Commutative ideals of BCI-algebras based on Łukasiewicz fuzzy sets

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Abstract

With the aim of applying the Łukasiewicz fuzzy set to commutative ideal in BCI-algebras, the concept of Łukasiewicz fuzzy commutative ideal is introduced, and its properties are investigated. The relationship between a Łukasiewicz fuzzy ideal and a Łukasiewicz fuzzy commutative ideal are discussed. After providing an example of a Łukasiewicz fuzzy ideal, not a Łukasiewicz fuzzy commutative ideal, conditions under which a Łukasiewicz fuzzy ideal can be a Łukasiewicz fuzzy commutative ideal are explored. Characterizations of Łukasiewicz fuzzy commutative ideals are displayed. Conditions under which ε-set, q-set, and O-set can be commutative ideals are found.

1 Introduction

Ideal concepts are a very important factor in studying BCK/BCI-algebras, and studies have been conducted on various types of ideals. The commutative ideal introduced by Meng [1] in 1993 is one of these ideals. The fuzzy set acts as a bridge so that algebra theory can be applied to applied sciences. Various kinds of fuzzy sets have been used in the study of substructures such as ideals in BCK/BCI-algebras (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). Łukasiewicz logic, which is the logic of the Łukasiewicz t-norm, is a non-classical and many-valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz as a three-valued logic. Using the idea of Łukasiewicz t-norm, Y. B. Jun [3] constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BCK-algebras and BCI-algebras. Y. B. Jun and S. Z. Song studied Łukasiewicz fuzzy (positive implicative) ideals in BCK/BCI-algebras (see [8, 9, 10, 11, 12, 13]).

For the purpose of applying the Łukasiewicz fuzzy set to a commutative ideal in BCI-algebras, we introduce the concept of Łukasiewicz fuzzy commutative ideal and study its properties. We discuss the relationship between Łukasiewicz fuzzy ideal and Łukasiewicz fuzzy commutative ideal. We give an example...
of a Łukasiewicz fuzzy ideal, not a Łukasiewicz fuzzy commutative ideal, and explore the conditions under which a Łukasiewicz fuzzy ideal can be a Łukasiewicz fuzzy commutative ideal. We discuss characterizations of Łukasiewicz fuzzy commutative ideals. We explore the conditions under which $\in$-set, $q$-set, and $O$-set can be commutative ideals.

2 Preliminaries

2.1 Basic concepts about BCK/BCI-algebras

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [2] and [3]) and was extensively investigated by several researchers. We recall the definitions and basic results required in this paper. See the books [1], [12] for further information regarding BCK-algebras and BCI-algebras.

If a set $X$ has a special element “0” and a binary operation “$*$” satisfying the conditions:

\((I_1)\) (\(\forall a, b, c \in X\)) ((\(a * b\) \(a * c\)) \(c * b\)) = 0),
\((I_2)\) (\(\forall a, b \in X\)) ((\(a * (a * b)\)) \(b = 0\)),
\((I_3)\) (\(\forall a \in X\)) (\(a * a = 0\)),
\((I_4)\) (\(\forall a, b \in X\)) (\(a * b = 0, b * a = 0 \Rightarrow a = b\)),

then we say that $X$ is a $BCI$-algebra. If a BCI-algebra $X$ satisfies the following identity:

\((K)\) (\(\forall a \in X\)) (\(0 * a = 0\)),

then $X$ is called a $BCI$-algebra. The BCI/BCK-algebra is written as $(X, 0)_*$. The order relation “$\leq$” in a BCK/BCI-algebra $(X, 0)_*$ is defined as follows:

\((\forall a, b \in X) (a \leq b \iff a * b = 0)\).

Every BCK/BCI-algebra $(X, 0)_*$ satisfies the following conditions (see [1], [12]):

\((\forall a \in X) (a * 0 = a)\),
\((\forall a, b, c \in X) (a \leq b \Rightarrow a * c \leq b * c, c * b \leq c * a)\),
\((\forall a, b, c \in X) ((a * b) * c = (a * c) * b)\).

Every BCI-algebra $(X, 0)_*$ satisfies (see [1]):

\((\forall a, b \in X) (a * (a * b)) = a * b)\),
\((\forall a, b \in X) (0 * (a * b) = (0 * a) * (0 * b))\).

A BCI-algebra $(X, 0)_*$ is said to be commutative (see [13]) if it satisfies:

\((\forall a, b \in X) (a \leq b \Rightarrow a = b * (b * a))\).

A subset $K$ of a BCK/BCI-algebra $(X, 0)_*$ is called

- a subalgebra of $X$ (see [1], [12]) if it satisfies:

\((\forall a, b \in K) (a * b \in K)\),

- an ideal of $X$ (see [1], [12]) if it satisfies:

\(0 \in K,\)
\((\forall a, b \in X) (a * b \in K, b \in K \Rightarrow a \in K)\).

A subset $K$ of a BCI-algebra $(X, 0)_*$ is called a commutative ideal of $X$ (see [11]) if it satisfies (10) and

\((\forall a, b, c \in X) \left( (a * b) * c \in K, c \in K \Rightarrow a * (((b * (b * a)) * (0 * (0 * (a * b)))) \in K) \right)\).
2.2 Basic concepts about (Łukasiewicz) fuzzy sets

A fuzzy set $\xi$ in a set $X$ of the form

$$\xi(b) := \begin{cases} t \in (0, 1) & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a fuzzy point with support $a$ and value $t$ and is denoted by $\langle a/t \rangle$.

For a fuzzy set $\xi$ in a set $X$, we say that a fuzzy point $\langle a/t \rangle$ is

1. contained in $\xi$, denoted by $\langle a/t \rangle \in \xi$, (see [14]) if $\xi(a) \geq t$.

2. quasi-coincident with $\xi$, denoted by $\langle a/t \rangle \approx q \xi$, (see [14]) if $\xi(a) + t > 1$.

If $\langle a/t \rangle \alpha \xi$ is not established for $\alpha \in \{e, q\}$, it is denoted by $\langle a/t \rangle \Pi \xi$.

