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A short note on categorical equivalences of proper weak pseudo EMV-algebras

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"This paper is dedicated to Professor Antonio Di Nola on the occasion of his 75th birthday."

Abstract

We study the class of weak pseudo EMV-algebras without top element that are a non-commutative generalization of MV-algebras, pseudo MV-algebras and generalized Boolean algebras. We present their categorical equivalences to a special category of pseudo MV-algebras with a fixed maximal and normal ideal as well as to a special category of unital ℓ-groups with a fixed maximal and normal ℓ-ideal.

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1 Introduction

The basic algebraic model for many valued Łukasiewicz logic is an MV-algebra introduced in the late fifties by Chang [3]. During the last decades MV-algebras penetrated into many areas of mathematics as well as they were very deeply studied. The fundamental result on representation of MV-algebras is due to Mundici, [21], saying that the category of MV-algebras is categorically equivalent to the category of unital Abelian ℓ-groups. For more information about MV-algebras, we recommend to consult with [4]. The importance of MV-algebras and corresponding unital ℓ-groups entails a whole variety of generalizations of MV-algebras: In [18] and independently in [22], pseudo MV-algebras were introduced as non-commutative MV-algebras. Pseudo BL-algebras as a non-commutative generalization of BL-algebras are defined in [7, 8], and in [19] pseudo hoops were studied, etc.

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Recently, [11, 13, 14] introduced common commutative and non-commutative generalizations of MV-algebras, pseudo MV-algebras and generalized Boolean algebras, called EMV-algebras (extended MV-algebras) and pseudo EMV-algebras, respectively, as algebras where top element is not assumed. Their basic representation theorem says that they are either equivalent to MV-algebras/pseudo MV-algebras if they possess a top element, or they can be embedded into an equivalent MV-algebra/pseudo MV-algebra as a maximal and normal ideal such that every element outside of the image is a complement of some element from the image. This generalizes the analogous result for generalized Boolean algebra, see [5, Thm 2.2]. For example, the Loomis–Sikorski type theorem for σ-complete EMV-algebras was established in [12].

We have to underline that not every maximal ideal of an MV-algebra can serve as an example for EMV-algebras.

Since the class of EMV-algebras and pseudo EMV-algebras, respectively, is not a variety, we have studied the least variety containing them and we have showed that this is the class of weak EMV-algebras and the class of weak pseudo EMV-algebras which were introduced and studied in [15] and [16, 17], respectively. The class of wPEMV-algebras is very reach, and the variety of wPEMV-algebras has uncountably many subvarieties whereas the variety of (commutative) wEMV-algebras has only countably many subvarieties. Every wPEMV-algebra admits a representation analogous to the one for pseudo EMV-algebras.

A wPEMV-algebra is not equivalent to a pseudo MV-algebra if and only if it does not admit a top element. Therefore, it is important to study the class of proper wPEMV-algebras, i.e. the class of wPEMV-algebras without top element. These structures can be studied also in frames of residuated lattices, see e.g. [1] which cover also semiclans [2] and Bosbach cone algebras [23].

In the paper, we present the categorical equivalence of proper wPEMV-algebras by a special category of pseudo MV-algebras as well as by a special category of unital ℓ-groups. As corollaries, we present also a subcategories of associated proper wPEMV-algebras. As a by-product, we obtain categorical equivalences for proper pseudo EMV-algebras.

The paper is organized as follows. Section 2 gives basic facts about studied non-commutative algebras - pseudo MV-algebras, pseudo EMV-algebras, and wPEMV-algebras. The main results, categorical equivalences, are presented in Section 3.

2 Basic Facts and Notions

The basic notion for our non-commutative many-valued reasoning is a pseudo MV-algebra defined in [18] (see also an equivalent structure called a generalized MV-algebra [22]). That is, a *pseudo MV-algebra* is an algebra $(M;\oplus,^-,^\sim,0,1)$ of type $(2,1,1,0,0)$ such that the following axioms hold for all $x,y,z\in M$ with an additional binary operation $\odot$ defined via

$$y\odot x=(x^-\oplus y^-)^\sim$$

(A1) $x\oplus(y\oplus z)=(x\oplus y)\oplus z$;

(A2) $x\oplus 0=0\oplus x=x$;

(A3) $x\oplus 1=1\oplus x=1$;

(A4) $1^\sim=0$; $1^-=0$;

(A5) $(x^-\oplus y^-)^\sim=(x^\sim\oplus y^\sim)^-$;

(A6) $x\oplus(x^\sim\odot y)=y\oplus(y^\sim\odot x)=(x\odot y^-)\oplus y=(y\odot x^-)\oplus x$;

(A7) $x\odot(x^-\oplus y)=(x\oplus y^\sim)\odot y$;

(A8) $(x^-)^\sim=x$.

If $\oplus$ is commutative, then a pseudo MV-algebra is an MV-algebra.

