Spherical fuzzy $K$-algebras

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“This paper is dedicated to Professor Young Bae Jun on the occasion of his 70th birthday.”

Abstract

In this research article, new fuzzy $K$-algebras, namely, spherical fuzzy $K$-algebras and $(\in, \in \lor q)$-spherical fuzzy $K$-algebras are constructed. Certain properties of these spherical fuzzy $K$-structures are investigated. The behavior of spherical fuzzy $K$-algebras under homomorphism is characterized. The spherical fuzzy $K$-algebra with thresholds is also delineated.

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1 Introduction

The notion of a $K$-algebra $(G, \cdot, \circ, e)$ was first introduced by Dar and Akram in [12, 11, 13, 14]. A $K$-algebra is an algebra built on a group $(G, \cdot, e)$ by adjoining an induced binary operation $\circ$ on $G$ which is attached to an abstract $K$-algebra $(G, \cdot, \circ, e)$. This system is, in general, non-commutative and non-associative adjoint with a right identity $e$, if $(G, \cdot, e)$ is non-commutative. For a given group $G$, the $K$-algebra is proper if $G$ is not an elementary Abelian 2-group. Thus, whether a $K$-algebra is Abelian or non-Abelian purely depends on the base group $G$. We call a $K$-algebra on a group $G$ as a $K(G)$-algebra due to its structural basis $G$. Akram and Kim [1] proved that the $K$-algebra on an Abelian group is equivalent to the $p$-semisimple $BCI$-algebra [10]. Recently, Naghibi et al. [23] have introduced generalized $K$-algebra (briefly, $gK$-algebra). They also have defined $gK$-subalgebras, (prime) $gK$-ideals and quotient $gK$-algebras. Algebraic structures play an emphatic role in Mathematics and the range of its applications are broad and multidisciplinary. Thus, algebraic structures provide a sufficient motivation to the researchers to review various concepts and stems out from the realm of abstract algebra in a broader framework of fuzzy setting. In 1965, Zadeh [27] proposed fuzzy sets for handling vague or hazy
information. Subsequently, intuitionistic fuzzy sets [8], picture fuzzy sets [10], Pythagorean fuzzy sets [25] and spherical fuzzy sets [15] were extended to overcome some drawbacks of fuzzy sets. Akram et al. [3] first introduced fuzzy structures on K-algebras and investigated some of their properties. We then developed different fuzzy K-algebras [1, 2, 4, 5, 7] with other researchers worldwide by applying generalizations of Zadeh’s fuzzy set theory. Further, Jun and Cho [18] and Jun and Park [19] investigated certain properties on fuzzy K-algebras. As a continuation study of K-algebras, we develop the concept of spherical fuzzy K-subalgebras, and present some of their properties. Moreover, we study the behavior of spherical fuzzy K-subalgebras under homomorphism. Finally, we discuss (∈, ∈ ∨ q)-spherical fuzzy K-algebras.

For other terminologies and results that are not mentioned in this article, the readers are referred to [9, 11, 21, 23, 24, 26].

2 Preliminaries

In this section, we review a class of K-algebras and spherical fuzzy sets.

Definition 2.1. [12] Let (G, ·, e) be a group such that each non-identity element is not of order 2. Let a binary operation ⊙ be introduced on the group G and defined by ⊙(x, y) = x ⊙ y = xy⁻¹ for all x, y ∈ G. If e is the identity of the group G, then:

1. e takes the shape of right ⊙-identity and not that of left ⊙-identity.
2. Each non-identity element (x ≠ e) is ⊙-involutionary because x ⊙ x = xx⁻¹ = e.
3. G is ⊙-nonassociative because (x ⊙ y) ⊙ z = x ⊙ (z ⊙ y⁻¹) ≠ x ⊙ (y ⊙ z) for all x, y, z ∈ G.
4. G is ⊙-noncommutative since x ⊙ y ≠ y ⊙ x for all x, y ∈ G.
5. If G is an elementary Abelian 2-group, then x ⊙ y = x · y.

Definition 2.2. [12] Let (G, ·, e) be a group in which each non-identity element is not of order 2. Then a K-algebra is a structure K = (G, ·, ⊙, e) on a group G in which induced binary operation ⊙ : G × G → G is defined by ⊙(x, y) = x ⊙ y = x.y⁻¹ and satisfies the following axioms:

(K1) (x ⊙ y) ⊙ (x ⊙ z) = (x ⊙ ((e ⊙ z) ⊙ (e ⊙ y))) ⊙ x,
(K2) x ⊙ (x ⊙ y) = (x ⊙ (e ⊙ y)) ⊙ x,
(K3) (x ⊙ x) = e,
(K4) (x ⊙ e) = x,
(K5) (e ⊙ x) = x⁻¹
for all x, y, z ∈ G.

Proposition 2.3. [12] Let G be an Abelian group which is not an elementary Abelian 2-group. Then a K-algebra K on G is represented by the following identities:

(K1) (x ⊙ y) ⊙ (x ⊙ z) = z ⊙ y,
(K2) \( x \odot (x \odot y) = y \),
(K3) \( x \odot x = e \),
(K4) \( x \odot e = x \),
(K5) \( e \odot x = x^{-1} \)
for all \( x, y, z \in G \).

We give here some examples of \( K \)-algebras.

Example 2.4. Let \( G = GL_n(\mathbb{R}) \subset M_n(\mathbb{R}) = \{ A = [a_{ij}] : \det(A) \neq 0 \} \) be the multiplicative group of all \( n \times n \) real non-singular matrices. Define the operation \( \odot \) on \( GL_n(\mathbb{R}) \) by \( A \odot B = AB^{-1} \) for all \( A, B \in GL_n(\mathbb{R}) \). Then \((G, \cdot, \odot, e)\) is a \( K \)-algebra \( \mathcal{K} \).

Example 2.5. Let \( G = V_2(\mathbb{R}) = \{ (x, y) : x, y \in \mathbb{R} \} \) be the set of all 2-dimensional real vectors which forms an additive (+) Abelian group. Define the operation \( \odot \) on \( V_2(\mathbb{R}) \) by \( x \odot y = x - y \) for all \( x, y \in V_2(\mathbb{R}) \). Then \((G, +, \odot, e)\) is a \( K \)-algebra \( \mathcal{K} \).

Example 2.6. \((\mathbb{Z}, +, \odot, 0), (\mathbb{Q}, +, \odot, 0), (\mathbb{R}, +, \odot, 0)\) form \( K \)-algebras by defining the operation \( \odot \) by \( x \odot y = x - y \) for all \( x, y \).

Remark 2.7. We remark here some facts about \( K \)-algebras:

1. Let \( G = \{ e, a, b, c \} \) be a Klein four group. Consider a \( K \)-algebra on \( G \) with the following Cayley Table:

\[
\begin{array}{c|cccc}
\odot & e & a & b & c \\
\hline
  e & e & a & b & c \\
  a & a & e & c & b \\
  b & b & c & e & a \\
  c & c & b & a & e \\
\end{array}
\]

This is an improper \( K \)-algebra on Klein four group since it is an elementary Abelian 2-group, i.e., \( x \odot y = x.y^{-1} = x.y \).

