Nilpotent soft polygroups

F. Mohammadzadeh¹ and E. Mohammadzadeh²

¹,² Department of Mathematics, Faculty of Science, Payame Noor University P.O. Box 19395-3697, Tehran, Iran
mohamadzadeh464@gmail.com, mohamadzadeh36@gmail.com.

Abstract

In this paper, we introduce nilpotent soft (sub)polygroups. In addition, nilpotency of intersection, extended intersection, restricted union of two nilpotent soft polygroups are studied. Especially, a necessary and sufficient condition between nilpotency of a polygroup and soft polygroups is obtained. Finally, we define two new soft polygroups \((S_\alpha)_{A\cup\{c\}}\) and \((Q_\alpha)_A\) derived from a soft polygroup \(\alpha_A\) and study on nilpotency of these structures. Also, we extend a soft homomorphism of groups to polygroups. This helps us to extend some properties of groups to polygroups.

Article Information

Corresponding Author:
F. Mohammadzadeh;
Received: May 2022;
Revised: May 2022;
Accepted: May 2022;
Paper type: Original.

Keywords:
Polygroups, nilpotent polygroups, soft polygroups.

1 Introduction

Some problems in engineering, medical science and social science are uncertain. One way for dealing with them is soft set theory. It was proposed by Molodtsov [20]. In addition, it has applications in Riemann integration, probability theory, game theory and etc (see [20, 21]). After that time it became an interesting topic for many authors and so they work on soft set theory. Maji et al. [17], introduced several operations on soft sets. Ali et al. [15], redefined complement of a soft set. Soft sets were used in lattice theory by Qin et al. [23]. Also, soft set theory was applied on research in BCI/BCK-algebras [10]. The studying of soft sets in groups began with the work of Aktaş and Çağman in [2], where the notion of soft groups were investigated and then Acar et al. in [1], extended the notion to rings. Recently, Wang et al. in [24], introduced soft polygroups.

In group theory, nilpotent group is an interesting subject and has been studied by many scholars. Abelian groups are an example of nilpotent groups. Hassanzadeh [13] introduced the
concept of nilpotency for pair of groups. Also, Ozkan and et al. in [22], investigated some applications of Fibonacci sequences in a finite nilpotent group.

An important branch in algebra is hyperstructures. It has applications in geometry, automata, probabilities, and so on. In 1934, Marty [19] introduced the concept of polygroups as a special hypergroup. In addition, polygroups have been discussed by Corsini [23], Borzooei [24], Davvaz [26] and so on. Some results of group theory are translated on polygroups such as nilpotent polygroup that has been studied in [15, 25].

Now, in this paper we study on nilpotent soft polygroup and investigate some properties of it. Especially, we obtain a necessary and sufficient condition between soft nilpotent polygroups and nilpotent polygroups. Finally, we define two new soft sets (\(S_F, A \cup \{c\}\)) and (\(Q_F, A\)) derived from a soft polygroup \((F, A)\). Then, we investigate some properties of them.

2 Preliminary

We begin our discussion with some fundamental definitions and results.

A hyperoperation \(\odot\) is a mapping from \(H \times H\) into the family of non-empty subsets of \(H\). A hypergroupoid \((H, \odot)\) is a non-empty set \(H\) with a hyperoperation \(\odot\). If \(A\) and \(B\) are non-empty subsets of \(H\), then \(A \odot B = \bigcup_{a \in A, b \in B} a \odot b\). Also, we use \(x \odot A\) instead of \(\{x\} \odot A\) and \(A \odot x\) for \(A \odot \{x\}\).

The structure \((H, \odot)\) is called a hypergroup if \(a \odot (b \odot c) = (a \odot b) \odot c\) and \(a \odot H = H \odot a = H\) for any \(a, b, c \in H\).

**Definition 2.1.** Let \(\cdot\) be a hyperoperation, \(e \in P\) and \(^{-1}\) be an unitary operation on \(P\). Then \((P, \cdot, e, ^{-1})\), is called a polygroup if for any \(x, y, z \in P\) the following conditions hold:

(i) \((x \cdot y) \cdot z = x \cdot (y \cdot z)\),

(ii) \(e \cdot x = x \cdot e = x\),

(iii) \(x \in y \cdot z \iff y \in x \cdot z^{-1} \iff z \in y^{-1} \cdot x\).

Let \((P_1, \cdot, e_1, ^{-1})\) and \((P_2, \ast, e_2, ^{-1})\) be two polygroups. Then \((P_1 \times P_2, \odot)\), where \(\odot\) is defined as follows, is a polygroup (see [8]).

\[
(x_1, y_1) \odot (x_2, y_2) = \{(x, y) \mid x \in x_1 \cdot x_2, \text{ and } y \in y_1 \ast y_2\}.
\]

**Note.** From now on, let \((H, \cdot)\) be a hypergroup and \((P, \cdot, e, ^{-1})\) be a polygroup. For \(x, y \in P\) we use \(xy\) instead of \(x \cdot y\).

