Journal of Algebraic Hyperstructures and Logical Algebras
Volume 2, Number 3, (2021), pp. 85-98

Spherical fuzzy K-algebras

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"This paper is dedicated to Professor Young Bae Jun on the occasion of his 70th birthday."

Abstract

In this research article, new fuzzy K-algebras, namely, spherical fuzzy K-algebras and $(\in, \in \vee q)$-spherical fuzzy K-algebras are constructed. Certain properties of these spherical fuzzy K- structures are investigated. The behavior of spherical fuzzy K-algebras under homomorphism is characterized. The spherical fuzzy K-algebra with thresholds is also delineated.

Article Information

Corresponding Author: M. Akram;
Received: August 2021;
Accepted: Invited paper;
Paper type: Original.

Keywords:

Spherical fuzzy sets, K-algebras, homomorphism, spherical fuzzy K-algebras.

1 Introduction

The notion of a K-algebra $(G, \cdot, \odot, e)$ was first introduced by Dar and Akram in [12, 11, 14, 13]. A K-algebra is an algebra built on a group $(G, \cdot, e)$ by adjoining an induced binary operation $\odot$ on G which is attached to an abstract K-algebra $(G, \cdot, \odot, e)$. This system is, in general, non-commutative and non-associative adjoint with a right identity e, if $(G, \cdot, e)$ is non-commutative. For a given group G, the K-algebra is proper if G is not an elementary Abelian 2-group. Thus, whether a K-algebra is Abelian or non-Abelian purely depends on the base group G. We call *a K-algebra on a group G* as a $K(G)$-algebra due to its structural basis G. Akram and Kim [6] proved that the K-algebra on an Abelian group is equivalent to the p-semisimple BCI-algebra [16]. Recently, Naghibi et al. [23] have introduced generalized K-algebra (briefly, gK-algebra). They also have defined gK-subalgebras, (prime) gK-ideals and quotient gK-algebras.
Algebraic structures play an emphatic role in Mathematics and the range of its applications are

https://doi.org/10.52547/HATEF.JAHLA.2.3.7

broad and multidisciplinary. Thus, algebraic structures provide a sufficient motivation to the researchers to review various concepts and stems out from the realm of abstract algebra in a broader framework of fuzzy setting. In 1965, Zadeh [27] proposed fuzzy sets for handling vague or hazy information. Subsequently, intuitionistic fuzzy sets [8], picture fuzzy sets [10], Pythagorean fuzzy sets [25] and spherical fuzzy sets [15] were extended to overcome some drawbacks of fuzzy sets. Akram et al. [3] first introduced fuzzy structures on K-algebras and investigated some of their properties. We then developed different fuzzy K-algebras [1, 2, 4, 5, 7] with other researchers worldwide by applying generalizations of Zadeh's fuzzy set theory. Further, Jun and Cho [18] and Jun and Park [19] investigated certain properties on fuzzy K-algebras. As a continuation study of K-algebras, we develop the concept of spherical fuzzy K-subalgebras, and present some of their properties. Moreover, we study the behavior of spherical fuzzy K-subalgebras under homomorphism. Finally, we discuss $(\in, \in \vee q)$-spherical fuzzy K-algebras.

For other terminologies and results that are not mentioned in this article, the readers are referred to [9, 17, 20, 21, 24, 26].

2 Preliminaries

In this section, we review a class of K-algebras and spherical fuzzy sets.

Definition 2.1. [12] *Let $(G, \cdot, e)$ be a group such that each non-identity element is not of order 2. Let a binary operation $\odot$ be introduced on the group G and defined by $\odot(x, y) = x \odot y = xy^{-1}$ for all $x,\ y \in G$. If e is the identity of the group G, then:*

(1) *e takes the shape of right $\odot$-identity and not that of left $\odot-$identity.*

(2) *Each non-identity element $(x \neq e)$ is $\odot$-involutionary because $x \odot x = xx^{-1} = e$.*

(3) *G is $\odot-$nonassociative because*
$(x \odot y) \odot z = x \odot (z \odot y^{-1}) \neq x \odot (y \odot z)$ for all $x,\ y,\ z \in G$.

(4) *G is $\odot-$noncommutative since $x \odot y \neq y \odot x$ for all $x,\ y \in G$.*

(5) *If G is an elementary Abelian 2-group, then $x \odot y\ = x \cdot y$.*

Definition 2.2. [12] *Let $(G, \cdot, e)$ be a group in which each non-identity element is not of order 2. Then a K- algebra is a structure $\mathcal{K} = (G, \cdot, \odot, e)$ on a group G in which induced binary operation $\odot : G \times G \to G$ is defined by $\odot(x, y) = x \odot y = x.y^{-1}$ and satisfies the following axioms:*

$(K1)$ $(x \odot y) \odot (x \odot z) = (x \odot ((e \odot z) \odot (e \odot y))) \odot x,$

$(K2)$ $x \odot (x \odot y) = (x \odot (e \odot y)) \odot x,$

$(K3)$ $(x \odot x) = e,$

$(K4)$ $(x \odot e) = x,$

$(K5)$ $(e \odot x) = x^{-1}$

for all $x,\ y,\ z \in G$.

Proposition 2.3. [12] *Let G be an Abelian group which is not an elementary Abelian 2-group. Then a K-algebra $\mathcal{K}$ on G is represented by the following identities:*

$(\overline{K1})$ $(x \odot y) \odot (x \odot z) = z \odot y$,

$(\overline{K2})$ $x \odot (x \odot y) = y$,

$(K3)$ $x \odot x = e$,

$(K4)$ $x \odot e = x$,

$(K5)$ $e \odot x = x^{-1}$

for all $x,\ y,\ z \in G$.

We give here some examples of K-algebras.

Example 2.4. *Let* $G = GL_n(\mathbb{R}) \subset M_n(\mathbb{R}) = \{A = [a_{ij}] : \det(A) \neq 0\}$ *be the multiplicative group of all* $n \times n$ *real non-singular matrices. Define the operation* $\odot$ *on* $GL_n(\mathbb{R})$ *by* $A \odot B = AB^{-1}$ *for all* $A, B \in GL_n(\mathbb{R})$. *Then* $(G, \cdot, \odot, e)$ *is a* K*-algebra* $\mathcal{K}$.

Example 2.5. *Let* $G = V_2(\mathbb{R}) = \{(x, y) : x, y \in \mathbb{R}\}$ *be the set of all 2-dimensional real vectors which forms an additive (+) Abelian group. Define the operation* $\odot$ *on* $V_2(\mathbb{R})$ *by* $x \odot y = x - y$ *for all* $x,\ y \in V_2(\mathbb{R})$. *Then* $(G, +, \odot, e)$ *is a* K*-algebra* $\mathcal{K}$.

Example 2.6. $(\mathbb{Z}, +, \odot, 0)$, $(\mathbb{Q}, +, \odot, 0)$, $(\mathbb{R}, +, \odot, 0)$ *form* K*-algebras by defining the operation* $\odot$ *by* $x \odot y = x - y$ *for all* $x,\ y$.

