



State hyper BE-algebras

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“This paper is dedicated to Professor Young Bae Jun on the occasion of his 70th birthday.”

Abstract

In this paper, state operators on hyper BE-algebras (correspondingly, state hyper BE-algebras) are introduced and studied. State hyper filters are introduced and generated state hyper filters are represented in state hyper BE-algebras. Also, maximal (prime) state hyper filters are characterised and the relations between state maximal hyper filters and state prime hyper filters are discussed. Moreover, some related results of state (compatible) hyper congruence are obtained. Especially, it follows that there is an isotone bijection between all state strong regular o-reflexive hyper filter $\mathcal{F}_s(H, \xi)$ and all state compatible hyper congruence $\mathcal{C}_s(H, \xi)$ on state commutative transitive RD-hyper BE-algebra (H, ξ) .

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1 Introduction

The hyper structure theory was introduced by Marty [16] at the 8th Congress of Scandinavian Mathematicians. In an algebraic hyper structure, the composition of two elements is not an element but a set. Since then many hyper algebraic structures have been extensively researched such as hyper BCK-algebras [15], hyper MV-algebras [13], hyper EQ-algebras [3, 7] and hyper equality algebras [4], etc. Also, in 2020, Borzooei, Aaly [1] and Davvaz [10] gave comprehensive overviews of hyper logical algebras, respectively. Now, hyper structure theory has been applied to many disciplines such as geometry, graphs, automata, cryptography, artificial intelligence, probability theory, dismutation reactions and inheritance, etc (see [9, 11]). Recently, Radfar, Rezaei and Saeid [18] introduced the notion of hyper BE-algebras as a generalization of BE-algebras where some types of hyper BE-algebras and hyper filters were given. Then they investigated commutative hyper BE-algebras in [19]. Cheng and Xin [4] systematically studied the filter theory of hyper BE-algebras and they also constructed quotient hyper BE-algebras via normal hyper filters.

The notion of states on MV-algebras was introduced by Mundici [17] in 1995 with the intent of capturing the notion of average degree of truth of a proposition in Lukasiewicz logic, and so the states have been used

as a semantical interpretation of the probability of fuzzy events a . That is, if s is a state and a is a fuzzy event, then $s(a)$ is presented as averaging of appearing the event a . State operators on MV-algebras (correspondingly, state MV-algebras) were introduced by Flaminio [12] in order to preserve some basic properties of the states on MV-algebras. In fact, state operators on logical algebras [2, 5, 8, 14] are able to cope with states in an universal algebraic setting. With the intent of representing the average of the true values of more fuzzy events in the logic, states on hyper MV-algebras [21] were introduced by Xin and Wang as a generalization of the states on MV-algebras. Then Xin and Davvaz [20, 22] applied the state theory to hyper BCK-algebras and introduced and systematically studied the states and state operators on hyper BCK-algebras. The above are our motivation to study state hyper BE-algebras.

This paper is organized as follows: In Section 2, we review some basic concepts and results on hyper BE-algebras. In Section 3, we introduce state operators (correspondingly, state hyper BE-algebras) and investigate some related properties of them. In Section 4, we introduce state hyper filters of hyper BE-algebras and give some representations of generated state hyper filters. Also, we present some characterizations of maximal (prime) state hyper filters and discuss the relationships between them. In Section 5, we introduce the concepts of state (compatible) hyper congruences and find some conditions that there is an isotone bijection between all state strong regular \circ -reflexive hyper filter $\mathcal{F}_s(H, \xi)$ and all state compatible hyper congruence $\mathcal{C}_s(H, \xi)$ on hyper BE-algebra (H, ξ) .

2 Preliminaries

Definition 2.1. [18] *Let H be a nonempty set and $\circ : H \times H \rightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ, 1)$ is called a hyper BE-algebra provided it satisfies the following axioms:*

(HBE1) $x \ll 1$ and $x \ll x$;

(HBE2) $x \circ (y \circ z) = y \circ (x \circ z)$;

(HBE3) $x \in 1 \circ x$;

(HBE4) $1 \ll x$ implies $x = 1$,

for all $x, y \in H$, where the relation \ll is defined by $x \ll y \Leftrightarrow 1 \in x \circ y$. For any two nonempty subsets A and B of H , $A \ll B$ means that there exist $a \in A, b \in B$ such that $a \ll b$.

Notice that in any hyper BE-algebra, $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ and $A \leq B$ means for any $a \in A$, there exists $b \in B$ such that $a \ll b$.

Example 2.2. [18] *Let $H = \{a, b, 1\}$. Define operations \circ_1, \circ_2 on H as follows:*

\circ_1	1	a	b	\circ_2	1	a	b
1	{1}	{a, b}	{b}	1	{1}	{a}	{b}
a	{1}	{1, a, b}	{b}	a	{1, a, b}	{1}	{a, b}
b	{1, b}	{1, a, b}	{1, a, b}	b	{1, a, b}	{1, a, b}	{1}

Then $(H, \circ_1, 1)$ and $(H, \circ_2, 1)$ are two hyper BE-algebras.

Proposition 2.3. [4, 18] *Let $(H, \circ, 1)$ be a hyper BE-algebra. Then for any $x, y \in H, A, B \subseteq H$:*

(P1) $y \in 1 \circ x$ implies $y \ll x$;

(P2) $x \ll y \circ x$ and $A \leq B \circ A$;

(P3) $x \ll (x \circ y) \circ y$ and $A \leq (A \circ B) \circ B$;

(P4) $A \leq B$ and $B \leq C$ imply $A \leq C$;

(P5) $A \leq B$ and $1 \in A$ imply $1 \in B$;

(P6) $A \ll B$ implies $1 \in A \circ B$.

Definition 2.4. [4, 18] *A hyper BE-algebra is said to be a/an*

(1) *R-hyper BE-algebra if $1 \circ x = \{x\}$, for all $x \in H$;*

(2) *D-hyper BE-algebra if $x \circ x = \{1\}$, for all $x \in H$;*

(3) *RD-hyper BE-algebra if it is both an R-hyper BE-algebra and a D-hyper BE-algebra;*

(4) *Transitive hyper BE-algebra if $y \circ z \leq (x \circ y) \circ (x \circ z)$ and $x \circ y \leq (y \circ z) \circ (x \circ z)$, for all $x, y, z \in H$.*

Definition 2.5. [18] Let H be a hyper BE-algebra. A nonempty subset F of H is said to be a hyper filter if it satisfies:

- (1) $1 \in F$;
- (2) $x \circ y \cap F \neq \emptyset$ and $x \in F$ imply $y \in F$, for all $x, y \in H$.

Lemma 2.6. [4] Let F be a hyper filter of a hyper BE-algebra $(H, \circ, 1)$. Then for any nonempty subset A, B of H , $A \cap F \neq \emptyset$ and $A \leq B$ imply $B \cap F \neq \emptyset$.

Definition 2.7. [4] Let H be a hyper BE-algebra and θ be an equivalence relation on H . Then

- (1) for any $A, B \subseteq H$, $A\theta B$ means for all $a \in A$ there exists $b \in B$ such that $a\theta b$ and for all $b \in B$ there exists $a \in A$ such that $a\theta b$;
- (2) θ is called a hyper congruence relation if for all $x, y, u, v \in H$, $x\theta y$ and $u\theta v$ imply $(x \circ u)\theta(y \circ v)$.

Lemma 2.8. Let θ be a hyper congruence on a hyper BE-algebra H . Then $A\theta B$ and $B\theta C$ imply $A\theta C$.

Proof. The proof is clear. □

One can easily prove that an equivalence relation on H is a hyper congruence relation if and only if $x\theta y$ implies $(x \circ u)\theta(y \circ u)$ and $(u \circ x)\theta(u \circ y)$, for all $u \in H$.

