



Minimal prime ideals in hoops

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“This paper is dedicated to Professor Young Bae Jun on the occasion of his 70th birthday.”

Abstract

In this paper, we define the notion of minimal prime ideals of hoops and investigate some properties of them. Then by using the notion of annihilators, we study the relation between minimal prime ideals and annihilators. Also, we introduce the notion of zero divisors elements of hoops and prove that the set of all zero divisors of hoops is a union of all minimal prime ideals of hoop. Finally, by using the notions of minimal prime ideals and maximal ideals of hoop, we introduce two new ideals as p -ideal and m -ideal. Then we study some properties of them and investigate the relation between them and prove that every p -ideal of semi-simple hoop is an m -ideal of it.

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1 Introduction

It is well known that logic gives a technique for the artificial intelligence to make the computers simulate human being in dealing with certainty and uncertainty in information. And as uncertain information processing, non-classical logic has become a formal and useful tool for computer science to deal with uncertain information, fuzzy information and intelligent system. Various logical algebras have been proposed and researched as the semantical systems of non-classical logical systems. Among these logical algebras, residuated lattices were introduced by Ward and Dilworth in 1939 to constitute the semantics of Höhle Monoidal Logic which are the basis for the majority of formal fuzzy logic. Apart from their logical interest, residuated lattices have interesting algebraic properties and include two important classes of algebras: BL-algebras and MV-algebras. In order to study the basic logic framework of fuzzy set system, based on continuous triangle module and

under the theoretical framework of residuated lattices theory, Hájek [9], proposed a new fuzzy logic system BL-system and the corresponding logical algebraic system BL-algebra. Hoops are naturally ordered commutative residuated integral monoids, which was introduced by Bosbach in [6, 7]. In recent decades, many mathematicians have worked on it and developed structure theory by using the notion of hoop (see [1, 2, 3, 4, 5, 10, 11, 12, 13]). Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic introduced by Hájek in [9]. The notion of ideals has been introduced in many algebraic structures such as lattices, rings, MV-algebras. Ideals theory is a very effective tool for studying various algebraic and logical systems. In the theory of MV-algebras the notion of ideals is at the center and deductive systems and ideals are dual notions, while in hoop, with the lack of a suitable algebraic addition, the focus is shifted to deductive systems also called filters. So the notion of ideals is missing in hoops. In [1], Aaly and et al., defined and characterized the notion of ideals and different kinds of ideals such as implicative, maximal and prime ideals on hoops. Then they investigated the relation between them and proved that every maximal implicative ideal of a \vee -hoop with (DNP) is a prime one. Also, they defined a congruence relation on hoops by ideals and studied the quotient that is made by it. This notion helped to show that an ideal is maximal if and only if the quotient hoop is a simple MV-algebra. Also, they investigated the relationship between ideals and filters by exploiting the set of complements. In addition, in [3], Aaly and et al., introduced the notions of annihilators in hoops, investigated some related properties of them and proved that annihilators are ideals of hoop. In addition, they showed that the set of all ideals of hoop is a bounded distributive pseudo-complement lattice, and by using this result, they proved that the set of all annihilators of hoop, is a Boolean algebra. Also, they used the notion of annihilator and introduced the special kind of ideal of hoop as α -ideal and showed that the set of all α -ideals of hoop is a complete distributive lattice and consequenced that under what condition it is a Boolean algebra. Now, in this paper, we define the notion of minimal prime ideals of hoops and investigate some properties of them. Then by using the notion of annihilators, we study the relation between minimal prime ideals and annihilators. Also, we introduce the notion of zero divisors elements of hoops and prove that the set of all zero divisors of hoops is a union of all minimal prime ideals of hoop. Finally, by using the notions of minimal prime ideals and maximal ideals of hoop, we introduce two new ideals as p -ideal and m -ideal. Then we study some properties of them and investigate the relation between them and prove that every p -ideal of semi-simple hoop is an m -ideal of it.

2 Preliminaries

In this section, we gather some basic notions relevant to hoop which will need in the next sections.

A hoop is an algebraic structure $(\mathcal{H}, \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that, for all $x, y, z \in \mathcal{H}$:

(HP1) $(\mathcal{H}, \odot, 1)$ is a commutative monoid,

(HP2) $x \rightarrow x = 1$,

(HP3) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,

(HP4) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$.

On hoop \mathcal{H} we define $x \leq y$ if and only if $x \rightarrow y = 1$. Obviously (\mathcal{H}, \leq) is a poset. A bounded hoop \mathcal{H} is a hoop which has the least element such as 0 such that $0 \leq x$, for all $x \in \mathcal{H}$. We let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. Let \mathcal{H} be a bounded hoop. Define a unary operation $'$ on \mathcal{H} by, $x' = x \rightarrow 0$, for all $x \in \mathcal{H}$. If $(x')' = x$, for all $x \in \mathcal{H}$, then the bounded hoop \mathcal{H} is said to have the double negation property, or (DNP) for short.