Remark 2.7. *We remark here some facts about* K*-algebras:*

(1) *Let* $G = \{e, a, b, c\}$ *be a Klein four group. Consider a* K*-algebra on* G *with the following Cayley Table:*

$\odot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

This is an improper K*-algebra on Klian four group since it is an elementary Abelian 2-group, i.e.,* $x \odot y = x.y^{-1} = x.y$.

(2) *Each non-identity element of a* K*-algebra is of order 2, i.e.,* $x \odot x = e$.

(3) *A* K*-algebra is proper if* G *is not an elementary Abelian 2-group.*

(4) *A* K*-algebra is Abelian or non-Abelian if the underline group* G *is Abelian or non-Abelian.*

Definition 2.8. [13] *A mapping* ϕ *from a* K*-algebra* $\mathcal{K}_1$ *into* $\mathcal{K}_2$ *is called a* K*-homomorphism if for every* $x_1,\ y_1 \in \mathcal{K}_1$, $\phi(x_1 \odot y_1) = \phi(x_1) \odot \phi(y_1)$, *where* $\phi(x_1),\ \phi(y_1) \in \mathcal{K}_2$.

Murali [22] proposed a definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy set. The idea of quasi-coincidence of a fuzzy point with a fuzzy set [24] played a vital role to produce different types of fuzzy algebraic structures.

Definition 2.9. [22] *A fuzzy set* μ *in a set* G *of the form*

$$\mu(y) = \begin{cases} t \in [0,1] & \text{for } x = y, \\ 0, & \text{for } x \neq y, \end{cases}$$

is said to be a fuzzy point with support x *and value* t *and is denoted by* x_t. *For a fuzzy point* x_t *and a fuzzy set* μ *in a set* G, *Pu and Liu* [24] *gave meaning to the symbol* $x_t\alpha\mu$, *where* $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. *We say that a fuzzy point* $x_t \in \mu$ *(resp.* $x_t q\mu$*) means that* $\mu(x) \geq t$ *(resp.*$\mu(x) + t > 1$*), and in this case,* x_t *is said to belong to (resp. be quasi-coincident with) a fuzzy set* μ. *Further,* $x_t \in \vee q\mu$ *means that* $x_t \in \mu$ *or* $x_t q\mu$, $x_t \in \wedge q\mu$ *means that* $x_t \in \mu$ *and* $x_t q\mu$, $x_t\overline{\alpha}\mu$ *means that* $x_t\alpha\mu$ *does not hold.*

In 2019, Gündogdu and Kahraman [15] presented the idea of spherical fuzzy sets.

Definition 2.10. [15] *Let* G *be a universe of discourse. An object of the form*

$$S = \{(x, \mu_S, \nu_S, \lambda_S) \mid x \in G\},$$

is called the spherical fuzzy set over the domain G. *The positive membership* μ_S, *neutral membership* ν_S *and the negative membership* λ_S *lie inside the unit interval* $[0,1]$ *and the condition* $\mu_S^2(x) + \nu_S^2(x) + \lambda_S^2(x) \leq 1$ *holds* $\forall x \in G$. *The* $\chi_S(x) = \sqrt{1 - \mu_S^2(x) - \nu_S^2(x) - \lambda_S^2(x)}$ *denotes the degree of refusal.*

The spherical fuzzy sets are more flexible then the existing models [8, 10, 25, 27] and has the ability to manage ambiguous data using three independent real-valued functions μ, ν and λ, with relatively relaxed condition. The graphical representation of spherical fuzzy sets is shown in Figure 1.

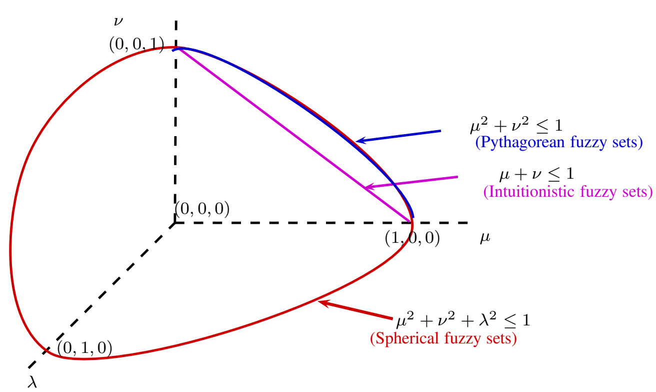

Figure 1: Graphical representation of spherical fuzzy sets

3 New fuzzy K-algebras

Definition 3.1. *A spherical fuzzy set* $S = (\mu_S, \nu_S, \lambda_S)$ *in a* K*-algebra* $\mathcal{K}$ *is called a spherical fuzzy* K*-subalgebra of* $\mathcal{K}$ *if it satisfies the following conditions:*

(a) $\mu_S(e) \geq \mu_S(x)$,

(b) $\nu_S(e) \geq \nu_S(x)$,

(c) $\lambda_S(e) \leq \lambda_S(x)$,

(d) $\mu_S(x \odot t) \geq \min\{\mu_S(x), \mu_S(y)\}$,

(e) $\nu_S(x \odot y) \geq \min\{\nu_S(x), \nu_S(y)\}$,

(f) $\lambda_S(x \odot y) \leq \max\{\lambda_S(x), \lambda_S(y)\}$

for all $x, y \in G$.

Example 3.2. *Consider a K-algebra* $\mathcal{K} = (G, \cdot, \odot, e)$ *on dihedral group* $G = \{e, a, b, c, x, y, u, v\}$, *where* $c = ab, x = a^2, y = a^3$, $u = a^2b, v = a^3b$ *and* $\odot$ *is given by the following Cayley's Table.*

$\odot$	e	a	b	c	x	y	u	v
e	e	y	b	c	x	a	u	v
a	a	e	c	u	y	x	v	b
b	b	c	e	y	u	v	x	a
c	c	u	a	e	v	b	y	x
x	x	a	u	v	e	y	b	c
y	y	x	v	b	a	e	c	u
u	u	v	x	a	b	c	e	y
v	v	b	y	x	c	u	a	e

We define a spherical fuzzy set $S = (\mu_S, \nu_S, \lambda_S)$ *in K-algebra as follows:*
$\mu_S(e) = 0.7, \nu_S(e) = 0.4, \lambda_S(e) = 0.1$,
$\mu_S(x) = 0.5, \nu_S(x) = 0.1, \lambda_S(x) = 0.3$ *for all* $x \neq e \in G$.
By direct calculations, it is easy to verify that S *is a spherical fuzzy K-subalgebra of* $\mathcal{K}$.