2. Each non-identity element of a \( K \)-algebra is of order 2, i.e., \( x \odot x = e \).

3. A \( K \)-algebra is proper if \( G \) is not an elementary Abelian 2-group.

4. A \( K \)-algebra is Abelian or non-Abelian if the underline group \( G \) is Abelian or non-Abelian.

Definition 2.8. [13] A mapping \( \phi \) from a \( K \)-algebra \( \mathcal{K}_1 \) into \( \mathcal{K}_2 \) is called a \( K \)-homomorphism if for every \( x_1, y_1 \in \mathcal{K}_1 \), \( \phi(x_1 \odot y_1) = \phi(x_1) \odot \phi(y_1) \), where \( \phi(x_1), \phi(y_1) \in \mathcal{K}_2 \).

Murali [22] proposed a definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy set. The idea of quasi-coincidence of a fuzzy point with a fuzzy set [24] played a vital role to produce different types of fuzzy algebraic structures.

Definition 2.9. [22] A fuzzy set \( \mu \) in a set \( G \) of the form

\[
\mu(y) = \begin{cases} 
  t \in [0, 1] & \text{for } x = y, \\
  0, & \text{for } x \neq y,
\end{cases}
\]

is said to be a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$. For a fuzzy point $x_t$ and a fuzzy set $\mu$ in a set $G$, Pu and Liu [24] gave meaning to the symbol $x_t\alpha\mu$, where $\alpha \in \{\in, q, \in \vee, q, \in \wedge q\}$. We say that a fuzzy point $x_t \in \mu$ (resp. $x_tq\mu$) means that $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, $x_t$ is said to belong to (resp. be quasi-coincident with) a fuzzy set $\mu$. Further, $x_t \in \vee q\mu$ means that $x_t \in \mu$ or $x_tq\mu$, $x_t \in \wedge q\mu$ means that $x_t \in \mu$ and $x_tq\mu$, $x_t\alpha\mu$ means that $x_t\alpha\mu$ does not hold.


**Definition 2.10.** [15] Let $G$ be a universe of discourse. An object of the form

$$S = \{(x, \mu_S, \nu_S, \lambda_S) \mid x \in G\},$$

is called the spherical fuzzy set over the domain $G$. The positive membership $\mu_S$, neutral membership $\nu_S$ and the negative membership $\lambda_S$ lie inside the unit interval $[0, 1]$ and the condition $\mu_S^2(x) + \nu_S^2(x) + \lambda_S^2(x) \leq 1$ holds $\forall x \in G$. The $\chi_S(x) = \sqrt{1 - \mu_S^2(x) - \nu_S^2(x) - \lambda_S^2(x)}$ denotes the degree of refusal.

The spherical fuzzy sets are more flexible than the existing models [8, 10, 25, 27] and has the ability to manage ambiguous data using three independent real-valued functions $\mu$, $\nu$ and $\lambda$, with relatively relaxed condition. The graphical representation of spherical fuzzy sets is shown in Figure 1.

![Graphical representation of spherical fuzzy sets](image)

**Figure 1:** Graphical representation of spherical fuzzy sets

### 3 New fuzzy $K$-algebras

**Definition 3.1.** A spherical fuzzy set $S = (\mu_S, \nu_S, \lambda_S)$ in a $K$-algebra $K$ is called a spherical fuzzy $K$-subalgebra of $K$ if it satisfies the following conditions:

(a) $\mu_S(e) \geq \mu_S(x)$,

(b) $\nu_S(e) \geq \nu_S(x)$,

(c) $\lambda_S(e) \leq \lambda_S(x)$,

(d) $\mu_S(x \odot t) \geq \min\{\mu_S(x), \mu_S(y)\}$,
(e) \( \nu_S(x \odot y) \geq \min\{\nu_S(x), \nu_S(y)\} \),

(f) \( \lambda_S(x \odot y) \leq \max\{\lambda_S(x), \lambda_S(y)\} \)

for all \( x, y \in G \).

**Example 3.2.** Consider a \( K \)-algebra \( K = (G, \cdot, \odot, e) \) on dihedral group \( G = \{e, a, b, c, x, y, u, v\} \), where \( c = ab, x = a^2, y = a^3, u = a^2b, v = a^3b \) and \( \odot \) is given by the following Cayley’s Table.

\[
\begin{array}{cccccccc}
\odot & e & a & b & c & x & y & u & v \\
\hline
\text{e} & e & y & b & c & x & a & u & v \\
\text{a} & a & e & c & u & y & x & v & b \\
\text{b} & b & c & e & y & u & v & x & a \\
\text{c} & c & u & a & e & v & b & y & x \\
\text{x} & x & a & u & v & e & y & b & c \\
\text{y} & y & x & v & b & a & e & c & u \\
\text{u} & u & v & x & a & b & c & e & y \\
\text{v} & v & b & y & x & c & u & a & e \\
\end{array}
\]

We define a spherical fuzzy set \( S = (\mu_S, \nu_S, \lambda_S) \) in \( K \)-algebra as follows:

\( \mu_S(e) = 0.7, \nu_S(e) = 0.4, \lambda_S(e) = 0.1 \),
\( \mu_S(x) = 0.5, \nu_S(x) = 0.1, \lambda_S(x) = 0.3 \) for all \( x \neq e \in G \).

By direct calculations, it is easy to verify that \( S \) is a spherical fuzzy \( K \)-subalgebra of \( K \).

**Proposition 3.3.** If \( S = (\mu_S, \nu_S, \lambda_S) \) is a spherical fuzzy \( K \)-subalgebra of \( K \), then

1. \( (\forall x, y \in G), (\mu_S(x \odot y) = \mu_S(y) \Rightarrow \mu_S(x) = \mu_S(e)) \).
   \( (\forall x, y \in G)(\mu_S(x) = \mu_S(e) \Rightarrow \mu_S(x \odot y) \geq \mu_S(y)) \).

2. \( (\forall x, y \in G), (\nu_S(x \odot y) = \nu_S(y) \Rightarrow \nu_S(x) = \nu_S(e)) \).
   \( (\forall x, y \in G)(\nu_S(x) = \nu_S(e) \Rightarrow \nu_S(x \odot y) \geq \nu_S(y)) \).

3. \( (\forall x, y \in G), (\lambda_S(x \odot y) = \lambda_S(y) \Rightarrow \lambda_S(x) = \lambda_S(e)) \).
   \( (\forall x, y \in G)(\lambda_S(x) = \lambda_S(e) \Rightarrow \lambda_S(x \odot y) \leq \lambda_S(y)) \).

**Proof.**

1. Assume that \( \mu_S(x \odot y) = \mu_S(y) \), for all \( x, y \in G \). Taking \( y = e \) and using (K4) of Definition \( K \), we have \( \mu_S(x) = \mu_S(x \odot e) = \mu_S(e) \). Let for \( x, y \in G \) be such that \( \mu_S(x) = \mu_S(e) \).
   Then \( \mu_S(x \odot y) \geq \min\{\mu_S(x), \mu_S(y)\} = \min\{\mu_S(e), \mu_S(y)\} = \mu_S(y) \).