Let θ be a hyper congruence relation on a hyper BE-algebra H . Denote $H/\theta = \{[x]_\theta : x \in H\}$ where $[x]_\theta = \{y \in H : y\theta x\}$. We define $\bar{\circ}$ by $[x]_\theta \bar{\circ} [y]_\theta = \{[a]_\theta : a \in x \circ y\}$, and define \ll_θ on H/θ by $[x]_\theta \ll_\theta [y]_\theta$ iff $[1]_\theta \in [x]_\theta \bar{\circ} [y]_\theta$ for any $[x]_\theta, [y]_\theta \in H/\theta$, where $[1]_\theta$ is said to be the kernel of θ and is denoted by $Ker(\theta)$. Clearly, $x \ll y$ implies that $[x]_\theta \ll_\theta [y]_\theta$. Moreover, for $A, B \in H/\theta$, $A \leq_\theta B$ means that for any $[x]_\theta \in A$, there exists $[y]_\theta \in B$ such that $[x]_\theta \ll_\theta [y]_\theta$.

Theorem 2.9. [4] Let $(H, \circ, 1)$ be a hyper BE-algebra and θ be a hyper congruence on H . Then $(H/\theta; \bar{\circ}, [1]_\theta)$ is a hyper BE-algebra, which is called as a quotient hyper BE-algebra with respect to θ .

3 State hyper BE-algebras

In this section, we introduce state operators on hyper BE-algebras and investigate some related properties of them. By H denote a hyper BE-algebra $(H, \circ, 1)$, unless otherwise specified.

Definition 3.1. A map $\xi : H \rightarrow H$ is said to be a state operator on H , if for all $x, y \in H$, it satisfies the following conditions:

- (S1) $\xi(1) = 1$;
- (S2) $\xi(\xi(x)) = \xi(x)$;
- (S3) $x \ll y$ implies $\xi(x) \ll \xi(y)$;
- (S4) $\xi(x \circ y) \leq \xi((x \circ y) \circ y) \circ \xi(y)$;
- (S5) $\xi(\xi(x) \circ \xi(y)) = \xi(x) \circ \xi(y)$.

Meanwhile, the pair (H, ξ) is said to be a state hyper BE-algebra.

Remark 3.2. (1) A state BE-algebra is a state hyper BE-algebra.

(2) By (P2), it is easy to see that (H, Id_H) is a state hyper BE-algebra, where $Id_H : H \rightarrow H$ is the identity map. Therefore a hyper BE-algebra can be seen as a state hyper BE-algebra.

Example 3.3. Let $H = \{a, b, 1\}$. Define an operation \circ on H as follows:

\circ	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1\}$	$\{a, b\}$	$\{1, b\}$

Then $(H, \circ, 1)$ is a hyper BE-algebras [18]. Consider the map $\xi : H \rightarrow H$:

$$\xi(x) = \begin{cases} b, & x = a, b, \\ 1, & x = 1. \end{cases}$$

One can check that (H, ξ) is a state hyper BE-algebra.

Definition 3.4. A state hyper BE-algebra (H, ξ) is called:

- *s-positively ordered* if $\xi(x) \ll \xi(y)$ implies $\xi(z) \circ \xi(x) \leq \xi(z) \circ \xi(y)$;
- *s-negatively ordered* if $\xi(x) \ll \xi(y)$ implies $\xi(y) \circ \xi(z) \leq \xi(x) \circ \xi(z)$;
- *s-ordered* if (H, ξ) is both *s-positively ordered* and *s-negatively ordered*, for all $x, y, z \in H$.

Definition 3.5. A hyper BE-algebra H is called:

- *positively ordered* if $x \ll y$ implies $z \circ x \leq z \circ y$;
- *negatively ordered* if $x \ll y$ implies $y \circ z \leq x \circ z$;
- *ordered* if H is both *positively ordered* and *negatively ordered*, for all $x, y, z \in H$. In particular, if ξ is a state operator on H , we call the state hyper BE-algebra (H, ξ) *positively ordered* (*negatively ordered*, *ordered*).

It is evident that if (H, ξ) is positively ordered (negatively ordered, ordered), then it is *s-positively ordered* (*s-negatively ordered*, *s-ordered*).

Example 3.6. In Example 3.3, we can check that the state hyper BE-algebra (H, ξ) is both *s-positively ordered* (*s-negatively ordered*, *s-ordered*) and *positively-ordered* (*negatively ordered*, *ordered*).

Proposition 3.7. Given a state hyper BE-algebra (H, ξ) and $x, y \in H$, we have:

- (SP1) $\xi(H)$ is a subalgebra of H ;
- (SP2) $\xi(H) \cap \text{Ker}(\xi) = \{1\}$;
- (SP3) $\xi(H) = \text{Fix}_\xi(H)$, where $\text{Fix}_\xi(H) = \{x \in H : \xi(x) = x\}$;
- (SP4) If (H, ξ) is *s-negatively ordered*, then $\xi(x \circ y) \leq \xi(x) \circ \xi(y)$. More generally, if (H, ξ) is ordered, then

$$\xi(x^n \circ (\cdots \circ (x^2 \circ (x^1 \circ x)) \cdots)) \leq \xi(x^n) \circ (\cdots \circ ((\xi(x^2) \circ (\xi(x^1) \circ \xi(x))) \cdots)).$$

Proof. (SP1) and (SP2) are obvious.

(SP3) If $x \in \xi(H)$, then there exists $a \in H$ such that $\xi(a) = x$. Hence $\xi(x) = \xi(\xi(a)) = \xi(a) = x$, and so $x \in \text{Fix}_\xi(H)$. Conversely, if $x \in \text{Fix}_\xi(H)$, then $\xi(x) = x$ and so $x \in \xi(H)$.

(SP4) Since $x \ll (x \circ y) \circ y$, we get $\xi(x) \ll \xi((x \circ y) \circ y)$. Considering (H, ξ) is *s-negatively ordered*, it follows that $\xi(x \circ y) \leq \xi((x \circ y) \circ y) \circ \xi(y) \leq \xi(x) \circ \xi(y)$. Therefore, $\xi(x \circ y) \leq \xi(x) \circ \xi(y)$. The second part can be easily seen. \square

4 State hyper filters of state hyper BE-algebras

In this section, we introduce state hyper filters in state hyper BE-algebras. We focus on giving the generated representations of state hyper filters and investigating maximal (prime) state hyper filters in state hyper BE-algebras.

Definition 4.1. Let (H, ξ) be a state hyper BE-algebra. A hyper filter F of H is said to be a state hyper filter of (H, ξ) if $x \in F$ implies $\xi(x) \in F$, for any $x \in H$.

Example 4.2. Consider the hyper BE-algebra $(H, \circ_1, 1)$ from Example 2.2. Define a map $\xi : H \rightarrow H$, where $\xi(b) = b, \xi(a) = \xi(1) = 1$, then it can be calculated that (H, ξ) is a state hyper BE-algebra, and $F = \{a, 1\}$ is a state hyper filter of (H, ξ) .

Proposition 4.3. Let (H, ξ) be an *s-negatively ordered* state hyper BE-algebra. Then $\text{Ker}(\xi)$ is a state hyper filter of (H, ξ) .

Proof. Clearly, $1 \in \text{Ker}(\xi)$. Let $x, y \in H$ such that $(x \circ y) \cap \text{Ker}(\xi) \neq \emptyset$ and $x \in \text{Ker}(\xi)$. Then $\xi(x) = 1$ and $1 \in \xi(x \circ y)$. Hence $\xi(x \circ y) \leq \xi(x) \circ \xi(y) = 1 \circ \xi(y)$ and so $1 \in 1 \circ \xi(y)$, which implies $1 \ll \xi(y)$. This results in $\xi(y) = 1$, that is, $y \in \text{Ker}(\xi)$. Therefore, $\text{Ker}(\xi)$ is a state hyper filter of (H, ξ) . \square

Proposition 4.4. Let (H, ξ) be an *s-negatively ordered* state hyper BE-algebra. If F in $\xi(H)$ is a state hyper filter, then $\xi^{-1}(F)$ in H is also a state hyper filter.

