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Ultra deductive systems and (nilpotent) Boolean elements in hoops

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"This paper is dedicated to Professor Antonio Di Nola on the occasion of his 75th birthday."

Abstract

In this paper, first we define the concept of nilpotent element on a hoop H, study some properties of them and investigate the relation with ultra deductive systems. Then by using this notion, we introduce cyclic hoops and prove that every cyclic hoop has a unique generator and is a local MV-algebra. In the follows, we introduce the notion of Boolean elements on hoops and investigate some of their properties and relation among Boolean elements with ultra deductive systems and nilpotent elements. Finally, we introduce a functor between the category of hoops and category of Boolean elements of them.

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1 Introduction

Non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered in a frame of residuated lattices [10]. For example, Hájek's BL (basic logic [12]), Lukasiewiczs MV (many-valued logic [9]) and MTL (monoidal t-norm based logic [14]) are determined by the class of BL-algebras, MV-algebras and MTL-algebras, respectively. All of these algebras have lattices with residuation as a common support set. Thus, it is very important to investigate properties of algebras with residuation. Hoops are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach in [9, 14]. In the last years, hoops theory was enriched with deep structure theorems(see [3, 9, 14]). Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops([3], Corollary 2.10) one obtains an elegant short proof of the completeness theorem for propositional basic logic(see [3], Theorem 3.8), introduced by Hájek in [12]. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops. The main example of BL-algebras in interval $[0, 1]$ endowed with the structure induced

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by a t-norm. In [5], the authors showed that there are relations among hoops and some of other logical algebras such as residuated lattices, MTL-algebras, BL-algebras, MV-algebras, BCK-algebras, equality algebras, EQ-algebras, R0-algebras, Hilbert algebras, Heyting algebras, Hertz algebras, lattice implication algebras and fuzzy implication algebras. The aim of this paper is to find that under what conditions hoops are equivalent to these logical algebras. For more study about hoops we suggest to study [1, 2, 5]

In this paper, we define the concept of order and nilpotent element of hoop H and we study some properties of them. Then by using this notion, we introduce cyclic hoops and prove that every cyclic hoop has a unique generator and is a local MV-algebra. Also, we introduce other notions such as dense and Boolean elements on hoops and investigate some of their properties and relation between them. Then by using the notion of Boolean element, we define a functor and prove some properties of hoop category.

2 Preliminaries

In this section, we will point out the concepts and conclusions that we will need throughout the article.

An algebraic structure $(H, \odot, \to, 1)$ is said to be a *hoop* if for all $x, y, z \in H$ the next conditions hold:
(H_1) $(H, \odot, 1)$ is a commutative monoid.
(H_2) $x \to x = 1$.
(H_3) $(x \odot y) \to z = x \to (y \to z)$.
(H_4) $x \odot (x \to y) = y \odot (y \to x)$.

Define a binary relation $\leq$ such that $x \leq y$ iff $x \to y = 1$ and $(H, \leq)$ is a poset. A *bounded hoop* is a hoop with the least element 0 such that, for all $x \in H$, $0 \leq x$. Consider $x^0 = 1$ and $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. In any bounded hoop, we can define the negation operation $\neg$ on H by, $\neg x = x \to 0$. We set $Mv(H) = \{x \in H \mid \neg(\neg x) = x\}$. If $Mv(H) = H$, then H has *double negation property*, or (DNP) for short(see [7, 8]).

A hoop H is said to be a $\vee$-hoop if the operation $\vee$ which is defined as $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$ be a join operation on H. It is easy to see that $\vee$-hoop is a distributive lattice (see [10]).

Note. From now on, in this paper we consider $(H, \odot, \to, 1)$ or H for short, as a bounded hoop.

Proposition 2.1. [10] *The following statements hold for any $x, y, z \in H$:*
(i) H is a meet-semilattice.
(ii) $x \odot y \leq x, y$ and for any $n \in \mathbb{N}$, $x^n \leq x$.
(iii) If $x \leq y$, then $z \to x \leq z \to y$, $y \to z \leq x \to z$ and $x \odot z \leq y \odot z$.
(iv) $x \to y \leq (z \to x) \to (z \to y)$ and $x \to y \leq (y \to z) \to (x \to z)$.
(v) If H is a $\vee$-hoop, then $(x \vee y)^n \to z = \bigvee\{(a_1 \odot a_2 \odot \cdots \odot a_n) \to z \mid a_i \in \{x, y\}\}$.
(vi) If H is a $\vee$-hoop, then $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$ and $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.
(vii) $x \leq \neg x \to y$, $\neg x \odot x = 0$, $\neg\neg\neg x = \neg x$ and $x \leq \neg\neg x$.
$(viii)$ $((x \to y) \to y) \to y = x \to y$.

Consider $\emptyset \neq F \subseteq H$. Then F is said to be a *deductive system* of H if $x, y \in F$, then $x \odot y \in F$ and if $x \leq y$ and $x \in F$, then $y \in F$.

All deductive systems of H showed by $\mathfrak{F}(H)$. Clearly, $1 \in F$ and F is *proper* if $F \neq H$. Obviously, $F \in \mathfrak{F}(H)$ iff $1 \in F$, and $x, x \to y \in F$ imply $y \in F$.

Let $F \in \mathfrak{F}(H)$. Define a relation $\sim_F$ on H as follows:

$$x \sim_F y \text{ iff } x \to y \in F \text{ and } y \to x \in F.$$

Then $\sim_F$ is a congruence relation on H. Consider $\frac{H}{F} = \{\frac{x}{F} \mid x \in H\}$. Then define the operations $\odot_F$ and $\to_F$ on $\frac{H}{F}$ as follows:

$$\frac{x}{F} \odot_F \frac{y}{F} = \frac{x \odot y}{F} \text{ and } \frac{x}{F} \to_F \frac{y}{F} = \frac{x \to y}{F}.$$

Then $(\frac{H}{F}, \odot_F, \to_F, \frac{1}{F})$ is a hoop.

If $X \subseteq H$, we denote by $\langle X \rangle$ the deductive system generated by X in H, that is $\langle X \rangle = \bigcap_{X \subseteq F} F$, where $F \in \mathfrak{F}(H)$. A description of $\langle X \rangle$ is easily obtained:

Proposition 2.2. [10] *Suppose $X \subseteq H$ and $F \in \mathfrak{F}(H)$. Then*

$$\begin{aligned} \langle X \rangle &= \{a \in H \mid x_1 \odot x_2 \odot \cdots \odot x_n \leq a \text{ for some } n \in \mathbb{N} \text{ and } x_1, \cdots, x_n \in X\} \\ &= \{a \in H \mid x_1 \to (x_2 \to (\cdots \to (x_n \to a) \cdots)) = 1 \text{ for some } n \in \mathbb{N} \text{ and } x_1, \cdots, x_n \in X\} \end{aligned}$$

In particular, for any element $x \in H$, we have

$$\langle x \rangle = \{a \in H \mid x^n \leq a \text{ for some } n \in \mathbb{N}\} = \{a \in H \mid x^n \to a = 1 \text{ for some } n \in \mathbb{N}\}.$$

Let $F \in \mathfrak{F}(H)$ and $x \in H$. Then

$$\langle F \cup \{x\} \rangle = \{a \in H \mid \exists\, n \in \mathbb{N},\ f \in F \text{ such that } f \odot x^n \leq a\} = \{a \in H \mid \exists\, n \in \mathbb{N} \text{ such that } x^n \to a \in F\}.$$

A proper deductive system U is called an *ultra deductive system* of H if U is the greatest deductive system of H which does not contain in any other proper deductive system of H. All ultra deductive systems of H are shown by $U(H)$. Let I be a non-empty subset of H. Then I is called an *ideal of H* if for any $x, y \in I$, $\neg x \to y \in I$, $x \leq y$ and $y \in I$ imply—- $x \in I$. Clearly, H and $\{0\}$ are the trivial ideals of H. The set of all ideals of H is denoted by $\mathcal{ID}(H)$. Also, I is called a *proper ideal* if I is an ideal of H such that $I \neq H$. Obviously, an ideal I is proper iff it is not containing 1. If H and K are two hoops, then $F : H \to K$ is a hoop homomorphism if for any $x, y \in H$, we have $f(x \to y) = f(x) \to f(y)$ and $f(x \odot y) = f(x) \odot f(y)$ (see [1]).

3 Nilpotent elements and ultra deductive systems in hoops

In the following, we define the concept of order and nilpotent element on H and we study some properties of them. Then we investigate relation between nilpotent elements and ultra deductive systems. Specially, we introduce a cyclic hoop and we prove that every cyclic hoop has a unique generator and is a local MV-algebra.

Definition 3.1. *A hoop H is said to be a* simple hoop *if $\mathfrak{F}(H) = \{\{1\}, H\}$.*

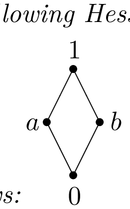
Example 3.2. (i) *Let $H = \{0, a, b, 1\}$ be a chain. Define the operations $\odot$ and $\to$ on H as follows:*

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

Then $(H, \odot, \to, 0, 1)$ is a simple hoop.

(ii) *Let $H = \{0, a, b, 1\}$ be a set with the following Hesse diagram.*



Define the operations $\odot$ and $\to$ on H as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Then $(H, \odot, \to, 0, 1)$ is a bounded hoop, where $\mathfrak{F}(H) = \{\{1\}, \{a, 1\}, \{b, 1\}, H\}$. Clearly, H is not simple.

Proposition 3.3. *Let U be a proper deductive system of H. Then the following statements are equivalent:*
(i) $U \in U(H)$,
(ii) $\frac{H}{U}$ is simple,
(iii) $x \in H \setminus U$ iff there exists $n \in \mathbb{N}$ such that $\neg(x^n) \in U$.