Proposition 2.1. [6, 7, 8] *Let $(\mathcal{H}, \odot, \rightarrow, 1)$ be a hoop. Then for all $x, y, z \in \mathcal{H}$ we have:*

- (i) (\mathcal{H}, \leq) is a \wedge -semilattice with $x \wedge y = x \odot (x \rightarrow y)$,
- (ii) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$,
- (iii) $x \odot y \leq x, y$,
- (iv) $x \rightarrow 1 = 1$ and $1 \rightarrow x = x$,
- (v) $x \odot (x \rightarrow y) \leq y$,
- (vi) $x \leq y$ implies $x \odot z \leq y \odot z$, $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$,
- (vii) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.
- (viii) If \mathcal{H} is bounded, then $x \leq x''$, $x \odot x' = 0$, $x''' = x'$ and $x \leq x' \rightarrow y$.

Proposition 2.2. [8] *Let \mathcal{H} be a hoop and for any $x, y \in \mathcal{H}$, we define, $x \sqcup y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$. Then the following conditions are equivalent:*

- (i) \sqcup is an associative operation on \mathcal{H} ,
- (ii) $x \leq y$ implies $x \sqcup z \leq y \sqcup z$, for all $x, y, z \in \mathcal{H}$,
- (iii) $x \sqcup (y \wedge z) \leq (x \sqcup y) \wedge (x \sqcup z)$, for all $x, y, z \in \mathcal{H}$,
- (iv) \sqcup is the join operation on \mathcal{H} .

A hoop \mathcal{H} is called a \sqcup -hoop if \sqcup is a join operation on \mathcal{H} and \sqcup -hoop is a distributive lattice.

Proposition 2.3. [8] *Let \mathcal{H} be a \sqcup -hoop. Then, for all $x, y, z \in \mathcal{H}$ we have:*

- (i) $(x \sqcup y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
- (ii) $x \odot (y \sqcup z) = (x \odot y) \sqcup (x \odot z)$.

Note. From now on, we suppose $\mathcal{H} = (\mathcal{H}, \odot, \rightarrow, 0, 1)$ or \mathcal{H} is a bounded hoop unless otherwise state. Let I be a non-empty subset of \mathcal{H} . Then I is called an ideal of \mathcal{H} if it satisfies the following conditions:

- (I1) $0 \in I$,
- (I2) for any $x, y \in I$, $x \odot y = x' \rightarrow y \in I$,
- (I3) for any $x, y \in \mathcal{H}$, $x \leq y$ and $y \in I$ imply $x \in I$.

Obviously, \mathcal{H} and $\{0\}$ are the trivial ideals of \mathcal{H} . The set of all ideals of \mathcal{H} is denoted by $\mathcal{I}d(\mathcal{H})$. An ideal I is called proper if $I \neq \mathcal{H}$. Clearly, an ideal I is proper if and only if $1 \notin I$ (see [1]). Let $\emptyset \neq X \subseteq \mathcal{H}$. We recall that the smallest ideal containing X in \mathcal{H} is called the generated ideal by X in \mathcal{H} and it is denoted by $\langle X \rangle$. It is also the intersection of all ideals of \mathcal{H} containing X .

Theorem 2.4. [1] *Let $\emptyset \neq X \subseteq \mathcal{H}$. Then*

$$\langle X \rangle = \{a \in \mathcal{H} \mid \exists n \in \mathbb{N} \text{ such that for } x_1, x_2, \dots, x_n \in X, a \leq x_1 \odot (x_2 \odot (\dots \odot (x_{n-1} \odot x_n) \dots))\}.$$

Note. Consider $a \odot a \odot \dots \odot a = na = (a')^{n-1} \rightarrow a$. If \mathcal{H} has (DNP), then $x \odot y = y \odot x$ and $na = ((a')^n)'$.

Proposition 2.5. [1] *Let $I \in \mathcal{I}d(\mathcal{H})$ and $a \in \mathcal{H}$. Then the following statements hold,*

- (i) $\langle a \rangle = \{x \in \mathcal{H} \mid \exists n \in \mathbb{N} \text{ such that } x \leq na\}$,
- (ii) if \mathcal{H} is a hoop with (DNP), then $(I \cup \{a\}) = \{x \in \mathcal{H} \mid \exists n \in \mathbb{N} \text{ such that } x \odot (na)' \in I\}$,
- (iii) if \mathcal{H} is a \sqcup -hoop with (DNP), then $(I \cup \{x\}) \cap (I \cup \{y\}) = (I \cup \{x \wedge y\})$.

Let P be a proper ideal of \mathcal{H} . Then P is called a prime ideal of \mathcal{H} if $x \wedge y \in P$ implies $x \in P$ or $y \in P$, for any $x, y \in \mathcal{H}$. The set of all prime ideals of \mathcal{H} is denoted by $\text{Spec}(\mathcal{H})$. Let U be a proper ideal of \mathcal{H} . Then U is called a maximal ideal of \mathcal{H} if no proper ideal of \mathcal{H} strictly containing U . It means that if there exists an ideal of \mathcal{H} such as J that $U \subseteq J \subseteq \mathcal{H}$, then $U = J$ or $J = \mathcal{H}$. The set of all maximal ideals of \mathcal{H} is denoted by $\text{Max}(\mathcal{H})$ (see [1]).