Pseudo MV-algebras are intimately connected with unital ℓ-groups. We note that an element u from the positive cone G^+ of an ℓ-group G is said to be a *strong unit* if given $x\in G$, there is an integer $n\geq 0$

such that $x \leq nu$. A couple (G,u), where u is a fixed strong unit of G, is said to be a *unital ℓ-group*. For more info on ℓ-groups, we recommend to consult [6].

The importance of ℓ-groups for non-commutative many-valued reasoning is due to the following two facts: (1) If (G,u) is a unital ℓ-group (not necessarily Abelian), we define

$$\Gamma(G,u) := [0,u]$$

and

$$\begin{aligned} x \oplus y &:= (x+y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then $\Gamma(G,u) := ([0,u]; \oplus, ^-, ^\sim, 0, u)$ is a pseudo MV-algebra [18].

(2) A basic result on representation of pseudo MV-algebras says that there is a one-to-one correspondence between pseudo MV-algebras and unital ℓ-groups, for more details see [10], moreover, there is a categorical equivalence between the category of pseudo MV-algebras and the category of unital ℓ-groups, generalizing a famous result on MV-algebras, see [21].

Pseudo MV-algebras and MV-algebras have many generalizations. One of them are pseudo EMV-algebras introduced in [13, 14]:

An algebra $(M; \vee, \wedge, \oplus, 0)$ of type $(2,2,2,0)$ is called a *pseudo EMV-algebra* (EMV stands for extended MV-algebras) if it satisfies the following conditions:

(E1) $(M; \vee, \wedge, 0)$ is a distributive lattice with the least element 0;

(E2) $(M; \oplus, 0)$ is an ordered monoid with a neutral element 0;

(E3) for each $a \in \mathcal{I}(M) := \{x \in M : x \oplus x = x\}$, (the set of idempotents) the elements

$$\lambda_a(x) := \min\{z \in [0,a] : z \oplus x = a\}, \quad \rho_a(x) := \min\{z \in [0,a] : x \oplus z = a\}$$

exist in M for all $x \in [0,a]$, and the algebra $([0,a]; \oplus, \lambda_a, \rho_a, 0, a)$ is a pseudo MV-algebra;

(E4) for each $x \in M$, there is $a \in \mathcal{I}(M)$ such that $x \leq a$.

If $\oplus$ is commutative, then $\lambda_a = \rho_a$ and $(M; \vee, \wedge, \oplus, 0)$ is an EMV-algebra introduced in [11].

In the same way as for pseudo MV-algebras, we can introduce a total binary operation $\odot$ in the following way: For all $x, y \in M$, we define

$$x \odot y = \rho_a(\lambda_a(y) \oplus \lambda_a(x)),$$

where $a \in \mathcal{I}(M)$ and $x, y \in [0,a]$. Then $x \odot y$ is correctly defined and it does not depend on $a \in \mathcal{I}(M)$. It is clear that if $(M; \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-algebra, then $(M; \vee, \wedge, \oplus, 0)$ is a pseudo EMV-algebra with top element. Conversely, if $(M; \vee, \wedge, \oplus, 0)$ is a pseudo EMV-algebra with top element 1, then $(M; \oplus, \lambda_1, \rho_1, 0, 1)$ is a pseudo MV-algebra. This shows that there is a one-to-one correspondence between pseudo MV-algebras and pseudo EMV-algebras with top element.

For example, (1) every generalized Boolean algebra gives a commutative pseudo EMV-algebra.

(2) Given an infinite system $\{M_i\}_i$ of pseudo MV-algebras, let M be the system of all sequences $(x_i)_i \in \prod_i M_i$, where $x_i \in M_i$ and each $x_i = 0$ for all but finitely many indices i. Then M is an example of a pseudo EMV-algebra without top element.

(3) For each pseudo EMV-algebra M without top element, there is a unique pseudo EMV-algebra N with top element such that M can be embedded into N as a maximal and normal ideal of N and every element of M is either in the image of M or is a complement of some element from the image of M. This basic representation theorem was established in [14, Thm 6.4].

(4) Not every maximal and normal ideal of a unital ℓ-group gives an example of a pseudo EMV-algebra. Indeed, take $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1,0))$, where $\mathbb{Z}$ is the set of integers, $\overrightarrow{\times}$ denotes the lexicographic product,

and G is a non-trivial ℓ-group. Then $I=\{(0,g)\colon g\in G^+\}$ is a unique maximal and normal ideal of M, but it cannot serve as an example of a pseudo EMV-algebra in view that it possesses only the zero idempotent element, and it does not dominate any non-zero element of I.

For more information about pseudo EMV-algebras consult with [13, 14].

Central algebras of the paper generalizing pseudo MV-algebras and pseudo EMV-algebras are weak pseudo EMV-algebras introduced in [16, 17].