**Definition 2.2.** Let \(K\) be a non-empty subset of \(P\). Then for any \(a, b \in K\), \(K\) is called a subpolygroup of \(P\) and we denote by \(K \trianglelefteq P\) if \(ab \subseteq K\) and \(a^{-1} \subseteq K\). Also, a subpolygroup \(N\) of \(P\) is called normal and we denote by \(N \trianglelefteq P\) if for any \(a \in P\), \(a^{-1}Na \subseteq N\).

For \(K \trianglelefteq P\) and \(x \in P\), let \(xK\) (\(Kx\)) be the left (right) coset of \(K\) and \(P/K\) be the set of all left (right) cosets of \(K\) in \(P\). We recall that for \(N \trianglelefteq P\), \(x, y \in P\) and every \(z \in xy\) we have \(Nx = xN\) and \(Nxy = Nz\). Also, \((P/N, \odot, N^{-1})\) is a polygroup, where

\[
(Nx) \odot (Ny) = \{Nz \mid z \in xy\} \text{ and } (Nx)^{-1} = Nx^{-1}.
\]

A polygroup is called commutative if for any \(x, y \in P\), \(xy = yx\). For two polygroups \((P, \bullet)\) and \((P, \ast)\), a map \(f : (P, \bullet) \to (P, \ast)\) is called a homomorphism if for any \(a, b \in P\), \(f(ab) \subseteq f(a) \ast f(b)\).
Also, $f$ is a good homomorphism if the equality holds. For an equivalence relation $\rho \subseteq P \times P$ and two non-empty subsets $X, Y$ of $P$ we have

$$X \beta Y \iff \forall x \in X, \forall y \in Y.$$ 

The relation $\rho$ is called strongly regular if for any $x, y, a \in P$ we have

$$x \beta y \iff a \cdot x \beta a \cdot y \text{ and } x \cdot a \beta y \cdot a.$$ 

We use $\text{SR}(H)$ for the set of all strongly regular relations on $H$.

In [16], Koskas defined the relation $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1$ is the diagonal relation and

$$a\beta_nb \iff \exists (x_1, \ldots, x_n) \in H^n, \{a, b\} \subseteq \prod_{i=1}^{n} x_i.$$ 

In addition $\beta^* \in \text{SR}(H)$, where $\beta^*$ is the transitive closure of $\beta$. In [11], Freni showed that if $H$ is a hypergroup, then $\beta = \beta^*$. The kernel of the canonical map $\pi : H \rightarrow H_{\beta^*}$, denote by $\omega_P$ or $\omega$, is called the core of $P$.

**Theorem 2.3.** Let $A$ be a non-empty subset of $P$. The intersection of any subpolygroups of $P$ containing $A$, denoted by $(A)$ is equal to $\bigcup \{x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k} | x_i \in A, k \in \mathbb{N}, \epsilon_i \in \{1, -1\}\}$.

**Definition 2.4.** The lower central series of $P$ is the sequence $\cdots \subseteq \gamma_1(P) \subseteq \gamma_0(P)$, where $\gamma_0(P) = P$ and for $k > 0$,

$$\gamma_{k+1}(P) = \langle \{h \in P | xy \cap hyx \neq \emptyset \text{ such that } x \in \gamma_k(P), y \in P\} > .$$

Also, $P$ is called a nilpotent polygroup (we write NP) if for some $n \in \mathbb{N}$, $\gamma_n(P) \subseteq \omega$. The smallest such $n$ is called class of $P$.

In [8] it is proved that for any $x, y \in P$ we have

$$\{h \in P | xy \cap hyx \neq \emptyset\} = \{h \in P | h \in [x, y]\},$$

where $[x, y] = \{t | t \in xyx^{-1}y^{-1}\}$ is the commutator of $x, y$.

**Theorem 2.5.** Let $P$ be an NP, $N \leq P$ and $K \leq P$. Then $K$ and $P/N$ are NP.

**Definition 2.6.** A pair $(\alpha, A) = \alpha_A$ is called a soft set over $U$, where $U$ refers to an initial universe set, $E$ is a set of parameters, $A \subseteq E$ and $\alpha$ is a map from $A$ to the power set $P(U)$.

We use $S(U)$ to show the set of all soft sets over $U$.

**Definition 2.7.** For $\alpha_A, \gamma_B \in S(U)$ we have the following statements:

(i) $\alpha_A \subseteq \gamma_B$, if $A \subseteq B$ and for any $a \in A$, $\alpha(a) \subseteq \gamma(a)$.

(ii) $\alpha_A = \gamma_B$, if $\alpha_A \subseteq \gamma_B$ and $\gamma_B \subseteq \alpha_A$.