Proposition 3.3. *If* $S = (\mu_S, \nu_S, \lambda_S)$ *is a spherical fuzzy K-subalgebra of* $\mathcal{K}$, *then*

1. $(\forall x, y \in G), (\mu_S(x \odot y) = \mu_S(y) \Rightarrow \mu_S(x) = \mu_S(e))$.
   $(\forall x, y \in G)(\mu_S(x) = \mu_S(e) \Rightarrow \mu_S(x \odot y) \geq \mu_S(y))$.
2. $(\forall x, y \in G), (\nu_S(x \odot y) = \nu_S(y) \Rightarrow \nu_S(x) = \nu_S(e))$.
   $(\forall x, y \in G)(\nu_S(x) = \nu_S(e) \Rightarrow \nu_S(x \odot y) \geq \nu_S(y))$.
3. $(\forall x, y \in G), (\lambda_S(x \odot y) = \lambda_S(y) \Rightarrow \lambda_S(x) = \lambda_S(e))$.
   $(\forall x, y \in G)(\lambda_S(x) = \lambda_S(e) \Rightarrow \lambda_S(x \odot y) \leq \lambda_S(y))$.

Proof. 1. Assume that $\mu_S(x \odot y) = \mu_S(y)$, for all $x, y \in G$. Taking $y = e$ and using (K4) of Definition 2.2, we have $\mu_S(x) = \mu_S(x \odot e) = \mu_S(e)$. Let for $x, y \in G$ be such that $\mu_S(x) = \mu_S(e)$.
Then $\mu_S(x \odot y) \geq \min\{\mu_S(x), \mu_S(y)\} = \min\{\mu_S(e), \mu_S(y)\} = \mu_S(y)$.

2. Again, assume that $\nu_S(x \odot y) = \nu_S(y)$, for all $x, y \in G$. Taking $y = e$ and by (K4) of Definition 2.2, we have $\nu_S(x) = \nu_S(x \odot e) = \nu_S(e)$. Also let $x, y \in G$ be such that $\nu_S(x) = \nu_S(e)$.
Then $\nu_S(x \odot y) \geq \min\{\nu_S(x), \nu_S(y)\} = \min\{\nu_S(e), \nu_S(y)\} = \nu_S(y)$.

3. Consider that $\lambda_S(x \odot y) = \lambda_S(y)$, for all $x, y \in G$. Taking $y = e$ and again by (K4) of Definition 2.2, we have $\lambda_S(x) = \lambda_S(x \odot e) = \lambda_S(e)$. Let $x, y \in G$ be such that $\lambda_S(x) = \lambda_S(e)$. Then $\lambda_S(x \odot y) \leq \max\{\lambda_S(x), \lambda_S(y)\} = \max\{\lambda_S(e), \lambda_S(y)\} = \lambda_S(y)$.
This completes the proof.

□

Definition 3.4. *Let $S = (\mu_S, \nu_S, \lambda_S)$ be a spherical fuzzy set in a K-algebra $\mathcal{K}$ and let $(\alpha, \beta, \gamma) \in [0,1] \times [0,1] \times [0,1]$ with $\alpha^2 + \beta^2 + \gamma^2 \leq 1$. Then level subset of S is defined by:*

$$\begin{aligned} S_{(\alpha,\beta,\gamma)} &= \{x \in G \mid \mu_S(x) \geq \alpha, \nu_S(x) \geq \beta, \lambda_S(x) \leq \gamma\} \\ &= \{x \in G \mid \mu_S(x) \geq \alpha\} \cap \{x \in G \mid \nu_S(x) \geq \beta\} \cap \{x \in G \mid \lambda_S(x) \leq \gamma\} \\ &= \cup(\mu_S, \alpha) \cap \cup^{'}(\nu_S, \beta) \cap L(\lambda_S, \gamma), \end{aligned}$$

is called (α, β, γ) -level subset of spherical fuzzy set S.

The set of all $(\alpha, \beta, \gamma) \in Im(\mu_S) \times Im(\nu_S) \times Im(\lambda_S)$ is known as an *image* of $S = (\mu_S, \nu_S, \lambda_S)$. The set $S_{(\alpha,\beta,\gamma)} = \{x \in G \mid \mu_S(x) > \alpha, \nu_S(x) > \beta, \lambda_S(x) < \gamma\}$ is known as *strong* (α, β, γ)- level subset of S.

Proposition 3.5. *If $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}$, then the level subsets $\cup(\mu_S, \alpha) = \{x \in G \mid \mu_S(x) \geq \alpha\}$, $\cup^{'}(\nu_S, \beta) = \{x \in G \mid \nu_S(x) \geq \beta\}$ and $L(\lambda_S, \gamma) = \{x \in G \mid \lambda_S(x) \leq \gamma\}$ are k-subalgebras of $\mathcal{K}$, for every $(\alpha, \beta, \gamma) \in Im(\mu_S) \times Im(\nu_S) \times Im(\lambda_S) \subseteq [0,1]$, where $Im(\mu_S)$, $Im(\nu_S)$ and $Im(\lambda_S)$ are sets of values of μ_S, ν_S and λ_S, respectively.*

Proof. Assume that $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}$ and let $(\alpha, \beta, \gamma) \in Im(\mu_S) \times Im(\nu_S) \times Im(\lambda_S)$ be such that $\cup(\mu_S, \alpha) \neq \emptyset, \cup^{'}(\nu_S, \beta) \neq \emptyset$ and $L(\lambda_S, \gamma) \neq \emptyset$. Now to prove that $\cup, \cup^{'}$ and L are level K-subalgebras. Let for $x, y \in \cup(\mu_S, \alpha)$, $\mu_S(x) \geq \alpha$ and $\mu_S(y) \geq \alpha$. It follows from Definition 3.1 that $\mu_S(x \odot y) \geq \min\{\mu_S(x), \mu_S(y)\} \geq \alpha$. It implies that $x \odot y \in \cup(\mu_S, \alpha)$. Hence $\cup(\mu_S, \alpha)$ is a level K-subalgebra of $\mathcal{K}$. Similar result can be proved for $\cup^{'}(\nu_S, \beta)$ and $L(\lambda_S, \gamma)$. □

Theorem 3.6. *Let $S = (\mu_S, \nu_S, \lambda_S)$ be a spherical fuzzy k-subalgebra and $(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2) \in Im(\mu_S) \times Im(\nu_S) \times Im(\lambda_S)$ with $\alpha_j^2 + \beta_j^2 + \gamma_j^2 \leq 1$ for $j = 1, 2$. If $(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)$, then $S_{(\alpha_1,\beta_1,\gamma_1)} = S_{(\alpha_2,\beta_2,\gamma_2)}$.*

Proof. If $(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)$, then clearly $S_{(\alpha_1,\beta_1,\gamma_1)} = S_{(\alpha_2,\beta_2,\gamma_2)}$.
Assume that $S_{(\alpha_1,\beta_1,\gamma_1)} = S_{(\alpha_2,\beta_2,\gamma_2)}$. Since $(\alpha_1, \beta_1, \gamma_1) \in Im(\mu_S) \times Im(\nu_S) \times Im(\lambda_S)$, there exists $x \in G$ such that $\mu_S(x) = \alpha_1, \nu_S(x) = \beta_1$ and $\lambda_S(x) = \gamma_1$. It follows that $x \in S_{(\alpha_1,\beta_1,\gamma_1)} = S_{(\alpha_2,\beta_2,\gamma_2)}$ so that $\alpha_1 = \mu_S(x) \geq \alpha_2, \beta_1 = \nu_S(x) \geq \beta_2$ and $\gamma_1 = \lambda_S(x) \leq \gamma_2$.
Also, $(\alpha_2, \beta_2, \gamma_2) \in Im(\mu_S) \times Im(\nu_S) \times Im(\lambda_S)$, there exists $y \in G$ such that $\mu_S(y) = \alpha_2, \nu_S(y) = \beta_2$ and $\lambda_S(y) = \gamma_2$. It follows that $y \in S_{(\alpha_2,\beta_2,\gamma_2)} = S_{(\alpha_1,\beta_1,\gamma_1)}$ so that $\alpha_2 = \mu_S(y) \geq \alpha_1$, $\beta_2 = \nu_S(y) \geq \beta_1$ and $\gamma_2 = \lambda_S(y) \leq \gamma_1$. Hence $(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)$. □