Proof. $(i \Rightarrow ii)$ Since $U \in U(H)$, we get $\frac{H}{U}$ is a non-obvious hoop. In addition, we know that there exists one-to-one corresponding relation between $\mathfrak{F}(H)$ and $\mathfrak{F}(\frac{H}{U})$ that containing U, we get that if $\frac{G}{U} \in \mathfrak{F}(\frac{H}{U})$, then $U \subsetneq G \subsetneq H$, a contradiction. Hence $\frac{H}{U}$ is simple.
$(ii \Rightarrow i)$ Suppose $G \in \mathfrak{F}(H)$ such that $U \subseteq G \subseteq H$. Thus $\frac{G}{U} \in \mathfrak{F}(\frac{H}{U})$. Since $\frac{H}{U}$ is simple, we have $\frac{G}{U} = \frac{1}{U}$ or $\frac{G}{U} = \frac{H}{U}$. It means $G = U$ or $G = U$, and so $U \in U(H)$.
$(i \Rightarrow iii)$ Let $x \in H \setminus U$. Since $U \in U(H)$, clearly $\langle U \cup \{x\} \rangle = H$, and so $0 \in \langle U \cup \{x\} \rangle$. Thus $x^n \to 0 \in U$. Hence $\neg(x^n) \in U$.

Conversely, if $x \in U$ and $\neg(x^n) \in U$, since $U \in \mathfrak{F}(H)$, then we have $0 \in U$, which is a contradiction. Hence, $x \notin U$.
$(iii \Rightarrow i)$ Consider $G \in \mathfrak{F}(H)$ such that $U \subset G \subseteq H$. Since $U \neq G$, there is $x \in G \setminus U$ such that $\neg(x^n) \in U$. Thus $\neg(x^n) \in G$. Since $x \in G$ and $G \in \mathfrak{F}(H)$, we get $0 \in G$, and so $G = H$. Therefore, $U \in U(H)$. □

Proposition 3.4. *For any proper deductive system F of H, there exists $U \in U(H)$ that contains F.*

Proof. Consider $\sum = \{P \in \mathfrak{F}(H) \mid P \neq H$ such that $F \subseteq P\}$. Since $F \in \sum$, we get $\sum \neq \emptyset$. By the simple way, we can see that any chain of elements in $(\sum, \subseteq)$ has a maximal in it. Hence, using Zorn's Lemma, there exists a maximal element $U \in \sum$ and it is easy to see that it is an ultra deductive system of H containing F. □

Definition 3.5. *A hoop H is called* local *if it has just one ultra deductive system. Obviously, any simple hoop is local.*

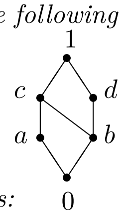
Example 3.6. *Let H be the hoop as in Example 3.2(i). Then $\{1\}$ is just an ultra deductive system of H.*

Proposition 3.7. *Every chain hoop is local.*

Proof. Suppose H is not local. Then there exist $U_1, U_2 \in U(H)$ such that $U_1 \neq U_2$. Thus there are $x \in U_1 \setminus U_2$ and $y \in U_2 \setminus U_1$. Since H is a chain, we have $x \leq y$ or $y \leq x$. If $x \leq y$, then from $U_1 \in \mathfrak{F}(H)$ and $x \in U_1$, we have $y \in U_1$, is a contradiction. By the similar way, if $y \leq x$, then $x \in U_2$, a contradiction. Hence $U_1 = U_2$, and so H is local. □

Definition 3.8. *If there exists the smallest $n \in \mathbb{N}$ such that $x^n = 0$, then n is called* order of *x and showed by $O(x)$ and x is called a* nilpotent element of *H. If for any $n \in \mathbb{N}$, $x^n \neq 0$, then $O(x) = \infty$. The set of all nilpotent elements of H is denoted by $Nil(H)$ and $Inf(H) = \{x \in H \mid O(x) = \infty\}$.*

Example 3.9. *Let $H = \{0, a, b, c, d, 1\}$ with the following Hesse diagram.*



Define the operations $\odot$ and $\to$ on H as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Then $(H, \odot, \to, 0, 1)$ is a bounded hoop. Then $Inf(H) = \{1, a, c, d\}$ and $Nil(H) = \{0, b\}$.

Notation. Since $1 \notin Nil(H)$, clearly $Nil(H)$ is not a deductive system of H.

Suppose $(X, \leq)$ is a lattice. A non-empty subset I of X is called a *lattice ideal* of X if for any $x, y \in X$, $x \leq y$ and $y \in I$ imply $x \in I$, and for any $x, y \in I$, $x \vee y \in I$.

Proposition 3.10. *The set $Nil(H)$ is a lattice ideal of H, where H is a bounded $\vee$-hoop.*

Proof. Clearly $0 \in Nil(H)$. Consider $y \in Nil(H)$ and $x \in H$ such that $x \leq y$. Then there exists $n \in \mathbb{N}$ such that $y^n = 0$. Since $x \leq y$, we have $x^n \leq y^n$ and so $x^n = 0$. Hence, $x \in Nil(H)$. Now, if $x, y \in Nil(H)$, then there are $n, m \in \mathbb{N}$ such that $x^n = y^m = 0$. Thus by Proposition 2.1(v), we have

$$(x \vee y)^{n+m} \to 0 = \bigwedge \{(a_1 \odot a_2 \odot \cdots a_k) \to 0 \mid \ a_i \in \{x, y\}\}.$$

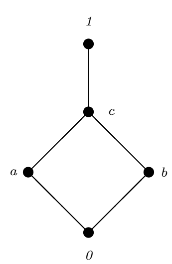
Clearly, if $i > n$, then $x^i = 0$, and so $(a_1 \odot a_2 \odot \cdots a_k) \to 0 = 1$, it means that $a_1 \odot a_2 \odot \cdots a_k = 0$. If $i \leq n$, then $j > m + n - i > m$, and so $y^j = 0$. Thus $a_1 \odot a_2 \odot \cdots a_k = 0$. Hence $(x \vee y)^{n+m} \to 0 = 1$, and so $(x \vee y)^{n+m} = 0$. Then $x \vee y \in Nil(H)$. Therefore, $Nil(H)$ is a lattice ideal of H. □

Next example shows that $Nil(H) \notin \mathcal{ID}(H)$.

Example 3.11. *Let H be a hoop as in Example 3.2(ii). Clearly $Nil(H) = \{0, a, b\}$ but it is not an ideal of H since $\neg b \to a = a \to a = 1 \notin Nil(H)$.*

Next example shows that $Inf(H) \notin \mathfrak{F}(H)$.

Example 3.12. *Let $A = \{0, a, b, c, 1\}$. Define the operations $\odot$ and $\to$ on H as follows,*



$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

Thus, $(H, \odot, \to, 0, 1)$ is a hoop. Since $Inf(H) = \{a, b, c, 1\}$ and $a \odot b = 0$, we consequence $Inf(H) \notin \mathfrak{F}(H)$.

Proposition 3.13. *Assume $U \in U(H)$. Then $U \subseteq Inf(H)$.*

Proof. Suppose $U \nsubseteq Inf(H)$. Then there is $x \in U \setminus Inf(H)$. Since $x \notin Inf(H)$, we get that there exists $n \in \mathbb{N}$ such that $x^n = 0$. As $U \in U(H)$, we have $0 \in U$, which is a contradiction. Therefore, $U \subseteq Inf(H)$. □

Next example shows that every subset of $Inf(H)$ is not an ultra deductive system of H, in general.

Example 3.14. *According to Example 3.12, since $Inf(H) = \{a, b, c, 1\} \notin \mathfrak{F}(H)$, we obtain $Inf(H) \notin U(H)$.*

Proposition 3.15. *Consider $U \in U(H)$. Then $U = Inf(H)$ iff for any $x, y \in H$, $x \odot y \in Nil(H)$ implies $x \in Nil(H)$ or $y \in Nil(H)$.*

Proof. Suppose $U = Inf(H)$ and for any $x, y \in H$, $x \odot y \in Nil(H)$. Thus $x \odot y \notin U$. Since $U \in U(H)$, we have $x \notin U$ or $y \notin U$. Because if $x, y \in U$, then $x \odot y \in U$, which is a contradiction. Hence, $x \notin Inf(H)$ or $y \notin Inf(H)$. Therefore, $x \in Nil(H)$ or $y \in Nil(H)$.

Conversely, by Proposition 3.13, clearly $U \subseteq Inf(H)$. Now, we prove $Inf(H)$ is a proper deductive system of H. For this, since $O(1) = \infty$, we get $1 \in Inf(H)$ and so $Inf(H) \neq \emptyset$. Assume $x \in Inf(H)$ and

$y \in H$ such that $x \leq y$. If $y \notin Inf(H)$, then there is $n \in \mathbb{N}$ such that $y^n = 0$ and from $x^n \leq y^n$ we obtain $x^n = 0$, which is a contradiction. Hence, $y \in Inf(H)$. Now, suppose $x, y \in Inf(H)$. If $x \odot y \notin Inf(H)$, then $x \odot y \in Nil(H)$ and by assumption, $x \in Nil(H)$ or $y \in Nil(H)$, a contradiction. Hence, $x \odot y \in Inf(H)$, and so $Inf(H) \in \mathfrak{F}(H)$. Moreover, from $0 \notin Inf(H)$, we get $Inf(H)$ is proper. Also, $U \in U(H)$ such that $U \subseteq Inf(H)$. Thus $U = Inf(H)$. □

Proposition 3.16. *A hoop H is simple iff for any $x \in H \setminus \{1\}$, $x \in Nil(H)$.*

Proof. (⇒) Consider $x \in H \setminus \{1\}$ such that $O(x) = \infty$. Let $F = \langle x \rangle$. Since $x \neq 1$, we get $\{1\} \subset F$. If $F = H$, then $0 \in F$, and so for $n \in \mathbb{N}$, $O(x) = n$, which is a contradiction. Hence, $\{1\} \subset F \subset H$. Thus $F \in \mathfrak{F}(H)$, a contradiction. Hence, for any $x \in H$, $x \in Nil(H)$.
(⇐) Suppose H is not simple. Then there is $F \in \mathfrak{F}(H)$ such that $\{1\} \subset F \subset H$. Consider $x \in F$ such that $\langle x \rangle = F$. Since $F \neq H$, we get $0 \notin F$, and so for any $n \in \mathbb{N}$, $x^n \neq 0$. Hence, $O(x) = \infty$, which is a contradiction. Therefore, H is simple. □

Proposition 3.17. *Suppose $\{H_i \mid i \in I\}$ is a family of hoops. Then*
(i) $x \in Nil(H)$ iff $\langle x \rangle = H$.
(ii) $Nil(\prod_{i\in I} H_i) = \prod_{i\in I} Nil(H_i)$.