(iii) If for any $a \in A$, $\alpha(a) = \emptyset$, then $\alpha_A$ is said a null soft set.

**Theorem 2.8.** Let $\alpha_A \in S(U)$ and $\text{Supp}(\alpha_A) = \{x \in A | \alpha(x) \neq \emptyset\}$. Then $\alpha_A$ is non-null if $\text{Supp}(\alpha_A) \neq \emptyset$. 
Definition 2.9. Let \( \alpha_A, \gamma_B \in S(U) \). Then for any \( x \in A \cap B \) and \( (x, y) \in A \times B \) we have

(i) the soft intersection \( (\alpha_A \cap \gamma_B, A \cap B) \) is defined by \( (\alpha_A \cap \gamma_B)(x) = \alpha(x) \cap \gamma(x) \).

(ii) the soft \( \tilde{\wedge} \)-product \( (\alpha_A \tilde{\wedge} \gamma_B, A \times B) \) is defined by \( (\alpha_A \tilde{\wedge} \gamma_B)(x, y) = \alpha(x) \cap \gamma(y) \).

(iii) the soft \( \tilde{x} \)-product \( (\alpha_A \tilde{x} \gamma_B, A \times B) \) is defined by \( (\alpha_A \tilde{x} \gamma_B)(x, y) = \alpha(x) \times \gamma(y) \).

Definition 2.10. Let \( \alpha_A \) be a non-null soft set over \( P \). Then \( \alpha_A \) is called a soft polygroup over \( P \) if \( \alpha(x) \leq P \) for any \( x \in \text{Supp}(\alpha_A) \).

Note. From now on, assume \( A \) is a non-empty subset of \( P \) and \( \alpha_A \in SP(P) \), where \( SP(P) \) is the set of all soft polygroups over \( P \). In addition, we use \( K \leq^a P \) when \( K \) is a nilpotent subpolygroup of \( P \).

3 Nilpotent soft polygroups

In this section first we define a nilpotent soft polyroup (we write NSP). Then, some examples are added to clarify the notion. Basically, for two soft polygroups \( \alpha_A \) and \( \gamma_B \) we study the nilpotency of derived soft sets such as \( \alpha_A \cap \gamma_B \) and \( \alpha_A \cap_R \gamma_B \) and so on. Finally, a relation between a nilpotent polyroup and its soft polygroups is obtained.

Definition 3.1. The soft polyroup \( \alpha_A \) is called a nilpotent soft polyroup over \( P \), we write NSP, if there is \( n \in \mathbb{N} \) such that for any \( a \in \text{Supp}(\alpha_A) \), \( \alpha(a) \leq^n P \).

We use \( \text{NSP}(P) \) for the set of all nilpotent soft polygroups over \( P \).

Example 3.2. Let \( P = \{a, b, c, e\} \). Then \( (P, \varnothing) \) is an NP (see [8]).

\[
\begin{array}{c|cccc}
\varnothing & a & b & c & e \\
\hline
a & \{e, a\} & c & \{b, c\} & a \\
b & c & e & a & b \\
c & \{b, c\} & a & \{e, a\} & c \\
e & a & b & c & e \\
\end{array}
\]

Assume \( A = P \) and define the soft set \( \alpha_A \in S(P) \) by \( \alpha(a) = \alpha(e) = P \) and \( \alpha(b) = \alpha(c) = \{a, e\} \).

Since \( \alpha(a), \alpha(e), \alpha(b), \alpha(c) \leq^n P \) we conclude that \( \alpha \in \text{NSP}(P) \).

In what follows we have a soft polyroup that is not an NSP.

Example 3.3. Assume \( P = \{a, b, c, d, f, g, e\} \) is a polyroup with the hyperoperation \( \bullet \) such that

\[
\begin{array}{cccccccc}
\bullet & e & a & b & c & d & f & g \\
\hline
a & a & e & b & c & d & f & g \\
b & b & b & \{e, a\} & g & f & d & c \\
c & c & c & f & \{e, a\} & g & b & d \\
d & d & d & g & f & \{e, a\} & c & b \\
f & f & f & c & d & b & g & \{e, a\} \\
g & g & g & d & b & c & \{e, a\} & f \\
e & e & a & b & c & d & f & g \\
\end{array}
\]

Assume \( A = P \) and define the soft set \( \alpha_A \in S(P) \) by \( \alpha(e) = \alpha(a) = \alpha(b) = \{e, a, b\} \) and \( \alpha(c) = \alpha(d) = \alpha(f) = \alpha(g) = P \).

Then \( P \) is not an NP. Because \( \omega_P = \{e, a\} \) and \( \gamma_n(P) = \{e, a, f, g\} \) and so \( \gamma_n(P) \notin \omega_P \). Therefore, \( \alpha_A \notin \text{NSP}(P) \).
Theorem 3.4. Assume $\alpha_A \in \text{NSP}(P)$ and $B \subseteq A$. If $(\alpha |_B)_B$ is non-null, then it is an NSP.