Theorem 3.7. *Let $\mathcal{K}$ be a K-algebra. Given a chain of K-subalgebras: $S_0 \subset S_1 \subset S_2 \subset ... \subset S_n = G$. Then there exists a spherical fuzzy K-subalgebra whose level K-subalgebras are exactly the K-subalgebras in this chain.*

Proof. Let $\{\alpha_k \mid k = 0, 1, \ldots, n\}, \{\beta_k \mid k = 0, 1, \ldots, n\}$ be finite decreasing sequences and $\{\gamma_k \mid k = 0, 1, \ldots, n\}$ be finite increasing sequence in $[0,1]$ such that $\alpha_i^2 + \beta_i^2 + \gamma_i^2 \leq 1$, for $i = 0, 1, 2, \ldots, n$. Let $S = (\mu_S, \nu_S, \lambda_S)$ be a spherical fuzzy set in $\mathcal{K}$ defined by $\mu_S(S_0) = \alpha_0, \nu_S(S_0) = \beta_0, \lambda_S(S_0) = \gamma_0$,

$\mu_S(S_k \setminus S_{k-1}) = \alpha_k, \nu_S(S_k \setminus S_{k-1}) = \beta_k$ and $\lambda_S(S_k \setminus S_{k-1}) = \gamma_k$, for $0 < k \leq n$. We claim that $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}$. Let $x, y \in G$. If $x, y \in S_k \setminus S_{k-1}$, then it implies that $\mu_S(x) = \alpha_k = \mu_S(y), \nu_S(x) = \beta_k = \nu_S(y)$ and $\lambda_S(x) = \gamma_k = \lambda_S(y)$. Since each S_k is a K-subalgebra, it follows that $x \odot y \in S_k$ so that either $x \odot y \in S_k \setminus S_{k-1}$ or $x \odot y \in S_{k-1}$. In any case, we conclude that:

$$\begin{aligned} \mu_S(x \odot y) &\geq \alpha_k = \min\{\mu_S(x), \mu_S(y)\}, \\ \nu_S(x \odot y) &\geq \beta_k = \min\{\nu_S(x), \nu_S(y)\}, \\ \lambda_S(x \odot y) &\leq \gamma_k = \max\{\lambda_S(x), \lambda_S(y)\}. \end{aligned}$$

For $i > j$, if $x \in S_i \setminus S_{i-1}$ and $y \in S_j \setminus S_{j-1}$, then $\mu_S(x) = \alpha_i$, $\mu_S(y) = \alpha_j, \nu_S(x) = \beta_i, \nu_S(y) = \beta_j$ and $\lambda_S(x) = \gamma_i, \lambda_S(y) = \gamma_j$ and $x \odot y \in S_i$ because S_i is a K-subalgebra and $S_j \subset S_i$. It follows that:

$$\begin{aligned} \mu_S(x \odot y) &\geq \alpha_i = \min\{\mu_S(x), \mu_S(y)\}, \\ \nu_S(x \odot y) &\geq \beta_i = \min\{\nu_S(x), \nu_S(y)\}, \\ \lambda_S(x \odot y) &\leq \gamma_i = \max\{\lambda_S(x), \lambda_S(y)\}. \end{aligned}$$

Thus, $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}$ and all its non-empty level subsets are level K-subalgebras of $\mathcal{K}$. Since $Im(\mu_S) = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}, Im(\nu_S) = \{\beta_0, \beta_1, \ldots, \beta_n\}, Im(\lambda_S) = \{\gamma_0, \gamma_1, \ldots, \gamma_n\}$. Therefore, the level K-subalgebras of $S = (\mu_S, \nu_S, \lambda_S)$ are given by the chain of K-subalgebras:

$$\begin{aligned} \cup(\mu_S, \alpha_0) \subset \cup(\mu_S, \alpha_1) \subset \ldots \subset \cup(\mu_S, \alpha_n) = G, \\ \cup^{'}(\nu_S, \beta_0) \subset \cup^{'}(\nu_S, \beta_1) \subset \ldots \subset \cup^{'}(\nu_S, \beta_n) = G, \\ L(\lambda_S, \gamma_0) \subset L(\lambda_S, \gamma_1) \subset \ldots \subset L(\lambda_S, \gamma_n) = G, \end{aligned}$$

respectively. Indeed,

$$\begin{aligned} \cup(\mu_S, \alpha_0) = \{x \in G \mid \mu_S(x) \geq \alpha_0\} = S_0, \\ \cup^{'}(\nu_S, \beta_0) = \{x \in G \mid \nu_S(x) \geq \beta_0\} = S_0, \\ L(\lambda_S, \gamma_0) = \{x \in G \mid \lambda_S(x) \leq \gamma_0\} = S_0. \end{aligned}$$

Now we prove that $\cup(\mu_S, \alpha_k) = S_k, \cup^{'}(\nu_S, \beta_k) = S_k$ and $L(\lambda_S, \gamma_k) = S_k$, for $0 < k \leq n$. Clearly, $S_k \subseteq \cup(\mu_S, \alpha_k), S_k \subseteq \cup^{'}(\nu_S, \beta_k)$ and $S_k \subseteq L(\lambda_S, \gamma_k)$. If $x \in \cup(\mu_S, \alpha_k)$, then $\mu_S(x) \geq \alpha_k$ and so $x \notin S_i$, for $i > k$. Hence $\mu_S(x) \in \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ which implies that $x \in S_i$, for some $i \leq k$ since $S_i \subseteq S_k$. It follows that $x \in S_k$.
Consequently, $\cup(\mu_S, \alpha_k) = S_k$ for some $0 < k \leq n$. Similar case can be proved for $\cup^{'}(\nu_S, \beta_k) = S_k$. Now if $y \in L(\lambda_S, \gamma_k)$, then $\lambda_S(x) \leq \gamma_k$ and so $y \notin S_i$, for some $j \leq k$. Thus, $\lambda_S(x) \in \{\gamma_0, \gamma_1, \ldots, \gamma_k\}$ which implies that $x \in S_j$, for some $j \leq k$. Since $S_j \subseteq S_k$. It follows that $y \in S_k$. Consequently, $L(\lambda_S, \gamma_k) = S_k$, for some $0 < k \leq n$. This completes the proof. □