A fuzzy set $\xi$ in a BCK/BCI-algebra $(X, 0)_*$ is called

- a fuzzy subalgebra of $(X, 0)_*$ (see [3]) if it satisfies:

$$\forall a, b \in X (\xi(a * b) \geq \min\{\xi(a), \xi(b)\}). \quad (12)$$

- a fuzzy ideal of $(X, 0)_*$ (see [4]) if it satisfies:

$$\forall a \in X (\xi(0) \geq \xi(a)), \quad (13)$$

$$\forall a, b \in X (\xi(a) \geq \min\{\xi(a * b), \xi(b)\}). \quad (14)$$

A fuzzy set $\xi$ in a BCI-algebra $(X, 0)_*$ is called

- a closed fuzzy ideal of $(X, 0)_*$ (see [4]) if it is a fuzzy ideal of $(X, 0)_*$ which satisfies:

$$\forall a \in X (\xi(0 * a) \geq \xi(a)). \quad (15)$$

- a fuzzy commutative ideal of $(X, 0)_*$ (see [5]) if it satisfies (13) and

$$\xi(a * ((b * (b * a)) * (0 * (0 * (a * b)))) \geq \min\{\xi((a * b) * c), \xi(c)\} \quad (16)$$

for all $a, b, c \in X$.

**Definition 2.1.** Let $\xi$ be a fuzzy set in a set $X$ and let $\delta \in (0, 1)$. A function

$$\delta : X \to [0, 1], \quad x \mapsto \max\{0, \xi(x) + \delta - 1\}$$

is called the Łukasiewicz fuzzy set of $\xi$ in $X$.

**Definition 2.2.** Let $\xi$ be a fuzzy set in $(X, 0)_*$ and $\delta$ an element of $(0, 1)$. Then its Łukasiewicz fuzzy set $\delta \xi$ in $X$ is called a Łukasiewicz fuzzy subalgebra of $(X, 0)_*$ if it satisfies:

$$\langle x/t_a \rangle \in \delta \xi, \quad \langle y/t_b \rangle \in \delta \xi \quad \Rightarrow \quad \langle (x/y)/\min\{t_a, t_b\} \rangle \in \delta \xi \quad (17)$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$.

**Lemma 2.3.** Let $\xi$ be a fuzzy set in $X$. Then its Łukasiewicz fuzzy set $\delta \xi$ in $X$ is a Łukasiewicz fuzzy subalgebra of $(X, 0)_*$ if and only if it satisfies:

$$\forall x, y \in X (\delta \xi(x * y) \geq \min\{\delta \xi(x), \delta \xi(y)\}). \quad (18)$$

**Definition 2.4.** Let $\xi$ be a fuzzy set in a BCK/BCI-algebra $X$. Then its Łukasiewicz fuzzy set $\delta \xi$ in $X$ is called a Łukasiewicz fuzzy ideal of $X$ if it satisfies:

$$\delta \xi(0) \text{ is an upper bound of } \{\delta \xi(x) \mid x \in X\}, \quad (19)$$

$$\langle (x * y)/t_a \rangle \in \delta \xi, \quad \langle y/t_b \rangle \in \delta \xi \quad \Rightarrow \quad \langle x/\min\{t_a, t_b\} \rangle \in \delta \xi \quad (20)$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$.
Lemma 2.5. Let $\xi$ be a fuzzy set in $(X,0)_*$. Then its Łukasiewicz fuzzy set $\delta^{\xi}$ is a Łukasiewicz fuzzy ideal of $(X,0)_*$ if and only if it satisfies:

\[(\forall x \in X)(\forall t_a \in (0,1]) ((x/t_a) \in \delta^{\xi} \Rightarrow (0/t_a) \in \delta^{\xi}),\]  

\[(\forall x,y \in X)(\xi(x) \geq \min\{\xi(x*y), \xi(y)\}).\]

Let $\xi$ be a fuzzy set in $X$. For the Łukasiewicz fuzzy set $\xi$ of $\xi$ in $X$ and $t \in (0,1]$, consider the sets

\[(\xi, t)_e := \{x \in X \mid (x/t) \in \delta^{\xi}\} \quad \text{and} \quad (\xi, t)_q := \{x \in X \mid (x/t) \in q^{\xi}\},\]

which are called the $\in$-set and $q$-set, respectively, of $\delta^{\xi}$ (with value $t$). Also, consider a set:

\[O(\xi) := \{x \in X \mid \xi(x) > 0\}\]

which is called an $O$-set of $\xi$. It is observed that

\[O(\xi) = \{x \in X \mid \xi(x) + \delta - 1 > 0\}.

3 Łukasiewicz fuzzy commutative ideals in BCI-algebras

In this section, let $(X,0)_*$ be a BCI-algebra, and $\delta$ be an element of $(0,1]$ unless otherwise specified.

For any elements $x$ and $y$ of $X$, let

\[x^n \ast y := x \ast \cdots \ast (x \ast (x \ast y)) \cdots,\]

where $x$ appears $n$ times.

Definition 3.1. Let $\xi$ be a fuzzy set in $X$. Then its Łukasiewicz fuzzy set $\delta^{\xi}$ in $X$ is called a Łukasiewicz fuzzy commutative ideal (briefly, LFC-ideal) of $X$ if it satisfies \((21)\) (or, equivalently \((21)\)) and

\[(\forall x,y,z \in X)(\forall t_a,t_c \in (0,1]) \left( (\xi(x \ast y) \ast z)/t_a \in \delta^{\xi}, (\xi(z/t_c) \in \delta^{\xi} \Rightarrow ((x \ast (y^2 \ast x) \ast (0^2 \ast (x \ast y))))/\min\{t_a,t_c\} \in \delta^{\xi} \right).\]

Example 3.2. Let $X = \{\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ be a set with a binary operation $\ast$ given as follows:

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Then $(X,\kappa_0)_*$ is a BCI-algebra (see \([3]\)). Define a fuzzy set $\xi$ in $X$ as follows:

\[\xi : X \rightarrow [0,1], \ x \mapsto \begin{cases} 0.97 & \text{if } x = \kappa_0, \\ 0.79 & \text{if } x = \kappa_1, \\ 0.59 & \text{if } x = \kappa_2, \\ 0.59 & \text{if } x = \kappa_3, \\ 0.59 & \text{if } x = \kappa_4. \end{cases}\]

Given $\delta := 0.58$, the Łukasiewicz fuzzy set $\delta^{\xi}$ of $\xi$ in $X$ is given as follows:

\[\delta^{\xi} : X \rightarrow [0,1], \ x \mapsto \begin{cases} 0.55 & \text{if } x = \kappa_0, \\ 0.37 & \text{if } x = \kappa_1, \\ 0.17 & \text{if } x = \kappa_2, \\ 0.17 & \text{if } x = \kappa_3, \\ 0.17 & \text{if } x = \kappa_4. \end{cases}\]

It is routine to verify that $\delta^{\xi}$ is a LFC ideal of $(X,\kappa_0)_*$. 