Definition 2.1. *An algebra $(M;\vee,\wedge,\oplus,\ominus,\oslash,0)$ of type (2,2,2,2,2,0) is called a* wPEMV-algebra *(w means weak) if it satisfies the following conditions:*

($W1$) $(M,\vee,\wedge,0)$ *is a distributive lattice with the least element* 0*;*

($W2$) $(M;\oplus,0)$ *is a monoid;*

($W3$) $(y\oplus x)\ominus x\le y$ *and* $x\oslash(x\oplus y)\le y$*;*

($W4$) $(y\ominus x)\oplus x=x\vee y=x\oplus(x\oslash y)$*;*

($W5$) $x\ominus(x\wedge y)=x\ominus y$ *and* $(x\wedge y)\oslash y=x\oslash y$*;*

($W6$) $y\ominus(x\oslash y)=x\wedge y=(y\ominus x)\oslash y$*;*

($W7$) $z\ominus(x\vee y)=(z\ominus x)\wedge(z\ominus y)$ *and* $(x\vee y)\oslash z=(x\oslash y)\wedge(y\oslash z)$*;*

($W8$) $(x\wedge y)\ominus z=(x\ominus z)\wedge(y\ominus z)$ *and* $z\oslash(x\wedge y)=(z\oslash x)\wedge(z\oslash y)$*;*

($W9$) $x\ominus(y\oplus z)=(x\ominus z)\ominus y$ *and* $(y\oplus z)\oslash x=z\oslash(y\oslash x)$*;*

($W10$) $x\oplus(y\vee z)=(x\oplus y)\vee(x\oplus z)$ *and* $(y\vee z)\oplus x=(y\oplus x)\vee(z\oplus x)$*.*

If $\oplus$ is commutative, then $x\ominus y=y\oslash x$, $x,y\in M$, and the subreduct $(M;\vee,\wedge,\oplus,\ominus,0)$ is said to be a *weak EMV-algebra* which is a commutative algebra introduced in [15].

An element $x\in M$ such that $x\oplus x=x$ is said to be *idempotent* or *Boolean*, and let $\mathcal{I}(M)$ be the set of idempotents of M. The basic properties of wPEMV-algebras are as follows, see [16, Prop 3.2].

Proposition 2.2. *Let $(M;\vee,\wedge,\oplus,\ominus,\oslash,0)$ be a wPEMV-algebra and $x,y,z\in M$. Then the following hold:*

(i) $(M;\oplus,0)$ *is an ordered monoid which is right and left naturally ordered (that is, $x\le y$ if and only if there is $u\in M$ such that $x\oplus u=y$, equivalently, there is $v\in M$ such that $v\oplus x=y$).*

(ii) $(a\ominus x)\oslash a=x=a\ominus(x\oslash a)$ *if* $x\le a$*.*

(iii) $x\wedge y=((a\ominus x)\vee(a\ominus y))\oslash a$ *and* $x\wedge y=a\ominus((x\oslash a)\vee(y\oslash a))$ *if* $x,y\le a$*.*

(iv) $x\le y$ *implies that* $x\ominus z\le y\ominus z$ *and* $z\oslash x\le z\oslash y$*. Also,* $z\ominus y\le z\ominus x$ *and* $y\oslash z\le x\oslash z$*.*

(v) $z\le x\oplus y$ *if and only if* $z\ominus y\le x$ *if and only if* $x\oslash z\le y$*.*

(vi) $(x\wedge y)\oplus z=(z\oplus z)\wedge(y\oplus z)$ *and* $z\oplus(x\wedge y)=(z\oplus x)\wedge(z\oplus y)$*.*

(vii) $z\ominus(x\wedge y)=(z\ominus x)\vee(z\ominus y)$ *and* $(x\wedge y)\oslash z=(x\oslash z)\vee(y\oslash z)$*.*

(viii) $x\ominus x=0=x\oslash x$ *and* $x\ominus 0=x=0\oslash x$*.*

(ix) $x\le y$ *if and only if* $x\ominus y=0$ *if and only if* $y\oslash x=0$*.*

(x) $(x\vee y)\ominus z=(x\ominus z)\vee(y\ominus z)$ *and* $z\oslash(x\vee y)=(z\oslash x)\vee(z\oslash y)$*.*

(xi) $x\ominus y\le x$ *and* $x\oslash y\le y$*.*

(xii) $(x\ominus y)\wedge(y\ominus x)=0=(x\oslash y)\wedge(y\oslash x)$*.*

(xiii) *If $a\oplus a=a$, then $a\oplus x=a\vee x=x\oplus a$.*

(xiv) $z \ominus (x \wedge y) = (z \ominus x) \vee (z \ominus y)$ *and* $(x \wedge y) \oslash z = (x \oslash z) \vee (y \oslash z)$.