Definition 3.8. *Let $\mathcal{K}_1 = (G_1, \cdot, \odot, e_1)$ and $\mathcal{K}_2 = (G_2, \cdot, \odot, e_2)$ be two K-algebras and let ϕ be a mapping from $\mathcal{K}_1$ into $\mathcal{K}_2$. If $B = (\mu_B, \nu_B, \lambda_B)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_2$, then the preimage of $B = (\mu_B, \nu_B, \lambda_B)$ under ϕ is a spherical fuzzy K-subalgebra of $\mathcal{K}_1$ defined by $\phi^{-1}(\mu_B)(x) = \mu_B(\phi(x))$, $\phi^{-1}(\nu_B)(x) = \nu_B(\phi(x))$ and $\phi^{-1}(\lambda_B)(x) = \lambda_B(\phi(x))$ for all $x \in G_1$.*

Theorem 3.9. *Let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $B = (\mu_B, \nu_B, \lambda_B)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_2$, then $\phi^{-1}(B)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_1$.*

Proof. It is easy to see that $\phi^{-1}(\mu_B)(e) \geq \phi^{-1}(\mu_B)(x)$, $\phi^{-1}(\nu_B)(e) \geq \phi^{-1}(\nu_B)(x)$ and $\phi^{-1}(\lambda_B)(e) \leq \phi^{-1}(\lambda_B)(x)$ for all $x \in G_1$. Let $x, y \in G_1$, then

$$\begin{aligned}\phi^{-1}(\mu_B)(x \odot y) &= \mu_B(\phi(x \odot y)) = \mu_B(\phi(x) \odot \phi(y)) \\ &\geq \min\{\mu_B(\phi(x)), \mu_B(\phi(y))\} = \min\{\phi^{-1}(\mu_B)(x), \phi^{-1}(\mu_B)(y)\}.\end{aligned}$$

In similar way, we can verify other conditions of Definition 3.1. Thus, we conclude that $\phi^{-1}(B)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_1$.

□

Theorem 3.10. *Let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $B = (\mu_B, \nu_B, \lambda_B)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_2$ and $S = (\mu_S, \nu_S, \lambda_S)$ is the preimage of B under ϕ. Then S is a spherical fuzzy K-subalgebra of $\mathcal{K}_1$.*

Proof. It is easy to see that $\mu_S(e) \geq \mu_S(x)$, $\nu_S(e) \geq \nu_S(x)$ and $\lambda_S(e) \leq \lambda_S(x)$, for all $x \in G_1$. Now for any $x, y \in G_1$, we have

$$\begin{aligned}\mu_S(x \odot y) &= \mu_B(\phi(x \odot y)) = \mu_B(\phi(x) \odot \phi(y)) \\ &\geq \min\{\mu_B(\phi(x)), \mu_B(\phi(y))\} = \min\{\mu_S(x), \mu_S(y)\}.\end{aligned}$$

In similar way, we can easily verify other conditions of Definition 3.1. Hence S is a spherical fuzzy K-subalgebra of $\mathcal{K}_1$. □

Definition 3.11. *Let a mapping $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ from $\mathcal{K}_1$ into $\mathcal{K}_2$ of K-algebras and let $S = (\mu_S, \nu_S, \lambda_S)$ be a spherical fuzzy set of $\mathcal{K}_2$. The map $S = (\mu_S, \nu_S, \lambda_S)$ is called the preimage of S under ϕ, if $\mu_S^\phi(x) = \mu_S(\phi(x))$, $\nu_S^\phi(x) = \nu_S(\phi(x))$ and $\lambda_S^\phi(x) = \lambda_S(\phi(x))$ for all $x \in G_1$.*

Proposition 3.12. *Let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_2$, then $S^\phi = (\mu_S^\phi, \nu_S^\phi, \lambda_S^\phi)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_1$.*

Proof. For any $x \in G_1$, we have

$$\mu_S^\phi(e_1) = \mu_S(\phi(e_1)) = \mu_S(e_2) \geq \mu_S(\phi(x)) = \mu_S^\phi(x).$$

For any $x, y \in G_1$, since S ia a spherical fuzzy K-subalgebra of $\mathcal{K}_2$,

$$\begin{aligned}\mu_S^\phi(x \odot y) &= \mu_S(\phi(x \odot y)) = \mu_S(\phi(x) \odot \phi(y)) \\ &\geq \min\{\mu_S(\phi(x)), \mu_S(\phi(y))\} = \min\{\mu_S^\phi(x), \mu_S^\phi(x)\}.\end{aligned}$$

Using the similar arguments, other conditions of Definition 3.1 can be verified. Hence $S^\phi = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_1$. □

Proposition 3.13. *Let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If $S^\phi = (\mu_S^\phi, \nu_S^\phi, \lambda_S^\phi)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_2$, then $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_1$.*

Proof. Using the similar arguments as used in Proposition 3.12, all conditions of Definition 3.1 can easily be verified. Hence we omit the proof. □

From Proposition 3.12 and Proposition 3.13, we conclude the following result.

Theorem 3.14. *Let $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. Then $S^\phi = (\mu_S^\phi, \nu_S^\phi, \lambda_S^\phi)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_1$ if and only if $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}_2$.*

Definition 3.15. *A spherical fuzzy K-subalgebra $S = (\mu_S, \nu_S, \lambda_S)$ of a K-algebra $\mathcal{K}$ is called characteristic if $\mu_S(\phi(x)) = \mu_S(x)$, $\nu_S(\phi(x)) = \nu_S(x)$ and $\lambda_S(\phi(x)) = \lambda_S(x)$ for all $x \in G$ and $\phi \in Aut(\mathcal{K})$.*

Definition 3.16. *A K-subalgebra S of a K-algebra $\mathcal{K}$ is said to be fully invariant if $\phi(S) \subseteq S$, for all $\phi \in End(\mathcal{K})$, where $End(\mathcal{K})$ is the set of all endomorphisms of a K-algebra $\mathcal{K}$. A spherical fuzzy K-subalgebra $S = (\mu_S, \nu_S, \lambda_S)$ of a K-algebra $\mathcal{K}$ is called fully invariant if $\mu_S(\phi(x)) \leq \mu_S(x)$, $\nu_S(\phi(x)) \leq \nu_S(x)$ and $\lambda_S(\phi(x)) \leq \lambda_S(x)$ for all $x \in G$ and $\phi \in End(\mathcal{K})$.*

Definition 3.17. *Let $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ and $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$ be spherical fuzzy K-subalgebras of $\mathcal{K}$. Then $S_1 = (\mu_{S_1}, \nu_{S_1}, _{S_1})$ is said to be the same type of $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$ if there exists $\phi \in Aut(\mathcal{K})$ such that $S_1 = S_2 \circ \phi$, i.e., $\mu_{S_1}(x) = \mu_{S_2}(\phi(x))$, $\nu_{S_1}(x) = \nu_{S_2}(\phi(x))$ and $\lambda_{S_1}(x) = \lambda_{S_2}(\phi(x))$ for all $x \in G$.*