Proof. For $b \in B$ since $B \subseteq A$, we have $\alpha |_B (b) = \alpha(b)$ and so by hypotheses $(\alpha |_B)_B \in \text{NSP}(P)$.

By the following example, we define a subset $B \subseteq A$ such that $\alpha_A$ is not an NSP but $\alpha |_B$ is an NSP.

Example 3.5. Assume $A$ and $P$ are as Example 3.3, and $B = \{e, a\}$. Define the soft set $\alpha_A \in S(P)$ by

$$\alpha(e) = \alpha(a) = \{e, a\}, \quad \alpha(b) = \alpha(c) = \{b, c\}.$$ 

If $\{b, c\} \notin P$ implies that $\alpha_A \notin \text{SP}(P)$. But $\{e, a\} \not\leq^n P$. It implies that $\alpha |_B \in \text{NSP}(P)$.

Example 3.6. Consider $P$, $A$ and $\alpha_A$ are as Example 3.3 and $B = \{e, a, b\}$. Since $\alpha(c) = P$ and $P$ is not nilpotent we have $\alpha_A \notin \text{NSP}(P)$ but every proper polygroup of order less than 7 is an NP (see §), thus $(\alpha |_B)_B \in \text{NSP}(P)$.

Definition 3.7. [24] For $\alpha_A, \gamma_B \in \text{SP}(U)$ and $x \in A \cup B$,

(i) the soft extended intersection $\alpha_A \cap \gamma_B$ is defined to be the soft set $(D, A \cup B)$, where

$$D(x) = \begin{cases} 
\alpha(x) & \text{if } x \in A - B, \\
\gamma(x) & \text{if } x \in B - A, \\
\alpha(x) \cap \gamma(x) & \text{if } x \in A \cap B.
\end{cases}$$

Replacing $\alpha(x) \cap \gamma(x)$ with $\alpha(x) \cup \gamma(x)$ in $D(x)$ we have the soft set $\alpha_A \cup \gamma_B = (D, A \cup B)$.

(ii) the restricted intersection $\alpha_A \cap \gamma_B$ is the soft set $(E, C)$ where $A \cap B \neq \emptyset$ and $C = A \cap B$ and for any $x \in C$, $E(x) = \alpha(x) \cap \gamma(x)$.

Theorem 3.8. Let $\alpha_A, \gamma_B \in \text{NSP}(P)$. Then

(i) $\alpha_A \cap \gamma_B \in \text{NSP}(P)$ if it is non-null.

(ii) $\alpha_A \cap \gamma_B \in \text{NSP}(P)$ if it is non-null and $A \cap B \neq \emptyset$.

(iii) $\alpha_A \cup \gamma_B \in \text{NSP}(P)$ if $A \cap B = \emptyset$.

(iv) $\alpha_A \wedge \gamma_B \in \text{NSP}(P)$.

Proof.

(i) Consider $\alpha_A \cap \gamma_B = (D, C)$ and $x \in \text{Supp}(D, C)$ and $x \in A - B$. By Definition 3.4, since $\alpha_A \in \text{NSP}(P)$ we obtain $D(x) = \alpha(x) \prec^n P$. For the case $x \in B - A$ by $\gamma_B \in \text{NSP}(P)$ we have $D(x) = \gamma(x) \prec^n P$. Finally, for $x \in A \cap B$ by Theorem 2.5, we have $D(x) = \alpha(x) \cap \gamma(x) \prec^n P$. Hence $(D, C) \in \text{NSP}(P)$.

(ii) By Definition 3.7 and the same manipulation of part (i), we have $\alpha_A \cap \gamma_B \in \text{NSP}(P)$.

(iii) By Definition 3.7 and $A \cap B = \emptyset$, we have

$$\text{Supp}(D, C) = \text{Supp}(\alpha_A) \cup \text{Supp}(\gamma_B) \neq \emptyset.$$ 

Then $(D, C)$ is non-null. For $x \in A - B$ we have $D(x) = \alpha(x)$ and $\alpha_A \in \text{NSP}(P)$ implies that $D(x) \prec^n P$. Also, for the case $x \in B - A$ we have $D(x) = \gamma(x) \prec^n P$. Therefore, $(D, C) \in \text{NSP}(P)$. 
(iv) Put \((H, A \times B)\) be the soft set \(\alpha_A \wedge \gamma_B\). By Definition 3.9 and Theorem 3.8, since \(\alpha_A\) and \(\gamma_B\) are non-null we have

\[
\text{Supp}(H, A \times B) = \text{Supp}(\alpha_A) \times \text{Supp}(\gamma_B) \neq \emptyset.
\]

Also, since \(\alpha_A, \gamma_B \in \text{NSP}(P)\) we conclude that for any \((x, y) \in A \times B\), \(\alpha(x) \cap \gamma(y) \preceq^n P\). Therefore, \((H, A \times B) \in \text{NSP}(P)\).