Proposition 3.3. Every LFC ideal \( \delta \) of \((X, 0)_*\) satisfies:

\[
(\forall x, y \in X)(\forall t \in (0, 1]) \left( (x \ast y)/t \right) \in \delta \Leftrightarrow \left( (x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y))))/t \right) \in \delta \tag{25}
\]

Proof. If we choose 0 instead of \( z \), and \( t := t_a = t_c \) from (24) and use (19), we will get (25). \( \square \)

We discuss the relationship between Łukasiewicz fuzzy ideals and LFC ideals.

Theorem 3.4. Every LFC ideal is a Łukasiewicz fuzzy ideal.

Proof. Let \( \delta \) be a LFC ideal of \((X, 0)_*\). Let \( x, y \in X \) and \( t_a, t_c \in (0, 1] \) be such that \((x \ast y)/t \in \delta \) and \((z/t_c) \in \delta \). Then \((x \ast y)/t \in \delta \) and so

\[
\langle x/\min\{t_a, t_c\} \rangle = \langle (x \ast 0)/\min\{t_a, t_c\} \rangle = \langle (x \ast ((0^2 \ast x) \ast (0^2 \ast (x \ast 0))))/\min\{t_a, t_c\} \rangle \in \delta
\]

by (I, 4) and (24). Hence \( \delta \) is a Łukasiewicz fuzzy ideal of \((X, 0)_*\). \( \square \)

The converse of Theorem 3.4 may not be true as shown in the following example.

Example 3.5. Let \( X = \{\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4\} \) be a set with a binary operation “\( \ast \)” given as follows:

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Then \((X, \kappa_0)_*\) is a BCK-algebra and so a BCI-algebra (see [12]). Define a fuzzy set \( \xi \) in \( X \) as follows:

\[
\xi : X \rightarrow [0, 1], \quad x \mapsto \begin{cases} 
0.89 & \text{if } x = \kappa_0, \\
0.77 & \text{if } x = \kappa_1, \\
0.43 & \text{if } x = \kappa_2, \\
0.59 & \text{if } x = \kappa_3, \\
0.43 & \text{if } x = \kappa_4.
\end{cases}
\]

Given \( \delta := 0.36 \), the Łukasiewicz fuzzy set \( \delta^{\xi} \) of \( \xi \) in \( X \) is given as follows:

\[
\delta^{\xi} : X \rightarrow [0, 1], \quad x \mapsto \begin{cases} 
0.25 & \text{if } x = \kappa_0, \\
0.13 & \text{if } x = \kappa_1, \\
0.00 & \text{if } x \in \{\kappa_2, \kappa_3, \kappa_4\}.
\end{cases}
\]

A simple calculation confirms that \( \delta^{\xi} \) is a Łukasiewicz fuzzy ideal of \((X, \kappa_0)_*\). If we take \( t_a \) and \( t_c \) in \((0, 0.23]\), then \((\kappa_2 \ast \kappa_3)/t_a \in \delta \) and \((\kappa_0/t_c) \in \delta \). But

\[
(\kappa_2 \ast ((\kappa_2 \ast \kappa_2) \ast (\kappa_3 \ast (\kappa_2 \ast \kappa_3))))/\min\{t_a, t_c\} = (\kappa_2/\min\{t_a, t_c\}) \in \delta.
\]

Hence \( \delta^{\xi} \) is not a LFC ideal of \((X, \kappa_0)_*\).

We explore the conditions under which a Łukasiewicz fuzzy ideal becomes LFC ideal.

Theorem 3.6. If a Łukasiewicz fuzzy ideal \( \delta^{\xi} \) of \((X, 0)_*\) satisfies the condition (25), then it is a LFC ideal of \((X, 0)_*\).
Let \( h \) which is equivalent to the following assertion.

\[
\delta \xi
\]

Hence \( \delta (x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y)))) \geq \delta (x \ast y) \) for all \( x, y \in X \). Let \( x, y, z \in X \) and \( t_a, t_c \in (0, 1] \) be such that \( \langle (x \ast y) \ast z) / t_a \rangle \in \delta \xi \) and \( (z / t_c) \in \delta \xi \). Then \( \langle (x \ast y) / \min\{t_a, t_c\} \rangle \in \delta \xi \) by (20), and so

\[
\delta (x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y)))) \geq \delta (x \ast y) \geq \min\{t_a, t_c\},
\]

that is, \( \langle (x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y)))) / \min\{t_a, t_c\} \rangle \in \delta \xi \). Therefore, \( \delta \xi \) is a LFC ideal of \((X, 0)_*\).

\[ \square \]

**Definition 3.7.** A Lukasiewicz fuzzy ideal \( \delta \xi \) of \((X, 0)_*\) is said to be closed if it is also a Lukasiewicz fuzzy subalgebra of \((X, 0)_*\).

**Theorem 3.8.** Every Lukasiewicz fuzzy ideal \( \delta \xi \) of \((X, 0)_*\) is closed if and only if it satisfies:

\[
(\forall x \in X) (\forall t \in (0, 1]) \left( (x / t) \in \delta \xi \Rightarrow (0 \ast x / t) \in \delta \xi \right).
\]

**Proof.** Assume that a Lukasiewicz fuzzy ideal \( \delta \xi \) of \((X, 0)_*\) is closed. Let \( x \in X \) and \( t \in (0, 1] \) be such that \( \langle x / t \rangle \in \delta \xi \). Then \( \langle 0 / t \rangle \in \delta \xi \) by (21), and so \( \langle (0 \ast x) / t \rangle = \langle (0 \ast x) / \min\{t, t\} \rangle \in \delta \xi \) by (17).