Proof. Clearly, $1 \in \xi^{-1}(F)$ from $\xi(1) = 1$. Let $x \in F$ and $(x \circ y) \cap F \neq \emptyset$. Hence $\xi(x) \in F$ and $\xi(x \circ y) \cap F \neq \emptyset$. Since (H, ξ) is s -negatively ordered, we have $\xi(x \circ y) \leq \xi(x) \circ \xi(y)$. It follows from Lemma 2.6 that $(\xi(x) \circ \xi(y)) \cap F \neq \emptyset$ and so $\xi(y) \in F$. Therefore $y \in \xi^{-1}(F)$, which implies that $\xi^{-1}(F)$ is a hyper filter of H . Let $x \in \xi^{-1}F$. Then $\xi(x) \in F$. Since F is a state hyper filter, we have $\xi(\xi(x)) \in F$ and so $\xi(x) \in \xi^{-1}(F)$. It shows that $\xi^{-1}(F)$ is a state hyper filter of (H, ξ) . \square

Let (H, ξ) be a state hyper BE-algebra and A be a nonempty subset of H . Denote by $[A]_s$ the least state hyper filter containing A of (H, ξ) , and call the state hyper filter generated by A . In particular, if $A = \{a\}$, then we write $[\{a\}]_s = [a]_s$. In addition, we denote by $[F \cup \{a\}]_s$ the generated state hyper filter by F and a , where $a \in H \setminus F$. In the following, we give some generated representation of state hyper filters in state hyper BE-algebras.

Theorem 4.5. *Given a state hyper BE-algebra (H, ξ) and $\emptyset \neq A \subseteq H$. Then, we have*

$$[A]_s = \{x \in H : 1 \in a_1 \circ (a_2 \circ (\cdots (a_n \circ (\xi(b_1) \circ (\cdots (\xi(b_m) \circ x) \cdots))) \cdots)), \text{ for some } a_1 \cdots, a_n, b_1, \cdots, b_m \in A, n, m \geq 1\}.$$

Proof. Denote the right side by B . Firstly, $B \subseteq [A]_s$. In fact, if $x \in B$, then there are $a_1 \cdots, a_n, b_1, \cdots, b_m \in A$ such that $1 \in a_1 \circ (a_2 \circ (\cdots (a_n \circ (\xi(b_1) \circ (\cdots (\xi(b_m) \circ x) \cdots))) \cdots))$. Hence

$$a_1 \circ (a_2 \circ (\cdots (a_n \circ (\xi(b_1) \circ (\cdots (\xi(b_m) \circ x) \cdots))) \cdots)) \cap [A]_s \neq \emptyset.$$

Since $a_1 \in A \subseteq [A]_s$ and $[A]_s$ is a state hyper filter of H , we have

$$a_2 \circ (\cdots (a_n \circ (\xi(b_1) \circ (\cdots (\xi(b_m) \circ x) \cdots))) \cdots)) \cap [A]_s \neq \emptyset.$$

Considering $\xi(b_i) \in [A]_s$, for all i and repeating the above process, we obtain $x \in [A]_s$. \square

Theorem 4.6. *Given an ordered state hyper BE-algebra (H, ξ) and $\emptyset \neq A \subseteq H$. Then, we have*

$$[A]_s = \{x \in H : 1 \in a_1 \circ (a_2 \circ (\cdots (a_n \circ (\xi(b_1) \circ (\cdots (\xi(b_m) \circ x) \cdots))) \cdots)) \text{ for some } a_1 \cdots, a_n, b_1, \cdots, b_m \in A, n, m \geq 1\}.$$

Proof. Denote the right side by B . Firstly, $B \subseteq [A]_s$. Next, we prove $[A]_s \subseteq B$, namely, B is a state hyper filter including A of H . To do this, we need have the following steps.

Step 1: $1 \in B$ and $A \subseteq B$, since $1 \in a \circ 1$ and $1 \in x \circ x$, for any $a, x \in A$.

Step 2: We show that B is a state hyper filter of H .

Let $x \in B$ and $(x \circ y) \cap B \neq \emptyset$. Then there are $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m, c_1, c_2, \cdots, c_p, d_1, d_2, \cdots, d_q \in A$ such that

$$1 \in a_1 \circ (a_2 \circ (\cdots (a_n \circ (\xi(b_1) \circ (\cdots (\xi(b_m) \circ x) \cdots))) \cdots)) \text{ and } 1 \in c_1 \circ (c_2 \circ (\cdots (c_p \circ (\xi(d_1) \circ (\cdots (\xi(d_q) \circ (x \circ y) \cdots))) \cdots))).$$

Hence using (HBE2) repeatedly, we get

$$c_1 \circ (c_2 \circ (\cdots (c_p \circ (\xi(d_1) \circ (\cdots (\xi(d_q) \circ (x \circ y) \cdots))) \cdots)) = x \circ (c_1 \circ (c_2 \circ (\cdots (c_p \circ (\xi(d_1) \circ (\cdots (\xi(d_q) \circ y) \cdots))) \cdots))).$$

This implies $1 \in x \circ (c_1 \circ (c_2 \circ (\cdots (c_p \circ (\xi(d_1) \circ (\cdots (\xi(d_q) \circ y) \cdots))) \cdots))$, and so

$$x \ll c_1 \circ (c_2 \circ (\cdots (c_p \circ (\xi(d_1) \circ (\cdots (\xi(d_q) \circ y) \cdots))) \cdots)).$$

Since H is ordered, we have

$$\begin{aligned} & a_1 \circ (a_2 \circ (\cdots (a_n \circ (\xi(b_1) \circ (\cdots (\xi(b_m) \circ x) \cdots))) \cdots)) \\ & \leq a_1 \circ (a_2 \circ (\cdots (a_n \circ (\xi(b_1) \circ (\cdots (\xi(b_m) \circ (c_1 \circ (c_2 \circ (\cdots (c_p \circ (\xi(d_1) \circ (\cdots (\xi(d_q) \circ y) \cdots))) \cdots))) \cdots))) \cdots)). \end{aligned}$$

It implies that

$$\begin{aligned} 1 & \in a_1 \circ (a_2 \circ (\cdots (a_n \circ (\xi(b_1) \circ (\cdots (\xi(b_m) \circ (c_1 \circ (c_2 \circ (\cdots (c_p \circ (\xi(d_1) \circ (\cdots (\xi(d_q) \circ y) \cdots))) \cdots))) \cdots))) \cdots)) \\ & = a_1 \circ (a_2 \circ (\cdots (a_n \circ (c_1 \circ (c_2 \circ (\cdots (c_p \circ (\xi(b_1) \circ (\cdots (\xi(b_m) \circ (\xi(d_1) \circ (\cdots (\xi(d_q) \circ y) \cdots))) \cdots))) \cdots))) \cdots)), \end{aligned}$$

which shows $y \in B$.

Step 3: We prove that $x \in B$ implies $\xi(x) \in B$.

By $x \in B$, there are $e_1, e_2, \dots, e_r, f_1, \dots, f_t \in A$ such that

$$1 \in e_1 \circ (e_2 \circ (\dots (e_n \circ (\xi(f_1) \circ (\dots (\xi(f_m) \circ x) \dots))).$$

It follows from (S1),(SP2) and (SP3) that

$$\begin{aligned} 1 &\in \xi(e_1 \circ (e_2 \circ (\dots (e_n \circ (\xi(f_1) \circ (\dots (\xi(f_m) \circ x) \dots)))) \\ &\leq \xi(e_1) \circ (\xi(e_2) \circ (\dots (\xi(e_n) \circ (\xi(\xi(f_1)) \circ (\dots (\xi(\xi(f_m)) \circ \xi(x)) \dots))) \\ &= \xi(e_1) \circ (\xi(e_2) \circ (\dots (\xi(e_n) \circ (\xi(f_1) \circ (\dots (\xi(f_m) \circ \xi(x)) \dots))). \end{aligned}$$

This shows $1 \in \xi(e_1) \circ (\xi(e_2) \circ (\dots (\xi(e_n) \circ (\xi(f_1) \circ (\dots (\xi(f_m) \circ \xi(x)) \dots)))$ and so $\xi(x) \in B$. \square

Corollary 4.7. *Given an ordered state hyper BE-algebra (H, ξ) and $a \in H$,*

$$[a]_s = \{x \in H : 1 \in a^n \circ (\xi(a)^m \circ x), n, m \geq 1\}.$$

Theorem 4.8. *Given an ordered state hyper BE-algebra (H, ξ) ,*

$$[F \cup \{a\}]_s = \{x \in H : a^n \circ (\xi(a)^m \circ x) \cap F \neq \emptyset, m, n \geq 1\},$$

where F is a state hyper filter of H and $a \in H \setminus F$.