Theorem 3.18. *Let $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ and $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$ be spherical fuzzy K-subalgebras of $\mathcal{K}$. Then $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ is a spherical fuzzy K-subalgebra having the same type of $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$ if and only if S_1 is isomorphic to S_2.*

Proof. The sufficient condition holds trivially, so we only need to prove the necessary condition. Let $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ be a spherical fuzzy K-subalgebra having same type of $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$. Then there exists $\phi \in Aut(\mathcal{K})$ such that $\mu_{S_1}(x) = \mu_{S_2}(\phi(x))$, $\nu_{S_1}(x) = \nu_{S_2}(\phi(x))$ and $\lambda_{S_1} = \lambda_{S_2}(\phi(x))$ for all $x \in G$.
Let $f : S_1(\mathcal{K}) \to S_2(\mathcal{K})$ be a mapping defined by $f(S_1(x)) = S_2(\phi(x))$ for all $x \in G$, that is, $f(\mu_{S_1}(x)) = \mu_{S_2}(\phi(x))$, $f(\nu_{S_1}(x)) = \nu_{S_2}(\phi(x))$ and $f(\lambda_{S_1}(x)) = \lambda_{S_2}(\phi(x))$, for all $x \in G$.
Clearly, f is surjective. f is injective because if $f(\mu_{S_1}(x)) = f(\mu_{S_1}(y))$ for all $x, y \in G$, then $\mu_{S_2}(\phi(x)) = \mu_{S_2}(\phi(y))$ and hence $\mu_{S_1}(x) = \mu_{S_1}(y)$. Similarly, we can prove $\nu_{S_1}(x) = \nu_{S_1}(y)$, $\lambda_{S_1}(x) = \lambda_{S_1}(y)$.
From Definition 2.8, it follows that f is a homomorphism as for $x, y \in G$,

$$\begin{aligned} f(\mu_{S_1}(x \odot y)) &= \mu_{S_2}(\phi(x \odot y)) = \mu_{S_2}(\phi(x) \odot \phi(y)), \\ f(\nu_{S_1}(x \odot y)) &= \nu_{S_2}(\phi(x \odot y)) = \nu_{S_2}(\phi(x) \odot \phi(y)), \\ f(\lambda_{S_1}(x \odot y)) &= \lambda_{S_2}(\phi(x \odot y)) = \lambda_{S_2}(\phi(x) \odot \phi(y)). \end{aligned}$$

Hence $S_1 = (\mu_{S_1}, \nu_{S_1}, \lambda_{S_1})$ is isomorphic to $S_2 = (\mu_{S_2}, \nu_{S_2}, \lambda_{S_2})$. □

We now develop $(\tilde{a}, \tilde{b})$-spherical fuzzy K-algebras.

Definition 3.19. *A spherical fuzzy set $S = (\mu_S, \nu_S, \lambda_S)$ in a set G is called an $(\tilde{a}, \tilde{b})-$spherical fuzzy K-subalgebra of $\mathcal{K}$ if it satisfies the following condition:*

$$u_{(\alpha_1,\beta_1,\gamma_1)}\ \tilde{a}S,\ v_{(\alpha_2,\beta_2,\gamma_2)}\ \tilde{a}S \Rightarrow (u \odot v)_{(\min(\alpha_1,\alpha_2),\min(\beta_1,\beta_2),\max(\gamma_1,\gamma_2))}\ \tilde{b}S$$

for all $u, v \in G, \alpha_1, \alpha_2 \in (0, 1], \beta_1, \beta_2 \in (0, 1], \gamma_1, \gamma_2 \in [0, 1)$.

Different spherical fuzzy K-subalgebras can be built by denoting $\tilde{a}$ and $\tilde{b}$ by any one of $\in, q, \in \vee q, \in \wedge q$ unless otherwise specified.

Remark 3.20. *Every $(\in, \in)$-spherical fuzzy K-subalgebra is, in fact, a spherical fuzzy K-subalgebra.*

Proposition 3.21. *Every* $(\in, \in)$*-spherical fuzzy* K*-subalgebra is an* $(\in, \in \vee q)$*-spherical fuzzy* K*-subalgebra.*

Proof. Let $S = (\mu_S, \nu_S, \lambda_S)$ be a spherical fuzzy K-subalgebra of $\mathcal{K}$. Let $u, v \in G$ and $\alpha_1, \alpha_2 \in (0, 1], \beta_1, \beta_2 \in (0, 1], \gamma_1, \gamma_2 \in [0, 1)$ be such that $u_{(\alpha_1,\beta_1,\gamma_1)} \in S$, $v_{(\alpha_2,\beta_2,\gamma_2)} \in S$. Then $u_{(\alpha_1,\beta_1,\gamma_1)} \in S$, $v_{(\alpha_2,\beta_2,\gamma_2)} \in S \Rightarrow (u \odot v)_{(\min(\alpha_1,\alpha_2),\min(\beta_1,\beta_2),\max(\gamma_1,\gamma_2))} \in \vee q\ S$. Hence S is an $(\in, \in \vee q)$-spherical fuzzy K-subalgebra of $\mathcal{K}$. □

Proposition 3.22. *Every* $(\in \vee q, \in \vee q)$*-spherical fuzzy* K*-subalgebra is an* $(\in, \in \vee q)$*-spherical fuzzy* K*-subalgebra of* $\mathcal{K}$*.*

Proof. By similar arguments as used in Proposition 3.21, it can be proved easily. □

Definition 3.23. *Let* $S = (\mu_S, \nu_S, \lambda_S)$ *be a spherical fuzzy set in* G*. The set* $\underline{S}=\{u \in G \mid \mu_S(u) \neq 0,\ \nu_S(u) \neq 0,\ \lambda_S(u) \neq 0\}$ *is called the support of* S*.*

Lemma 3.24. *If* S *is a non-zero* $(\in, \in)$*-spherical fuzzy* K*-subalgebra of* $\mathcal{K}$*, then* $\underline{S}$ *is a* K*-subalgebra of* $\mathcal{K}$*.*

Proof. Let $S = (\mu_S, \nu_S, \lambda_S)$ be a non-zero $(\in, \in)$-spherical fuzzy K-subalgebra of $\mathcal{K}$ and let $u, v \in \underline{S}$. Then $\mu_S(u) \neq 0$ and $\mu_S(v) \neq 0$, $\nu_S(u) \neq 0$ and $\nu_S(v) \neq 0$ and $\lambda_S(u) \neq 0$, $\lambda_S(v) \neq 0$. If $\mu_S(u \odot v) = 0, \nu_S(u \odot v) = 0$ and $\lambda_S(u \odot v) = 0$. Since $u_{\mu_S}(u) \in S$, $v_{\mu_S}(v) \in S$, $u_{\nu_S}(u) \in S$ and $v_{\nu_S}(v) \in S$, $u_{\lambda_S}(u) \in S$ and $v_{\lambda_S}(v) \in S$ but

$$(u \odot v)_{(\min(\mu_S(u),\mu_S(v)),\min(\nu_S(u),\nu_S(v)),\max(\lambda_S(u),\lambda_S(v)))} \notin S.$$

Since $\mu_S(u \odot v) = 0, \nu_S(u \odot v) = 0$ and $\lambda_S(u \odot v) = 0$, a contradiction. Hence $\mu_S(u \odot v) \neq 0, \nu_S(u \odot v) \neq 0$ and $\lambda_S(u \odot v) \neq 0$ which shows that $(u \odot v) \in \underline{S}$, consequently $\underline{S}$ is a K-subalgebra of S. □

The proofs of the following results are straightforward, hence we omit.