Conversely, let \( \delta \xi \) be a Lukasiewicz fuzzy ideal of \((X, 0)_*\) that satisfies (20). Let \( x, y \in X \) and \( t_a, t_b \in (0, 1] \) be such that \( \langle x / t_a \rangle \in \delta \xi \) and \( \langle y / t_b \rangle \in \delta \xi \). Then \( \langle (x \ast y) / t_a \rangle = \langle (0 \ast y) / t_b \rangle \in \delta \xi \) by (I3), (4) and (21). It follows from (21) that \( \langle (x \ast y) / \min\{t_a, t_b\} \rangle \in \delta \xi \). Consequently, \( \delta \xi \) is a closed Lukasiewicz fuzzy ideal of \((X, 0)_*\).

\[ \square \]

**Lemma 3.9.** Every Lukasiewicz fuzzy ideal \( \delta \xi \) of \( X \) satisfies:

\[
(\forall x, y, z \in X) (\forall t_b, t_c \in (0, 1]) \left( x \ast y \leq z, \langle y / t_b \rangle \in \delta \xi, \langle z / t_c \rangle \in \delta \xi \Rightarrow \langle x / \min\{t_b, t_c\} \rangle \in \delta \xi \right),
\]

which is equivalent to the following assertion.

\[
(\forall x, y, z \in X) (x \ast y \leq z \Rightarrow \delta \xi (x) \geq \min\{\delta \xi (y), \delta \xi (z)\}).
\]

**Theorem 3.10.** Let \( \delta \xi \) be a closed Lukasiewicz fuzzy ideal of \((X, 0)_*\). Then it is a LFC ideal of \((X, 0)_*\) if and only if it satisfies:

\[
(\forall x, y \in X) (\forall t \in (0, 1]) \left( \langle (x \ast y) / t \rangle \in \delta \xi \Rightarrow \langle (x \ast (y^2 \ast x)) / t \rangle \in \delta \xi \right).
\]

**Proof.** Let \( \delta \xi \) be a closed Lukasiewicz fuzzy ideal of \((X, 0)_*\). Assume that \( \delta \xi \) is a LFC ideal of \((X, 0)_*\). Let \( x, y \in X \) and \( t \in (0, 1] \) be such that \( \langle (x \ast y) / t \rangle \in \delta \xi \). Since \( \langle (x \ast y) / \xi (x \ast y) \rangle \in \delta \xi \), we have \( \langle (x \ast (y^2 \ast x)) \ast (0^2 \ast (x \ast y))) / \xi (x \ast y) \rangle \in \delta \xi \) by Proposition 3.3. That is,

\[
\delta (x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y)))) \geq \delta (x \ast y).
\]

Since

\[
(x \ast (y^2 \ast x)) \ast (x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y))))
\]

\[
\leq ((y^2 \ast x) \ast (0^2 \ast (x \ast y))) \ast (y^2 \ast x)
\]

\[
= (y^2 \ast x) \ast (y^2 \ast x) \ast (0^2 \ast (x \ast y))
\]

\[
= 0 \ast (0^2 \ast (x \ast y)) = 0 \ast (x \ast y),
\]

\[ \square \]
it follows from Theorem 3.8 and Lemma 3.4 that
\[ \delta_\xi(x \ast (y^2 \ast x)) \geq \min\{\delta_\xi(x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y)))), \delta_\xi(0 \ast (x \ast y))\} \]
\[ \geq \min\{\delta_\xi(x \ast y), \delta_\xi(0 \ast (x \ast y))\} \]
\[ = \delta_\xi(x \ast y) \geq t, \]

i.e., \((x \ast (y^2 \ast x))/t) \in \delta_\xi\).

Conversely, let \(\delta_\xi\) be a closed Łukasiewicz fuzzy ideal of \((X, 0)_\ast\) satisfying the condition (29). Let \(x, y \in X\) and \(t \in (0, 1]\) be such that \(\langle x \ast y\rangle/t) \in \delta_\xi\). Then \(\delta_\xi(x \ast y) \geq t\). Since \(\langle x \ast y\rangle/\delta_\xi(x \ast y) \in \delta_\xi\), we get
\[ \langle (x \ast (y^2 \ast x))/\delta_\xi(x \ast y) \rangle \in \delta_\xi\) by (29), and so \(\delta_\xi(x \ast (y^2 \ast x)) \geq \delta_\xi(x \ast y)\). Since
\[ (x \ast (y^2 \ast x)) \ast (0^2 \ast (x \ast y))) \ast (x \ast (y^2 \ast x)) \]
\[ \leq (y^2 \ast x) \ast (y^2 \ast x) \ast (0^2 \ast (x \ast y))) \]
\[ \leq 0^2 \ast (x \ast y), \]

we have
\[ \delta_\xi(x \ast (y^2 \ast x) \ast (0^2 \ast (x \ast y)))) \geq \min\{\delta_\xi(x \ast (y^2 \ast x)), \delta_\xi(0^2 \ast (x \ast y))\} \]
\[ \geq \min\{\delta_\xi(x \ast y), \delta_\xi(0 \ast (x \ast y))\} = \delta_\xi(x \ast y) \geq t \]

by Theorem 3.8 and Lemma 3.4. Hence \(\langle (x \ast (y^2 \ast x) \ast (0^2 \ast (x \ast y))))/t) \in \delta_\xi\), and therefore \(\delta_\xi\) is a LFC ideal of \((X, 0)_\ast\) by Theorem 3.6. \(\square\)

**Lemma 3.11.** \(\square\) A BCI-algebra is commutative if and only if it satisfies:
\[ (\forall x, y \in X)(x^2 \ast y = y^2 \ast (x^2 \ast y)). \] (30)

**Theorem 3.12.** In a commutative BCI-algebra, every closed Łukasiewicz fuzzy ideal is a LFC ideal.

*Proof.* Let \(\delta_\xi\) be a closed Łukasiewicz fuzzy ideal of a commutative BCI-algebra \((X, 0)_\ast\). Let \(x, y \in X\) and \(t \in (0, 1]\) be such that \(\langle x \ast y\rangle/t) \in \delta_\xi\). Using \((I_1), (I_3), (4), (6)\), and Lemma 3.11, leads to
\[ (x \ast (y^2 \ast x)) \ast (x \ast y) = (y^2 \ast (x \ast y)) \ast (y^2 \ast x) \]
\[ = (y^3 \ast x) \ast (y \ast (x^2 \ast y)) \ast (y \ast (x^2 \ast y)) \leq (x^2 \ast y) \ast x = 0 \ast (x \ast y). \]

It follows from Theorem 3.8 and Lemma 3.4 that
\[ \delta_\xi(x \ast (y^2 \ast x)) \geq \min\{\delta_\xi(x \ast y), \delta_\xi(0 \ast (x \ast y))\} = \delta_\xi(x \ast y) \geq t, \]

that is, \((x \ast (y^2 \ast x))/t) \in \delta_\xi\). Therefore, \(\delta_\xi\) is a LFC ideal of \((X, 0)_\ast\) by Theorem 3.10. \(\square\)

The theorem below reveals that an LFC ideal can be derived from fuzzy commutative ideal.

**Theorem 3.13.** If \(\xi\) is a fuzzy commutative ideal of \((X, 0)_\ast\), then its Łukasiewicz fuzzy set \(\delta_\xi\) is a LFC ideal of \((X, 0)_\ast\).