Proof. Set $B = \{x \in H : a^n \circ (\xi(a)^m \circ x) \cap F \neq \emptyset, n, m \geq 1\}$. Firstly, $B \subseteq [F \cup \{a\}]_s$. If $x \in H$, then $a^n \circ (\xi(a)^m \circ x) \cap F \neq \emptyset$ and so $a^n \circ (\xi(a)^m \circ x) \cap [F \cup \{a\}]_s \neq \emptyset$. Since $a \in [F \cup \{a\}]_s$, $\xi(a) \in [F \cup \{a\}]_s$ and $[F \cup \{a\}]_s$ is a state hyper filter of H , we get $x \in [F \cup \{a\}]_s$.

Conversely, we show $[F \cup \{a\}]_s \subseteq B$. By $(a \circ 1) \cap F \neq \emptyset$ and $(a \circ a) \cap F \neq \emptyset$, we have $1 \in B$ and $a \in B$. Let $x \in F$. It follows from $x \ll \xi(a) \circ x$ that $(\xi(a) \circ x) \cap F \neq \emptyset$. By using $\xi(a) \circ x \leq \xi(a) \circ (\xi(a) \circ x) = \xi(a)^2 \circ x$, we get $\xi(a)^2 \circ x \cap F \neq \emptyset$. Repeating the above process, $a^n \circ (\xi(a)^m \circ x) \cap F \neq \emptyset$, which shows $x \in B$ and so $F \subseteq B$. Next let $x \in [F \cup \{a\}]_s$. Then according to Theorem 4.6, there are $a_1, a_2, \dots, a_n, b_1, \dots, b_m \in F \cup \{a\}$ such that $1 \in a_1 \circ (a_2 \circ (\dots (a_n \circ (\xi(b_1) \circ (\dots (\xi(b_m) \circ x) \dots))))$.

Case 1. If there are some i, j such that $a_i = a, b_j = a$, then $1 \in a_1 \circ (a_2 \circ (\dots (a^r \circ (\xi(a)^t \circ x) \dots)))$. Since $a_p, \xi(b_q) \in F (p \neq i, q \neq j)$ and F is a state hyper filter of H , we have $a^r \circ (\xi(a)^t \circ x) \cap F \neq \emptyset$, which implies $x \in B$.

Case 2. If $a_i, b_j \neq a$ for all i, j , then $a_i, \xi(b_j) \in F$. Since F is a state hyper filter of H , we get $x \in F$ and so $x \in B$. \square

A state hyper filter F of a state hyper BE-algebra (H, ξ) is said to be proper if $F \neq H$. In the following, we introduce maximal (prime) state hyper filters in state hyper BE-algebras.

Definition 4.9. *Let (H, ξ) be a state hyper BE-algebra. A proper state hyper filter F of (H, ξ) is called*

- *maximal provided that $H = [F \cup \{a\}]_s$ for any $a \in H \setminus F$.*
- *prime provided that $F_1 \cap F_2 \subseteq F$ implies $F_1 \subseteq F$ or $F_2 \subseteq F$ for any state hyper filters F_1, F_2 of (H, ξ) .*

In the following, we investigate characterizations of maximal (prime) state hyper filters in a state hyper BE-algebra.

Theorem 4.10. *Let (H, ξ) be a state hyper BE-algebra and P is a proper state hyper filter of (H, ξ) . Then F is maximal if and only if $M \subseteq F \subseteq H$ implies $M = F$ or $F = H$ for any state hyper filter of (H, ξ) .*

Proof. Assume that M is a maximal state hyper filter and F is a state hyper filter of (H, ξ) such that $M \subseteq F \subseteq H$. If $M \neq F$, then $M \subset F$ and there exists $a \in F$ but $a \notin M$. Since M is maximal, we have $[M \cup \{a\}]_s = H$. Hence for any $x \in H$, $x \in [M \cup \{a\}]_s$, that is, there is $m, n \in N$ such that $a^n \circ (\xi(a)^m \circ x) \cap M \neq \emptyset$ and thus $a^n \circ (\xi(a)^m \circ x) \cap F \neq \emptyset$. Since $a \in F$ and F is a state hyper filter of H , we get $\xi(a) \in F$. According to the definition of the hyper filter of H , we obtain $x \in F$ and so $H \subseteq F$. It follows that $F = H$.

Conversely, assume that the condition is true and M is a proper state hyper filter of (H, ξ) . Let $a \in H \setminus M$. If $[M \cup \{a\}]_s \neq H$, then there exists $x \in H$ such that $x \notin [M \cup \{a\}]_s$. Hence $M \subseteq [M \cup \{a\}]_s \subset H$ and thus $M = [M \cup \{a\}]_s$. Therefore, $a \in M$, a contradiction. This shows that $[M \cup \{a\}]_s = H$, namely, M is a maximal state hyper filter of (H, ξ) . \square

Theorem 4.11. *Let (H, ξ) be a state hyper BE-algebra and P be a proper state hyper filter of (H, ξ) . Then P is prime if and only if $[x]_s \cap [y]_s \subseteq P$ implies $x \in P$ or $y \in P$, for all $x, y \in H$.*

Proof. Assume that P is a prime state hyper filter of (H, ξ) . Set $[x]_s \cap [y]_s \subseteq P$ for $x, y \in H$. Then $x \in [x]_s \subseteq P$ or $y \in [y]_s \subseteq P$. Conversely, if F_1 and F_2 are two state hyper filters of (H, ξ) such that $F_1 \cap F_2 \subseteq P$. Let $x \in F_1$ and $y \in F_2$. Then $[x]_s \subseteq F_1$ and $[y]_s \subseteq F_2$. Hence $[x]_s \cap [y]_s \subseteq F_1 \cap F_2 \subseteq P$ and so $x \in P$ or $y \in P$. Therefore, $F_1 \subseteq P$ or $F_2 \subseteq P$. \square

Next, we deduce some properties about maximal (prime) state hyper filters in a state hyper BE-algebra.

Proposition 4.12. *Suppose that (H, ξ) is an ordered state hyper BE-algebra. If $\xi^{-1}(F)$ in H and F in $\xi(H)$ are two state hyper filters, then $\xi^{-1}(F)$ is maximal implies F is maximal.*

Proof. Assume that $\xi^{-1}(F)$ is a maximal state hyper filter of H . Then for any $x \in H$ and $a \in H \setminus F$, there are $n, m \geq 1$ such that $a^n \circ (\xi(a)^m \circ x) \cap \xi^{-1}(F) \neq \emptyset$. Hence it follows from (SP4) that there is $y \in \xi^{-1}(F)$, i.e., $\xi(y) \in F$ such that $\xi(y) \in \xi(a^n \circ (\xi(a)^m \circ x)) \leq \xi(a)^n \circ (\xi(\xi(a)^m) \circ \xi(x))$. Since $\xi(y) \in F$ and F is a state hyper filter of H , we get $\xi(a)^n \circ (\xi(\xi(a)^m) \circ \xi(x)) \cap F \neq \emptyset$. This shows $\xi(H) = [F \cup \{\xi(a)\}]_s$ for any $\xi(a) \in \xi(H) \setminus \xi(F)$. \square

Theorem 4.13. *Let F be a state hyper filter of a state hyper BE-algebra (H, ξ) . Then for any $a, b \in H$, $[a]_s \cap [b]_s \subseteq F$ if and only if $[F \cup \{a\}]_s \cap [F \cup \{b\}]_s = F$.*

Proof. Assume that $[F \cup \{a\}]_s \cap [F \cup \{b\}]_s = F$. Since $a \in [a]_s$ and $b \in [b]_s$, we have $[a]_s \subseteq [F \cup \{a\}]_s$ and $[b]_s \subseteq [F \cup \{b\}]_s$. Hence $[a]_s \cap [b]_s \subseteq [F \cup \{a\}]_s \cap [F \cup \{b\}]_s = F$.