(xv) *The binary operation* $\oplus$ *is commutative if and only if* $x \ominus y = y \oslash x$.

We note that every pseudo MV-algebra and every pseudo EMV-algebra can be converted into a weak pseudo EMV-algebra $(M; \vee, \wedge, \oplus, \ominus, \oslash, 0)$, where

$$x \ominus y \quad = \quad x \odot \lambda_a(y), \quad \text{where } x, y \leq a \in \mathcal{I}(M), \tag{1}$$

$$x \oslash y \quad = \quad \rho_a(x) \odot y, \quad \text{where } x, y \leq a \in \mathcal{I}(M). \tag{2}$$

If $(M; \vee, \wedge, \oplus, \ominus, \oslash, 0)$ is a wPEMV-algebra such that its reduct $(M; \vee, \wedge, \oplus, 0)$ is a pseudo EMV-algebra, then the wPEMV-algebra M is said to be an *associated wPEMV-algebra*.

Finally, let G^+ be the positive cone of an ℓ-group G. Set $x \oplus y = x + y$, $x \ominus y = (x - y) \vee 0$, and $x \oslash y = (-x + y) \vee 0$ for all $x, y \in G^+$. Then $(G^+; \vee, \wedge, \oplus, \ominus, \oslash, 0)$ is a wPEMV-algebra called a *wPEMV-algebra of a positive cone* or a *conic algebra*. Then it is cancellative, i.e. if $x \oplus y_1 = x \oplus y_2$, then $y_1 = y_2$ and if $x_1 \oplus y = x_2 \oplus y$, then $x_1 = x_2$, and vice versa, every cancellative wPEMV-algebra is isomorphic to some conic algebra, [16, Thm 5.8].

As for pseudo EMV-algebras, we have also a basic representation of wPEMV-algebras by wPEMV-algebras with top element:

Theorem 2.3. [Basic Representation Theorem for wPEMV-algebras]
Every wPEMV-algebra M either has a top element and so it is an associated wPEMV-algebra or it can be embedded into an associated wPEMV-algebra N with top element as a maximal and normal ideal of N. Moreover, every element of N is either in the image of $x \in M$ or is a right complement of the image of some element $x \in M$.

3 Categorical equivalences of proper wPEMV-algebras

A wPEMV-algebra M is said to be *proper* if M does not contain a top element. We note that the class of proper wPEMV-algebras is not a variety. The class is closed under direct products but neither under subalgebras and nor under homomorphic images. Indeed, take a proper associated wPEMV-algebra M and let $a \in \mathcal{I}(M)$ be non-zero. Then the interval $[0, a]$ is a bounded subalgebra of M and it is a homomorphic image of M under the mapping $f_a : M \to [0, a]$ defined by $f_a(x) = x \wedge a$, $x \in M$. We recall that every proper wPEMV-algebra has infinitely many elements.

We introduce a category PwPEMV such that its objects are proper wPEMV-algebras and morphisms are homomorphisms of wPEMV-algebras. Now, we introduce a special category PPMV of pseudo MV-algebras whose objects are couples (M, I), where M is a pseudo MV-algebra and I is a fixed maximal and normal ideal of M such that every element of M belongs either to I or to its complement; we know that $I^- := \{x^- \colon x \in I\} = \{x^\sim \colon x \in I\} =: I^\sim$. If (M_1, I_1) and (M_2, I_2) are two objects of PPMV, then a morphism from (M_1, I_1) to (M_2, I_2) is any homomorphism $\phi : M_1 \to M_2$ of pseudo MV-algebras such that $\phi(I_1) \subseteq I_2$. It is easy to see that PwPEMV and PPMV are categories.

For example, (1) the MV-algebra $M = [0, 1]$ has a unique maximal ideal, namely $I = \{0\}$, but (M, I) is not an object of PPMV. Similarly, if G is any ℓ-group not necessarily Abelian, then $M := \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ is a pseudo MV-algebra (not necessarily an MV-algebra) whose $I := \{(0, g) \colon g \in G^+\}$ is a unique maximal and normal ideal of M, so that (M, I) is an object of PPMV.

(2) Let $\mathbb{N}$ be the set of all natural numbers and let M be the Boolean algebra of all finite and co-finite subsets of $\mathbb{N}$. For each $n \in \mathbb{N}$, let $I_n = \{A \subset M \colon n \notin A\}$ and $I_\infty = \{A \subset N \colon |A| < \infty\}$. Then I_n $(n \geq 1)$ and I_∞ are only maximal ideals of M and each (M, I_n) and (M, I_∞) are objects of PPMV.

In what follows, we show that both categories are categorically equivalent. The proofs will depend on the following propositions.