2. Again, assume that \( \nu_S(x \odot y) = \nu_S(y) \), for all \( x, y \in G \). Taking \( y = e \) and by (K4) of Definition \( K \), we have \( \nu_S(x) = \nu_S(x \odot e) = \nu_S(e) \). Also let \( x, y \in G \) be such that \( \nu_S(x) = \nu_S(e) \).
   Then \( \nu_S(x \odot y) \geq \min\{\nu_S(x), \nu_S(y)\} = \min\{\nu_S(e), \nu_S(y)\} = \nu_S(y) \).

3. Consider that \( \lambda_S(x \odot y) = \lambda_S(y) \), for all \( x, y \in G \). Taking \( y = e \) and again by (K4) of Definition \( K \), we have \( \lambda_S(x) = \lambda_S(x \odot e) = \lambda_S(e) \). Let \( x, y \in G \) be such that \( \lambda_S(x) = \lambda_S(e) \).
   Then \( \lambda_S(x \odot y) \leq \max\{\lambda_S(x), \lambda_S(y)\} = \max\{\lambda_S(e), \lambda_S(y)\} = \lambda_S(y) \).
   This completes the proof. \( \square \)
Definition 3.4. Let \( S = (\mu_S, \nu_S, \lambda_S) \) be a spherical fuzzy set in a \( K \)-algebra \( K \) and let \((\alpha, \beta, \gamma) \in [0, 1] \times [0, 1] \times [0, 1] \) with \( \alpha^2 + \beta^2 + \gamma^2 \leq 1 \). Then level subset of \( S \) is defined by:

\[
S_{(\alpha, \beta, \gamma)} = \{ x \in G \mid \mu_S(x) \geq \alpha, \nu_S(x) \geq \beta, \lambda_S(x) \leq \gamma \} \\
= \{ x \in G \mid \mu_S(x) \geq \alpha \} \cap \{ x \in G \mid \nu_S(x) \geq \beta \} \cap \{ x \in G \mid \lambda_S(x) \leq \gamma \}
\]

\[= \cup(\mu_S, \alpha) \cap \cup(\nu_S, \beta) \cap L(\lambda_S, \gamma),\]

is called \((\alpha, \beta, \gamma)\)-level subset of spherical fuzzy set \( S \).

The set of all \((\alpha, \beta, \gamma) \in \text{Im}(\mu_S) \times \text{Im}(\nu_S) \times \text{Im}(\lambda_S)\) is known as an image of \( S = (\mu_S, \nu_S, \lambda_S) \). The set \( S_{(\alpha, \beta, \gamma)} = \{ x \in G \mid \mu_S(x) > \alpha, \nu_S(x) > \beta, \lambda_S(x) < \gamma \} \) is known as strong \((\alpha, \beta, \gamma)\)-level subset of \( S \).

Proposition 3.5. If \( S = (\mu_S, \nu_S, \lambda_S) \) is a spherical fuzzy \( K \)-subalgebra of \( K \), then the level subsets \( \cup(\mu_S, \alpha) \) and \( \cup(\nu_S, \beta) \) are \( K \)-subalgebras of \( K \), for every \((\alpha, \beta, \gamma) \in \text{Im}(\mu_S) \times \text{Im}(\nu_S) \times \text{Im}(\lambda_S) \subseteq [0, 1] \), where \( \text{Im}(\mu_S) \), \( \text{Im}(\nu_S) \) and \( \text{Im}(\lambda_S) \) are strong \((\alpha, \beta, \gamma)\)-level subsets of \( S \).

Proof. Assume that \( S = (\mu_S, \nu_S, \lambda_S) \) is a spherical fuzzy \( K \)-subalgebra of \( K \) and let \((\alpha, \beta, \gamma) \in \text{Im}(\mu_S) \times \text{Im}(\nu_S) \times \text{Im}(\lambda_S)\) be such that \( \cup(\mu_S, \alpha) \neq \emptyset, \cup(\nu_S, \beta) \neq \emptyset \) and \( L(\lambda_S, \gamma) \neq \emptyset \). Now to prove that \( \cup(\mu_S, \alpha) \cap \cup(\nu_S, \beta) \) and \( L(\lambda_S, \gamma) \) are \( K \)-subalgebras. Let for \( x, y \in \cup(\mu_S, \alpha), \mu_S(x) \geq \alpha \) and \( \mu_S(y) \geq \alpha \). It follows from Definition 3.1 that \( \mu_S(x \odot y) \geq \min\{\mu_S(x), \mu_S(y)\} \geq \alpha \). It implies that \( x \odot y \in \cup(\mu_S, \alpha) \). Hence \( \cup(\mu_S, \alpha) \) is a level \( K \)-subalgebra of \( K \). Similar result can be proved for \( \cup(\nu_S, \beta) \) and \( L(\lambda_S, \gamma) \).

\[\square\]

Theorem 3.6. Let \( S = (\mu_S, \nu_S, \lambda_S) \) be a spherical fuzzy \( k \)-subalgebra and \((\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2) \in \text{Im}(\mu_S) \times \text{Im}(\nu_S) \times \text{Im}(\lambda_S) \) with \( \alpha_j^2 + \beta_j^2 + \gamma_j^2 \leq 1 \) for \( j = 1, 2 \). If \((\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)\), then \( S_{(\alpha_1, \beta_1, \gamma_1)} = S_{(\alpha_2, \beta_2, \gamma_2)} \).

Proof. If \((\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)\), then clearly \( S_{(\alpha_1, \beta_1, \gamma_1)} = S_{(\alpha_2, \beta_2, \gamma_2)} \). Assume that \( S_{(\alpha_1, \beta_1, \gamma_1)} = S_{(\alpha_2, \beta_2, \gamma_2)} \). Since \((\alpha_1, \beta_1, \gamma_1) \in \text{Im}(\mu_S) \times \text{Im}(\nu_S) \times \text{Im}(\lambda_S)\), there exists \( x \in G \) such that \( \mu_S(x) = \alpha_1, \nu_S(x) = \beta_1 \) and \( \lambda_S(x) = \gamma_1 \). It follows that \( x \in S_{(\alpha_1, \beta_1, \gamma_1)} = S_{(\alpha_2, \beta_2, \gamma_2)} \) so that \( \alpha_1 = \mu_S(x) \geq \alpha_2, \beta_1 = \nu_S(x) \geq \beta_2 \) and \( \gamma_1 = \lambda_S(x) \leq \gamma_2 \).