Conversely, assume that $[a]_s \cap [b]_s \subseteq F$ for $a, b \in H$. Clearly, $F \subseteq [F \cup \{a\}]_s \cap [F \cup \{b\}]_s$. Now, set $x \in [F \cup \{a\}]_s \cap [F \cup \{b\}]_s$. Then there are $m, n, r, t \in N$ such that $a^n \circ (\xi(a)^m \circ x) \cap M \neq \emptyset$ and $b^r \circ (\xi(b)^t \circ x) \cap M \neq \emptyset$. Hence there exist $p, q \in F$ such that $p \in a^n \circ (\xi(a)^m \circ x)$ and $q \in b^r \circ (\xi(b)^t \circ x)$. Thus

$$1 \in p \circ p \subseteq p \circ (a^n \circ (\xi(a)^m \circ x)) = a^n \circ (\xi(a)^m \circ (p \circ x)),$$

and so $a^n \circ (\xi(a)^m \circ (p \circ x)) \cap [a]_s \neq \emptyset$. Since $[a]_s$ is a state hyper filter of H and $a \in [a]_s$, we get $(p \circ x) \cap [a]_s \neq \emptyset$. By the similar way, $(q \circ x) \cap [b]_s \neq \emptyset$. Also $p \circ x \leq q \circ (p \circ x) = p \circ (q \circ x)$ and $q \circ x \leq p \circ (q \circ x)$. It follows from Lemma 2.6 that $p \circ (q \circ x) \cap [a]_s \neq \emptyset$ and $p \circ (q \circ x) \cap [b]_s \neq \emptyset$. Thus $p \circ (q \circ x) \cap ([a]_s \cap [b]_s) \neq \emptyset$, which implies $F \cap (p \circ (q \circ x)) \neq \emptyset$. Considering $p, q \in F$, we have $x \in F$. This shows that $[F \cup \{a\}]_s \cap [F \cup \{b\}]_s \subseteq F$. \square

Theorem 4.14. *Let (H, ξ) be an s -negatively ordered state hyper BE-algebra with $\xi(x)^{n+m} \circ \xi(a) \leq x^n \circ (\xi(x)^m \circ a)$ for all $m, n \in N, a, x \in H$, and $\xi(H)$ be a state hyper filter of H . If F in $\xi(H)$ is a prime state hyper filter and $\xi^{-1}(F) \neq H$, then $\xi^{-1}(F)$ in H is also a prime state hyper filter.*

Proof. By Proposition 4.4, $\xi^{-1}(F)$ in H is a state hyper filter. Now, let $[x]_s \cap [y]_s \subseteq \xi^{-1}(F), x, y \in H$ and $p \in [\xi(x)]_s \cap [\xi(y)]_s$. Then there exist $m, n, r, t \in N$ such that

$$1 = \xi(1) \in \xi(x)^n \circ (\xi(\xi(x))^m \circ p) = \xi(x)^n \circ (\xi(x)^m \circ p) = \xi(x)^{n+m} \circ p$$

and

$$1 = \xi(1) \in \xi(y)^r \circ (\xi(\xi(y))^t \circ p) = \xi(y)^t \circ (\xi(y)^r \circ p) = \xi(y)^{t+r} \circ p.$$

Hence $\xi(x)^{n+m} \circ p \cap \xi(H) \neq \emptyset$. Since $\xi(x) \in \xi(H)$ and $\xi(H)$ is a hyper filter, we have $p \in \xi(H)$. Thus there is $a \in H$ such that $p = \xi(a)$ and so $1 \in \xi(x)^{n+m} \circ p = \xi(x)^{n+m} \circ \xi(a) \leq x^n \circ (\xi(x)^m \circ a)$. It implies that $x^n \circ (\xi(x)^m \circ a) \cap \xi^{-1}(F) \neq \emptyset$ and hence $a \in [\xi^{-1}(F) \cup \{x\}]_s$. Similarly, $a \in [\xi^{-1}(F) \cup \{y\}]_s$. This shows $a \in [\xi^{-1}(F) \cup \{y\}]_s \cap [\xi^{-1}(F) \cup \{x\}]_s$. Considering $[x]_s \cap [y]_s \subseteq \xi^{-1}(F)$ and Theorem 4.13, we get $a \in [\xi^{-1}(F) \cup \{y\}]_s \cap [\xi^{-1}(F) \cup \{x\}]_s = \xi^{-1}(F)$. Therefore, $p = \xi(a) \in F$, which implies $[\xi(x)]_s \cap [\xi(y)]_s \subseteq F$. Since F in $\xi(H)$ is a prime state hyper filter, we obtain $\xi(x) \in F$ or $\xi(y) \in F$. It follows from Theorem 4.11 that $\xi^{-1}(F)$ in H is a prime state hyper filter. \square

The following theorem delivers the relationship between maximal state hyper filters and prime state hyper filters in a state hyper BE-algebra.

Theorem 4.15. *Every maximal state hyper filter of a state hyper BE-algebra (H, ξ) is a prime state hyper filter.*

Proof. Let M be a maximal state hyper filter of (H, ξ) and $[a]_s \cap [b]_s \subseteq M, a, b \in H$. Assume that $a, b \notin M$, we have $[M \cup \{a\}]_s = H, [M \cup \{b\}]_s = H$. Hence $[M \cup \{a\}]_s \cap [M \cup \{b\}]_s = H$. By Theorem 4.13 $[a]_s \cap [b]_s \not\subseteq M$, a contradiction. Therefore $a \in M$ or $b \in M$ and so M is a state prime hyper filter. \square

5 State hyper congruences on state hyper BE-algebras

In this section, we introduce the concept of state hyper congruences on state hyper BE-algebras.

Let H be a hyper BE-algebra and F be a hyper filter of H . Denote by θ_F the binary relation generated by F . Define Δ_F as follows: $x\Delta_F y$ iff $(x \circ y) \cap F \neq \emptyset$ and $(y \circ x) \cap F \neq \emptyset$. If Δ_F is a hyper congruence, then we denote the quotient algebra H/Δ_F by $H/F = \{[x]_F : x \in H\}$. Thus $(H/F, \bar{\circ}, [1]_F)$ is a hyper BE-algebra, where $\{[1]_F\}$ is a hyper filter of H/F .

Definition 5.1. *A hyper congruence θ on a hyper BE-algebra H is called compatible if for all $x, y \in H$, there exists $z \in H$ such that $x \circ y \theta z$.*

Let θ be an equivalence relation on a hyper BE-algebra H and $A, B, C \subseteq H$. Then $A\bar{\theta}B$ is defined by $a\theta b$ for all $a \in A$ and $b \in B$. Thus $A\bar{\theta}B$ and $B\bar{\theta}C$ imply $A\bar{\theta}C$.

Lemma 5.2. *Let $(H, \circ, 1)$ be a hyper BE-algebra and θ be an equivalence relation on H . Then the following are equivalent:*

- (1) θ is a compatible hyper congruence on H ;
- (2) θ satisfies: $x\theta y$ and $u\theta v$ imply $x \circ u\theta y \circ v$.

Proof. (1) \Rightarrow (2) Assume that θ is a compatible hyper congruence on H . Let $x\theta_F y$. Then $(x \circ a)\theta(y \circ a)$ and $(a \circ x)\theta(a \circ y)$ for all $a \in H$. Hence for all $t \in x \circ a$, there exists $s \in y \circ a$ such that $s\theta t$ and for all $m \in y \circ a$, there exists $n \in x \circ a$ such that $m\theta n$. On the other hand, since θ is compatible, there are $u, v \in H$ such that $(x \circ a)\theta u$ and $(y \circ a)\theta v$. Based on the two parts, we deduce $t\theta_F u\theta_F n\theta_F m$. Thus for all $t \in x \circ a$ and $m \in y \circ a$, we have $t\theta_F m$. That is, $(x \circ a)\bar{\theta}(y \circ a)$, for all $a \in H$. Similarly, we can obtain $(a \circ x)\bar{\theta}(a \circ y)$, for all $a \in H$. It is not difficult to see that $(x \circ u)\bar{\theta}(y \circ v)$.