Proof. (i) Suppose $x \in Nil(H)$ iff $O(x) = n$, for $n \in \mathbb{N}$ iff $x^n = 0$ iff $0 \in \langle x \rangle$ iff $\langle x \rangle = H$.
(ii) We set $\overline{x} = (x_1, x_2, \cdots, x_i, \cdots)$. Then $\overline{x} \in Nil(\prod_{i\in I} H_i)$ iff there is $n \in \mathbb{N}$ such that $\overline{x}^n = \overline{0}$ iff $(x_1^n, x_2^n, \cdots, x_i^n, \cdots) = (0, 0, \cdots, 0, \cdots)$ iff for any $i \in I$, $x_i^n = 0$ iff for any $i \in I$, $x_i^n \in Nil(H_i)$ iff $\overline{x} \in \prod_{i\in I} Nil(H_i)$. □

Definition 3.18. *Suppose H is finite. If there is an element $x \in H$ such that $O(x) = \mid H \mid -1$, then H is said to be* cyclic *and x is a* generator of *H.*

Example 3.19. (*i*) *Every non-zero subalgebra of a cyclic hoop is cyclic.*
(*ii*) *Let H be a hoop as in Example 3.2(ii). Then $Nil(H) = \{0, a, b\}$ and $O(b) = 3 = \mid H \mid -1$. Hence, b is a generator of H and H is cyclic.*

Theorem 3.20. *Consider H is cyclic such that $\mid H \mid= n + 1$. Then*
(*i*) *there is $x \in H$ such that $O(x) = n$ and $H = \{x^i \to 0 \mid 0 \leq i \leq n\}$.*
(*ii*) *H is a chain.*
(*iii*) *the generator is the greatest element of $H \setminus \{1\}$.*
(*iv*) *the generator of H is unique.*
(*v*) *H has (DNP).*

Proof. (i) Since H is cyclic, by Definition 3.18, we get H is finite and there is an element $x \in H$ such that $O(x) = \mid H \mid -1 = n+1-1 = n$. Set $K = \{x^i \to 0 \mid 0 \leq i \leq n\}$. From $O(x) = n$, we obtain $x^0 \to 0 = 0$ and $x^n \to 0 = 1$, thus $0, 1 \in K$. Now, we prove that every both members of K are distinct. For this, suppose $x^i \to 0, x^j \to 0 \in K$, for any $1 \leq i, j \leq n-1$. If $i < j$ and $x^i \to 0 = x^j \to 0$, then

$$\begin{aligned}1 = x^n \to 0 = x^{n-j+j} \to 0 &= (x^{n-j} \odot x^j) \to 0 = x^{n-j} \to (x^j \to 0)\\ &= x^{n-j} \to (x^i \to 0) = (x^{n-j} \odot x^i) \to 0\\ &= x^{n-j+i} \to 0.\end{aligned}$$

Since $i < j$, we have $n - j + i < n$ and so $x^{n-j+i} = 0$, which is a contradiction with $O(x) = n$. Hence, $x^i \to 0 \neq x^j \to 0$, for any $1 \leq i, j \leq n-1$. Also, obviously $\mid K \mid= n+1$. Since $K \subseteq H$ and $\mid K \mid=\mid H \mid$, we have $K = H$.
(ii) By (i), $H = \{x^i \to 0 \mid 0 \leq i \leq n\}$. Suppose $a, b \in H$. Then there are $0 \leq i, j \leq n$ such that $a = x^i \to 0$ and $b = x^j \to 0$. With out loss of generality, suppose $j \leq i$. Then $x^i \leq x^j$, by Proposition 2.1(iii), $x^j \to 0 \leq x^i \to 0$, and so $b \leq a$. By the similar way, if $i \leq j$, then $a \leq b$. Hence, H is a chain.

(iii) By (i), $H = \{x^i \to 0 \mid 0 \leq i \leq n\}$. Since $x \neq 1$ is a generator of H, we have $x \in \{x^i \to 0 \mid 0 \leq i \leq n\}$. Thus there is $1 \leq i \leq n$ such that $x = x^i \to 0$. Hence

$$x^{i+1} \to 0 = x \to (x^i \to 0) = x \to x = 1$$

and so $x^{i+1} = 0$. From $O(x) = n$, we have $n \leq i+1$, and so $n-1 \leq i$. If $i = n$, then $x = x^n \to 0 = 1$, is a contradiction. Thus $i = n-1$ and so $x = x^{n-1} \to 0$. Therefore, the generator is the greatest element of $H \setminus \{1\}$.
(iv) Suppose there are two generators for H. By (ii), H is a chain, so $x \leq y$ or $y \leq x$. In addition, by (iii), the generator is the greatest element of $H \setminus \{1\}$. Thus $x = y$.
(v) Consider $a \in H$. By (i), for $0 \leq i \leq n$, $a = x^i \to 0$. Then by Proposition 2.1(viii) we have

$$a = x^i \to 0 = ((x^i \to 0) \to 0) \to 0 = \neg(\neg a).$$

Hence, H has (DNP). □

Corollary 3.21. *Every cyclic hoop is an MV-algebra.*

Proof. By Theorem 3.20(v) and [5, Theorem 3.12 and Corollary 3.13] the proof is clear. □

Next example shows that the converse of above theorem does not hold.

Example 3.22. *(i) Let H be a hoop as in Example 3.2(i). Clearly H has DNP property but H is not cyclic.*
(ii) Let H be a hoop as in Example 3.2(ii). This example confirm Theorem 3.20.
(iii) Let $H = \{0, a, b, 1\}$ be a chain. Define the operations $\odot$ and $\to$ on H as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

Then $(H, \odot, \to, 0, 1)$ is a bounded hoop. Obviously H is generated by 0 that is not the greatest element of H. Hence, H is not cyclic.

Corollary 3.23. *If H is a cyclic hoop such that $\mid H \mid = n+1$, then H is a local hoop.*

Proof. By Theorem 3.20, H is a chain and by Proposition 3.7, H is local. □

Note. Define $Ds(H) = \{x \in H \mid \neg x = 0\}$. Since $\neg 0 = 1$, obviously, we get $0 \notin Ds(H)$ and so $Ds(H) \notin \mathcal{ID}(H)$.

Example 3.24. *Let H be the hoop as in Example 3.22(iii). Then $Ds(H) = \{a, b, 1\}$.*

Proposition 3.25. *The set $Ds(H)$ is a deductive system of H.*

Proof. Clearly $1 \in Ds(H)$. Consider $x, y \in Ds(H)$. Then $\neg x = \neg y = 0$. Thus

$$\neg(x \odot y) = x \to \neg y = x \to 0 = 0,$$

and so $x \odot y \in Ds(H)$. If $x \in Ds(H)$ and $x \leq y$, since $\neg y \leq \neg x = 0$, we have $\neg y = 0$. Hence $y \in Ds(H)$. Therefore, $Ds(H) \in \mathfrak{F}(H)$. □

Proposition 3.26. *If $F \subseteq Ds(H)$ and F is a proper deductive system of H, then $Ds(\frac{H}{F}) = \frac{Ds(H)}{F}$*

Proof. Consider $x \in H$. Then $\frac{x}{F} \in \frac{Ds(H)}{F}$ iff $x \in Ds(H)$ iff $\neg x = 0$ iff $\neg\neg x = 1 \in F$ iff $0 \to \neg x \in F$ and $\neg x \to 0 \in F$ iff $\frac{\neg x}{F} = \frac{0}{F}$ iff $\neg\left(\frac{x}{F}\right) = \frac{0}{F}$ iff $\frac{x}{F} \in Ds(\frac{H}{F})$. □

Proposition 3.27. *The following statements are equivalent:*
(i) $\frac{H}{Ds(H)}$ *implies (DNP) property.*
(ii) For any $x \in H$, $\neg\neg(\neg\neg x \to x) = 1$.

Proof. $(i \Rightarrow ii)$ Since $\frac{H}{Ds(H)}$ implies (DNP) property, we get that for any $\frac{x}{Ds(H)} \in \frac{H}{Ds(H)}$, we have $\neg\neg(\frac{x}{Ds(H)}) = \frac{x}{Ds(H)}$. Thus $x \to \neg\neg x = 1 \in Ds(H)$ and $\neg\neg x \to x \in Ds(H)$. Then $\neg(\neg\neg x \to x) = 0$, and so $\neg\neg(\neg\neg x \to x) = 1$.
$(ii \to i)$ Since for any $x \in H$, $\neg\neg(\neg\neg x \to x) = 1$, we get $\neg(\neg\neg x \to x) = \neg\neg\neg(\neg\neg x \to x) = 0$, and so $\neg\neg x \to x \in Ds(H)$. Also, from $x \to \neg\neg x = 1$, we obtain $x \to \neg\neg x \in Ds(H)$. Thus $\neg\neg(\frac{x}{Ds(H)}) = \frac{x}{Ds(H)}$. Therefore, $\frac{H}{Ds(H)}$ implies (DNP) property. □

Note. Define $R(H) = \bigcap_{U \in U(H)} U$.

Example 3.28. *Let H be a hoop as in Example 3.2(ii). Then* $R(H) = \{a,1\} \cap \{b,1\} = \{1\}$.

Proposition 3.29. $R(H) = \{x \in H \mid \forall n \in \mathbb{N},\ \exists\ k \in \mathbb{N}\ s.t.\ (\neg x^n)^k = 0\}$.

Proof. Let $B = \{x \in H \mid \forall n \in \mathbb{N},\ \exists\ k \in \mathbb{N}\ s.t.\ (\neg x^n)^k = 0\}$. Suppose $x \in B$ such that $x \notin R(H)$. Since $x \notin R(H)$, we get that there exists $U \in U(H)$ such that $x \notin U$. By Proposition 3.3(iii), there is $n \in \mathbb{N}$ such that $\neg x^n \in U$. Since $U \in \mathfrak{F}(H)$, for any $k \in \mathbb{N}$, $(\neg x^n)^k \in U$. On the other side, from $x \in B$ we have $(\neg x^n)^k = 0$, and so $0 \in U$, which is a contradiction. Hence, $B \subseteq R(H)$.