Define a mapping $\Phi : \mathsf{PPMV} \to \mathsf{PwPEMV}$ as follows: For any object $(M, I) \in \mathsf{PPMV}$, let

$$\Phi(M, I) := I$$

and if (M_1,I_1) and (M_2,I_2) are objects of PPMV and $\phi:(M_1,I_1)\to(M_2,I_2)$ is a morphism, then

$$\Phi(\phi)(x):=\phi(x),\quad x\in I_1.$$

We note that if (M,I) is an object of PPMV, then the restriction of the operations $\vee,\wedge,\oplus,\ominus,\oslash$ from M onto I gives that I is an object of PwPEMV, i.e. I is a proper wPEMV-algebra.

Proposition 3.1. *The mapping Φ is a well-defined functor that is faithful and full from the category* PPMV *into the category* PwPEMV*.*

Proof. We start with showing that Φ is a well-defined functor. That is, we show that if $\phi:(M_1,I_1)\to(M_2,I_2)$ is a morphism in the category PPMV, then $\Phi(\phi)$ is a morphism in PwPEMV. The mapping $\Phi(\phi)$ is a wPEMV-homomorphism from the proper wPEMV-algebra I_1 into the proper wPEMV-algebra I_2.

Let ϕ_1 and ϕ_2 be two morphisms from (M_1,I_1) into (M_2,I_2) such that $\Phi(\phi_1)=\Phi(\phi_2)$. Then $\phi_1(x)=\phi_2(x)$ for each $x\in I_1$. If $x\in M_1\setminus I_1$, there is an element $x_0\in I_1$ such that $x=x_0^-$. Whence $\phi_1(x)=\phi_1(x_0^-)=(\phi_1(x_0))^-=(\phi_2(x_0))^-=\phi_2(x)$ which implies $\phi_1=\phi_2$, i.e. Φ is a faithful functor.

Now, we show that Φ is a full functor, let $h:I_1\to I_2$ be a morphism from PPMV, i.e. h is a homomorphism of wPEMV-algebras. By Basic Representation Theorem 2.3 for wPEMV-algebras, there are associated wPEMV-algebras M_1 and M_2 with top element such that I_1 and I_2 can be embedded into M_1 and M_2, respectively, as their maximal and normal ideals. Without loss of generality, we can assume that I_i is a wPEMV-subalgebra of M_i for $i=1,2$. We assert that there is a morphism $\phi:(M_1,I_1)\to(M_2,I_2)$ such that $\Phi(\phi)=h$. That is, h can be extended to a homomorphism ϕ of pseudo MV-algebras from M_1 into M_2 for some objects (M_1,I_1) and (M_2,I_2) from PPMV. We set $\phi(x)=h(x)$ and $\phi(x^-)=(h(x))^-$ if $x\in I_1$. Then $\phi(1)=\phi(0^-)=h(0)^-=1$. Clearly, $\phi(x)=h(x)$ if $x\in I_1$, and $\phi(x^-)=(\phi(x))^-$, $\phi(x^\sim)=h(x)^\sim$, $x\in M_1$.

We notify that using the Basic Representation Theorem, in each wPEMV-algebra I we can define a binary operation $\odot$ by $x\odot y=(y^\sim\oplus x^\sim)^-$; equivalently $x\oplus y=(y^-\oplus x^-)^\sim$ for all $x,y\in I$. We note that $x\odot y\in I$; indeed, if $x\odot y=(y^\sim\oplus x^\sim)^-=z^-$ for some $z\in I$, then $y^\sim,x^\sim\le y^\sim\oplus x^\sim=z$ which proves $x^\sim,y^\sim\in I$, a contradiction. In addition, we have $\phi(x\odot y)=\phi(x)\odot\phi(y)$.

Now let $x,y\in M_1$. We show $\phi(x\oplus y)=\phi(x)\oplus\phi(y)$. There are four cases: (1) $x,y\in I_1$, then clearly $\phi(x\oplus y)=\phi(x)\oplus\phi(y)$. (2) $x=x_0^-$ and $y=y_0^-$ for some $x_0,y_0\in I_1$. Then $\phi(x\oplus y)=\phi(x_0^-\oplus y_0^-)=\phi((y_0\odot x_0)^-)=(\phi(y_0\odot x_0))^-=(\phi(y_0)\odot\phi(x_0))^-=\phi(x_0)^-\oplus\phi(y_0)^-=\phi(x)\oplus\phi(y)$. (3) $x=x_0^-$ and $y=y_0$ for some $x_0,y_0\in I_1$. We get

$$\begin{aligned}\phi(x\oplus y)&=\phi(x_0^-\oplus y)=\phi((y^\sim\odot x_0)^-)=\phi((y\oslash x_0)^-)\\&=(\phi(y)\oslash\phi(x_0))^-=\phi(x)\oplus\phi(y).\end{aligned}$$

(4) $x=x_0$ and $y=y_0^-$. This case is dual to (3).