(a) If S is a non-zero $(\in, q)$-spherical fuzzy K-subalgebra of $\mathcal{K}$, then $\underline{S}$ is a K-subalgebra of $\mathcal{K}$.

(b) If S is a non-zero $(q, \in)$-spherical fuzzy K-subalgebra of $\mathcal{K}$, then $\underline{S}$ is a K-subalgebra of $\mathcal{K}$.

(c) If S is a non-zero (q, q)-spherical fuzzy K-subalgebra of $\mathcal{K}$, then $\underline{S}$ is a K-subalgebra of $\mathcal{K}$.

(d) If S is a non-zero $(\tilde{a}, \tilde{b})$-spherical fuzzy K-subalgebra of $\mathcal{K}$, then $\underline{S}$ is a K-subalgebra of $\mathcal{K}$.

Definition 3.25. *A spherical fuzzy set* $S = (\mu_S, \nu_S, \lambda_S)$ *in a* K*-algebra* $\mathcal{K}$ *is called an* $(\in, \in \vee q)$*-spherical fuzzy* K*-subalgebra of* $\mathcal{K}$ *if it satisfies the following conditions:*

(a) $e_{(\alpha,\beta,\gamma)} \in S \Rightarrow (u)_{(\alpha,\beta,\gamma)} \in \vee q\ S,$

(b) $u_{(\alpha_1,\beta_1,\gamma_1)} \in S, v_{(\alpha_2,\beta_2,\gamma_2)} \in S \Rightarrow (u \odot v)_{(\min(\alpha_1,\alpha_2),\min(\beta_1,\beta_2),\max(\gamma_1,\gamma_2)} \in \vee q S$

for all $u, v \in G, \alpha, \alpha_1, \alpha_2 \in (0, 1], \beta, \beta_1, \beta_2 \in (0, 1], \gamma, \gamma_1, \gamma_2 \in [0, 1)$*.*

Example 3.26. *Consider a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6\}$ is the cyclic group of order 7 and $\odot$ is given by the following Cayley's Table.*

$\odot$	e	x	x^2	x^3	x^4	x^5	x^6
e	e	x^6	x^5	x^4	x^3	x^2	x
x	x	e	x^6	x^5	x^4	x^3	x^2
x^2	x^2	x	e	x^6	x^5	x^4	x^3
x^3	x^3	x^2	x	e	x^6	x^5	x^4
x^4	x^4	x^3	x^2	x	e	x^6	x^5
x^5	x^5	x^4	x^3	x^2	x	e	x^6
x^6	x^6	x^5	x^4	x^3	x^2	x	e

We define a spherical fuzzy set $S = (\mu_S, \nu_S, \lambda_S)$ in $\mathcal{K}$ as follows:

$$\mu_S(u)= \begin{cases} 1 & \text{when } u = e, \\ 0.7 & \textit{otherwise}, \end{cases} \quad \nu_S(u)= \begin{cases} 1 & \text{when } u = e, \\ 0.6 & \textit{otherwise}, \end{cases} \quad \lambda_S(u)= \begin{cases} 0 & \text{when } u = e, \\ 0.5 & \textit{otherwise}. \end{cases}$$

Taking $\alpha = 0.4, \alpha_1 = 0.5, \alpha_2 = 0.3,\ \beta = 0.5, \beta_1 = 0.6, \beta_2 = 0.3,\ \gamma = 0.6, \gamma_1 = 0.6, \gamma_2 = 0.5$, where $\alpha, \alpha_1, \alpha_2 \in (0, 1], \beta, \beta_1, \beta_2 \in (0, 1], \gamma, \gamma_1, \gamma_2 \in [0, 1)$.
By direct calculations, it is easy to see that S is an $(\in, \in \vee q)$-spherical fuzzy K-subalgebra of $\mathcal{K}$.

We formulate the following results without their proofs.

Theorem 3.27. *Let S be a spherical fuzzy set in $\mathcal{K}$. Then S is an $(\in, \in \vee q)$-spherical fuzzy K-subalgebra of $\mathcal{K}$ if and only if*

(i) $\mu_S(u) \geq \min(\mu_S(e), 0.5)$,

(ii) $\nu_S(u) \geq \min(\nu_S(e), 0.5)$,

(iii) $\lambda_S(u) \leq \max(\lambda_S(e), 0.5)$,

(iv) $\mu_S(u \odot v) \geq \min(\mu_S(u), \mu_S(v), 0.5)$,

(v) $\nu_S(u \odot v) \geq \min(\nu_S(u), \nu_S(v), 0.5)$,

(vi) $\lambda_S(u \odot v) \leq \max(\lambda_S(u), \lambda_S(v), 0.5)$

for all $u, v \in G$.

Theorem 3.28. *Let S be a spherical fuzzy set in $\mathcal{K}$. Then S is an $(\in, \in \vee q)$-spherical fuzzy K-subalgebra of $\mathcal{K}$ if and only if each non-empty $S_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$. For $\alpha, \beta \in (0.5, 1], \gamma \in [0.5, 1)$.*

Theorem 3.29. *Let S be a spherical fuzzy set in $\mathcal{K}$. Then $S_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$ if and only if*

$$\begin{aligned} (a)\ & \max(\mu_S(u \odot v), 0.5) \geq \min(\mu_S(u), \mu_S(v)), \\ & \max(\nu_S(u \odot v), 0.5) \geq \min(\nu_S(u), \nu_S(v)), \\ & \min(\lambda_S(u \odot v), 0.5) \leq \max(\lambda_S(u), \lambda_S(v)), \end{aligned}$$

$$\begin{aligned} (b)\ & \max(\mu_S(e), 0.5) \geq (\mu_S(u), \\ & \max(\nu_S(e), 0.5) \geq (\nu_S(u), \\ & \min(\lambda_S(e), 0.5) \leq (\lambda_S(u), \ \textit{for all } u, v \in G. \end{aligned}$$