Conversely, suppose $x \in R(H)$ such that $x \notin B$. Then for any $U \in U(H)$, $x \in U$. Since $U \in \mathfrak{F}(H)$, for any $n \in \mathbb{N}$, $x^n \in U$, and so $\neg(x^n) \notin U$. In addition, $x \notin B$, then there exists $n \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $(\neg x^n)^k \neq 0$. So the generated deductive system that is made by $(\neg x^n)^k$ is proper. By Proposition 3.4, there exists $U \in U(H)$ such that $\langle(\neg x^n)^k\rangle \subseteq U$, which is a contradiction. Hence, $R(H) \subseteq B$. Therefore,

$$R(H) = \{x \in H \mid \forall n \in \mathbb{N},\ \exists\ k \in \mathbb{N}\ s.t.\ (\neg x^n)^k = 0\}.$$

□

Proposition 3.30. *The next statements hold:*
(i) $Ds(H) \subseteq R(H)$.
(ii) $\frac{x}{Ds(H)} \in \frac{U}{Ds(H)}$ *if and only if* $x \in U$.
(iii) $U(\frac{H}{Ds(H)}) = \{\frac{U}{Ds(H)} \mid U \in U(H)\}$.
(iv) $R(\frac{H}{Ds(H)}) = \frac{R(H)}{Ds(H)}$.
(v) $\frac{x}{Ds(H)} \in \frac{R(H)}{Ds(H)}$ *if and only if* $x \in R(H)$.
(vi) $U(H)$ *and* $U(\frac{H}{Ds(H)})$ *are homeomorphic topological space.*
(vii) if $x \in R(H)$, *then* $\neg\neg x \in R(H)$.

Proof. (i) Suppose $Ds(H) \nsubseteq R(H)$. Then there exists $x \in Ds(H)$ such that $\neg x = 0$ and $x \notin R(H)$. Thus there is $U \in U(H)$ such that $x \notin U$. By Proposition 3.3(iii), there is $n \in \mathbb{N}$ such that $\neg(x^n) \in U$. Hence

$$x^n \to 0 = x^{n-1} \to (x \to 0) = x^{n-1} \to \neg x = x^{n-1} \to 0 = \cdots = x \to 0 = 0.$$

Thus $0 \in U$, which is a contradiction. Therefore, $Ds(H) \subseteq R(H)$.
(ii) Suppose $\frac{x}{Ds(H)} \in \frac{U}{Ds(H)}$. Then there is $y \in U$ such that $\frac{x}{Ds(H)} = \frac{y}{Ds(H)}$, and so $x \to y, y \to x \in Ds(H)$. By (i), $Ds(H) \subseteq R(H) \subseteq U$. From $y \to x \in U$, $U \in \mathfrak{F}(H)$ and $y \in U$, we get $x \in U$. The proof of converse is obvious.
(iii) Assume $U \in \mathfrak{F}(H)$ such that $Ds(H) \subseteq U$. Then $U \in U(H)$ iff for any deductive system of H such as F such that $U \subseteq F \subseteq H$, we have $U = F$ or $H = F$ iff $\frac{U}{Ds(H)} = \frac{F}{Ds(H)}$ or $\frac{H}{Ds(H)} = \frac{F}{Ds(H)}$ iff $\frac{U}{Ds(H)} \in U(\frac{H}{Ds(H)})$.
(iv)

$$R(\frac{H}{Ds(H)}) = \bigcap_{i \in I}\left(\frac{U_i}{Ds(H)}\right) = \frac{\bigcap_{i \in I} U_i}{Ds(H)} = \frac{R(H)}{Ds(H)}.$$

(v) By (iii) and (iv), we have $\frac{x}{Ds(H)} \in \frac{R(H)}{Ds(H)} = R(\frac{H}{Ds(H)})$ iff by (iv) for any $\frac{U}{Ds(H)} \in U\left(\frac{H}{Ds(H)}\right)$, $\frac{x}{Ds(H)} \in \frac{U}{Ds(H)}$ iff for any $U \in U(H)$, $x \in U$ if and only if $x \in R(H)$.
(vi) Define $\Omega : U(H) \to U\left(\frac{H}{Ds(H)}\right)$ where $\Omega(U) = \frac{U}{Ds(H)}$. Clearly Ω is an epimorphism. Consider $U_1, U_2 \in U(H)$. Then $\Omega(U_1) = \Omega(U_2)$ iff $\frac{U_1}{Ds(H)} = \frac{U_2}{Ds(H)}$ iff for any $x \in U_1$, $\frac{x}{Ds(H)} \in \frac{U_1}{Ds(H)}$ iff there is $y \in U_2$ such that $\frac{x}{Ds(H)} = \frac{y}{Ds(H)}$ iff by (i), $x \to y, y \to x \in Ds(H) \subseteq R(H) \subseteq U_2$. Then $y \to x \in U_2$. Since $U_2 \in \mathfrak{F}(H)$ and $y \in U_2$, we obtain $x \in U_2$ and so $U_1 \subseteq U_2$. In addition, from $U_1, U_2 \in U(H)$, we have $U_1 = U_2$. Hence, Ω is an isomorphism. For proving being topological space we use the definition of Zarisky topology. It means that the open sets of this topology are like $T(a) = \{U \in \mathrm{Max}(H) \mid a \notin U\}$. Thus

$$\begin{aligned} S_{\text{Ultra}}(\frac{x}{Ds(H)}) &= \{\frac{U}{Ds(H)} \mid U \in U(H),\ \frac{x}{Ds(H)} \notin \frac{U}{Ds(H)}\} \\ &= \{\frac{U}{Ds(H)} \mid U \in U(H),\ x \notin U\} \\ &= \{\frac{U}{Ds(H)} \mid U \in S_{\text{Ultra}}(x)\} \\ &= \{\Omega(U) \mid U \in S_{\text{Ultra}}(x)\}. \end{aligned}$$

Thus the image of any open set is an open set. Clearly Ω is surjective and so $\Omega^{-1}(\frac{U}{Ds(H)}) = U$ or $\Omega^{-1}(S_{\text{Ultra}}(\frac{x}{Ds(H)})) = S_{\text{Ultra}}(x)$. Therefore, $U(H)$ and $U(\frac{H}{Ds(H)})$ are homeomorphic topological space.
(vii) The proof is straightforward. □

In the following proposition we investigate some conditions for finding the relation between $R(H)$, $Nil(H)$ and $Inf(H)$.

Proposition 3.31. *(i) $R(H) = Inf(H)$ iff for any $x, y \in H$, $x \odot y \in Nil(H)$ implies $x \in Nil(H)$ or $y \in Nil(H)$.*
(ii) If H is a chain, then $x \notin R(H)$ iff $x \in Nil(H)$.
(iii) If H is a chain, then $R(H) = Inf(H)$.
(iv) If H is a chain, then $x \in Nil(H)$ iff $\frac{x}{R(H)} \in Nil(\frac{H}{R(H)})$.
(v) $\neg(x^n) \in Nil(H)$ iff $x \notin Nil(H)$.
(vi) If H is a chain, then $\neg(x^n) \in Nil(H)$ implies $x \in R(H)$.
(vii) $Ds(H) \subseteq \{x \in H \mid \neg(x^n) \in Nil(H)\}$.
(viii) If H is a chain and $\neg(x^n) \in Nil(H)$, then $\neg x < x$.

Proof. (i) By Proposition 3.15, the proof is clear.
(ii) If $x \in Nil(H)$ and $x \in R(H)$, then there is $n \in \mathbb{N}$ such that $x^n = 0$ and $x^n \in R(H)$, which is a contradiction. Thus $x \notin R(H)$. Now, suppose $x \notin R(H)$. Then there is $U \in U(H)$ such that $x \notin U$. Then by Proposition 3.3(iii), there is $n \in \mathbb{N}$ such that $\neg(x^n) \in U$. Since H is a chain, we have $x \leq \neg(x^n)$ or $\neg(x^n) \leq x$. If $\neg(x^n) \leq x$, then since $\neg(x^n) \in U$, we get $x \in U$, a contradiction. Thus $x \leq \neg(x^n)$, and so $x^{n+1} = 0$. Hence, $x \in Nil(H)$.
(iii) By Proposition 3.13, obviously $R(H) \subseteq Inf(H)$. Assume $x \in Inf(H)$ where $x \notin R(H)$. As H is a chain, by (ii), $x \in Nil(H)$, a contradiction. Hence, $Inf(H) \subseteq R(H)$, and so $R(H) = Inf(H)$.
(iv) Consider $x \in Nil(H)$. Then there is $n \in \mathbb{N}$ such that $x^n = 0$. Thus $0 \to x^n = x^n \to 0 = 1 \in R(H)$ and so $\left(\frac{x}{R(H)}\right)^n = \frac{x^n}{R(H)} = \frac{0}{R(H)}$. Hence, $\frac{x}{R(H)} \in Nil(\frac{H}{R(H)})$.

Conversely, assume $\frac{x}{R(H)} \in Nil(\frac{H}{R(H)})$. Then there is $m \in \mathbb{N}$ such that $\frac{x^m}{R(H)} = \frac{0}{R(H)}$. Thus $\neg(x^m) \in R(H)$. If $x \in R(H)$, then for any $m \in \mathbb{N}$, $x^m \in R(H)$, and so $0 \in R(H)$, a contradiction, since $R(H) \in \mathfrak{F}(H)$. Thus $x \notin R(H)$, and by (ii) $x \in Nil(H)$.
(v) ($\Rightarrow$) Suppose $\neg(x^n) \in Nil(H)$. Then there exists $m \in \mathbb{N}$ such that $(\neg(x^n))^m = 0$. If $x \in Nil(H)$, then there is $n \in \mathbb{N}$ where $x^n = 0$ and $\neg(x^n) = 1$. Thus for any $m \in \mathbb{N}$, we have $(\neg(x^n))^m = 1 \neq 0$, which is a contradiction. Hence, $x \notin Nil(H)$. The proof of other side is similar.
(vi) Consider H is a chain such that $\neg(x^n) \in Nil(H)$. Then by (v) we get $x \notin Nil(H)$. Thus by (ii), we obtain $x \in R(H)$.