Assume \(I\) is an index set and \((\alpha_i)_{i \in I} \in \text{NSP}(P)\). Then by extending Theorem 3.8, we have the following corollary.

**Corollary 3.9.** The soft set \((\bigcap_y)_{i \in I}(\alpha_i)_{A_i} \in \text{NSP}(P)\) if it is non-null. Also, if \(\bigcap_{i \in I} A_i \neq \emptyset\), then \((\bigcap_{i \in I}(\alpha_i)_{A_i}) \in \text{NSP}(P)\), whenever it is non-null.

**Corollary 3.10.** Let \((\alpha_i)_{i \in I} \in \text{NSP}(P)\) such that for any \(i, j \in I\), \(A_i \cap A_j = \emptyset\). Then \(\bigcup_{i \in I}(\alpha_i)_{A_i} \in \text{NSP}(P)\). Also, \(\bigcap_{i \in I}(\alpha_i)_{A_i} \in \text{NSP}(P)\).

**Proof.** The proof is clear by Theorem 3.8.

In what follows we show that \(A \cap B = \emptyset\) is a vital condition in Theorem 3.8.(iii).

**Example 3.11.** Let \(P\) and \(A\) be as Example 3.2, and \(B = \{a\}\). Define two soft sets \(\alpha_A\), \(\gamma_B \in \text{SP}(P)\) by \(\alpha(e) = P\), \(\alpha(a) = \alpha(b) = \alpha(c) = \{e, a\}\) and \(\gamma(a) = \{e, b\}\), respectively. Then \(\gamma_B \in \text{NSP}(P)\). But \(D(a) = \alpha(a) \cup \gamma(a) = \{b, a, e\} \neq P\) and so \((D, C) \notin \text{NSP}(P)\).

**Theorem 3.12.** Let \(f : P_1 \rightarrow P_2\) be a one to one and good homomorphism of polygroups \(P_1\) and \(P_2\). If \(A \preceq^n P_1\), then \(\alpha(a) \preceq^n P_2\).

**Theorem 3.13.** Let \(f : P_1 \rightarrow P_2\) be a good homomorphism, \(\alpha_A \in \text{SP}(P_1)\). Then the soft set \(f\alpha_A \in \text{SP}(P_2)\), where \(f\alpha_A(x) = f(\alpha_A(x))\) for any \(x \in A\).

**Proof.** Let \(x \in A\) and \(y_1, y_2 \in f\alpha_A(x)\). Then there exist \(x_1, x_2 \in \alpha_A(x)\) such that \(y_1 = f(x_1), y_2 = f(x_2)\). Since \(f\) is a good homomorphism we get that \(y_1 y_2 \subseteq f\alpha_A(x)\) and \(y_1^{-1} \in f\alpha_A(x)\). This complete the proof.

**Theorem 3.14.** Assume \(f : P_1 \rightarrow P_2\) is a one to one and good homomorphism. If \(\alpha_A \in \text{NSP}(P_1)\), then \(f\alpha_A \in \text{NSP}(P_2)\).

**Proof.** By Theorem 3.13, \(f\alpha_A \in \text{SP}(P_2)\) and

\[
\text{Supp}(f\alpha_A) = \{x \in A \mid (f\alpha_A)(x) \neq \emptyset\} = \{x \mid f(\alpha_A)(x) \neq \emptyset\} = \{x \mid \alpha_A(x) \neq \emptyset\} = \text{Supp}(\alpha_A).
\]

Since \(\alpha_A \in \text{NSP}(P_1)\) we conclude that for any \(x \in \text{Supp}(\alpha_A)\), \(\alpha(x) \preceq^n P_1\). It follows by Theorem 3.12 and \((f\alpha_A)(x) = f(\alpha_A(x))\) that for any \(x \in \text{Supp}(f\alpha_A)\), \((f\alpha_A)(x) \preceq^n P_2\). Therefore, \(f\alpha_A \in \text{NSP}(P_2)\).

**Definition 3.15.** Assume \(\alpha_A, \gamma_B \in \text{SP}(P)\). Then \(\gamma_B\) is called a nilpotent soft subpolygroup of \(\alpha_A\), denote by \(\gamma_B \preceq^n \alpha_A\), if \(B \subseteq A\) and for any \(x \in \text{Supp}(\gamma_B)\), \(\gamma(x) \preceq^n \alpha(x)\) for some \(n \in \mathbb{N}\).
Example 3.16. Assume $A, P$ are as Example 3.2. Define $\alpha_A \in SP(P)$ by $\alpha(e) = \alpha(b) = P$ and $\alpha(c) = \alpha(a) = \{b, e\}$. Let $B = \{a, b, e\}$ and define $\gamma_B \in SP(P)$ by $\gamma(e) = \{b, e\} = \gamma(b)$ and $\gamma(a) = \{e\}$. Since $B \subseteq A$ and

$$\gamma(e) = \gamma(b) = \{b, e\} \leq^a P = \alpha(e) = \alpha(b), \quad \gamma(a) = \{e\} \leq^a \alpha(a) = \{e, b\},$$

we conclude that $\gamma_B \bowtie^{ns} \alpha_A$.