*Proof.* Let \(\xi\) be a fuzzy commutative ideal of \((X, 0)_\ast\). Then
\[ \delta_\xi(0) = \max\{0, \xi(0) + \delta - 1\} \geq \max\{0, \xi(x) + \delta - 1\} = \delta_\xi(x) \]

for all \(x \in X\). Hence \(\delta_\xi(0)\) is an upper bound of \(\{\xi_\xi(x) \mid x \in X\}\). Let \(x, y, z \in X\) and \(t_a, t_c \in (0, 1]\) be such that \(\langle (x \ast y \ast z)/t_a\rangle \in \delta_\xi\) and \(z/t_c) \in \delta_\xi\). Then \(\delta_\xi((x \ast y \ast z) \geq t_a\) and \(\delta_\xi(z) \geq t_c\), which imply that
\[ \delta_\xi(x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y))))) = \max\{0, \xi(x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y)))) \ast \delta - 1\} \]
\[ \geq \max\{0, \min\{\xi((x \ast y \ast z), \xi(z)) \ast \delta - 1\} \]
\[ \geq \max\{0, \min\{\xi((x \ast y \ast z) \ast (\xi(z) + \delta - 1\} \}
\[ \geq \min\{0, \xi((x \ast y \ast z) + \delta - 1, \max\{0, \xi(z) + \delta - 1\}\} \}
\[ \geq \min\{\delta_\xi((x \ast y \ast z), \delta_\xi(z)) \geq \min\{t_a, t_c\}. \]

Hence \((x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y))))/\min\{t_a, t_c\}) \in \delta_\xi\), and therefore \(\delta_\xi\) is a LFC ideal of \(X\). \(\square\)
We explore the conditions under which \( \varepsilon \)-set, \( q \)-set, and \( O \)-set can be commutative ideals.

**Theorem 3.14.** Let \( \delta \xi \) be the Lukasiewicz fuzzy set of a fuzzy set \( \xi \) in \( X \). Then the \( \varepsilon \)-set \((\xi_t)_{t \in \varepsilon}\) of \( \delta \xi \) is a commutative ideal of \((X, 0)\), for all \( t \in (0.5, 1) \) if and only if the following assertions are valid.

\[
(\forall x \in X) \left( \delta^{(\xi_t)}(x) \leq \max(\delta^{(\xi_t)}(0), 0.5) \right), \\
(\forall x, y, z \in X) \left( \min(\delta^{(\xi_t)}((x+y) * z), \delta^{(\xi_t)}(z)) \leq \max(\delta^{(\xi_t)}((y^2 * x) + (0^2 * (x+y))), 0.5) \right).
\]

**Proof.** Assume that \((\xi_t)_{t \in \varepsilon}\) is a commutative ideal of \((X, 0)\) for \( t \in (0.5, 1) \). If

\[
\delta^{(\xi_t)}(a) > \max(\delta^{(\xi_t)}(0), 0.5),
\]

for some \( a \in X \), then \( \delta^{(\xi_t)}(a) > \delta^{(\xi_t)}(0) \). If we take \( t := \delta^{(\xi_t)}(a) \), then \((a/t) = \delta^{(\xi_t)}(a) \), that is, \( a \in \delta^{(\xi_t)}(t)_{t \in \varepsilon} \), and \( 0 \notin \delta^{(\xi_t)}(t)_{t \in \varepsilon} \). This is a contradiction, and so \( \delta^{(\xi_t)}(x) \leq \max(\delta^{(\xi_t)}(0), 0.5) \) for all \( x \in X \). Now, suppose that the condition \((31)\) is not valid. Then there exist \( x, y, z \in X \) such that

\[
\min(\delta^{(\xi_t)}((x+y) * z), \delta^{(\xi_t)}(z)) > \max(\delta^{(\xi_t)}((y^2 * x) + (0^2 * (x+y))), 0.5).
\]

If we take \( t := \min(\delta^{(\xi_t)}((x+y) * z), \delta^{(\xi_t)}(z)) \), then \( t = (0.5, 1) \) and \( ((x+y) * z)/t, \ (z/t) \in \delta^{(\xi_t)}(t)_{t \in \varepsilon} \), i.e., \( (x+y) * z, z \in \delta^{(\xi_t)}(t)_{t \in \varepsilon} \). Since \((\xi_t)_{t \in \varepsilon}\) is a commutative ideal of \( X \), we have \( x * ((y^2 * x) + (0^2 * (x+y))) \in \delta^{(\xi_t)}(t)_{t \in \varepsilon} \). But \( \delta^{(\xi_t)}((x+y) * z) > \min(\delta^{(\xi_t)}((x+y) * z), \delta^{(\xi_t)}(z)) \) implies \( x * ((y^2 * x) + (0^2 * (x+y))) \notin \delta^{(\xi_t)}(t)_{t \in \varepsilon} \), a contradiction. Hence the condition \((32)\) is valid.

Conversely, suppose that \( \delta^{(\xi)} \) satisfies \((31)\) and \((32)\). Let \( t \in (0.5, 1) \). For every \( x \in \delta^{(\xi_t)}(t)_{t \in \varepsilon} \), we have \( 0.5 < t \leq \delta^{(\xi_t)}(x) \leq \max(\delta^{(\xi_t)}(0), 0.5) \) by \((31)\). Thus \( 0 \notin \delta^{(\xi_t)}(t)_{t \in \varepsilon} \). Let \( x, y, z \in X \) be such that \( (x+y) * z \in \delta^{(\xi_t)}(t)_{t \in \varepsilon} \) and \( z \in \delta^{(\xi_t)}(t)_{t \in \varepsilon} \). Then \( \delta^{(\xi_t)}((x+y) * z) \geq t \) and \( \delta^{(\xi_t)}(z) \geq t \), which imply from \((32)\) that

\[
0.5 < t \leq \min(\delta^{(\xi_t)}((x+y) * z), \delta^{(\xi_t)}(z)) \leq \max(\delta^{(\xi_t)}((y^2 * x) + (0^2 * (x+y))), 0.5).
\]

Hence \( ((x+y) * z) \in \delta^{(\xi_t)}(t)_{t \in \varepsilon} \), i.e., \( x * ((y^2 * x) + (0^2 * (x+y))) \in \delta^{(\xi_t)}(t)_{t \in \varepsilon} \). Therefore \( \delta^{(\xi_t)}(t)_{t \in \varepsilon} \) is a commutative ideal of \( X \) for \( t \in (0.5, 1) \).