Also, \((\alpha_2, \beta_2, \gamma_2) \in \text{Im}(\mu_S) \times \text{Im}(\nu_S) \times \text{Im}(\lambda_S)\), there exists \( y \in G \) such that \( \mu_S(y) = \alpha_2, \nu_S(y) = \beta_2 \) and \( \lambda_S(y) = \gamma_2 \). It follows that \( y \in S_{(\alpha_2, \beta_2, \gamma_2)} = S_{(\alpha_1, \beta_1, \gamma_1)} \) so that \( \alpha_2 = \mu_S(y) \geq \alpha_1, \beta_2 = \nu_S(y) \geq \beta_1 \) and \( \gamma_2 = \lambda_S(y) \leq \gamma_1 \). Hence \((\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)\).

\[\square\]

Theorem 3.7. Let \( K \) be a \( K \)-algebra. Given a chain of \( K \)-subalgebras: \( S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n = G \). Then there exists a spherical fuzzy \( K \)-subalgebra whose level \( K \)-subalgebras are exactly the \( K \)-subalgebras in this chain.

Proof. Let \( \{\alpha_k \mid k = 0, 1, \ldots, n\}, \{\beta_k \mid k = 0, 1, \ldots, n\} \) be finite decreasing sequences and \( \{\gamma_k \mid k = 0, 1, \ldots, n\} \) be finite increasing sequence in \([0, 1]\) such that \( \alpha_i^2 + \beta_i^2 + \gamma_i^2 \leq 1 \), for \( i = 0, 1, 2, \ldots, n \). Let \( S = (\mu_S, \nu_S, \lambda_S) \) be a spherical fuzzy set in \( K \) defined by \( \mu_S(S_0) = \alpha_0, \nu_S(S_0) = \beta_0, \lambda_S(S_0) = \gamma_0, \mu_S(S_k \setminus S_{k-1}) = \alpha_k, \nu_S(S_k \setminus S_{k-1}) = \beta_k \) and \( \lambda_S(S_k \setminus S_{k-1}) = \gamma_k \), for \( 0 < k \leq n \). We claim that \( S = (\mu_S, \nu_S, \lambda_S) \) is a spherical fuzzy \( K \)-subalgebra of \( K \). Let \( x, y \in G \). If \( x, y \in S_k \setminus S_{k-1} \), then it implies that \( \mu_S(x) = \alpha_k, \nu_S(x) = \beta_k = \nu_S(y) \) and \( \lambda_S(x) = \gamma_k = \lambda_S(y) \). Since each \( S_k \) is a \( K \)-subalgebra, it follows that \( x \odot y \in S_k \) so that either \( x \odot y \in S_k \) or \( x \odot y \in S_{k-1} \). In any
case, we conclude that:

\[
\begin{align*}
\mu_S(x \circ y) &\geq \alpha_k = \min\{\mu_S(x), \mu_S(y)\}, \\
\nu_S(x \circ y) &\geq \beta_k = \min\{\nu_S(x), \nu_S(y)\}, \\
\lambda_S(x \circ y) &\leq \gamma_k = \max\{\lambda_S(x), \lambda_S(y)\}.
\end{align*}
\]

For \(i > j\), if \(x \in S_i \setminus S_{i-1}\) and \(y \in S_j \setminus S_{j-1}\), then \(\mu_S(x) = \alpha_i, \mu_S(y) = \alpha_j, \nu_S(x) = \beta_i, \nu_S(y) = \beta_j\) and \(\lambda_S(x) = \gamma_i, \lambda_S(y) = \gamma_j\). Hence \(x \circ y \in S_i\) because \(S_i\) is a \(K\)-subalgebra and \(S_j \subset S_i\). It follows that:

\[
\begin{align*}
\mu_S(x \circ y) &\geq \alpha_i = \min\{\mu_S(x), \mu_S(y)\}, \\
\nu_S(x \circ y) &\geq \beta_i = \min\{\nu_S(x), \nu_S(y)\}, \\
\lambda_S(x \circ y) &\leq \gamma_i = \max\{\lambda_S(x), \lambda_S(y)\}.
\end{align*}
\]

Thus, \(S = (\mu_S, \nu_S, \lambda_S)\) is a spherical fuzzy \(K\)-subalgebra of \(K\) and all its non-empty level subsets are level \(K\)-subalgebras of \(K\). Since \(\text{Im}(\mu_S) = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}, \text{Im}(\nu_S) = \{\beta_0, \beta_1, \ldots, \beta_n\}, \text{Im}(\lambda_S) = \{\gamma_0, \gamma_1, \ldots, \gamma_n\}\). Therefore, the level \(K\)-subalgebras of \(S = (\mu_S, \nu_S, \lambda_S)\) are given by the chain of \(K\)-subalgebras:

\[
\begin{align*}
\cup(\mu_S, \alpha_0) &\subset \cup(\mu_S, \alpha_1) \subset \cdots \subset \cup(\mu_S, \alpha_n) = G, \\
\cup(\nu_S, \beta_0) &\subset \cup(\nu_S, \beta_1) \subset \cdots \subset \cup(\nu_S, \beta_n) = G, \\
L(\lambda_S, \gamma_0) &\subset L(\lambda_S, \gamma_1) \subset \cdots \subset L(\lambda_S, \gamma_n) = G,
\end{align*}
\]

respectively. Indeed,

\[
\begin{align*}
\cup(\mu_S, \alpha_0) &= \{x \in G \mid \mu_S(x) \geq \alpha_0\} = S_0, \\
\cup(\nu_S, \beta_0) &= \{x \in G \mid \nu_S(x) \geq \beta_0\} = S_0, \\
L(\lambda_S, \gamma_0) &= \{x \in G \mid \lambda_S(x) \leq \gamma_0\} = S_0.
\end{align*}
\]

Now we prove that \(\cup(\mu_S, \alpha_k) = S_k, \cup(\nu_S, \beta_k) = S_k\) and \(L(\lambda_S, \gamma_k) = S_k\), for \(0 < k \leq n\). Clearly, \(S_k \subseteq \cup(\mu_S, \alpha_k), S_k \subseteq \cup(\nu_S, \beta_k)\) and \(S_k \subseteq L(\lambda_S, \gamma_k)\). If \(x \in \cup(\mu_S, \alpha_k)\), then \(\mu_S(x) \geq \alpha_k\) and so \(x \notin S_i\), for \(i > k\). Hence \(\mu_S(x) \in \{\alpha_0, \alpha_1, \ldots, \alpha_k\}\) which implies that \(x \in S_i\), for some \(i \leq k\) since \(S_i \subseteq S_k\). It follows that \(x \in S_i\).

Consequently, \(\cup(\mu_S, \alpha_k) = S_k\) for some \(0 < k \leq n\). Similar case can be proved for \(\cup(\nu_S, \beta_k) = S_k\).