(vii) Assume $x \in Ds(H)$. Then $\neg x = 0$. We prove that for $m \in \mathbb{N}$, $(\neg(x^n))^m = 0$. For this, since $\neg x = 0$, we have

$$\neg(x^n) = x^n \to 0 = x^{n-1} \to (x \to 0) = x^{n-1} \to 0 = \cdots = x \to 0 = \neg x = 0. \tag{1}$$

Thus by (1), we consequence

$$\begin{aligned}(\neg(x^n))^m \to 0 &= \big((\neg(x^n))^{m-1} \odot (\neg(x^n))\big) \to 0 = (\neg(x^n))^{m-1} \to ((\neg(x^n)) \to 0)\\ &= (\neg(x^n))^m \to (0 \to 0) = (\neg(x^n))^m \to 1\\ &= 1.\end{aligned}$$

Thus $(\neg(x^n))^m = 0$. Hence, $\neg(x^n) \in Nil(H)$.
(viii) Since H is a chain, we have $x \leq \neg x$ or $\neg x \leq x$. If $x \leq \neg x$, since $\neg(x^n) \in Nil(H)$, by (v) we obtain $x \notin Nil(H)$, then $1 = x \to \neg x = x^2 \to 0$. Thus $x^2 = 0$, and so $x \in Nil(H)$ which is a contradiction. Hence, $\neg x \leq x$. Now, if $x = \neg x$, then

$$1 = x \to \neg x = x \to (x \to 0) = x^2 \to 0.$$

Thus $x^2 = 0$ and so $x \in Nil(H)$. By (v) we have $\neg(x^n) \notin Nil(H)$, a contradiction. Therefore, $\neg x < x$. □

Theorem 3.32. *H is local iff $H = Nil(H) \cup R(H)$ iff for any $x, y \in H$, $x \odot y \in Nil(H)$ implies $x \in Nil(H)$ or $y \in Nil(H)$.*

Proof. Suppose $x \in H$. Then $O(x) \leq \infty$ or $O(x) = \infty$. If $O(x) \leq \infty$, then $x \in Nil(H)$. If $O(x) = \infty$, then $x \in Inf(H)$. By Proposition 3.31(i), $R(H) = Inf(H)$ iff for any $x, y \in H$, $x \odot y \in Nil(H)$ implies $x \in Nil(H)$ or $y \in Nil(H)$. Hence, $x \in R(H)$, therefore, $H = Nil(H) \cup R(H)$. Now, suppose H is not local. Then there exist $U_1, U_2 \in U(H)$ and $R(H) \subseteq U_1, U_2$. Thus

$$H \subseteq Nil(H) \cup R(H) \subseteq Nil(H) \cup U_1 \subseteq H \quad , \quad H \subseteq Nil(H) \cup R(H) \subseteq Nil(H) \cup U_2 \subseteq H.$$

Thus $Nil(H) \cup U_1 = Nil(H) \cup U_2$. Since $Nil(H) \cap U_1 = Nil(H) \cap U_2 = \emptyset$, then $U_1 = U_2$, which is a contradiction. Therefore, H is local.

Conversely, suppose H is local. Then there exists just one ultra deductive system such as U. Thus $R(H) = U$. Clearly, $Nil(H) \cup R(H) \subseteq H$. Conversely, suppose $x \in H$. If $O(x) \leq \infty$, then $x \in Nil(H)$. If $O(x) = \infty$, then $x \in Inf(H)$. Suppose $x \in Inf(H) \setminus U$. Thus $x \notin U$. So $U \subseteq \langle U \cup \{x\} \rangle \subseteq H$. Since $U \in U(H)$, we get $\langle U \cup \{x\} \rangle = H$. On the other side, $U \subseteq Inf(H)$ and $x \in Inf(H)$, then $H = \langle U \cup \{x\} \rangle \subseteq Inf(H)$ and so $0 \in Inf(H)$, which is a contradiction. Therefore, $Inf(H) = U$ and so $H = Nil(H) \cup R(H)$. □

Proposition 3.33. *Suppose $F \in \mathfrak{F}(H)$ such that $F \subseteq Ds(H)$. Then for any $x, y \in H$ we have*
(i) $\frac{x}{F} = \frac{0}{F}$ iff $x = 0$ and $\frac{x}{F} \leq \frac{\neg y}{F}$ iff $x \leq \neg y$.
(ii) $O(x) = O(\frac{x}{F})$.
(iii) $Ds(\frac{H}{F}) = \frac{Ds(H)}{F}$.

Proof. (i) Suppose $\frac{x}{F} = \frac{0}{F}$. Then $\neg x \in F$. Since $F \subseteq Ds(H)$ and $x \leq \neg\neg x$, we get $\neg\neg x = 0$ and so $x = 0$. The proof of converse is clear. Now, assume $\frac{x}{F} \leq \frac{\neg y}{F}$. Then $x \to \neg y \in F$. Since $F \subseteq Ds(H)$, we obtain $\neg(x \odot y) = x \to \neg y \in Ds(H)$ and so $\neg\neg(x \odot y) = 0$. As $x \odot y \leq \neg\neg(x \odot y)$, then $x \odot y \leq 0$, so $x \leq \neg y$. The proof of converse is clear.
(ii) By (i), for all $x \in H$ and $n \in \mathbb{N}$, $x^n = 0$ iff $\frac{x^n}{F} = \frac{0}{F}$. Thus $O(x) = O(\frac{x}{F})$.
(iii) Assume $\frac{x}{F} \in Ds(\frac{H}{F})$ iff $\neg(\frac{x}{F}) = 0$ iff $\neg\neg x = \neg x \to 0 \in F \subseteq Ds(H)$. Thus $\neg x = 0$, and so $x \in Ds(H)$. Hence $\frac{x}{F} \in \frac{Ds(H)}{F}$. The proof of other side is similar. □

Proposition 3.34. *The hoop H is local iff $\frac{H}{Ds(H)}$ is local.*

Proof. Suppose $\frac{H}{Ds(H)}$ is not local. Then $\frac{U_1}{Ds(H)}, \frac{U_2}{Ds(H)} \in U(\frac{H}{Ds(H)})$. Since there is a one-to-one correspondence between $\mathfrak{F}(H)$ and $\mathfrak{F}(\frac{H}{Ds(H)})$ which contain $Ds(H)$, we have $U_1, U_2 \in U(H)$, a contradiction.

The proof of converse is similar. □

Let $f : H \to K$ be a hoop homomorphism. Define $\overline{f} : \frac{H}{Ds(H)} \to \frac{K}{Ds(K)}$ such that for any $\frac{x}{Ds(H)} \in \frac{H}{Ds(H)}$, we have $\overline{f}\left(\frac{x}{Ds(H)}\right) = \frac{f(x)}{Ds(K)}$. In the following we show $\overline{f}$ is well-defined. For this, suppose $\frac{x}{Ds(H)}, \frac{y}{Ds(H)} \in \frac{H}{Ds(H)}$ we have $\frac{x}{Ds(H)} = \frac{y}{Ds(H)}$ iff $x \to y, y \to x \in Ds(H)$ iff $\neg(x \to y) = 0$ and $\neg(y \to x) = 0$ iff $f(\neg(x \to y)) = 0$ and $f(\neg(y \to x)) = 0$ iff $\neg(f(x) \to f(y)) = 0$ and $\neg(f(y) \to f(x)) = 0$ iff $f(x) \to f(y), f(y) \to f(x) \in Ds(K)$ iff $\frac{f(x)}{Ds(K)} = \frac{f(y)}{Ds(K)}$. Therefore, $\overline{f}$ is well-defined.

Moreover, we can see that the following diagram is commutative:

$$\begin{array}{ccc} H & \xrightarrow{\ f\ } & K \\ \Big\downarrow{\scriptstyle \pi_H} & & \Big\downarrow{\scriptstyle \pi_K} \\ \frac{H}{Ds(H)} & \xrightarrow{\ \overline{f}\ } & \frac{K}{Ds(K)} \end{array} \tag{2}$$

It means that $\pi_K \circ f = \overline{f} \circ \pi_H$. As a consequence, we can define a functor $\mathfrak{T} : \mathbf{Hoop} \to \mathbf{Hoop}$ where for any $H \in \mathcal{O}bj(\mathbf{Hoop})$, $\mathfrak{T}(H) = \frac{H}{Ds(H)} \in \mathcal{O}bj(\mathbf{Hoop})$ and for any $f \in \mathcal{M}or(\mathbf{Hoop})$, $\mathfrak{T}(f) = \overline{f} \in \mathcal{M}or(\mathbf{Hoop})$.

Proposition 3.35. *(i) If f is an epimorphism, then $\overline{f}$ is an epimorphism, too.*
(ii) If f is one-to-one, then $\overline{f}$ is one-to-one, too.