(2) \Rightarrow (1) Assume that the condition (2) holds. Then one can easily deduce that θ is a hyper congruence on H . Moreover, it follows from $x\theta x, y\theta y$ that $(x \circ y)\bar{\theta}(x \circ y)$. This implies that there exists $t \in x \circ y \subseteq H$ such that $(x \circ y)\theta t$. Consequently, θ is a compatible hyper congruence on H . \square

Definition 5.3. *Let $(H, \circ, 1)$ be a hyper BE-algebra. For all $x, y, u, v \in H$, a hyper filter F of H is called*

- \circ -reflexive if $x \circ y \cap F \neq \emptyset$ implies $x \circ y \subseteq F$;
- regular if $x\Delta_F y$ and $u\Delta_F v$ imply $(x \circ u)\Delta_F(y \circ v)$;
- strong regular if $x\Delta_F y$ and $u\Delta_F v$ imply $(x \circ u)\bar{\Delta}_F(y \circ v)$.

It is easy to see that, a hyper filter F of a hyper BE-algebra H is strong regular if and only if $x\Delta_F y$ implies $(x \circ u)\bar{\Delta}_F(y \circ u)$ and $(u \circ x)\bar{\Delta}_F(u \circ y)$, for all $u \in H$.

Definition 5.4. *Let (H, ξ) be a state hyper BE-algebra. A hyper congruence θ on H is said to be a state hyper congruence on (H, ξ) if $x\theta y$ implies $\xi(x)\theta\xi(y)$, for all $x, y \in H$.*

Proposition 5.5. *Let (H, ξ) be an s -negatively ordered state hyper BE-algebra and θ be a state hyper congruence on (H, ξ) . Then F is a state hyper filter on (H, ξ) if and only if $\bar{F} = \{[x] : x \in F\}$ is a state hyper filter on $(H/\theta, \hat{\xi})$.*

Proof. Assume that F is a state hyper filter on (H, ξ) . Firstly, $1 \in F$ results in $[1] \in \bar{F}$. Let $[x] \in \bar{F}$ and $[x]\bar{\circ}[y] \cap \bar{F} \neq \emptyset$. Then $x \in F$ and there is $[a] \in [x]\bar{\circ}[y]$ such that $[a] \in a \in \bar{F}$, where $a \in x \circ y$. Hence $(x \circ y) \cap F \neq \emptyset$. Since F is a hyper filter on H , we have $y \in F$ and so $[y] \in \bar{F}$. Therefore, \bar{F} is a hyper filter on H/θ . Next let $[x] \in \bar{F}$. Then $x \in F$ and so $\xi(x) \in F$. Thus $\hat{\xi}([x]) = [\xi(x)] \in \bar{F}$. It shows that \bar{F} is a state hyper filter on $(H/\theta, \hat{\xi})$.

Conversely, assume that \overline{F} is a state hyper filter on $(H/\theta, \widehat{\xi})$. Then $[1] \in \overline{F}$ implies $1 \in F$. Let $x \in F$ and $(x \circ y) \cap F \neq \emptyset$. Then $[x] \in \overline{F}$ and there is $a \in x \circ y$ such that $a \in F$. Hence, there is $a \in x \circ y$ such that $[a] \in \overline{F}$ but $[a] \in [x] \overline{\circ} [y]$. It implies that $([x] \overline{\circ} [y]) \cap \overline{F} \neq \emptyset$. Since \overline{F} is a hyper filter on H/θ , we get $[y] \in \overline{F}$, which derives $y \in F$. Therefore, F is a state hyper filter on H . Now, let $[x] \in \overline{F}$. Then $[\xi(x)] = \widehat{\xi}([x]) \in \overline{F}$. Hence $\xi(x) \in F$, which proves that F is a state hyper filter on (H, ξ) . \square

Theorem 5.6. *Let (H, ξ) be an s -negatively ordered state transitive hyper BE-algebra. If F is a state strong regular \circ -reflexive hyper filter of (H, ξ) , then Δ_F is a state compatible hyper congruence on (H, ξ) .*

Proof. The reflexivity and the symmetry of Δ_F are evident. Now, let $x \Delta_F y$ and $y \theta_F z$. Then $(x \circ y) \cap F \neq \emptyset$ and $(y \circ x) \cap F \neq \emptyset$. Since H is transitive, it follows from $y \circ z \leq (x \circ y) \circ (x \circ z)$ that $((x \circ y) \circ (x \circ z)) \cap F \neq \emptyset$. Also, since F is \circ -reflexive, we have $x \circ y \subseteq F$. Hence $(x \circ z) \cap F \neq \emptyset$. Similarly, $(z \circ x) \cap F \neq \emptyset$. Consequently, $x \Delta_F z$ and so the transitivity holds. It shows that θ_F is an equivalence relation. Considering that F is strong regular hyper filter, by Lemma 5.2 we deduce that Δ_F is a compatible hyper congruence on H . Next, let $x \Delta_F y$. Then $(x \circ y) \cap F \neq \emptyset$ and $(y \circ x) \cap F \neq \emptyset$. Since F is state hyper filter, we get $\xi(x \circ y) \cap F \neq \emptyset$. Combining $\xi(x \circ y) \leq \xi(x) \circ \xi(y)$, it follows from Lemma 2.6 that $(\xi(x) \circ \xi(y)) \cap F \neq \emptyset$. Similarly, $(\xi(y) \circ \xi(x)) \cap F \neq \emptyset$. Therefore, $\xi(x) \theta \xi(y)$, that is, Δ_F is a state hyper congruence on (H, ξ) . Thus Δ_F is a state compatible hyper congruence on (H, ξ) . Combing the above arguments, we obtain Δ_F is a state compatible hyper congruence on (H, ξ) . \square

Theorem 5.7. *Let (H, ξ) be a state RD-hyper BE-algebra such that $\xi(x \circ y) \circ \xi(y) = \xi(y \circ x) \circ \xi(x)$, for any $x, y \in H$, and θ be a hyper congruence on H . Then θ is a state hyper congruence on (H, ξ) if and only if $1 \theta x$ implies $1 \theta \xi(x)$, for all $x \in H$.*

Proof. The necessity is clear. Now, assume that the condition holds. Let $x \theta y$ for any $x, y \in H$. Then $(x \circ x) \theta (x \circ y)$ and hence $1 \theta (x \circ y)$. Thus there exists $b \in x \circ y$ such that $1 \theta b$ and for any $c \in x \circ y$ such that $c \theta 1$. It follows from the condition that for the above b, c there exists $\xi(b) \in \xi(x \circ y)$ such that $1 \in \xi(b)$ and for any $\xi(c) \in \xi(x \circ y)$ such that $1 \in \xi(c)$. According to Definition 2.7, we get $1 \theta \xi(x \circ y)$, which shows $1 \circ \xi(y) \theta \xi(x \circ y) \circ \xi(y)$. That is, $\xi(y) \theta \xi(x \circ y) \circ \xi(y)$. Similarly, $\xi(x) \theta \xi(y \circ x) \circ \xi(x)$. Since $\xi(x \circ y) \circ \xi(y) = \xi(y \circ x) \circ \xi(x)$, we obtain $\xi(x) \theta \xi(y)$. Therefore, θ is a state hyper congruence on (H, ξ) . \square

Theorem 5.8. *Let (H, ξ) be an s -negatively ordered state hyper BE-algebra and θ be a state congruence on (H, ξ) . Then $(H/\theta, \widehat{\xi})$ is a state hyper BE-algebra, where the map $\widehat{\xi} : H/\theta \rightarrow H/\theta$ is defined by $\widehat{\xi}([x]) = [\xi(x)]$.*