Now if \(y \notin L(\lambda_S, \gamma_k)\), then \(\lambda_S(x) \leq \gamma_k\) and so \(y \notin S_i\), for some \(j \leq k\). Thus, \(\lambda_S(x) \in \{\gamma_0, \gamma_1, \ldots, \gamma_k\}\) which implies that \(x \in S_j\), for some \(j \leq k\) since \(S_j \subseteq S_k\). It follows that \(y \notin S_k\). Consequently, \(L(\lambda_S, \gamma_k) = S_k\), for some \(0 < k \leq n\). This completes the proof. \(\square\)

**Definition 3.8.** Let \(K_1 = (G_1, \cdot, \circ, e_1)\) and \(K_2 = (G_2, \cdot, \circ, e_2)\) be two \(K\)-algebras and let \(\phi\) be a mapping from \(K_1\) into \(K_2\). If \(B = (\mu_B, \nu_B, \lambda_B)\) is a spherical fuzzy \(K\)-subalgebra of \(K_2\), then the preimage of \(B = (\mu_B, \nu_B, \lambda_B)\) under \(\phi\) is a spherical fuzzy \(K\)-subalgebra of \(K_1\) defined by \(\phi^{-1}(\mu_B)(x) = \mu_B(\phi(x)), \phi^{-1}(\nu_B)(x) = \nu_B(\phi(x))\) and \(\phi^{-1}(\lambda_B)(x) = \lambda_B(\phi(x))\) for all \(x \in G_1\).

**Theorem 3.9.** Let \(\phi : K_1 \to K_2\) be an epimorphism of \(K\)-algebras. If \(B = (\mu_B, \nu_B, \lambda_B)\) is a spherical fuzzy \(K\)-subalgebra of \(K_2\), then \(\phi^{-1}(B)\) is a spherical fuzzy \(K\)-subalgebra of \(K_1\).

**Proof.** It is easy to see that \(\phi^{-1}(\mu_B)(e) \geq \phi^{-1}(\nu_B)(e) \geq \phi^{-1}(\lambda_B)(e)\) and \(\phi^{-1}(\lambda_B)(e) \leq \phi^{-1}(\nu_B)(e)\) for all \(x \in G_1\). Let \(x, y \in G_1\), then

\[
\phi^{-1}(\mu_B)(x \circ y) = \mu_B(\phi(x \circ y)) = \mu_B(\phi(x) \circ \phi(y)) \geq \min\{\mu_B(\phi(x)), \mu_B(\phi(y))\} = \min\{\phi^{-1}(\mu_B)(x), \phi^{-1}(\mu_B)(y)\}.
\]
In similar way, we can verify other conditions of Definition 3.11. Thus, we conclude that $\phi^{-1}(B)$ is a spherical fuzzy $K$-subalgebra of $K_1$.

**Theorem 3.10.** Let $\phi : K_1 \to K_2$ be an epimorphism of $K$-algebras. If $B = (\mu_B, \nu_B, \lambda_B)$ is a spherical fuzzy $K$-subalgebra of $K_2$ and $S = (\mu_S, \nu_S, \lambda_S)$ is the preimage of $B$ under $\phi$. Then $S$ is a spherical fuzzy $K$-subalgebra of $K_1$.

**Proof.** It is easy to see that $\mu_S(e) \geq \mu_S(x)$, $\nu_S(e) \geq \nu_S(x)$ and $\lambda_S(e) \leq \lambda_S(x)$, for all $x \in G_1$. Now for any $x, y \in G_1$, we have

$$
\mu_S(x \circ y) = \mu_B(\phi(x \circ y)) = \mu_B(\phi(x) \odot \phi(y)) \\
\geq \min\{\mu_B(\phi(x)), \mu_B(\phi(y))\} = \min\{\mu_S(x), \mu_S(y)\}.
$$

In similar way, we can easily verify other conditions of Definition 3.11. Hence $S$ is a spherical fuzzy $K$-subalgebra of $K_1$.

**Definition 3.11.** Let a mapping $\phi : K_1 \to K_2$ from $K_1$ into $K_2$ of $K$-algebras and let $S = (\mu_S, \nu_S, \lambda_S)$ be a spherical fuzzy set of $K_2$. The map $S' = (\mu_S', \nu_S', \lambda_S')$ is called the preimage of $S$ under $\phi$, if $\mu_S'(x) = \mu_S(\phi(x))$, $\nu_S'(x) = \nu_S(\phi(x))$ and $\lambda_S'(x) = \lambda_S(\phi(x))$ for all $x \in G_1$.

**Proposition 3.12.** Let $\phi : K_1 \to K_2$ be an epimorphism of $K$-algebras. If $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy $K$-subalgebra of $K_2$, then $S' = (\mu_S', \nu_S', \lambda_S')$ is a spherical fuzzy $K$-subalgebra of $K_1$.

**Proof.** For any $x \in G_1$, we have

$$
\mu_S'(e_1) = \mu_S(\phi(e_1)) = \mu_S'(e_2) \geq \mu_S(\phi(x)) = \mu_S'(x).
$$

For any $x, y \in G_1$, since $S$ ia a spherical fuzzy $K$-subalgebra of $K_2$,

$$
\mu_S'(x \circ y) = \mu_S(\phi(x \circ y)) = \mu_S(\phi(x) \odot \phi(y)) \\
\geq \min\{\mu_S(\phi(x)), \mu_S(\phi(y))\} = \min\{\mu_S'(x), \mu_S'(y)\}.
$$

Using the similar arguments, other conditions of Definition 3.11 can be verified. Hence $S' = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy $K$-subalgebra of $K_1$.

**Proposition 3.13.** Let $\phi : K_1 \to K_2$ be an epimorphism of $K$-algebras. If $S' = (\mu_S', \nu_S', \lambda_S')$ is a spherical fuzzy $K$-subalgebra of $K_2$, then $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy $K$-subalgebra of $K_1$.

**Proof.** Using the similar arguments as used in Proposition 3.12, all conditions of Definition 3.11 can easily be verified. Hence we omit the proof.

From Proposition 3.12 and Proposition 3.13, we conclude the following result.

**Theorem 3.14.** Let $\phi : K_1 \to K_2$ be an epimorphism of $K$-algebras. Then $S' = (\mu_S', \nu_S', \lambda_S')$ is a spherical fuzzy $K$-subalgebra of $K_1$ if and only if $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy $K$-subalgebra of $K_2$.

**Definition 3.15.** A spherical fuzzy $K$-subalgebra $S = (\mu_S, \nu_S, \lambda_S)$ of a $K$-algebra $K$ is called characteristic if $\mu_S(\phi(x)) = \mu_S(x)$, $\nu_S(\phi(x)) = \nu_S(x)$ and $\lambda_S(\phi(x)) = \lambda_S(x)$ for all $x \in G$ and $\phi \in \text{Aut}(K)$. 
Definition 3.16. A $K$-subalgebra $S$ of a $K$-algebra $K$ is said to be fully invariant if $\phi(S) \subseteq S$, for all $\phi \in \text{End}(K)$, where $\text{End}(K)$ is the set of all endomorphisms of a $K$-algebra $K$. A spherical fuzzy $K$-subalgebra $S = (\mu_S, \nu_S, \lambda_S)$ of a $K$-algebra $K$ is called fully invariant if $\mu_S(\phi(x)) \leq \mu_S(x)$, $\nu_S(\phi(x)) \leq \nu_S(x)$ and $\lambda_S(\phi(x)) \leq \lambda_S(x)$ for all $x \in G$ and $\phi \in \text{End}(K)$.