Proof. (i) Assume $\frac{y}{Ds(K)} \in \frac{K}{Ds(K)}$. Since π_K is an epimorphism, there is $x \in K$ such that $\pi_K(x) = \frac{y}{Ds(K)}$. By hypothesis, f is an epimorphism, then there is $z \in H$, where $f(z) = x$. Thus $(\pi_K \circ f)(z) = \pi_K(f(z)) = \frac{y}{Ds(K)}$. Since Diagram (2) is commutative, we have $(\overline{f} \circ \pi_H)(z) = \frac{y}{Ds(K)}$. Hence, $\overline{f}(\frac{z}{Ds(H)}) = \frac{y}{Ds(K)}$. Therefore, $\overline{f}$ is an epimorphism.
(ii) Suppose $\frac{x}{Ds(H)}, \frac{y}{Ds(H)} \in \frac{H}{Ds(H)}$. Then

$$\begin{aligned} \overline{f}(\frac{x}{Ds(H)}) = \overline{f}(\frac{y}{Ds(H)}) &\iff \frac{f(x)}{Ds(K)} = \frac{f(y)}{Ds(K)} \\ &\iff f(x) \to f(y), f(y) \to f(x) \in Ds(K) \\ &\iff \neg\,(f(x) \to f(y)) = \neg\,(f(y) \to f(x)) = 0 \\ &\iff f(\neg(x \to y)) = f(\neg(y \to x)) = 0 \\ &\iff \neg(x \to y) = \neg(y \to x) = 0 \\ &\iff x \to y, y \to x \in Ds(H) \\ &\iff \frac{x}{Ds(H)} = \frac{y}{Ds(H)} \end{aligned}$$

Therefore, $\overline{f}$ is one-to-one. □

By Proposition 3.30(i) we prove $Ds(H) \subseteq R(H)$. Now, we show that the following diagram is commutative:

$$\begin{array}{ccc} H & \xrightarrow{\ \pi_H\ } & \frac{H}{Ds(H)} \\ & \searrow{\scriptstyle \Psi_H} & \Big\downarrow{\scriptstyle \varphi_H} \\ & & \frac{H}{R(H)} \end{array} \tag{3}$$

Let $x \in H$. Then we define φ_H and Ψ_H on H as follows:

$$\varphi_H\left(\frac{x}{Ds(H)}\right) = \frac{x}{R(H)} \quad \text{and} \quad \Psi_H(x) = \frac{x}{R(H)}.$$

Clearly, φ_H and Ψ_H are well-defined and $\varphi_H \circ \pi_H = \Psi_H$. Obviously, φ_H is an epimorphism but not one-to-one. Also, it is easy to see that Ψ_H is surjective if φ_H is surjective.

Proposition 3.36. *Consider $Ds(H) = R(H)$. Then φ_H is one-to-one.*

Proof. Suppose $\varphi_H\left(\frac{x}{Ds(H)}\right) = \varphi_H\left(\frac{y}{Ds(H)}\right)$, for any $\frac{x}{Ds(H)}, \frac{y}{Ds(H)} \in \frac{H}{Ds(H)}$. Then $\frac{x}{R(H)} = \frac{y}{R(H)}$. Thus $x \to y, y \to x \in R(H)$ and by hypothesis $x \to y, y \to x \in Ds(H)$, and so $\frac{x}{Ds(H)} = \frac{y}{Ds(H)}$. Hence, φ_H is one-to-one. □

As we see, we define $Mv(H) = \{x \in H \mid \neg(\neg x) = x\}$. Clearly, $0, 1 \in Mv(H)$ and we can see that $Mv(H)$ is closed under $\to$. For this, suppose $x, y \in Mv(H)$. Then $\neg(\neg x) = x$ and $\neg(\neg y) = y$. Thus

$$\neg\neg(x \to y) = \neg\neg(\neg(\neg x) \to \neg(\neg y)) = \neg\neg\neg(\neg(\neg x) \odot \neg y) = \neg(\neg(\neg x) \odot \neg y) = \neg(\neg x) \to \neg(\neg y) = x \to y. \tag{4}$$

Hence, $x \to y \in Mv(H)$.

Now, for any $x, y \in Mv(H)$ we define the operations $\odot'$ and $\to'$ on $Mv(H)$ by $x \odot' y = \neg\neg(x \odot y)$ and $x \to' y = x \to y$. Clearly, if $Mv(H) = H$, then all these operation concide with the operations $\odot$ and $\to$ in H.

Theorem 3.37. *The algebraic structure $(Mv(H), \odot', \to', 0, 1)$ is a bounded hoop*

Proof. Suppose $x, y, z \in Mv(H)$. Then $x \to' x = x \to x = 1$ and

$$x \odot' (x \to' y) = \neg\neg(x \odot (x \to y)) = \neg\neg(y \odot (y \to x)) = y \odot' (y \to' x)$$

Moreover,

$$\begin{aligned}(x \odot' y) \to' z &= (\neg\neg(x \odot y)) \to z = (\neg\neg(x \odot y)) \to \neg(\neg z) = \neg z \to \neg\neg\neg(x \odot y)) = \neg z \to \neg(x \odot y)\\ &= \neg z \to (x \to \neg y) = x \to (\neg z \to \neg y) = x \to (y \to \neg(\neg z)) = x \to (y \to z)\\ &= x \to' (y \to' z)\end{aligned}$$

Now, it is enough to prove $(Mv(H), \odot', 1)$ is a commutative monoid. For this, for any $x, y, z \in Mv(H)$, we have $x \odot' y = \neg\neg(x \odot y) = \neg\neg(y \odot x) = y \odot' x$ and $x \odot' 1 = \neg\neg(x \odot 1) = \neg\neg x = x$, finally,

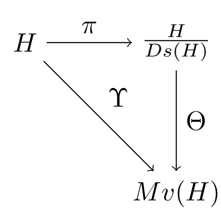
$$\begin{aligned}x \odot' (y \odot' z) &= \neg\neg(x \odot \neg\neg(y \odot z)) = \neg\left[(x \odot \neg\neg(y \odot z)) \to 0\right] = \neg\left[x \to (\neg\neg(y \odot z) \to 0)\right]\\ &= \neg\left[x \to \neg(y \odot z)\right] = \neg\left[x \to (z \to \neg y)\right] = \neg\left[z \to (x \to \neg y)\right] = \neg\left[z \to \neg(x \odot y)\right]\\ &= \neg\left[z \to \neg\neg\neg(x \odot y)\right] = \neg\left[\neg\neg(x \odot y) \to \neg z\right] = \neg\neg(\neg\neg(x \odot y) \odot z)\\ &= (x \odot' y) \odot' z\end{aligned}$$

Therefore, $(Mv(H), \odot', \to', 0, 1)$ is a bounded hoop. □

Corollary 3.38. *The algebraic structure $(Mv(H), \oplus, \neg, 0, 1)$ is an MV-algebra, where for any $x, y \in Mv(H)$, $\neg x \oplus \neg y = \neg(x \odot' y)$.*

Proof. The proof is straightforward. □

Theorem 3.39. *Define the maping $\Theta : \frac{H}{Ds(H)} \to Mv(H)$, where for any $\frac{x}{Ds(H)} \in \frac{H}{Ds(H)}$, we have $\Theta(\frac{x}{Ds(H)}) = \neg\neg x$. Then Θ is an isomorphism and the following diagram is commutative, where $\Upsilon(x) = \neg\neg x$, for any $x \in H$.*



(5)

Proof. We have to attention that in this diagram we suppose $Mv(H) = (Mv(H), \odot', \rightarrow', 0, 1)$. Suppose $\frac{a}{Ds(H)}, \frac{b}{Ds(H)} \in \frac{H}{Ds(H)}$. Then

$$\begin{aligned}\Theta(\frac{a}{Ds(H)} \odot \frac{b}{Ds(H)}) &= \Theta(\frac{a \odot b}{Ds(H)}) = \neg\neg(a \odot b) = \neg(a \rightarrow \neg b) = \neg(a \rightarrow \neg\neg\neg b) = \neg(\neg\neg b \rightarrow \neg a)\\ &= \neg(\neg\neg b \rightarrow \neg\neg\neg a) = \neg\neg(\neg\neg b \odot \neg\neg a) = \neg\neg a \odot' \neg\neg b\\ &= \Theta(\frac{a}{Ds(H)}) \odot' \Theta(\frac{b}{Ds(H)}).\end{aligned}$$

Also, by (4), we have

$$\Theta(\frac{a}{Ds(H)} \rightarrow \frac{b}{Ds(H)}) = \Theta(\frac{a \rightarrow b}{Ds(H)}) = \neg\neg(a \rightarrow b) = \neg\neg a \rightarrow \neg\neg b = \Theta(\frac{a}{Ds(H)}) \rightarrow \Theta(\frac{b}{Ds(H)}).$$

Hence, Θ is a hoop homomorphism. Moreover, $\frac{a}{Ds(H)} \in ker\Theta$ iff $\Theta(\frac{a}{Ds(H)}) = \frac{1}{Ds(H)}$ iff $\neg\neg a = 1$ iff by Proposition 2.1(vii), $\neg a = \neg\neg\neg a = 0$ iff $a \in Ds(H)$ iff $ker\Theta = Ds(H)$. Hence $ker\Theta = \{\frac{1}{Ds(H)}\}$. Therefore, Θ is monomorphism. Also, for any $x \in Mv(H)$, since $\neg\neg x = x$, we have $\Theta(\frac{x}{Ds(H)}) = x$. Thus, Θ is a hoop isomorphism. Moreover, for any $x \in H$, $\Theta \circ \pi(x) = \Theta(\frac{x}{Ds(H)}) = \neg\neg x = \Upsilon(x)$. Therefore, $\Theta \circ \pi = \Upsilon$, and so the diagram is commutative. □

Corollary 3.40. *$R(Mv(H)) = R(H) \cap Mv(H)$.*

Proof. By Diagram (5), we have

$$R(Mv(H)) = R(\Theta(\frac{H}{Ds(H)})) = \Theta(R(\frac{H}{Ds(H)})) = \Theta(\frac{R(H)}{Ds(H)}) = \Theta(\pi(R(H))) = \Upsilon(R(H)) = \neg\neg(R(H)). \tag{6}$$

Now, suppose $y \in \neg\neg(R(H))$. Then there exists $x \in R(H)$ such that $\neg\neg x = y$. Since $\neg\neg y = \neg\neg\neg\neg x = \neg\neg x = y$, we get $y \in Mv(H)$. Also, since $x \in R(H)$, by Proposition 3.30(vii) we obtain $\neg\neg x \in R(H)$, and so $y \in R(H)$. Hence, $y \in R(H) \cap Mv(H)$, and so $\neg\neg(R(H)) \subseteq R(H) \cap Mv(H)$. On the other side, consider $x \in R(H) \cap Mv(H)$. Then $x \in R(H)$ and $x \in Mv(H)$. Since $x \in R(H)$, by Proposition 3.30(vii) we get $\neg\neg x \in R(H)$ and from $x \in Mv(H)$ we have $\neg\neg x = x$. Thus $x \in \neg\neg(R(H))$, and so $R(H) \cap Mv(H) \subseteq \neg\neg(R(H))$. Hence, $R(H) \cap Mv(H) = \neg\neg(R(H))$. Therefore, by (6), $R(Mv(H)) = R(H) \cap Mv(H)$. □

4 Boolean elements in hoops

In this section, we introduce the notion of Boolean elements and investigate some properties of them. Then we study the relation among Boolean elements with ultra deductive systems and nilpotent elements. Finally, we introduce a functor between the category of hoops and category of Boolean elements of them.