Theorem 3.17. Assume $\alpha_A, \gamma_B \in NSP(P)$. If $B \subseteq A$ and for any $x \in \text{Supp}(\gamma_B)$, $\gamma(x) \subseteq \alpha(x)$, then $\gamma_B \bowtie^{ns} \alpha_A$.

Proof. It is straightforward.

Theorem 3.18. Assume $\alpha_A \in NSP(P)$ and $(\gamma_i)_{B_i \in I} \bowtie^{ns} \alpha_A$. Then

(i) $\bigcap_{i \in I} (\gamma_i)_{B_i} \bowtie^{ns} \alpha_A$.

(ii) If $\bigcap_{i \in I} B_i \neq \emptyset$, then $(\bigcap_{i \in I} (\gamma_i)_{B_i}) \bowtie^{ns} \alpha_A$ when it is non-null.

(iii) If for any $i, j \in I$, $B_i \cap B_j = \emptyset$, then $\bigcup_{i \in I} (\gamma_i)_{B_i} \bowtie^{ns} \alpha_A$.

(iv) $\tilde{\Lambda}_{i \in I} (\gamma_i)_{B_i} \bowtie^{ns} \tilde{\Lambda}_{i \in I} \alpha_A$.

Proof. By Theorems 3.8 and 3.17, we get (ii). Other parts are proved similarly.

Definition 3.19. The soft set $\alpha_A$ is called a whole soft polygroup over $P$ if for any $x \in A$, $\alpha(x) = P$.

Theorem 3.20. $P$ is an NP if and only if every soft polygroup of $P$ is nilpotent.

Proof. $(\Rightarrow)$ By Theorem 2.3, we get the result.

$(\Leftarrow)$ Consider every soft polygroup of $P$ is nilpotent. Put $\alpha_A$ be the whole soft polygroup. Then for any $x \in \text{Supp}(\alpha_A)$, $P = \alpha(x)$ and so $P$ is an NP.

4 Soft homomorphism

In this section first we clarify the notion of soft homomorphism by an example. Also, we define two new soft sets $(S_a)_{A \cup \{e\}}$ and $(Q_a)_{A}$ derived from a soft polygroup $\alpha_A$. Then, we investigate some properties of them.

Definition 4.1. Suppose $\alpha_A \in SP(P_1)$ and $\gamma_B \in SP(P_2)$. Then

(i) $(f, g)$ is called a soft homomorphism between $\alpha_A$ and $\gamma_B$ if $f : P_1 \rightarrow P_2$ is a good epimorphism, $g : A \rightarrow B$ is a surjective map and for any $x \in A$, $f(\alpha(x)) = \gamma(g(x))$.

(ii) we write $\alpha_A \simeq \gamma_B$ if there is a soft homomorphism.

(iii) we write $\alpha_A \preceq \gamma_B$ if $\alpha_A \simeq \gamma_B$ such that $f$ is a good isomorphism and $g$ is a bijective map.

Theorem 4.2. Let $(G, \cdot)$ be a group. Then $(P_G, \circ, e, e^{-1})$ is a polygroup, where $P_G = G \cup \{a\}$, $a \notin G$ and $\circ$ is defined as follows:

(1) $a \circ a = e$,
(2) $e \circ x = x \circ e = x, \forall x \in P_G$,
(3) $a \circ x = x \circ a = x, \forall x \in P_G - \{e, a\}$,
(4) $x \circ y = x.y, \forall (x, y) \in G^2; y \neq x^{-1}$,
(5) $x \circ x^{-1} = x^{-1} \circ x = \{e, a\}, \forall x \in P_G - \{e, a\}$.

In addition, $P_G$ is an NP if and only if $G$ is a nilpotent group.
Assume $G$ is the quaternion group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$. Since $G$ is nilpotent by Theorem 4.2, we conclude that $(P_G, \circ, e, -1)$ is an NSP.