Definition 3.17. Let $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ and $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$ be spherical fuzzy $K$-subalgebras of $K$. Then $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ is said to be the same type of $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$ if there exists $\phi \in \text{Aut}(K)$ such that $S_1 = S_2 \circ \phi$, i.e., $\mu_{S_1}(x) = \mu_{S_2}(\phi(x))$, $\nu_{S_1}(x) = \nu_{S_2}(\phi(x))$ and $\lambda_{S_1}(x) = \lambda_{S_2}(\phi(x))$ for all $x \in G$.

Theorem 3.18. Let $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ and $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$ be spherical fuzzy $K$-subalgebras of $K$. Then $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ is a spherical fuzzy $K$-subalgebra having the same type of $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$ if and only if $S_1$ is isomorphic to $S_2$.

Proof. The sufficient condition holds trivially, so we only need to prove the necessary condition. Let $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ be a spherical fuzzy $K$-subalgebra having same type of $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$. Then there exists $\phi \in \text{Aut}(K)$ such that $\mu_{S_1}(x) = \mu_{S_2}(\phi(x))$, $\nu_{S_1}(x) = \nu_{S_2}(\phi(x))$ and $\lambda_{S_1}(x) = \lambda_{S_2}(\phi(x))$ for all $x \in G$.

Let $f : S_1(K) \to S_2(K)$ be a mapping defined by $f(S_1(x)) = S_2(\phi(x))$ for all $x \in G$, that is, $f(\mu_{S_1}(x)) = \mu_{S_2}(\phi(x))$, $f(\nu_{S_1}(x)) = \nu_{S_2}(\phi(x))$ and $f(\lambda_{S_1}(x)) = \lambda_{S_2}(\phi(x))$, for all $x \in G$.

Clearly, $f$ is surjective since if $f(\mu_{S_1}(x)) = f(\mu_{S_1}(y))$ for all $x, y \in G$, then $\mu_{S_1}(x) = \mu_{S_1}(y)$. Similarly, we can prove $\nu_{S_1}(x) = \nu_{S_1}(y)$, $\lambda_{S_1}(x) = \lambda_{S_1}(y)$. Hence $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ is isomorphic to $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$.

We now develop $(\tilde{a}, \tilde{b})$-spherical fuzzy $K$-algebras.

Definition 3.19. A spherical fuzzy set $S = (\mu_S, \nu_S, \lambda_S)$ in a set $G$ is called an $(\tilde{a}, \tilde{b})$-spherical fuzzy $K$-subalgebra of $K$ if it satisfies the following condition:

$$u_{(\alpha_1, \beta_1, \gamma_1)} \tilde{a}S, \ v_{(\alpha_2, \beta_2, \gamma_2)} \tilde{a}S \Rightarrow (u \odot v)_{(\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2))} \tilde{b}S$$

for all $u, v \in G, \alpha_1, \alpha_2 \in (0, 1], \beta_1, \beta_2 \in (0, 1], \gamma_1, \gamma_2 \in [0, 1)$.

Different spherical fuzzy $K$-subalgebras can be built by denoting $\tilde{a}$ and $\tilde{b}$ by any one of $\in, \notin, \forall q, \in \forall q$ unless otherwise specified.

Remark 3.20. Every $(\in, \notin)$-spherical fuzzy $K$-subalgebra is, in fact, a spherical fuzzy $K$-subalgebra.

Proposition 3.21. Every $(\in, \notin)$-spherical fuzzy $K$-subalgebra is an $(\in, \notin)$-spherical fuzzy $K$-subalgebra.

Proof. Let $S = (\mu_S, \nu_S, \lambda_S)$ be a spherical fuzzy $K$-subalgebra of $K$. Let $u, v \in G$ and $\alpha_1, \alpha_2 \in (0, 1], \beta_1, \beta_2 \in (0, 1], \gamma_1, \gamma_2 \in [0, 1)$ be such that $u_{(\alpha_1, \beta_1, \gamma_1)} \in S, \ v_{(\alpha_2, \beta_2, \gamma_2)} \in S$. Then $u_{(\alpha_1, \beta_1, \gamma_1)} \in S, \ v_{(\alpha_2, \beta_2, \gamma_2)} \in S \Rightarrow (u \odot v)_{(\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2))} \in \forall q \ S$. Hence $S$ is an $(\in, \notin)$-spherical fuzzy $K$-subalgebra of $K$. 

\[\square\]
Every $(\in,q,\in,q)$-spherical fuzzy $K$-subalgebra is an $(\in,\in,q)$-spherical fuzzy $K$-subalgebra of $\mathcal{K}$.

Proof. By similar arguments as used in Proposition 3.22, it can be proved easily. \hfill \qed

**Definition 3.23.** Let $S=(\mu_S,\nu_S,\lambda_S)$ be a spherical fuzzy set in $G$. The set $\mathcal{S} = \{u \in G \mid \mu_S(u) \neq 0, \nu_S(u) \neq 0, \lambda_S(u) \neq 0\}$ is called the support of $S$.

**Lemma 3.24.** If $S$ is a non-zero $(\in,\in)$-spherical fuzzy $K$-subalgebra of $\mathcal{K}$, then $\mathcal{S}$ is a $K$-subalgebra of $\mathcal{K}$.

Proof. Let $S=(\mu_S,\nu_S,\lambda_S)$ be a non-zero $(\in,\in)$-spherical fuzzy $K$-subalgebra of $\mathcal{K}$ and let $u,v \in \mathcal{S}$. Then $\mu_S(u) \neq 0$ and $\mu_S(v) \neq 0$, $\nu_S(u) \neq 0$ and $\nu_S(v) \neq 0$, and $\lambda_S(u) \neq 0$, $\lambda_S(v) \neq 0$. If $\mu_S(u \circ v) = 0, \nu_S(u \circ v) = 0$ and $\lambda_S(u \circ v) = 0$. Since $u_{\mu_S}(u) \in S$, $v_{\mu_S}(v) \in S$, $u_{\nu_S}(u) \in S$, and $v_{\nu_S}(v) \in S$ but

$$(u \circ v)_{(\min(\mu_S(u),\mu_S(v)),\min(\nu_S(u),\nu_S(v)),\max(\lambda_S(u),\lambda_S(v)))} \notin S.$$  

Since $\mu_S(u \circ v) = 0, \nu_S(u \circ v) = 0$ and $\lambda_S(u \circ v) = 0$, a contradiction. Hence $\mu_S(u \circ v) \neq 0, \nu_S(u \circ v) \neq 0$ and $\lambda_S(u \circ v) \neq 0$ which shows that $(u \circ v) \in \mathcal{S}$, consequently $\mathcal{S}$ is a $K$-subalgebra of $\mathcal{S}$.

The proofs of the following results are straightforward, hence we omit.

(a) If $S$ is a non-zero $(\in,q)$-spherical fuzzy $K$-subalgebra of $\mathcal{K}$, then $\mathcal{S}$ is a $K$-subalgebra of $\mathcal{K}$.