Proof. Suppose that $S_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$ and let $\max(\mu_S(u\odot v), 0.5) < \min(\mu_S(u), \mu_S(v)) = \alpha$, $\max(\nu_S(u \odot v), 0.5) < \min(\nu_S(u), \nu_S(v)) = \beta$, $\min(\lambda_S(u \odot v), 0.5) > \max(\lambda_S(u), \lambda_S(v)) = \gamma$. Then for $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0.5, 1)$ and $u, v \in S_{(\alpha,\beta,\gamma)}$, $\mu_S(u \odot v) < \alpha, \nu_S(u \odot v) < \beta, \lambda_S(u \odot v) > \gamma$. Since $u, v \in S_{(\alpha,\beta,\gamma)}$ and $S_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$, so $u, v \in S_{(\alpha,\beta,\gamma)}$ or $\mu_S(u \odot v) \geq \alpha, \nu_S(u \odot v) \geq \beta, \lambda_S(u \odot v) \leq \gamma$, which is a contradiction.
Conversely, suppose that conditions (a) and (b) holds. Assume that $\alpha, \beta \in (0.5, 1], \gamma \in [0.5, 1)$, for $u, v \in S_{(\alpha,\beta,\gamma)}$. Then we have
$0.5 < \alpha \leq \min(\mu_S(u), \mu_S(v)) \leq \max(\mu_S(u \odot v), 0.5) \Rightarrow \mu_S(u \odot v) \geq \alpha$,
$0.5 < \beta \leq \min(\nu_S(u), \nu_S(v)) \leq \max(\nu_S(u \odot v), 0.5) \Rightarrow \nu_S(u \odot v) \geq \beta$,
$0.5 > \gamma \geq \max(\lambda_S(u), \lambda_S(v)) \geq \min(\lambda_S(u \odot v), 0.5) \Rightarrow \lambda_S(u \odot v) \leq \gamma$,
$0.5 < \alpha \leq \mu_S(u) \leq \max(\mu_S(e), 0.5) \Rightarrow \mu_S(u) \geq \alpha$,
$0.5 < \beta \leq \nu_S(u) \leq \max(\nu_S(e), 0.5) \Rightarrow \nu_S(u) \geq \beta$,
$0.5 > \gamma \geq \lambda_S(u) \geq \min(\lambda_S(e), 0.5) \Rightarrow \lambda_S(u) \leq \gamma$ for some $u, v \in G$ $u \odot v \in S_{(\alpha,\beta,\gamma)}$. Thus $S_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$. □

Definition 3.30. *Let $\epsilon_1, \epsilon_2 \in [0, 1]$ and $\epsilon_1 < \epsilon_2$. Suppose $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy K-subalgebra of $\mathcal{K}$. Then S is called a spherical fuzzy K-subalgebra with thresholds (ϵ_1, ϵ_2) of $\mathcal{K}$ if*

$$\begin{aligned}&\max(\mu_S(u \odot v), \epsilon_1) \geq \min(\mu_S(u), \mu_S(v), \epsilon_2),\\ &\max(\nu_S(u \odot v), \epsilon_1) \geq \min(\nu_S(u), \nu_S(v), \epsilon_2),\\ &\min(\lambda_S(u \odot v), \epsilon_1) \leq \max(\lambda_S(u), \lambda_S(v), \epsilon_2) \textit{ for all } u, v \in G.\end{aligned}$$

Example 3.31. *Using Example 3.26, it is easy to see that $S = (\mu_S, \nu_S, \lambda_S)$ is a spherical fuzzy K-subalgebra with thresholds $(\epsilon_1 = 0.3, \epsilon_2 = 0.52)$ and for $(\epsilon_1 = 0.55, \epsilon_2 = 0.64)$.*

Remark 3.32. *Let for $\epsilon_1, \epsilon_2 \in [0, 1]$ and $\epsilon_1 < \epsilon_2$ unless otherwise specified.*
(i) When $\epsilon_1 = 0$ and $\epsilon_2 = 1$ in spherical fuzzy K-subalgebra with thresholds (ϵ_1, ϵ_2), S is an ordinary spherical fuzzy K-subalgebra.
(2) When $\epsilon_1 = 0$ and $\epsilon_2 = 0.5$ in spherical fuzzy K-subalgebra with thresholds (ϵ_1, ϵ_2), S is an $(\in, \in \vee q)-$ spherical fuzzy K-subalgebra.

Theorem 3.33. *A spherical fuzzy set S in $\mathcal{K}$ is a spherical fuzzy K-subalgebra with thresholds (ϵ_1, ϵ_2) if and only if $\cup(\mu_S, \alpha)$, $\cup'(\nu_S, \beta)$, $L(\lambda_S, \gamma)(\neq \phi), \alpha, \beta, \gamma \in (\epsilon_1, \epsilon_2]$ is a K-subalgebra of $\mathcal{K}$.*

Proof. Assume that S is a spherical fuzzy K-subalgebra with thresholds (ϵ_1, ϵ_2). First, we prove that $\cup(\mu_S, \alpha)$ is a K-subalgebra of $\mathcal{K}$, let $u, v \in \cup(\mu_S, \alpha)$. Then $\mu_S(u) \geq \alpha$ and $\mu_S(v) \geq \alpha$, $\alpha \in (\epsilon_1, \epsilon_2]$. Since S is a spherical fuzzy K-subalgebra. It follows that

$$\max(\mu_S(u \odot v), \epsilon_1) \geq \min(\mu_S(u), \mu_S(v), \epsilon_2) = \alpha,$$

so that $u \odot v \in \cup(\mu_S, \alpha)$. So, $\cup(\mu_S, \alpha)$ is a K-subalgebra of $\mathcal{K}$. Similarly, we can proof for $\cup'(\nu_S, \beta)$ and $L(\lambda_S, \gamma)$. Thus, $S_{(\alpha,\beta,\gamma)}$ is a K-subalgebra of $\mathcal{K}$.
Conversely, consider that a spherical fuzzy set S be such that $S_{(\alpha,\beta,\gamma)} \neq \phi$ is a K-subalgebra of $\mathcal{K}$ for $(\epsilon_1, \epsilon_2) \in [0, 1]$ and $(\epsilon_1 < \epsilon_2)$. Suppose that $\max(\mu_S(u \odot v), \epsilon_1) < \min(\mu_S(u), \mu_S(v), \epsilon_2) = \alpha$, then $\mu_S(u \odot v) < \alpha$, $u \in \cup(\mu_S, \alpha), v \in \cup(\mu_S, \alpha), \alpha \in (\epsilon_1, \epsilon_2]$. Since $u, v \in \cup(\mu_S, \alpha)$ and $\cup(\mu_S, \alpha)$ is a K-subalgebra, $u \odot v \in \cup(\mu_S, \alpha)$, i.e., $\mu_S(u \odot v) \geq \alpha$, a contradiction. Similar results can be obtained for $\cup'(\nu_S, \beta)$ and $L(\lambda_S, \gamma)$. This completes the proof. □

4 Conclusion

The classical approach to formulate a mathematical model of a problem is based on the assumption of precise data. On the contrary, real-life data is not always crisp, and all descriptions can not always be expressed or measured precisely. A fuzzy set theory is a useful mathematical tool for handling real-life imprecise data. Since Zadeh has proposed the fuzzy set theory, it has aroused a lot of attention. The spherical fuzzy fuzzy set is one of the most important extensions of a fuzzy set. We have applied the concept of spherical fuzzy sets to K-algebras and have constructed new spherical fuzzy K-algebras. The study of K-algebras can be extended to: (i) Spherical fuzzy soft K-algebras, (ii) T-spherical fuzzy K-algebras, (iii) (α, β)-spherical fuzzy soft K-algebras, (iv) Type 2 fuzzy K-algebras, and (v) Rough K-algebras.

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