**Theorem 3.15.** If a Lukasiewicz fuzzy set \( \delta^{\xi} \) in \( X \) satisfies:

\[
(\forall x \in X) (\forall t \in (0.5, 1)) \left( (x/t) q^{\delta^{\xi}} \Rightarrow (0/t) \leq \delta^{\xi}(t) \right), \\
(\forall x, y, z \in X)(\forall t_0, t_1 \in (0.5, 1)) \left( ((x+y) * z)/t_0 q^{\delta^{\xi}}, (z/t_1) q^{\delta^{\xi}} \Rightarrow (x * ((y^2 * x) + (0^2 * (x+y))))/\max(t_0, t_1) \in \delta^{\xi} \right),
\]

then the non-empty \( \varepsilon \)-set \((\delta^{\xi}, \max(t_0, t_1))_{t_0, t_1 \in \varepsilon} \) of \( \delta^{\xi} \) is a commutative ideal of \((X, 0)\), for all \( t_0, t_1 \in (0.5, 1) \).

**Proof.** Let \( t_0, t_1 \in (0.5, 1) \) and assume that the \( \varepsilon \)-set \((\delta^{\xi}, \max(t_0, t_1))_{t_0, t_1 \in \varepsilon} \) of \( \delta^{\xi} \) is non-empty. Then there exists \( x \in \delta^{\xi}, \max(t_0, t_1))_{t_0, t_1 \in \varepsilon} \), and so \( \delta^{\xi}(x) \geq \max(t_0, t_1) > 1 - \max(t_0, t_1) \), i.e., \( x/\max(t_0, t_1) \in \delta^{\xi} \). Hence \( (0/\max(t_0, t_1)) \in \delta^{\xi} \) by \((33)\), and thus \( 0 \in \delta^{\xi}, \max(t_0, t_1))_{t_0, t_1 \in \varepsilon} \). Let \( x, y, z \in X \) be such that \( (x+y) * z \in \delta^{\xi}, \max(t_0, t_1))_{t_0, t_1 \in \varepsilon} \) and \( \delta^{\xi}(z) \geq \max(t_0, t_1) > 1 - \max(t_0, t_1) \), that is, \( ((x+y) * z)/\max(t_0, t_1) q^{\delta^{\xi}} \) and \( z/\max(t_0, t_1) q^{\delta^{\xi}} \). It follows from \((34)\) that

\[
((x+y) * (y^2 * x) + (0^2 * (x+y)))/\max(t_0, t_1) \in \delta^{\xi}.
\]

Hence \( ((x+y) * (y^2 * x) + (0^2 * (x+y))) \in \delta^{\xi}, \max(t_0, t_1))_{t_0, t_1 \in \varepsilon} \), and therefore \((\delta^{\xi}, \max(t_0, t_1))_{t_0, t_1 \in \varepsilon} \) is a commutative ideal of \((X, 0)\), for all \( t_0, t_1 \in (0.5, 1) \).

**Theorem 3.16.** If a Lukasiewicz fuzzy set \( \delta^{\xi} \) in \( X \) satisfies \((33)\) and

\[
(\forall x, y, z \in X)(\forall t_0, t_1 \in (0.5, 1)) \left( ((x+y) * z)/t_0 q^{\delta^{\xi}}, (z/t_1) q^{\delta^{\xi}} \Rightarrow (x * ((y^2 * x) + (0^2 * (x+y))))/\min(t_0, t_1) \in \delta^{\xi} \right),
\]

then the non-empty \( \varepsilon \)-set \((\delta^{\xi}, \min(t_0, t_1))_{t_0, t_1 \in \varepsilon} \) of \( \delta^{\xi} \) is a commutative ideal of \((X, 0)\), for all \( t_0, t_1 \in (0.5, 1) \).
Proof. It can be verified through a process similar to the proof in Theorem 3.18.

Lemma 3.17. Every LFC ideal $\delta_\xi$ of $(X,0)_\ast$ satisfies:

$$\forall x, y, z \in X, (x * ((y^2 * x) * (0^2 * (x * y)))) \geq \min\{\delta_\xi((x * y) * z), \delta_\xi(z)\}. \quad (36)$$

Proof. Note that $\langle ((x * y) * z) / \delta_\xi, \langle (x * y) * z \rangle \rangle \in \delta_\xi$ and $\langle z / \delta_\xi, \langle z \rangle \rangle \in \delta_\xi$ for all $x, y, z \in X$. It follows from (24) that

$$\langle (x * ((y^2 * x) * (0^2 * (x * y)))) / \delta_\xi, \langle (x * y) * z \rangle \rangle \rangle \in \delta.$$  

Hence $\delta_\xi(x * ((y^2 * x) * (0^2 * (x * y)))) \geq \min\{\delta_\xi((x * y) * z), \delta_\xi(z)\}$ for all $x, y, z \in X$.

Theorem 3.18. If the Lukasiewicz fuzzy set $\delta_\xi$ of a fuzzy set $\xi$ in $X$ is a LFC ideal of $X$, then its q-set $(\delta_\xi, t)_q$ is a commutative ideal of $X$ for all $t \in (0, 1]$.

Proof. Assume that $\delta_\xi$ is a LFC ideal of $(X,0)_\ast$, and let $t \in (0, 1]$. If $0 \notin (\delta_\xi, t)_q$, then $\langle 0/t, \pi^q_\xi \rangle$, that is, $\delta_\xi(0) + t \leq 1$. Since $\delta_\xi(0) \geq \delta_\xi(x)$ for $x \in (\delta_\xi, t)_q$, it follows that $\delta_\xi(x) \leq \delta_\xi(0) \leq 1 - t$. Hence $\langle x/t, \pi^q_\xi \rangle$, and so $x \notin (\delta_\xi, t)_q$. This is a contradiction, and therefore $0 \notin (\delta_\xi, t)_q$.

Corollary 3.19. If $\xi$ is a fuzzy commutative ideal of $(X,0)_\ast$, then the q-set $(\delta_\xi, t)_q$ of $\delta_\xi$ is a commutative ideal of $X$ for all $t \in (0, 1]$.

Theorem 3.20. Let $\xi$ be a fuzzy set in $X$. For the Lukasiewicz fuzzy set $\delta_\xi$ of $\xi$ in $X$, if the q-set $(\delta_\xi, t)_q$ of $\delta_\xi$ is a commutative ideal of $X$, then the following assertions are valid.

$$0 \in (\delta_\xi, t_a)_\xi,$$

$$\langle (x * (y^2 * x) * (0^2 * (x * y))) / q_\xi, \langle (x / t_c) \rangle q_\xi \Rightarrow x * ((y^2 * x) * (0^2 * (x * y))) \in (\delta_\xi, \max\{t_a, t_b\})_\xi \quad (37)$$

for all $x, y \in X$ and $t_a, t_c \in (0, 0.5)$.