Summarizing, ϕ is a homomorphism of pseudo MV-algebras which is a unique extension of h onto M_1. Whence, $\Phi(\phi)=h$, $\phi(I_1)\subseteq I_2$, and Φ is a full functor. □

Proposition 3.2. *Let M be a proper wPEMV-algebra and $h_i:M\to M_i$ be a wPEMV-embedding of M into a wPEMV-algebra M_i with top element for $i=1,2$. Define*

$$M_i^0=\{x\in M_i:\ \text{either } x=h_i(x_0) \text{ or } x=h_i(x_0)^- \text{ for } x_0\in M\}$$

for $i=1,2$. Then $(M_1^0;\oplus,^-,^\sim,0,1)$ and $(M_2^0;\oplus,^-,^\sim,0,1)$ are isomorphic pseudo MV-algebras such that $(M_i^0,h_i(M))$ are objects in PPMV *for $i=1,2$.*

Proof. Let $h_i:M\to N_i$ be an embedding for $i=1,2$. Let us define $\phi:N_1^0\to N_2^0$ such that $\phi(x)=h_2(x_0)$ if $x=h_1(x_0)$ for $x_0\in M_1$ and $\phi(x)=(h_2(x_0))^-$ if $x=h_1(x_0)^-$ for $x_0\in M_1$. Then, due to the proof of the Proposition 3.1 that Φ is a full functor, we can prove that ϕ is a homomorphism of pseudo MV-algebras. In addition, ϕ is a bijection, so that it is an isomorphism. Clearly, $(N_i^0,h_i(M))\in$ PPMV for $i=1,2$. □

Proposition 3.3. *The functor Φ from the category* PPMV *into the category* PwPEMV *has a left-adjoint.*

Proof. We show that for a proper wPEMV-algebra M, there is a universal arrow $((N,I),f)$ i.e., (N,I) is an object in PPMV and f is a morphism from M into $\Phi(N,I)=I$ such that if (N',I') is an object from PwPEMV and f' is a morphism from M into $\Phi(N',I')$, then there exists a unique morphism $f^*:(N,I)\to(N',I')$ such that $\Phi(f^*)\circ f=f'$.

Indeed, by the Basic Representation Theorem 2.3 and Proposition 3.2, there are a unique (up to isomorphism) associated wPEMV-algebra N with top element and an injective wPEMV-homomorphism $f:M\to N$ such that $f(M)$ is a maximal and normal ideal of N with $f(M)\cup(f(M))^-=N$. We assert that $((N,I),f)$ is a universal arrow for M. Let (N',I') be an object from PEMV and let f' be a morphism from M into $\Phi(N',I')$. We can define a mapping $f^*:N\to N'$ such that $f^*(f(x)):=f'(x)$ if $x\in M$ and if $y\in N\setminus f(M)$, there is $y_0\in M$ such that $y=(f(y_0))^-$, and we set $f^*(y)=(f'(y_0))^-$. Then $f^*:N\to N'$ is a unique pseudo MV-homomorphism such that $\Phi(f^*)\circ f=f'$.

Define a mapping $\Psi:\mathsf{PwPEMV}\to\mathsf{PPMV}$ by $\Psi(M):=(N,I)$ whenever $((N,I),f)$ is a universal arrow for M and if $f':M\to M'$ is a wPEMV-homomorphism, there is a unique morphism $f^*:(N,I)\to(N',I')$, where $\Phi(N',I')=M'$, then we set $\Psi(f'):=f^*$. Using the Basic Representation Theorem, we have that Ψ is a left-adjoint functor of the functor Φ. ☐

Theorem 3.4. *The functor Φ defines a categorical equivalence of the category* PPMV *and the category of proper wPEMV-algebras* PwPEMV.

In addition, if $h:\Phi(N,I)\to\Phi(N',I')$ is a morphism of proper wPEMV-algebras, then there is a unique homomorphism $\phi:N\to N'$ of pseudo MV-algebras with $\phi(I)\subseteq I'$ such that $h=\Psi(\phi)$, and

(i) *if h is surjective, so is ϕ;*

(ii) *if h is injective, so is ϕ.*

Proof. According to [20, Thm IV.4.1 (i),(iii)], since Φ is faithful and full, by Proposition 3.1, and it has a left-adjoint, by Proposition 3.3, then it is necessary to show that, for any proper wPEMV-algebra M, there is an object (N,I) in PPMV such that $\Phi(N,I)$ is isomorphic to M. For this, it is sufficient to take any universal arrow $((N,I),f)$ of M. ☐

We introduce two subcategories of PwPEMV and of PPMV, respectively, corresponding to associated proper wPEMV-algebras. Thus, let PwPEMV_a be the set of associated proper wPEMV-algebras; it is equivalent to the set of proper pseudo EMV-algebras. In addition, let PwPEMV_a be the set of $(N,I)\in\mathsf{PwPEMV}$ such that I is a maximal and normal ideal of the pseudo MV-algebra N having enough idempotents, i.e. given $x\in I$, there is an idempotent $a\in I$ such that $x\le a$. Then PwPEMV_a and PPMV_a are proper subcategories of PwPEMV and PPMV, respectively. Theorem 3.4 entails the following categorical equivalence.