(b) If $S$ is a non-zero $(q,\in)$-spherical fuzzy $K$-subalgebra of $\mathcal{K}$, then $\mathcal{S}$ is a $K$-subalgebra of $\mathcal{K}$.

(c) If $S$ is a non-zero $(q,q)$-spherical fuzzy $K$-subalgebra of $\mathcal{K}$, then $\mathcal{S}$ is a $K$-subalgebra of $\mathcal{K}$.

(d) If $S$ is a non-zero $(\tilde{a},\tilde{b})$-spherical fuzzy $K$-subalgebra of $\mathcal{K}$, then $\mathcal{S}$ is a $K$-subalgebra of $\mathcal{K}$.

**Definition 3.25.** A spherical fuzzy set $S=(\mu_S,\nu_S,\lambda_S)$ in a $K$-algebra $\mathcal{K}$ is called an $(\in,\in,q)$-spherical fuzzy $K$-subalgebra of $\mathcal{K}$ if it satisfies the following conditions:

(a) $e_{(\alpha,\beta,\gamma)} \in S \Rightarrow (u)_{(\alpha,\beta,\gamma)} \in \forall q S$,

(b) $u_{(\alpha_1,\beta_1,\gamma_1)} \in S, v_{(\alpha_2,\beta_2,\gamma_2)} \in S \Rightarrow (u \circ v)_{(\min(\alpha_1,\alpha_2),\min(\beta_1,\beta_2),\max(\gamma_1,\gamma_2))} \in \forall q S$

for all $u,v \in G, \alpha,\alpha_1,\alpha_2 \in (0,1], \beta,\beta_1,\beta_2 \in (0,1], \gamma,\gamma_1,\gamma_2 \in [0,1]$.

**Example 3.26.** Consider a $K$-algebra $\mathcal{K} = (G, \cdot, \circ, e)$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6\}$ is the cyclic group of order 7 and $\circ$ is given by the following Cayley’s Table.

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>$e$</th>
<th>$x$</th>
<th>$x^2$</th>
<th>$x^3$</th>
<th>$x^4$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>$x^6$</td>
<td>$x^5$</td>
<td>$x^4$</td>
<td>$x^3$</td>
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<tr>
<td>$x$</td>
<td>$x$</td>
<td>$e$</td>
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<td>$x^3$</td>
<td>$x^2$</td>
<td>$x$</td>
<td>$e$</td>
</tr>
</tbody>
</table>
We define a spherical fuzzy set $S = (\mu_S, \nu_S, \lambda_S)$ in $K$ as follows:

$$
\mu_S(u) = \begin{cases} 
1 & \text{when } u = e, \\
0.7 & \text{otherwise,}
\end{cases}
\nu_S(u) = \begin{cases} 
1 & \text{when } u = e, \\
0.6 & \text{otherwise,}
\end{cases}
\lambda_S(u) = \begin{cases} 
0 & \text{when } u = e,
\end{cases}
$$

Taking $\alpha = 0.4, \alpha_1 = 0.5, \alpha_2 = 0.3, \beta = 0.5, \beta_1 = 0.6, \beta_2 = 0.3, \gamma = 0.6, \gamma_1 = 0.6, \gamma_2 = 0.5$, where $\alpha, \alpha_1, \alpha_2 \in (0, 1], \beta, \beta_1, \beta_2 \in (0, 1], \gamma, \gamma_1, \gamma_2 \in [0, 1]$.

By direct calculations, it is easy to see that $S$ is an $(\in, \in \cap)$-spherical fuzzy $K$-subalgebra of $K$.

We formulate the following results without their proofs.

**Theorem 3.27.** Let $S$ be a spherical fuzzy set in $K$. Then $S$ is an $(\in, \in \cap)$-spherical fuzzy $K$-subalgebra of $K$ if and only if

(i) $\mu_S(u) \geq \min(\mu_S(e), 0.5)$,

(ii) $\nu_S(u) \geq \min(\nu_S(e), 0.5)$,

(iii) $\lambda_S(u) \leq \max(\lambda_S(e), 0.5)$,

(iv) $\mu_S(u \circ v) \geq \min(\mu_S(u), \mu_S(v), 0.5)$,

(v) $\nu_S(u \circ v) \geq \min(\nu_S(u), \nu_S(v), 0.5)$,

(vi) $\lambda_S(u \circ v) \leq \max(\lambda_S(u), \lambda_S(v), 0.5)$

for all $u, v \in G$.

**Theorem 3.28.** Let $S$ be a spherical fuzzy set in $K$. Then $S$ is an $(\in, \in \cap)$-spherical fuzzy $K$-subalgebra of $K$ if and only if each non-empty $S_{(\alpha, \beta, \gamma)}$ is a $K$-subalgebra of $K$. For $\alpha, \beta \in (0.5, 1], \gamma \in [0.5, 1)$.

**Theorem 3.29.** Let $S$ be a spherical fuzzy set in $K$. Then $S_{(\alpha, \beta, \gamma)}$ is a $K$-subalgebra of $K$ if and only if

(a) $\max(\mu_S(u \circ v), 0.5) \geq \min(\mu_S(u), \mu_S(v))$,

(b) $\max(\mu_S(e), 0.5) \geq (\mu_S(u))$,

for all $u, v \in G$.

**Proof.** Suppose that $S_{(\alpha, \beta, \gamma)}$ is a $K$-subalgebra of $K$ and let $\max(\mu_S(u \circ v), 0.5) < \min(\mu_S(u), \mu_S(v)) = \alpha, \max(\nu_S(u \circ v), 0.5) < \min(\nu_S(u), \nu_S(v)) = \beta, \min(\lambda_S(u \circ v), 0.5) > \max(\lambda_S(u), \lambda_S(v)) = \gamma$.

Then for $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0.5, 1)$ and $u, v \in S_{(\alpha, \beta, \gamma)}$, $\mu_S(u \circ v) < \alpha, \nu_S(u \circ v) < \beta, \lambda_S(u \circ v) > \gamma$. Since $u, v \in S_{(\alpha, \beta, \gamma)}$ and $S_{(\alpha, \beta, \gamma)}$ is a $K$-subalgebra of $K$, so $u, v \in S_{(\alpha, \beta, \gamma)}$ or $\mu_S(u \circ v) \geq \alpha, \nu_S(u \circ v) \geq \beta, \lambda_S(u \circ v) \leq \gamma$, which is a contradiction.

Conversely, suppose that conditions (a) and (b) holds. Assume that $\alpha, \beta \in (0.5, 1], \gamma \in [0.5, 1)$, for $u, v \in S_{(\alpha, \beta, \gamma)}$. Then we have

$0.5 < \alpha \leq \min(\mu_S(u), \mu_S(v)) \leq \max(\mu_S(u \circ v), 0.5) \Rightarrow \mu_S(u \circ v) \geq \alpha$,

$0.5 < \beta \leq \min(\nu_S(u), \nu_S(v)) \leq \max(\nu_S(u \circ v), 0.5) \Rightarrow \nu_S(u \circ v) \geq \beta$. 