Proof. Let $x, y \in X$ and $t_a, t_c \in (0, 0.5)$. If $0 \notin (\delta_\xi, t_a)_\xi$, then $\langle 0/t_a, \pi^q_\xi \rangle$ and so $\delta_\xi(0) < t_a \leq 1 - t_a$ since $t_a \leq 0.5$. Hence $\langle 0/t_a, \pi^q_\xi \rangle$ and thus $0 \notin (\delta_\xi, t_a)_\xi$. This is a contradiction, and therefore $0 \notin (\delta_\xi, t_a)_\xi$.

Let $\langle (x * (y * z) / t_a) q_\xi \rangle \rangle \langle (z / t_c) q_\xi \rangle$. Then $\langle (x * (y * z) / t_a) q_\xi \rangle$ and $\langle (z / t_c) q_\xi \rangle$. Hence $x * ((y^2 * x) * (0^2 * (x * y))) / (\delta_\xi, \max\{t_a, t_c\})_q$, and so

$$\delta_\xi(x * ((y^2 * x) * (0^2 * (x * y)))) \geq 1 - \max\{t_a, t_c\}.$$  

that is, $\langle x * ((y^2 * x) * (0^2 * (x * y))) / \max\{t_a, t_c\} \rangle \in (\delta_\xi, \max\{t_a, t_c\})_\xi$. Therefore $x * ((y^2 * x) * (0^2 * (x * y))) \in (\delta_\xi, \max\{t_a, t_c\})_\xi$.

Theorem 3.21. If a Lukasiewicz fuzzy set $\delta_\xi$ in $X$ satisfies:

$$\forall x \in X, (\forall t \in (0, 0.5)) (\langle x/t \rangle \in (\delta_\xi, t)_q,$$

and

$$\langle (x, y, z \in X) (\forall t_a, t_c \in (0, 0.5)) \left( \langle (x * (y^2 * x) * (0^2 * (x * y))) / (\min\{t_a, t_c\}) q_\xi \rangle \right), \quad (40)$$

then the non-empty q-set $(\delta_\xi, \min\{t_a, t_c\})_q$ of $\delta_\xi$ is a commutative ideal of $(X,0)_\ast$ for all $t_a, t_c \in (0, 0.5)$.
Proof. Let \( t_a, t_c \in (0, 0.5] \). If \( (\xi, \min\{t_a, t_c\}) \) is non-empty, then there exists \( x \in (\xi, \min\{t_a, t_c\}) \). Hence \( \xi(x) < x \). Let \( x, y, z \in X \) be such that \( x * y \in (\xi, \min\{t_a, t_c\}) \) and \( z \in (\xi, \min\{t_a, t_c\}) \). Thus \( \xi(x * y) * z = x * (y * z) \). Hence \( (x * (y * z)) / \min\{t_a, t_c\} \) \( (x * y) \). It follows from \( \xi \) that \( (x * y) \in (\xi, \min\{t_a, t_c\}) \). Therefore \( (\xi, \min\{t_a, t_c\}) \) is a commutative ideal of \((X, 0)_o\).

Theorem 3.22. If a Łukasiewicz fuzzy set \( \xi \) in \( X \) satisfies (37) and (38) for all \( x, y, z \in X \) and \( t_a, t_c \in \{0, 0.5\} \), then the q-set \((\xi, t_a, t_c) \) of \( \xi \) is a commutative ideal of \((X, 0)_o\), for all \( t \in (0, 0.5] \).

Proof. Let \( t \in (0, 0.5, 1] \). Assume that \( \xi \) satisfies (37) and (38) for all \( x, y, z \in X \). The condition \( \xi \) induces \( \xi(0) * t \geq 2t > 1 \), i.e., \( (0/t) \) \( (\xi, t) \) \( q \). Hence \( 0 \in (\xi, t) \). Let \( x, y, z \in X \) be such that \( x * y \in (\xi, t) \) and \( z \in (\xi, t) \). Then \( (x * y) * z \) / \( (\xi, t) ) \). It follows from (38) that \( x * y \in (\xi, t) \). Hence \( \xi(x * (y * z)) \) \( (0/t) \) \( (\xi, t) \). Therefore \( (\xi, t) \) is a commutative ideal of \((X, 0)_o\).

Theorem 3.23. If \( \xi \) is a fuzzy commutative ideal of \((X, 0)_o\), then the non-empty O-set of \( \xi \) is a commutative ideal of \((X, 0)_o\).

Proof. If \( \xi \) is a fuzzy commutative ideal of \((X, 0)_o\), then \( \xi \) is a LFC ideal of \((X, 0)_o\) (see Theorem 1.13). It is clear that \( 0 \in O(\xi) \). Let \( x, y, z \in X \) be such that \( x \in O(\xi) \) and \( x * y \) \( (\xi, t) \) \( z \). Then \( \xi((x * y) * z) > 0 \) and \( \xi(z) > 0 \). Since \( ((x * y) * z) / \xi(z) \) \( \xi(z) \) \( \xi(z) \), we have
\[ ((x * (y * z)) / \min\{\xi((x * y) * z), \xi(z)\}) \in \xi \]
by (24). It follows that
\[ \xi(x * (y * z)) \geq \min\{\xi((x * y) * z), \xi(z)\} > 0. \]
Hence \( x * (y * z) \in O(\xi) \), and therefore \( O(\xi) \) is a commutative ideal of \((X, 0)_o\).

Theorem 3.24. If a Łukasiewicz fuzzy set \( \xi \) in \( X \) satisfies (21) and
\[ (\forall x, y, z \in X)(\forall t_a, t_c \in (0, 1])\left( ((x * y) * z) / t_a \in (\xi, t) \Rightarrow ((x * (y * z)) / \max\{t_a, t_c\}) \right) \quad (41) \]
then the non-empty O-set of \( \xi \) is a commutative ideal of \((X, 0)_o\).