Definition 4.1. *Consider H is a $\vee$-hoop. Then $e \in H$ is said to be a* Boolean element *if $e \vee \neg e = 1$ and $e \wedge \neg e = 0$. All Boolean elements of H is showed by $Bo(H)$.*

Example 4.2. *Let H be the hoop as in Example 3.9. Obviously, H is a $\vee$-hoop and $Bo(H) = \{0, a, d, 1\}$.*

Proposition 4.3. *If H is a $\vee$-hoop, then $e \in Bo(H)$ implies $e = e^2$, $e = \neg(\neg e)$ and $\neg e \rightarrow e = e$.*

Proof. Suppose $e \in Bo(H)$. By Proposition 2.1(ii), $e^2 \leq e$. Since $e \in Bo(H)$, we have $e \vee \neg e = 1$. Then

$$e \rightarrow e^2 = (1 \odot e) \rightarrow e^2 = ((e \vee \neg e) \odot e) \rightarrow e^2.$$

By Proposition 2.1(vi) and (vii),

$$((e \vee \neg e) \odot e) \rightarrow e^2 = ((e \odot e) \vee (\neg e \odot e)) \rightarrow e^2 = e^2 \rightarrow e^2 = 1.$$

Hence, $e \leq e^2$, and so $e = e^2$. Now, we prove $e = \neg(\neg e)$. For this, by Proposition 2.1(vii), $e \leq \neg(\neg e)$. Since $e \in Bo(H)$, we have $e \vee \neg e = 1$, then by Proposition 2.1(vi) and (vii),

$$\neg(\neg e) \to e = ((e \vee \neg e) \odot \neg(\neg e)) \to e = ((e \odot \neg(\neg e)) \vee (\neg e \odot \neg(\neg e))) \to e = (e \odot \neg(\neg e)) \to e.$$

Thus, by Proposition 2.1(ii), $(e \odot \neg(\neg e)) \to e = 1$. Hence, $\neg(\neg e) \to e = 1$, and so $e = \neg(\neg e)$. Finally, for proving $\neg e \to e = e$, by Proposition 2.1(vii), we have $e \leq \neg e \to e$. It is enough to prove $\neg e \to e \leq e$. For this, since $e \in Bo(H)$, we get $\neg e \in Bo(H)$. In addition, $e = \neg(\neg e)$ and $e^2 = e$, so we consequence that

$$(\neg e \to e) \to e = (\neg e \to \neg(\neg e)) \to \neg(\neg e) = \neg(\neg e \odot \neg e) \to \neg(\neg e) = \neg(\neg e) \to \neg(\neg e) = 1.$$

Thus, $\neg e \to e = e$. □

Proposition 4.4. *Let H be a $\vee$-hoop. For any $e, f \in Bo(H)$ and $x, y \in H$ we have:*
(i) if $e \leq x$, then $\neg e \to x = x$.
(ii) $e \to x = e \to (e \to x)$.
(iii) $e \to (x \to y) = (e \to x) \to (e \to y)$.
(iv) $\neg e \to x = e \vee x$.

Proof. (i) Suppose $e \leq x$. Clearly $x \leq \neg e \to x$. Conversely, by Proposition 2.1(vi),

$$x = 1 \to x = (e \vee \neg e) \to x = (e \to x) \wedge (\neg e \to x) = \neg e \to x.$$

(ii) Since $e \in Bo(H)$, by Proposition 4.3, $e^2 = e$. Then $e \to x = (e \odot e) \to x = e \to (e \to x)$.
(iii) Since $x \leq e \to x$, by Proposition 2.1(iii), we have $(e \to x) \to y \leq x \to y$ and so $e \to ((e \to x) \to y) \leq e \to (x \to y)$. Hence, $(e \to x) \to (e \to y) \leq e \to (x \to y)$. Conversely, by Proposition 2.1(iv), we get $x \to y \leq (e \to x) \to (e \to y)$. Then by (ii)

$$\begin{aligned} e \to (x \to y) &\leq e \to ((e \to x) \to (e \to y)) \\ &= (e \to x) \to (e \to (e \to y)) \\ &= (e \to x) \to (e^2 \to y) \\ &= (e \to x) \to (e \to y). \end{aligned}$$

Therefore, $e \to (x \to y) = (e \to x) \to (e \to y)$.
(iv) By Proposition 2.1(vii), we have $e \leq \neg e \to x$ and $x \leq \neg e \to x$. Thus $x \vee e \leq \neg e \to x$. Now, consider there is $z \in H$ such that $x, e \leq z$. We prove $\neg e \to x \leq z$. For this, since $e, x \leq z$, by (i) and (iii) we have

$$(\neg e \to x) \to z = (\neg e \to x) \to (\neg e \to z) = \neg e \to (x \to z) = \neg e \to 1 = 1.$$

So $(\neg e \to x) \leq z$. Hence, $\neg e \to x = e \vee x$. □

Proposition 4.5. *Let H and H_i, where $i \in I$ be $\vee$-hoops. Then the following statements hold:*
(i) $Bo(H) \cap Ds(H) = R(H) \cap Bo(H) = \{1\}$.
(ii) $Bo(H) \cap Nil(H) = \{0\}$.
(iii) $Bo(\prod_{i \in I} H_i) = \prod_{i \in I} Bo(H_i)$.
(iv) For any $e \in Bo(H)$, $\langle \boldsymbol{e} \rangle = \{x \in H \mid e \leq x\}$.
(v) For any $e, f \in Bo(H)$, $e \odot f = e \wedge f \in Bo(H)$ and $e \to f = \neg e \vee f \in Bo(H)$.

Proof. By Proposition 4.4, the proof of (iii) and (iv) is clear.
(i) Obviously, $\{1\} \subseteq Bo(H) \cap Ds(H)$. Suppose $x \in Bo(H) \cap Ds(H)$. Then by Proposition 4.3, since $x \in Bo(H)$, we have $\neg\neg x = x$. Also, from $x \in Ds(H)$, we have $\neg x = 0$ and so $\neg\neg x = 1$. Thus $x = 1$. Hence, $Bo(H) \cap Ds(H) \subseteq \{1\}$. Therefore, $Bo(H) \cap Ds(H) = \{1\}$. Moreover, clearly, $\{1\} \subseteq R(H) \cap Bo(H)$. Suppose $x \in R(H) \cap Bo(H)$. From $x \in R(H)$, by Proposition 3.29, we have for any $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $(\neg x^n)^k = 0$. Since $x \in Bo(H)$ we get $x^2 = x$ and $\neg x \in Bo(H)$. Thus $\neg x = 0$ and so $x = \neg\neg x = 1$. Hence $R(H) \cap Bo(H) \subseteq \{1\}$. Therefore, $R(H) \cap Bo(H) = \{1\}$.

(ii) Clearly $\{0\} \subseteq Bo(H) \cap Nil(H)$. Consider $x \in Bo(H) \cap Nil(H)$. Then $x \in Bo(H)$ and $x \in Nil(H)$, so $x^2 = x$ and there is $n \in \mathbb{N}$ such that $x^n = 0$, respectively. Thus $x^n = x$ and so $x = 0$. Therefore, $Bo(H) \cap Nil(H) = \{0\}$.
(v) Suppose $e, f \in Bo(H)$. Since $\neg e \leq e \to f$ and $f \leq e \to f$, we get $\neg e \vee f \leq e \to f$. On the other side, from $e, f \in Bo(H)$, by Proposition 4.3 we have $\neg\neg e = e$ and $\neg\neg f = f$. Thus $\neg e \vee f = (f \to \neg e) \to \neg e$. Then

$$(e \to f) \to ((f \to \neg e) \to \neg e) = ((e \to f) \odot (f \to \neg e)) \to \neg e \geq (e \to \neg e) \to \neg e = e \vee \neg e = 1.$$

Hence, $e \to f \leq \neg e \vee f$ and so $e \to f = \neg e \vee f$. Also, by Propositions 4.4(iv) and 2.1(vi), we have

$$e \wedge f = e \odot (e \to f) = e \odot (\neg e \vee f) = (e \odot \neg e) \vee (e \odot f) = 0 \vee (e \odot f) = e \odot f.$$

In addition, from $\vee$-hoop is a distributive lattice, we have

$$(e \wedge f) \wedge (\neg e \vee \neg f) = ((e \wedge f) \wedge \neg e) \vee ((e \wedge f) \wedge \neg f) = 0,$$
$$(e \wedge f) \vee (\neg e \vee \neg f) = ((e \wedge f) \vee \neg e) \vee \neg f = (f \vee \neg e) \vee \neg f = 1.$$

Therefore, $e \odot f = e \wedge f \in Bo(H)$ and by similar way $e \to f = \neg e \vee f \in Bo(H)$. □

Proposition 4.6. *Consider H is a $\vee$-hoop. Then $Bo(H) = Bo(Mv(H))$.*

Proof. Since $Mv(H) \subseteq H$, obviously $Bo(Mv(H)) \subseteq Bo(H)$. Suppose $x \in Bo(H)$. By Proposition 4.3 we have $\neg\neg x = x$ and so $x \in Mv(H)$. Thus $x \in Bo(H) \cap Mv(H)$. Moreover, from $\neg\neg\neg x = \neg x$, we get $\neg x \in Mv(H)$. Since $x \wedge \neg x = 0$ and $x \vee \neg x = 1$ we have $x \in Bo(Mv(H))$. Therefore, $Bo(H) = Bo(Mv(H))$. □

Proposition 4.7. *If H is a local $\vee$-hoop, then $Bo(H) = \{0, 1\}$ and $H = Nil(H) \cup \{x \in H \mid \neg x \in Nil(H)\}$.*