Let \( K \) be expressed or measured precisely. A fuzzy set theory is a useful mathematical tool for handling real-life data and all descriptions cannot always be expressed or measured precisely. The classical approach to formulate a mathematical model of a problem is based on the assumption that the data is always of the form \((\alpha, \beta, \gamma)\) and \(\beta \leq \gamma\).

\[ 0.5 > \gamma \geq \max(\lambda_S(u), \lambda_S(v)) \geq \min(\lambda_S(u \odot v), 0.5) \Rightarrow \lambda_S(u \odot v) \leq \gamma, \]
\[ 0.5 < \alpha \leq \mu_S(u) \leq \max(\mu_S(e), 0.5) \Rightarrow \mu_S(u) \geq \alpha, \]
\[ 0.5 < \beta \leq \nu_S(u) \leq \max(\nu_S(e), 0.5) \Rightarrow \nu_S(u) \geq \beta, \]
\[ 0.5 > \gamma > \lambda_S(u) \geq \min(\lambda_S(e), 0.5) \Rightarrow \lambda_S(u) \leq \gamma \text{ for some } u, v \in G \text{ if } u \odot v \in S(\alpha, \beta, \gamma). \]

Thus \( S(\alpha, \beta, \gamma) \) is a \( K \)-subalgebra of \( K \).

**Definition 3.30.** Let \( \epsilon_1, \epsilon_2 \in [0,1] \) and \( \epsilon_1 < \epsilon_2 \). Suppose \( S = (\mu_S, \nu_S, \lambda_S) \) is a spherical fuzzy \( K \)-subalgebra of \( K \). Then \( S \) is called a spherical fuzzy \( K \)-subalgebra with thresholds \((\epsilon_1,\epsilon_2)\) of \( K \) if

\[
\begin{align*}
\max(\mu_S(u \odot v), \epsilon_1) &\geq \min(\mu_S(u), \mu_S(v), \epsilon_2), \\
\max(\nu_S(u \odot v), \epsilon_1) &\geq \min(\nu_S(u), \nu_S(v), \epsilon_2), \\
\min(\lambda_S(u \odot v), \epsilon_1) &\leq \max(\lambda_S(u), \lambda_S(v), \epsilon_2) \text{ for all } u, v \in G.
\end{align*}
\]

**Example 3.31.** Using Example 3.20, it is easy to see that \( S = (\mu_S, \nu_S, \lambda_S) \) is a spherical fuzzy \( K \)-subalgebra with thresholds \((\epsilon_1 = 0.3, \epsilon_2 = 0.52)\) and for \((\epsilon_1 = 0.55, \epsilon_2 = 0.64)\).

**Remark 3.32.** Let for \( \epsilon_1, \epsilon_2 \in [0,1] \) and \( \epsilon_1 < \epsilon_2 \) unless otherwise specified.

(i) When \( \epsilon_1 = 0 \) and \( \epsilon_2 = 1 \) in spherical fuzzy \( K \)-subalgebra with thresholds \((\epsilon_1, \epsilon_2)\), \( S \) is an ordinary spherical fuzzy \( K \)-subalgebra.

(ii) When \( \epsilon_1 = 0 \) and \( \epsilon_2 = 0.5 \) in spherical fuzzy \( K \)-subalgebra with thresholds \((\epsilon_1, \epsilon_2)\), \( S \) is an \((\epsilon, \in \lor \forall)\)-spherical fuzzy \( K \)-subalgebra.

**Theorem 3.33.** A spherical fuzzy set \( S \) in \( K \) is a spherical fuzzy \( K \)-subalgebra with thresholds \((\epsilon_1, \epsilon_2)\) if and only if \( \cup(\mu_S, \alpha), \cup(\nu_S, \beta), L(\lambda_S, \gamma)(\neq \phi), \alpha, \beta, \gamma \in (\epsilon_1, \epsilon_2) \) is a \( K \)-subalgebra of \( K \).

**Proof.** Assume that \( S \) is a spherical fuzzy \( K \)-subalgebra with thresholds \((\epsilon_1, \epsilon_2)\). First, we prove that \( \cup(\mu_S, \alpha) \) is a \( K \)-subalgebra of \( K \), let \( u, v \in \cup(\mu_S, \alpha) \). Then \( \mu_S(u) \geq \alpha \) and \( \mu_S(v) \geq \alpha \), \( \alpha \in (\epsilon_1, \epsilon_2) \). Since \( S \) is a spherical fuzzy \( K \)-subalgebra. It follows that

\[
\max(\mu_S(u \odot v), \epsilon_1) \geq \min(\mu_S(u), \mu_S(v), \epsilon_2) = \alpha,
\]

so that \( u \odot v \in \cup(\mu_S, \alpha) \). So, \( \cup(\mu_S, \alpha) \) is a \( K \)-subalgebra of \( K \). Similarly, we can proof for \( \cup(\nu_S, \beta) \) and \( L(\lambda_S, \gamma) \). Thus, \( S(\alpha, \beta, \gamma) \) is a \( K \)-subalgebra of \( K \).

Conversely, consider that a spherical fuzzy set \( S \) be such that \( S(\alpha, \beta, \gamma) \neq \phi \) is a \( K \)-subalgebra of \( K \) for \((\epsilon_1, \epsilon_2) \in [0,1] \) and \( \epsilon_1 < \epsilon_2 \). Suppose that \( \max(\mu_S(u \odot v), \epsilon_1) < \min(\mu_S(u), \mu_S(v), \epsilon_2) = \alpha \), then \( \mu_S(u \odot v) < \alpha, u \in \cup(\mu_S, \alpha), v \in \cup(\mu_S, \alpha), \alpha \in (\epsilon_1, \epsilon_2] \). Since \( u, v \in \cup(\mu_S, \alpha) \) and \( \cup(\mu_S, \alpha) \) is a \( K \)-subalgebra, \( u \odot v \in \cup(\mu_S, \alpha) \), i.e., \( \mu_S(u \odot v) \geq \alpha \), a contradiction. Similar results can be obtained for \( \cup(\nu_S, \beta) \) and \( L(\lambda_S, \gamma) \). This completes the proof.

4 Conclusion

The classical approach to formulate a mathematical model of a problem is based on the assumption of precise data. On the contrary, real-life data is not always crisp, and all descriptions cannot always be expressed or measured precisely. A fuzzy set theory is a useful mathematical tool for handling real-life imprecise data. Since Zadeh has proposed the fuzzy set theory, it has aroused a lot of attention. The spherical fuzzy set theory is one of the most important extensions of a fuzzy set. We have applied the concept of spherical fuzzy sets to \( K \)-algebras and have constructed new spherical fuzzy \( K \)-algebras. The study of \( K \)-algebras can be extended to: (i) Spherical fuzzy soft \( K \)-algebras, (ii) \( T \)-spherical fuzzy \( K \)-algebras, (iii) \((\alpha, \beta)\)-spherical fuzzy soft \( K \)-algebras, (iv) Type 2 fuzzy \( K \)-algebras, and (v) Rough \( K \)-algebras.
References