**Example 4.4.** Consider $P = \mathbb{Z} \cup \{a\}$, $P' = (\{0\} \otimes \mathbb{Z}) \cup \{(0, r)\}$ be two polygroups as Definition 4.2. Take $A = 2\mathbb{Z} \cup \{a\}$, $B = (\{0\} \otimes 6\mathbb{Z}) \cup \{(0, r)\}$ and define $\delta_A \in SP(P)$ and $\eta_B \in SP(P')$ by
\[
\delta(x) = \begin{cases} \{0, a\} & x = a, \\ \{0\} \otimes 6y\mathbb{Z} & y \in 6\mathbb{Z} \end{cases} \quad \text{and} \quad \eta(0, y) = \begin{cases} \{0\} \otimes 6y\mathbb{Z} & y \in 6\mathbb{Z} \\ \{(0, r), (0, 0)\} & y = r \end{cases}
\]

Then the functions
\[
f : P \to P', \quad g : A \to B
\]
\[
f(x) = \begin{cases} (0, x) & x \in \mathbb{Z} \\ (0, r) & x = a \end{cases} \quad \text{and} \quad g(y) = \begin{cases} (0, 3y) & y \in 2\mathbb{Z} \\ (0, r) & y = a \end{cases}
\]

are isomorphism and bijective map, respectively. Also, for any $x \in A$, $f(\delta(x)) = \eta(g(x))$. Consequently, $\delta_A \simeq \eta_B$.

**Definition 4.5.** Assume $\alpha_A$ is a soft group over the group $G$ with identity element $e$ and $a \notin G$. We define the soft set $(S_\alpha)_{A \cup \{a\}} \in S(P_G)$ by
\[
S_\alpha(x) = \begin{cases} \alpha(x) & x \in A \\ \{e, a\} & x = a \end{cases}
\]

In what follows we extend a soft group to an NSP.

**Theorem 4.6.** Consider $\alpha_A$ is a soft group over a nilpotent group $G$. Then $(S_\alpha)_{A \cup \{a\}} \in NSP(P_G)$.

**Proof.** Since $\alpha(x), \{e, a\} \simeq P_G$ we conclude that $(S_\alpha)_{A \cup \{a\}} \in SP(P_G)$. Also, by the nilpotency of $G$ and Theorems 4.2 and 2.5, we get $\alpha(x), \{a, e\} \preceq^n P_G$. Consequently, $(S_\alpha)_{A \cup \{a\}} \in NSP(P_G)$.

**Theorem 4.7.** Consider $\alpha_A$ and $\gamma_B$ are two soft polygroups over $P_1$ and $P_2$, respectively. If $\alpha_A \simeq \gamma_B$ and $\alpha_A \in NSP(P_1)$, then $\gamma_B \in NSP(P_2)$.

**Proof.** Since $\alpha_A \in NSP(P_1)$ we have for any $x \in \text{Supp}(\alpha_A)$, $\alpha(x) \preceq^n P_1$ and so by Theorem 3.12, $f(\alpha(x)) \preceq^n P_2$. On the other hand, for any $y \in \text{Supp}(\gamma_B)$, there exists $x \in \text{Supp}(\alpha_A)$ with $\gamma(x) = y$. Thus, $\alpha_A \simeq \gamma_B$ implies that $\gamma(y) = \gamma(g(x)) = f(\alpha(x)) \preceq^n P_2$. Therefore, $\gamma_B \in NSP(P_2)$.

**Definition 4.8.** Let $\alpha_A \in SP(P)$ and $N \preceq P$ such that for any $x \in A$, $N \subseteq \alpha(x)$. Then the soft set $Q_\alpha : A \to P(\frac{P}{N})$ defined by $Q_\alpha(x) = \frac{\alpha(x)}{N}$ is called the quotient soft polygroup of $\alpha_A$.

**Example 4.9.** Assume $P$ and $A$ are an Example 3.2, $N = \{e, a\}$ and $\alpha_A$ is the whole soft polygroup of $P$. Then $Q_\alpha(x) = \frac{P}{N}$ is the whole soft polygroup of $\alpha_A$.

**Theorem 4.10.** Assume $\alpha_A \in NSP(P)$. Then $(Q_\alpha)_{A \subseteq P(\frac{P}{N})}$.

**Proof.** By $\alpha_A \in NSP(P)$, for any $x \in \text{Supp}(\alpha_A)$, we have $\alpha(x) \preceq^n P$ of class say $n$. Since
\[
\emptyset \neq \text{supp}(Q_\alpha) = \{x \in A \mid Q_\alpha(x) \neq \emptyset\} = \{x \in A \mid \frac{\alpha(x)}{N} \neq \emptyset\},
\]
we conclude that $F(x) \neq \emptyset$, i.e $x \in \text{Supp}(\alpha_A)$. Then by Definition 4.8 and Theorem 3.4, for any $x \in \text{Supp}(Q_\alpha)$, $Q_\alpha(x) = \frac{\alpha(x)}{N} \preceq^n \frac{P}{N}$ and so $(Q_\alpha)_{A \subseteq P(\frac{P}{N})}$. 