Corollary 3.5. *The subcategories PwPEMV_a and PPMV_a are categorically equivalent.*

Proof. Let G be a non-zero ℓ-group and define the pseudo MV-algebra $N=\Gamma(\mathbb{Z}\overrightarrow{\times}G,(1,0))$ with the unique maximal and normal ideal $I=\{(0,g)\colon g\in G^+\}$. Then $(N,I)\in\mathsf{PPMV}\setminus\mathsf{PPMV}_a$ and I defines a non-associated proper wPEMV-algebra. If Φ_a and Ψ_a denote the restrictions of Φ and Ψ onto PPMV and PwPEMV, respectively, then they are functors, and by Theorem 3.4, Φ_a defines a categorical equivalence in question. ☐

We note that the latter corollary gives also a categorical equivalence for proper pseudo EMV-algebras.

We introduce the following partial addition $+$ on every pseudo MV-algebra N: $x+y$ is defined iff $y\odot x=0$, and in such a case, $x+y:=x\oplus y$; if $N=\Gamma(G,u)$, then $+$ coincides with the group addition restricted to $[0,u]$. We say that a couple (G,f) is a *universal group* for a pseudo MV-algebra N if (i) f is a mapping from N into a po-group G which preserves partial addition $+$ on N such that $G=G^+-G^+$, $f(N)$ generates G^+ as a semigroup, (ii) for any group K and any $+$-preserving mapping $h:N\to K$, there is a group homomorphism $\phi:G\to K$ such that $h=\phi\circ f$. Due to [10, Thm 5.3], if $N\cong\Gamma(G,u)$, then (G,f) is a universal group for N, where f is an isomorphism $f:N\to\Gamma(G,u)$.

Now, we define a special category MNLUG of unital ℓ-groups whose objects are triples (G,u,I), where (G,u) is a unital ℓ-group and I is a fixed maximal and normal ℓ-ideal of G such that the ℓ-group generated by I and u is G. A morphism $f:(G_1,u_1,I_1)\to(G_2,u_2,I_2)$ is any homomorphism of unital ℓ-groups $f:G_1\to G_2$ such that $f(I_1)\subseteq I_2$.

Let us define a functor $\Gamma_I : \mathsf{MNLUG} \to \mathsf{PPMV}$ as follows: If (G,u,I) is an object of MNLUG, then

$$\Gamma_I(G,u,I) := (\Gamma(G,u), I \cap [0,u]),$$

and if f is a morphism from an object (G_1,u_1,I_1) into another object (G_2,u_2,I_2), then

$$\Gamma_I(f)(x) := f(x), \quad x \in \Gamma(G,u).$$

Proposition 3.6. Γ_I *is a well-defined functor that is faithful and full. Moreover, it has a left-adjoint.*

Proof. Clearly $\Gamma_I(G,u,I) = (\Gamma(G,u), I \cap [0,u]) \in \mathsf{PPMV}$. If $f : (G_1,u_1,I_1) \to (G_2,u_2,I_2)$ is a morphism, then the restriction of f onto $\Gamma(G_1,u_1)$ is in fact a homomorphism of pseudo MV-algebras with $f(I_1) \subseteq I_2$, so that $\Gamma_I(f)(I_1 \cap [0,u_1]) \subseteq I_2 \cap [0,u_2]$, and, Γ_I is a correctly defined functor.

Let f_1 and f_2 be two morphisms from (G_1,u_1,I_1) into (G_2,u_2,I_2) such that $\Gamma_I(f_1) = \Gamma_I(f_2)$. Then $f_1(x) = f_2(x)$ for each $x \in \Gamma(G_1,u_1)$, so that $f_1(x) = f_2(x)$ for each $x \in G_1$ and $f_1 = f_2$.

Now, let $\phi : \Gamma_I(G_1,u_1,I_1) = (\Gamma(G_1,u_1), I_1 \cap [0,u_1]) \to \Gamma_I(G_2,u_2,I_2) = (\Gamma(G_2,u_2), I_2 \cap [0,u_2])$ be a morphism. According to the proof of [10, Prop 6.1], we can uniquely extend ϕ to a homomorphism of unital ℓ-groups $f : G_1 \to G_2$. Applying [9, Thm 6.1], $I_i \cap [0,u_i]$ can be uniquely extended to the ℓ-ideal I_i, $i = 1,2$, we have $f(I_1) \subseteq I_2$ and f is a morphism from (G_1,u_1,I_1) into (G_2,u_2,I_2). Finally, we have established Γ_I is a full functor because $\Gamma_I(f) = \phi$.

