 \in \varphi^{-1}(D)$ and $\widehat{\varphi}([x, \alpha]) = \widehat{\varphi}([x_1, \alpha_1]) = [\varphi(x_1), \alpha_1] \in \widehat{D}$ and $[x, \alpha] \in \widehat{\varphi}^{-1}(D)$. Hence, $\widehat{\varphi^{-1}(D)} \subseteq \widehat{\varphi}^{-1}(\widehat{D})$.

□

Proposition 3.13. *Let* $\varphi : A_1 \longrightarrow A_2$ *be a homomorphism. Then,* $\varphi^{-1}(C^{'}) = (\widehat{\varphi}^{-1}(C))^{'}$

Proof. Suppose that $\varphi : A_1 \longrightarrow A_2$ is a homomorphism and $\widehat{\varphi} : M_1 \longrightarrow M_2$ is an associated homomorphism. Hence,

$$\begin{aligned} x \in \varphi^{-1}(C^{'}) \Longleftrightarrow \varphi(x) \in C^{'} &\Longleftrightarrow \forall \alpha \in \Gamma, \quad [\varphi(x), \alpha] \in C \\ &\Longleftrightarrow \forall \alpha \in \Gamma, \quad \widehat{\varphi}([x, \alpha]) \in C \\ &\Longleftrightarrow \forall \alpha \in \Gamma, \ [x, \alpha] \in \widehat{\varphi}^{-1}(C) \\ &\Longleftrightarrow x \in (\widehat{\varphi}^{-1}(C))^{'} \end{aligned}$$

Therefore, $\varphi^{-1}(C^{'}) = (\widehat{\varphi}^{-1}(C))^{'}$. □

Definition 3.14. *Let* ρ *be an equivalence relation on* Γ *UP-algebra* A*. Then, we define the relation* $\widehat{\rho}$ *on* M *as follows:*

$$[x_1, \alpha_1]\widehat{\rho}[x_2, \alpha_2] \Longleftrightarrow \ \forall \ z \in A, \quad (x_1 \oplus_{\alpha_1} z)\rho(x_2 \oplus_{\alpha_2} z).$$

In view of the above definition, our aim is to establish the following fundamental theorem of a ΓUP-algabras.

Theorem 3.15. *Let* ρ *be an equivalence relation on* Γ *UP-algebra. Then,* $\widehat{A/\rho} \cong M/\widehat{\rho}$*.*

Proof. Suppose that $\varphi : \widehat{A/\rho} \longrightarrow M/\widehat{\rho}$, defined by

$$\varphi([\rho(x), \alpha]) = \widehat{\rho}([x, \alpha]).$$

Let $[\rho(x_1), \alpha_1] = [\rho(x_2), \alpha_2]$. Then, for every $\rho(z) \in A/\rho$, $\rho(x_1) \oplus_{\alpha_1} \rho(z) = \rho(x_2) \oplus_{\alpha_2} \rho(z)$. Hence, $(x_1 \oplus_{\alpha_1} z)\rho(x_2 \oplus_{\alpha_2} z)$ and $\varphi([x_1, \alpha_1]) = \varphi([x_2, \alpha_2])$. Then, φ is well-defined. Also, for every $[\rho(x_1), \alpha_1], [\rho(x_2), \alpha_2] \in \widehat{A/\rho}$

$$\begin{aligned} \varphi([\rho(x_1), \alpha_1] \circ [\rho(x_2), \alpha_2]) &= \varphi([\rho(x_1) \oplus_{\alpha_1} \rho(x_2), \alpha_2]) \\ &= \varphi([\rho(x_1 \oplus_{\alpha_1} x_2), \alpha_2]) \\ &= \widehat{\rho}([x_1 \oplus_{\alpha_1} x_2, \alpha_2]) \\ &= \widehat{\rho}([x_1, \alpha_1])\widehat{\rho}([x_2, \alpha_2]) \\ &= \widehat{\varphi}([\rho(x_1), \alpha_1])\widehat{\varphi}([\rho(x_2), \alpha_2]). \end{aligned}$$

Clearly, φ is onto. To show that φ is one to one, suppose that $\varphi([\rho(x_1), \alpha_1]) = \varphi([\rho(x_1), \alpha_2])$. Hence, for every $z \in H$, $\rho(x_1 \oplus_{\alpha_1} z) = \rho(x_2 \oplus_{\alpha_2} z)$. Then, $[\rho(x_1), \alpha_1] = [\rho(x_2), \alpha_2]$ and φ is an isomorphism. □

Corollary 3.16. *Let* $\mathcal{C}_{\Gamma UP}$ *be a category of* Γ *UP-algebras and* $\mathcal{C}_{UP}$ *be a category of* UP*-algebras and onto homomorphisms. Then, by Theorem 3.2 and Proposition 3.5, we define a functor* ψ *between these category such that* $\psi(A) = M$ *and* $\psi(\varphi) = \widehat{\varphi}$*, where* M*,* M_1 *and* M_2 *are* Γ *UP-algebras and* $\varphi : M_1 \longrightarrow M_2$ *is a onto homomorphism.*

4 Conclusion

Our aim of this paper is to introduce and consider the algebraic properties of a new algebra, that is, the ΓUP-algebra. In addition, we also observe the connections between the ΓUP-algebras and UP algabras. As a consequence, we establish a convariant functor between the categories of ΓUP and UP-algebras. We hope that our research in this paper will bring some attention of the study and applications of this new algebraic structures. Would serve for further study in ΓUP- algebras and its related topics.

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