Proof. From Theorem 2.9 $(H/\theta, \overline{\circ}, [1])$ is a hyper BE-algebra. Let $[x], [y] \in H/\theta$. Then $\widehat{\xi}$ is well defined. In fact, if $[x] = [y]$, we have $x \theta y$ and hence $\xi(x) \theta \xi(y)$. This implies $[\xi(x)] = [\xi(y)]$, namely, $\widehat{\xi}([x]) = \widehat{\xi}([y])$. In the following, it suffice to prove that $\widehat{\xi}$ is a state operator on H/θ .

$$(S1) \quad \widehat{\xi}([1]) = [\xi(1)] = [1].$$

$$(S2) \quad \widehat{\xi}(\widehat{\xi}([x])) = \widehat{\xi}([\xi(x)]) = [\xi(\xi(x))] = [\xi(x)].$$

(S3) Let $[x] \ll_{\theta} [y]$ for any $x, y \in H$. Then $[1] \in [x] \overline{\circ} [y]$ and so $[1] \in \{[a] : a \in x \circ y\}$. It implies that $1 \in x \circ y$ and thus $1 = \xi(1) \in \xi(x \circ y)$. Since $\xi(x \circ y) \leq \xi(x) \circ \xi(y)$, from (P5) we have $1 \in \xi(x) \circ \xi(y)$, which shows $[1] \in \{[\xi(a)] : \xi(a) \in \xi(x) \circ \xi(y)\}$. That is, $[1] \in [\xi(x)] \overline{\circ} [\xi(y)]$. Therefore $[\xi(x)] \ll_{\theta} [\xi(y)]$ and so $\widehat{\xi}([x]) \ll_{\theta} \widehat{\xi}([y])$.

$$(S4) \quad \widehat{\xi}([x] \overline{\circ} [y]) = \{\widehat{\xi}([a]) : a \in x \circ y\} = \{[\xi(a)] : \xi(a) \in \xi(x \circ y)\}, \text{ and}$$

$$\begin{aligned} \widehat{\xi}([x] \overline{\circ} [y]) \overline{\circ} \widehat{\xi}([y]) &= \widehat{\xi}([x] \overline{\circ} [y]) \overline{\circ} [\xi(y)] \\ &= \{[\xi(b)] : b \in (x \circ y) \circ y\} \overline{\circ} [\xi(y)] \\ &= \{[\xi(c)] : c \in ((x \circ y) \circ y) \circ y\} \\ &= \{[\xi(c)] : \xi(c) \in \xi(((x \circ y) \circ y) \circ y)\}. \end{aligned}$$

Denote $A = \{[\xi(a)] : \xi(a) \in \xi(x \circ y)\}$ and $B = \{[\xi(c)] : \xi(c) \in \xi(((x \circ y) \circ y) \circ y)\}$. By (P3) $x \circ y \leq ((x \circ y) \circ y) \circ y$, we have $\xi(x \circ y) \leq \xi(((x \circ y) \circ y) \circ y)$. Since $\xi(a) \ll \xi(c)$, we get $[\xi(a)] \ll_{\theta} [\xi(c)]$. Hence for any $[\xi(a)] \in A$ and $\xi(a) \in \xi(x \circ y)$, there is $[\xi(c)] \in B$ and $\xi(c) \in \xi(((x \circ y) \circ y) \circ y)$ such that $[\xi(a)] \ll_{\theta} [\xi(c)]$. That is, $A \leq_{\theta} B$, which shows $\widehat{\xi}([x] \overline{\circ} [y]) \leq_{\theta} \widehat{\xi}([x] \overline{\circ} [y]) \overline{\circ} \widehat{\xi}([y])$.

(S5)

$$\begin{aligned}
\widehat{\xi}(\widehat{\xi}([x])\overline{\circ}\widehat{\xi}([y])) &= \widehat{\xi}([\xi(x)]\overline{\circ}[\xi(y)]) \\
&= \{\widehat{\xi}([\xi(a)]) : \xi(a) \in \xi(x) \circ \xi(y)\} \\
&= \{[\xi(\xi(a))] : \xi(a) \in \xi(x) \circ \xi(y)\} \\
&= \{[\xi(a)] : \xi(a) \in \xi(x) \circ \xi(y)\} \\
&= [\xi(x)]\overline{\circ}[\xi(y)] \\
&= \widehat{\xi}([x])\overline{\circ}\widehat{\xi}([y])
\end{aligned}$$

□

In the following, we deliver characterizations of state strong regular \circ -reflexive hyper filters of a state hyper BE-algebra. To do this, we first investigate a related result.

Lemma 5.9. *Let H be a transitive hyper BE-algebra and F be a hyper filter of H . Then $[1]_F$ is a hyper filter and $\text{Ker}(\Delta_F) = [1]_F = F$.*

Proof. Let $x \in \text{Ker}(\Delta_F) = [1]_F$. Then $x\Delta_F 1$ and so $1 \circ x \cap F \neq \emptyset$. Since F is a hyper filter, we have $x \in F$, which implies that $\text{Ker}(\Delta_F) \subseteq F$. On the other hand, let $x \in F$. It follows from $x \in 1 \circ x$ that $(1 \circ x) \cap F \neq \emptyset$. This together with $(x \circ 1) \cap F \neq \emptyset$ imply $x\theta_F 1$, namely, $x \in [1]_F = \text{Ker}(\theta_F)$. Consequently, $\text{Ker}(\Delta_F) = [1]_F = F$ and $[1]_F$ is hyper filter. □

Theorem 5.10. *Let (H, ξ) be an s -negatively ordered state transitive hyper BE-algebra such that $1 \circ 1 = \{1\}$. If F is a strong regular \circ -reflexive hyper filter of H , then the following are equivalent:*

- (1) F is a state strong regular \circ -reflexive hyper filter of (H, ξ) ;
- (2) There exists a state compatible hyper congruence on (H, ξ) whose kernel is F ;
- (3) $x, y \in F$ implies $\xi(x) \circ \xi(y) \subseteq F$.

Proof. (1) \Rightarrow (2) Assume that F is a state strong regular \circ -reflexive hyper filter of (H, ξ) . Then from Theorem 5.6, Δ_F is a state compatible hyper congruence on (H, ξ) and moreover from Lemma 5.9, $\text{Ker}(\Delta_F) = F$.

(2) \Rightarrow (3) Assume that there exists a state compatible hyper congruence θ on (H, ξ) such that $F = \text{Ker}(\theta)$. Let $x, y \in F$. Then we have $x\theta 1$ and $y\theta 1$ thus $\xi(x)\theta 1$ and $\xi(y)\theta 1$. Hence $(\xi(x) \circ \xi(y))\theta(1 \circ 1) = \{1\}$. This shows that for any $a \in \xi(x) \circ \xi(y)$, $a\theta 1$, namely, $a \in \text{Ker}(\theta) = F$. Therefore, $\xi(x) \circ \xi(y) \subseteq F$.

(3) \Rightarrow (1) Assume that the condition (3) holds. Let $x \in F$. Since $x, 1 \in F$, we derive $\xi(x) \in 1 \circ \xi(x) = \xi(1) \circ \xi(x) \subseteq F$ and so $\xi(x) \in F$. It shows that F is a state regular \circ -reflexive hyper filter of (H, ξ) . □

Lemma 5.11. *Let (H, ξ) be a state R -hyper BE-algebra and θ is a state compatible hyper congruence on (H, ξ) . Then $\text{Ker}(\theta) = [1]_\theta$ is a state strong regular \circ -reflexive hyper filter of (H, ξ) .*

Proof. It is clear that $1 \in \text{Ker}(\theta)$. Let $x \in \text{Ker}(\theta)$ and $(x \circ y) \cap \text{Ker}(\theta) \neq \emptyset$. Then $x\theta 1$ and there exists $a \in x \circ y$ such that $a \in \text{Ker}(\theta)$. Since $x\theta 1$, we have $(x \circ y)\theta(1 \circ y) = y$. Hence for the above element $a \in x \circ y$, we get $a\theta y$, namely, $a \in [y]_\theta$. Thus $[y]_\theta = [1]_\theta$, which implies $y \in \text{Ker}(\theta)$. Therefore, $\text{Ker}(\theta)$ is a hyper filter of H . Now, let $x \in \text{Ker}(\theta)$. Then $x\theta 1$. Taking into consideration that θ is a state hyper congruence, we deduce that $\xi(x)\theta\xi(1)$ and so $\xi(x)\theta 1$. Consequently $\xi(x) \in \text{Ker}(\theta)$, proving that $\text{Ker}(\theta)$ is a state hyper filter of (H, ξ) .