Proof. Let \( O(\xi) \) be a non-empty O-set of \( \xi \). Then there exists \( x \in O(\xi) \), and so \( t := \xi(x) > 0 \), i.e., \( (\xi, t) \in \xi \) for \( t > 0 \). Hence \( (0/t) \in \xi \) by (21), and thus \( \xi(0) \geq t > 0 \). Hence \( 0 \in O(\xi) \). Let \( x, y, z \in X \) be such that \( x * y \) \( z \) \( O(\xi) \) and \( z \in O(\xi) \). Then \( \xi((x * y) * z) \geq \delta > 0 \) and \( \xi(z) \geq \delta > 0 \). Since \( ((x * y) * z) / \xi(z) \) \( \xi(z) \) \( \xi(z) \), and \( (z / \xi(z)) \in \xi \), it follows from (41) that
\[ ((x * (y * z)) / \max\{\xi((x * y) * z), \xi(z)\}) / q_{\xi} \]
If \( x * (y * z) \) \( O(\xi) \), then \( \xi(x * (y * z)) = 0 \), and so
\[ \xi(x * (y * z)) = \max\{\xi((y * z)), \xi(z)\} = \max\{\xi((x * y) * z), \xi(z)\} \]
\[ = \max\{\xi((x * y) * z) + \delta - 1, \xi(z) + \delta - 1\} \]
\[ = \max\{\xi((x * y) * z) + \delta - 1, \xi(z) + \delta - 1\} \]
\[ \leq 1 + \delta - 1 \leq 1. \]
Hence \( ((x * (y * z)) / \max\{\xi((x * y) * z), \xi(z)\}) / q_{\xi} \), a contradiction. Thus \( x * (y * z) \) \( O(\xi) \), and therefore \( O(\xi) \) is a commutative ideal of \((X, 0)_o\).
Theorem 3.25. If a Łukasiewicz fuzzy set $\xi_\delta$ in $X$ satisfies $(0/\delta) q_\xi$ and 
\[
(\forall x, y, z \in X) \left( \left( ((x \ast y) \ast z) /\delta \right) q_\xi, (z/\delta) q_\xi \Rightarrow ((x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y)))) /\delta) \in \delta_\xi \right),
\]
then the $O$-set of $\xi_\delta$ is a commutative ideal of $(X, 0)_*$. 

Proof. Let $O(\xi_\delta)$ be the $O$-set of $\xi_\delta$. If $(0/\delta) q_\xi$, then $\xi(0) + \delta > 1$ and so 
\[
\delta_\xi(0) = \max\{0, \xi(0) + \delta - 1\} = \xi(0) + \delta - 1 > 0.
\]
Hence $0 \in O(\xi_\delta)$. Let $x, y, z \in X$ be such that $(x \ast y) \ast z \in O(\xi_\delta)$ and $z \in O(\xi_\delta)$. Then $\xi((x \ast y) \ast z) + \delta > 1$ and
\[
\xi(z) + \delta > 1, \text{ i.e., } ((x \ast y) \ast z) /\delta) q_\xi \text{ and } (z/\delta) q_\xi.
\]
It follows from (42) that $\xi((x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y)))) /\delta) \in \delta_\xi$, which shows $\xi((x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y)))) \geq \delta > 0$. Hence $x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y))) \in O(\xi_\delta)$, and therefore $O(\xi_\delta)$ is a commutative ideal of $(X, 0)_*$. \hfill $\square$

Theorem 3.26. Let $\xi_\delta$ be a Łukasiewicz fuzzy set in $X$ that satisfies:
\[
(\forall y \in X)(\forall t \in [\delta, 1]) \left( \langle y/t \rangle q_\xi \Rightarrow (0/\delta) \in \xi_\delta \right),
\]
\[
(\forall x, y, z \in X)(\forall t_a, t_c \in [\delta, 1]) \left( \left( ((x \ast y) \ast z) /t_a \right) q_\xi, (z/t_c) q_\xi \Rightarrow x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y))) \in \xi_\delta \right).
\]
Then the $O$-set of $\delta_\xi$ is a commutative ideal of $(X, 0)_*$. 

Proof. Let $t \in [\delta, 1]$ and $y \in O(\xi_\delta)$. Then $\xi(y) + t \geq \xi(y) + \delta > 1$, and so $\langle y/t \rangle q_\xi$, which implies that $\langle 0/\delta \rangle \in \delta_\xi$ by (43). Hence $\delta_\xi(0) \geq \delta > 0$, i.e., $0 \in O(\xi_\delta)$. Let $t_a, t_c \in [\delta, 1]$ and $x, y, z \in X$ be such that $(\langle (x \ast y) \ast z \rangle/t_a) q_\xi$ and $(z/t_c) q_\xi$. Then $\xi((x \ast y) \ast z) + t_a \geq \xi((x \ast y) \ast z) + \delta > 1$ and $\xi(z) + t_c \geq \xi(z) + \delta > 1$. Thus $\langle (x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y))) \rangle \in \xi_\delta$. Hence $\delta_\xi((x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y)))) \geq \delta > 0$, and so $x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y))) \in O(\xi_\delta)$. Consequently, $O(\xi_\delta)$ is a commutative ideal of $(X, 0)_*$. \hfill $\square$

Corollary 3.27. Let $\xi_\delta$ be a Łukasiewicz fuzzy set in $X$ that satisfies:
\[
(\forall x, y \in X) \left( \langle y/\delta \rangle q_\xi \Rightarrow (0/\delta) \in \delta_\xi \right),
\]
\[
(\forall x, y, z \in X) \left( \langle ((x \ast y) \ast z)/\delta \rangle q_\xi, (z/\delta) q_\xi \Rightarrow x \ast ((y^2 \ast x) \ast (0^2 \ast (x \ast y))) \in \xi_\delta \right).
\]
Then the $O$-set of $\delta_\xi$ is a commutative ideal of $(X, 0)_*$. 

4 Conclusion

The concept of Łukasiewicz fuzzy sets using Łukasiewicz $t$-norm was first introduced by Y. B. Jun, and it was applied to BCK/BCI-algebras. For the purpose of applying the Łukasiewicz fuzzy set to a commutative ideal in BCI-algebras, we introduced the concept of Łukasiewicz fuzzy commutative ideals and study its properties. We established the relationship between a Łukasiewicz fuzzy ideal and a Łukasiewicz fuzzy commutative ideal, and provided an example to show that a Łukasiewicz fuzzy ideal may not be a Łukasiewicz fuzzy commutative ideal. We explored the conditions under which a Łukasiewicz fuzzy ideal can be a Łukasiewicz fuzzy commutative ideal. We considered characterizations of Łukasiewicz fuzzy commutative ideals, and explored the conditions under which $\xi$-set, $q$-set, and $O$-set can be commutative ideals.

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References