To show that Φ_I has a left-adjoint, we show that for an object (N,I_0) in PPMV, there is a universal arrow $((G,u,I),f)$, i.e. (G,u,I) is an object from MNLUG and f is a morphism from (N,I_0) into $\Gamma_I(G,u,I) = (\Gamma(G,u), I \cap [0,u])$ such that if (G',u',I') is an object from MNLUG and f' is a morphism from (N,I_0) into $\Gamma_I(G',u',I') = (\Gamma(G',u'), I' \cap [0,u'])$, then there is a unique morphism $f^* : (G,u,I) \to (G',u,I')$ such that $\Gamma_I(f^*) \circ f = f'$.

Take the universal group (G,f) for the pseudo MV-algebra N. Then f is a pseudo MV-bijection from N onto $\Gamma(G,u)$. We assert that $((G,u,I),f)$ is a universal arrow for (N,I_0), where I is an ℓ-ideal of G generated by $f(I_0)$. Indeed, take an object (G',u',I') from MNLUG and let f' be a morphism from $(N,I_0) \to \Gamma_I(G',u',I') = (\Gamma(G',u'), I'_0)$, where $I'_0 = I' \cap [0,u']$. Since $f : M \to \Gamma(G,u) \subset G^+$ is a $+$-preserving mapping and $f' : M \to \Gamma(G',u') \subseteq G'$ is also a $+$-preserving mapping, then there is a unique homomorphism of unital ℓ-groups $f^* : G \to G'$ such that $f^* \circ f = f'$. First, if $x \in I_0$, then $f(x) \in I$ and thus $f^*(f(x)) = f'(x) \in I'_0$.

We recall that if I_0 is an ideal of $\Gamma(G,u)$, then the ℓ-ideal I of G generated by an ideal I_0 of $\Gamma(G,u)$ is $I = \{x \in G \colon \exists x_i, y_j \in I_0, x = x_1 + \cdots + x_n - y_1 - \cdots - y_m\}$.

Therefore, if $x \in I$ is arbitrary, $f'(x) \in I'$, so that f^* is also a morphism from (G,u,I) to $(G',u,'I')$, i.e. $((G,u,I),f)$ is the required universal arrow.

Define a mapping $\Xi_I : \mathsf{PPMV} \to \mathsf{MNLUG}$ by $\Xi_I(N,I_0) = (G,u,I)$ if $((G,u,I),f)$ is a universal arrow for (N,I_0) and I is a maximal ℓ-ideal of G generated by $f(I_0)$. If f' is a morphism from (N,I_0) into (N',I'_0), there is a unique morphism $f^* : (G,u,I) \to (G',u',I')$, where $N' \cong \Gamma(G',u')$ and I' is a maximal ℓ-ideal of G' generated by $f'(I'_0)$, consequently $\Xi_I(f') := f^*$. Therefore, Ξ_I is a left-adjoint of Γ_I. ☐

The last results entail the following categorical equivalence:

Theorem 3.7. *The categories* MNLUG*,* PPMV *and* PwPEMV *are mutually categorically equivalent.*

Proof. The categorical equivalence of PPMV and PwPEMV follows from Theorem 3.4, and using ideas analogous to the proof of Theorem 3.4, we can show that MNLUG and PPMV are categorical equivalent, too. ☐

Finally, let MNLUG_a be the set of objects $(G,u,I) \in \mathsf{MNLUG}$ such that, for each $x \in I \cap [0,u]$, there is an idempotent $a \in I \cap [0,u]$ with $x \leq a$. Then MNLUG_a is a proper subcategory of MNLUG_a and using Corollary 3.5 and ideas from proof of Corollary 3.5, we can establish the next corollary.

Corollary 3.8. *The categories* PvPEMV_a*,* PPMV_a *and* MNLUG_a *are mutually categorical equivalent.*

We note that the latter result gives also a categorical equivalences for the set of proper pseudo EMV-algebras because every proper associated wPEMV-algebra is in fact a proper pseudo EMV-algebra.

4 Conclusion

The class of proper weak pseudo EMV-algebras is the class of weak pseudo EMV-algebras that are not equivalent to pseudo MV-algebras. Therefore, we characterize the categorical equivalence of the category of proper wPEMV-algebras by a special category of pseudo MV-algebras as couples consisting of a pseudo MV-algebra and a fixed maximal and normal ideal such that every element of the pseudo MV-algebra belongs either to the ideal or is a complement of some element from the ideal, Theorem 3.4. An analogous result is given by a special class of unital ℓ-groups with a fixed maximal and normal ℓ-ideal, see Theorem 3.7. In addition, we characterize also a subcategory of associated proper wPEMV-algebras and we found for them categorical representations, see Corollary 3.5 and Corollary 3.8. As a by-product, we have obtained categorical equivalences for proper pseudo EMV-algebras.

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