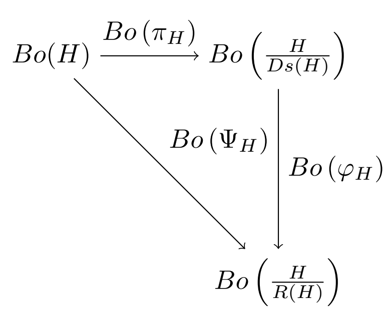
Proof. Consider $x \in Bo(H) \setminus \{0, 1\}$. Since $x \vee \neg x = 1$, we obtain $\neg x \in Bo(H)$. By assumption H is local, thus it has just one ultra deductive system such as U such that $\langle x \rangle \subseteq U$ and $\langle \neg x \rangle \subseteq U$. Hence $0 \in U$, a contradiction. Therefore, $Bo(H) = \{0, 1\}$. By Theorem 3.32, since H is local, we have $H = Nil(H) \cup R(H) = Nil(H) \cup U$, where U is the only ultra deductive system of H. Let $B = \{x \in H \mid \neg x \in Nil(H)\}$. Suppose $x \in U$, then $\neg x \notin U$, thus $O(\neg x) < \infty$ and so $\neg x \in Nil(H)$. Hence $x \in B$ and so $U \subseteq B$. Thus $H = Nil(H) \cup U \subseteq Nil(H) \cup B \subseteq H$. Therefore, $H = Nil(H) \cup \{x \in H \mid \neg x \in Nil(H)\}$. □

Corollary 4.8. *Let H be cyclic such that $\mid H \mid = n + 1$. If H is $\vee$-hoop, then $Bo(H) = \{0, 1\}$.*

Proof. By Theorem 3.20, H is a chain and by Proposition 4.7, $Bo(H) = \{0, 1\}$. □

Now, we define another functor between the category of hoops and the category of Boolean elements of them, where $\mathfrak{T} : \mathbf{Hoop} \to \mathbf{Bool}$ such that for any $H \in \mathcal{O}bj(\mathbf{Hoop})$, $\mathfrak{T}(H) = Bo(H) \in \mathcal{O}bj(\mathbf{Bool})$ and for any $f \in \mathcal{M}or(\mathbf{Hoop})$, $\mathfrak{T}(f) = Bo(f) \in \mathcal{M}or(\mathbf{Bool})$.

Hence, according to Diagram (3), the next diagram in the category of Boolean algebras is commutative.



(7)

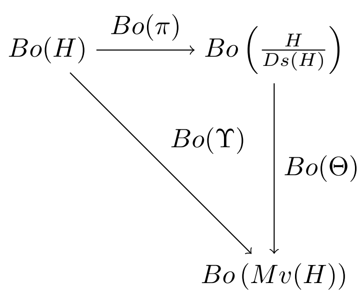
Proposition 4.9. *Two homomorphism $Bo(\pi_H)$ and $Bo(\Psi_H)$ are injective.*

Proof. Let $x, y \in Bo(H)$. Then by Proposition 4.5(v), $x \to y, y \to x \in Bo(H)$. In addition, $Bo(\Psi_H)(x) = Bo(\Psi_H)(y)$ iff $x \to y, y \to x \in R(H)$ iff $x \to y, y \to x \in Bo(H) \cap R(H)$, by Proposition 4.5(i), $Bo(H) \cap R(H) = \{1\}$ iff $x \to y, y \to x \in \{1\}$ iff $x = y$. Hence, $Bo(\Psi_H)$ is injective. Also, by commutativity of diagram, since $Bo(\varphi_H) \circ Bo(\pi_H) = Bo(\Psi_H)$ and we show $Bo(\Psi_H)$ is injective and so $Bo(\pi_H)$ is injective, too. □

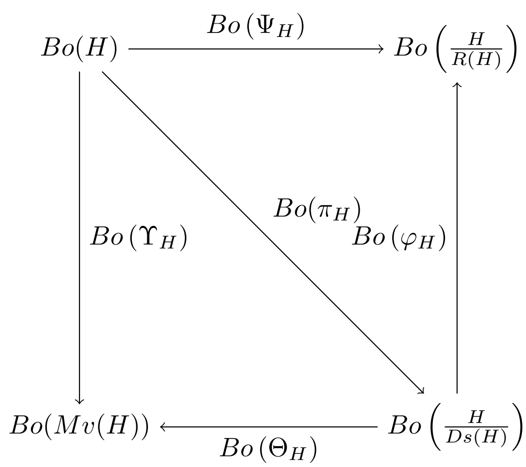
Proposition 4.10. *(i) If $Ds(H) = R(H)$, then $Bo(\Psi_H)$ is surjective.*
(ii) If $Bo(\Psi_H)$ is surjective, then $Bo(\varphi_H)$ is surjective.

Proof. (i) If $Ds(H) = R(H)$, then $Bo(\varphi_H)$ is an isomorphism. By commutativity of diagram, obviously $Bo(\Psi_H)$ is surjective.
(ii) Suppose $Bo(\Psi_H)$ is surjective and $\overline{y} \in Bo(\frac{H}{R(H)})$. Then there exists $x \in Bo(H)$ such that $Bo(\Psi_H)(x) = \overline{y}$. By commutativity of diagram, we have $Bo(\varphi_H) \circ Bo(\pi_H)(x) = Bo(\Psi_H)(x) = \overline{y}$, and so $Bo(\varphi_H)(\frac{x}{Ds(H)}) = \overline{y}$. Therefore, $Bo(\varphi_H)$ is surjective. □

By considering Diagram (5), we affect the Boolean functor on this diagram as follows:



By using these diagrams we make a new diagram as follows:



(8)

Theorem 4.11. *According to Diagram (8), $Bo(\Psi_H)$ is surjective iff $Bo(\varphi_H)$ is surjective.*

Proof. ($\Rightarrow$) By Proposition 4.10(ii), the proof is clear.
($\Leftarrow$) According to commutativity of Diagram (8), we prove that homomorphism $Bo(\Upsilon_H)$ is an isomorphism. For this, by Proposition 4.3, we know that for any $x \in Bo(H)$, $\neg\neg x = x$. Assume $x, y \in Bo(H)$ such that $Bo(\Upsilon_H)(x) = Bo(\Upsilon_H)(y)$ and so $\neg\neg x = \neg\neg y$. Since $x, y \in Bo(H)$ we have $x = \neg\neg x = \neg\neg y = y$. Hence, $Bo(\Upsilon_H)$ is injective. Now, suppose $y \in Bo(Mv(H))$. Clearly, $Bo(Mv(H)) \subseteq Bo(H) \cap Mv(H)$, and so $y \in Bo(H)$. Thus, $y = \neg\neg y = Bo(\Upsilon_H)(y)$. Hence, $Bo(\Upsilon_H)$ is surjective. Also, $Bo(\Theta_H)$ is an isomorphism since by Theorem 3.26, Θ is an isomorphism. Then by commutativity of diagram we have $Bo(\Theta) \circ Bo(\pi_H) = Bo(\Upsilon_H)$. Hence, $Bo(\pi_H)$ is an isomorphism and from $Bo(\varphi_H) \circ Bo(\pi_H) = Bo(\Psi_H)$, $Bo(\varphi_H)$ is surjective and $Bo(\pi_H)$ is an isomorphism, we consequence that $Bo(\varphi_H)$ is surjective. □

Corollary 4.12. *$Bo(\Psi_H)$ is surjective iff $Bo(\varphi_H)$ is an isomorphism.*

5 Conclusions and future works

In this paper, we define the concept of order and nilpotent element of H and we study some properties of them. Then by using this notion, we introduce cyclic hoops and prove that every cyclic hoop has a unique generator and is a local MV-algebra. Also, we introduce other notions such as dense and Boolean elements on hoops and investigate some of their properties and relation between them. Then by using the notion of Boolean element, we define a functor and prove some properties of hoop category.

References

[1] M. Aaly Kologani, R.A. Borzooei, *On ideal theory on hoops*, Mathematica Bohemica, 145(2) (2019), 1–22.

[2] R.A. Borzooei, M. Aaly Kologani, *Filter theory of hoop algebras*, Journal of Advanced Research in Pure Mathematics, 6(4) (2014), 72–86.

[3] R.A. Borzooei, M. Aaly Kologani, *Stabilizer topology of hoops*, Algebraic Structures and Their Applications, 1(1) (2014), 35–48.

[4] R.A. Borzooei, M. Aaly Kologani, *Local and perfect semihoops*, Journal of Intelligent and Fuzzy Systems, 29 (2015), 223–234.

[5] R.A. Borzooei, M. Aaly Kologani, *Results on hoops*, Journal of Algebraic Hyperstructures and Logical Algebras, 1(1) (2020), 61–77.

[6] R.A. Borzooei, O. Zahiri, *Radical and it's applications in BCH-algebras*, Iranian Journal of Mathematical Sciences and Informatics, 8(1) (2013), 15–29.

[7] B. Bosbach, *Komplementäre Halbgruppen. Axiomatik und Arithmetik*, Fundamenta Mathematicae, 64(3) (1969), 257–287.

[8] B. Bosbach, *Komplementäre Halbgruppen. Kongruenzen und Quatienten*, Fundamenta Mathematicae, 69(1) (1970), 1–14.

[9] F. Esteva, L. Godo, *Monoidal t-norm based logic, towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems, 124(3) (2001), 271–288.

[10] G. Georgescu, L. Leustean, V. Preoteasa, *Pseudo-hoops*, Journal of Multiple-Valued Logic and Soft Computing, 11(1-2) (2005), 153–184.

[11] G. Georgescu, C. Muresan, *Boolean lifting property for residuated lattices*, Soft Computing, 18 (2014), 2075–2089.

[12] P. Hájek, *Mathematics of fuzzy logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[13] M. Kondo, W.A. Dudek, *Filter theory of BL-algebras*, Soft Computing, 12(5) (2008), 419–423.

[14] T. Kowalski, H. Ono, *Residuated lattices: An algebraic glimpse at logic without contraction*, Japan Advanced Institute of Science and Technology, 2001.

[15] E. Mohammadzadeh, R.A. Borzooei, *Engel, nilpotent and solvable BCI-algebras*, An. St. Univ. Ovidius Constantsa, 27(1) (2019), 169–192.

[16] C. Muresan, *Dense elements and classes of residuated lattices*, arXiv:0901.1630vl, (2009).

[17] F. Xie, H. Liu, *Ideals in pseudo-hoop algebras*, Journal of Algebraic Hyperstructures and Logical Algebras, 1(4) (2020), 39–53.