Nilpotent soft polygroups

Theorem 4.11. Consider $\alpha_A \in SP(P_1)$, $\gamma_B \in SP(P_2)$ and $\alpha_A \sim \gamma_B$ with a soft homomorphism $(f, g)$. If $N \leq P_1$, $N \leq \alpha(x)$ for any $x \in \text{Supp}(\alpha_A)$ and $g$ is a bijective map, then $(Q_\alpha)_A \simeq \gamma_B$, where $Q_\alpha(x) = \frac{\alpha(x)}{N}$.

Corollary 4.12. Assume $\alpha_A$ and $\gamma_B$, $N$ and $(Q_\alpha)_A$ are as Theorem 4.11. If $\gamma_B \in NSP(P_2)$, then $(Q_\alpha)_A \in NSP(P_1/N)$.

Proof. By Theorem 4.11, $(Q_\alpha)_A \simeq \gamma_B$. Since $\gamma_B \in NSP(P_2)$ by Theorems 4.7 and 4.10, we conclude that $(Q_\alpha)_A \in NSP(P_1/N)$.

By the following theorem we extend a soft homomorphism of groups to polygroups.

Theorem 4.13. If $\alpha_{A_i}$, $\gamma_{A_2}$ are two soft groups of $G_1, G_2$, $c_i \notin G_i$ ($i=1,2$) and $\alpha_{A_1} \sim \gamma_{A_2}$, then

$$(S_{\alpha})_{A_1 \cup \{c_1\}} \sim (S_{\gamma})_{A_2 \cup \{c_2\}}.$$  

Proof. The proof of Theorem 4.4 implies that $(S_{\alpha})_{A_1 \cup \{c_1\}} \in SP(P_{G_1})$, $(S_{\gamma})_{A_2 \cup \{c_2\}} \in SP(P_{G_2})$. Since $\alpha_{A_1} \sim \gamma_{A_2}$ by Definition 4.1, $f : G_1 \rightarrow G_2$ is a homomorphism of groups, $g : A_1 \rightarrow A_2$ is a surjective map and for any $x \in A_1$, $f(\alpha_{A_1})(x) = (\gamma_{A_2})(g(x))$. Define $g_1 : A_1 \cup \{c_1\} \rightarrow A_2 \cup \{c_2\}$ by

$$g_1(x) = \begin{cases} x \in A_1, & x \in A_1, \\ c_2 & x = c_1. \end{cases}$$  

Now, it is easy to see that $f_1$ is a good epimorphism of polygroups and $g_1$ is a surjective map. In addition, $(f_1(S_{\alpha})_{A_1 \cup \{c_1\}}(c_1)) = (S_{\gamma})_{A_2 \cup \{c_2\}}(g_1(c_1))$ and for any $x \in B_1$,

$$(f_1(S_{\alpha})_{A_1 \cup \{c_1\}}(x)) = (S_{\gamma})_{A_2 \cup \{c_2\}}(g_1(x)).$$

Therefore, $(f_1, g_1)$ is a soft homomorphism between $(S_{\alpha})_{A_1 \cup \{c_1\}}$ and $(S_{\gamma})_{A_2 \cup \{c_2\}}$. Consequently, $(S_{\alpha})_{A_1 \cup \{c_1\}} \sim (S_{\gamma})_{A_2 \cup \{c_2\}}$.

Corollary 4.14. Consider $\alpha_{A_1}$ and $\gamma_{A_2}$ are as Theorem 4.13. If $G_1$ is a nilpotent group and $(S_{\alpha})_{A_1} \simeq (S_{\gamma})_{A_2}$, then $(S_{\gamma})_{A_2 \cup \{c_2\}} \in NSP(P_{G_2})$.

Proof. By the same manipulation of Theorem 4.13, we have if $(\alpha_{A_1}) \simeq (\gamma)_{A_2}$, then $(S_{\alpha})_{A_1 \cup \{c_1\}} \simeq (S_{\gamma})_{A_2 \cup \{c_2\}}$. Also, by Theorem 4.2, $P_{G_2}$ is an NP and so Theorem 3.20 implies that $(S_{\alpha})_{A_1 \cup \{c_1\}} \in NSP(P_{G_1})$. Therefore, by Theorem 4.7, we have $(S_{\gamma})_{A_2 \cup \{c_2\}} \in NSP(P_{G_2})$.

5 Conclusion

In this paper, for a polygroup $P$ and a soft set $\alpha_A$ the notion of nilpotent soft (sub)polygroups were defined. Some examples have been used to clarify the concept of nilpotent soft polygroup. In addition, a connection between nilpotency of soft polygroup and polygroup was obtained. Especially, the quotient of a soft polygroup was defined and a relation between nilpotency of a soft polygroup and its quotient was obtained. Also, by the notion of soft homomorphism we extend a soft homomorphic of groups to get a soft homomorphic of polygroups. Then, some new nilpotent soft polygroups were attained. This work can be used on Engel and solvabel soft polygroups, too.
References