Next we prove that $\text{Ker}(\theta)$ is \circ -reflexive. Let $x, y \in H$ such that $(x \circ y) \cap \text{Ker}(\theta) \neq \emptyset$. Then there exists $a \in x \circ y$ such that $a\theta 1$. Since θ is compatible, we have $(x \circ y)\theta t$ for some $t \in H$. Hence $a\theta t$ and so $1\theta t$. It shows that $x \circ y \subseteq [1]_\theta = \text{Ker}(\theta)$.

Finally, we shows that $\text{Ker}(\theta) = [1]_\theta$ is strong regular. Let $x\Delta_{\text{Ker}(\theta)}y$. Then $(x \circ y) \cap [1]_\theta \neq \emptyset$. Since H is transitive, we have $x \circ y \leq (u \circ x) \circ (u \circ y)$ and so $(u \circ x) \circ (u \circ y) \cap [1]_\theta \neq \emptyset$. It follows that there exist $m \in u \circ x, n \in u \circ y$ such that $m \circ n \cap [1]_\theta \neq \emptyset$. Considering that $[1]_\theta$ is \circ -reflexive, we get that $m \circ n \subseteq [1]_\theta$, that is, $m \circ n\theta 1$. On the other hand, since θ is compatible, there exist $p, q \in H$ such that $u \circ x\theta p, u \circ y\theta q$. It implies that for all $s \in u \circ x, r \in u \circ y, s\theta p, r\theta q$ and hence $s \circ r\theta p \circ q$. Also we can see that $m\theta p, n\theta q$ and thus $m \circ n\theta p \circ q$. So we have $s \circ r\theta p \circ q\theta m \circ n\theta 1$. It deduce that $s \circ r\theta 1$ and thus $s \circ r \cap [1]_\theta \neq \emptyset$ for all

$s \in u \circ x, r \in u \circ y$. Also, it is similar that we can get the other part $r \circ s \cap [1]_\theta \neq \emptyset$ for all $s \in u \circ x, r \in u \circ y$. This implies $(u \circ x)\overline{\Delta_{\text{Ker}(\theta)}}(u \circ y)$. Meanwhile in a similar way, we can deliver that $(x \circ v)\overline{\Delta_{\text{Ker}(\theta)}}(y \circ v)$. Therefore, $\text{Ker}(\theta) = [1]_\theta$ is strong regular.

Combing the above arguments, $\text{Ker}(\theta) = [1]_\theta$ is a state strong regular \circ -reflexive hyper filter of (H, ξ) . \square

Corollary 5.12. *Let (H, ξ) be a state transitive R-hyper BE-algebra and θ is a state compatible hyper congruence on (H, ξ) . Then*

(1) $\Delta_{\text{Ker}(\theta)}$ is a compatible hyper congruence on H ;

(2) $H/\text{Ker}(\theta), \widehat{\xi}$ is a state hyper BE-algebra, where the map $\widehat{\xi} : H/\theta \rightarrow H/\theta$ is defined by $\widehat{\xi}([x]_{\text{Ker}(\theta)}) = [\xi(x)]_{\text{Ker}(\theta)}$.

Proof. (1) According to Lemma 5.11, we know that $\text{Ker}(\theta)$ is a state strong regular \circ -reflexive hyper filter of (H, ξ) . It suffice to prove that $\Delta_{\text{Ker}(\theta)}$ is an equivalence relation on H . In fact, the reflexivity and symmetry of $\Delta_{\text{Ker}(\theta)}$ are obvious. Let $x, y, z \in H$ such that $x\Delta_{\text{Ker}(\theta)}y$ and $y\Delta_{\text{Ker}(\theta)}z$. Then $(x \circ y) \cap \text{Ker}(\theta) \neq \emptyset$ and $(y \circ z) \cap \text{Ker}(\theta) \neq \emptyset$. Since H is transitive, we have $x \circ y \leq (y \circ z) \circ (x \circ z)$ and so $(y \circ z) \circ (x \circ z) \cap \text{Ker}(\theta) \neq \emptyset$. Considering that $\text{Ker}(\theta)$ is \circ -reflexive, it follows from $x \circ y \subseteq \text{Ker}(\theta)$ that $(x \circ z) \cap \text{Ker}(\theta) \neq \emptyset$. Similarly $(z \circ x) \cap \text{Ker}(\theta) \neq \emptyset$. Therefore, $x\Delta_{\text{Ker}(\theta)}z$, which implies the transitivity is true. The proof is completed.

(2) It can be seen immediately according to Theorem 5.8. \square

Given a state hyper BE-algebra (H, ξ) , denote by $\mathcal{C}_s(H, \xi)$ all state compatible hyper congruence on (H, ξ) and denote by $\mathcal{F}_s(H, \xi)$ all state strong regular \circ -reflexive hyper filter of (H, ξ) .

Theorem 5.13. *Let (H, ξ) be a state commutative transitive RD-hyper BE-algebra. Then there is an isotone bijection between $\mathcal{C}_s(H, \xi)$ and $\mathcal{F}_s(H, \xi)$.*

Proof. Define $\Phi : \mathcal{C}_s(H, \xi) \rightarrow \mathcal{F}_s(H, \xi)$ by $\Phi(\theta) = \text{Ker}(\theta)$. According to Lemma 5.11, $\text{Ker}(\theta)$ is a state strong regular \circ -reflexive hyper filter, namely, $\text{Ker}(\theta) \in \mathcal{F}_s(H, \xi)$. It is obvious that the map Φ is well-defined.

Assume that $\theta_1, \theta_2 \in \mathcal{C}_s(H, \xi)$ such that $\Phi(\theta_1) = \Phi(\theta_2)$. Let $x\theta_1y, x, y \in H$. Then $(x \circ y)\theta_1(y \circ y)$. This together with $y \circ y = \{1\}$ implies that $(x \circ y)\theta_11$. Hence $x \circ y \subseteq \text{Ker}(\theta_1) = \text{Ker}(\theta_2)$ and so $(x \circ y)\theta_21$. Thus $(x \circ y) \circ y\theta_21 \circ y$. Combining $1 \circ y = \{y\}$, we have $(x \circ y) \circ y\theta_2y$. In a similar, we can get $(y \circ x) \circ x\theta_2x$. It follows from $(x \circ y) \circ y = (y \circ x) \circ x$ that $x\theta_2y$. Therefore, $\theta_1 \subseteq \theta_2$. Similarly, $\theta_2 \subseteq \theta_1$. It leads to Φ is one-to-one.

Let F be a state strong regular \circ -reflexive hyper filter of (H, ξ) . Then by Theorem 5.10, Δ_F is a state compatible hyper congruence on (H, ξ) and $\text{Ker}(\Delta_F) = F$. Therefore, $\Phi(\Delta_F) = \text{Ker}(\Delta_F) = F$. This shows that Φ is onto.

Finally, we prove Φ is isotone. Set $\theta_1 \subseteq \theta_2$, for $\theta_1, \theta_2 \in \mathcal{C}_s(H, \xi)$ and $x \in \Phi(\theta_1) = \text{Ker}(\theta_1)$. Then $(x, 1) \in \theta_1 \subseteq \theta_2$, which implies $x \in \text{Ker}(\theta_2) = \Phi(\theta_2)$. It follows that $\Phi(\theta_1) \subseteq \Phi(\theta_2)$.

Combing the above arguments, we deduce that there is an isotone bijection between $\mathcal{C}_s(H, \xi)$ and $\mathcal{F}_s(H, \xi)$. \square

6 Conclusions

States play an important role in studying fuzzy logics and the related algebraic structures. In this paper, we introduce Bosbach states on hyper BE-algebras and obtain some important results. In future work, we shall further study state theory, especially on quotient hyper BE-algebras.

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