Theorem 2.6. [1] Let \mathcal{H} be a \sqcup -hoop with (DNP), I be a proper ideal of \mathcal{H} and $\emptyset \neq S \subseteq \mathcal{H}$ such that $I \cap S = \emptyset$. If S is \wedge -closed, then there exists $P \in \text{Spec}(\mathcal{H})$ such that $I \subseteq P$ and $P \cap S = \emptyset$.

Theorem 2.7. [1] Let \mathcal{H} be a \sqcup -hoop with (DNP). Then $\text{Max}(\mathcal{H}) \subseteq \text{Spec}(\mathcal{H})$.

Theorem 2.8. [1] Let \mathcal{H} be a \sqcup -hoop with (DNP) and P be a proper ideal of \mathcal{H} . Then P is a prime ideal of \mathcal{H} if and only if, for any $I, J \in \text{Id}(\mathcal{H})$ such that $I \cap J \subseteq P$, we get $I \subseteq P$ or $J \subseteq P$.

Definition 2.9. [3] Let X be a non-void subset of \mathcal{H} and set $X^\perp = \{a \in \mathcal{H} \mid a \wedge x = 0, \text{ for any } x \in X\}$. Then X^\perp is called an annihilator of X .

Proposition 2.10. [3] Let X be a non-empty subset of \mathcal{H} . Then X^\perp is a proper ideal of \mathcal{H} .

3 Minimal prime ideals on hoops

In this section, we define the notion of minimal prime ideals and zero divisors elements of hoops and investigate some properties of them. Then by using the notion of annihilators, we study the relation between minimal prime ideals and annihilators. Also, we prove that the set of all zero divisors of hoops is a union of all minimal prime ideals of hoop. But before that we need some properties about prime ideals and we prove them.

Theorem 3.1. Every proper ideal of \mathcal{H} is contained in a maximal ideal of \mathcal{H} .

Proof. Suppose I is a proper ideal of \mathcal{H} . Define

$$\sum = \{Q \in \text{Id}(\mathcal{H}) \mid Q \text{ is a proper ideal of } \mathcal{H} \text{ such that } I \subseteq Q\}.$$

Since $I \in \sum$, we have $\sum \neq \emptyset$. Consider (\sum, \subseteq) . If $\{Q_i\}_{i \in I}$ is a family of proper ideals of \mathcal{H} containing I , then by Zorn's Lemma, $U = \bigcup_{i \in I} Q_i$ is a maximal element of \sum . Clearly, U is a maximal ideal of \mathcal{H} containing I . Because if there exists a proper ideal J of \mathcal{H} such that $U \subseteq J$, then $I \subseteq J$ and so $J \in \sum$ which is a contradiction with being maximal element U . Therefore, every proper ideal of \mathcal{H} is contained in a maximal ideal of \mathcal{H} . \square

Proposition 3.2. If \mathcal{H} is a chain, then every proper ideal of \mathcal{H} is prime. In addition, $\{0\} \in \text{Spec}(\mathcal{H})$.

Proof. Suppose P is a proper ideal of \mathcal{H} such that $x \wedge y \in P$, for $x, y \in \mathcal{H}$. By assumption, \mathcal{H} is a chain, thus $x \leq y$ or $y \leq x$. If $x \leq y$, then $x = x \wedge y \in P$. By the similar way, if $y \leq x$, then $y \in P$. Hence, $P \in \text{Spec}(\mathcal{H})$. In addition, $\{0\}$ is a proper ideal of \mathcal{H} , hence $\{0\} \in \text{Spec}(\mathcal{H})$. \square

Proposition 3.3. Let \mathcal{H} be a \sqcup -hoop with (DNP). Then

(i) $\bigcap \{P \mid P \in \text{Spec}(\mathcal{H})\} = \{0\}$.

(ii) If $0 \neq x \in \mathcal{H}$, then there exists $P \in \text{Spec}(\mathcal{H})$ such that $x \notin P$.

Proof. (i) Clearly, $\{0\} \subseteq \bigcap \{P \mid P \in \text{Spec}(\mathcal{H})\}$. Suppose $0 \neq x \in \bigcap \{P \mid P \in \text{Spec}(\mathcal{H})\}$. Consider $I = \{0\}$ and $S = \{x\}$. Then by Theorem 2.6, there exists $P \in \text{Spec}(\mathcal{H})$ such that $I \subseteq P$ and $x \notin P$, which is a contradiction. Therefore, $x = 0$, and so $\bigcap \{P \mid P \in \text{Spec}(\mathcal{H})\} = \{0\}$.

(ii) Let $0 \neq x \in \mathcal{H}$ such that for all $P \in \text{Spec}(\mathcal{H})$, $x \in P$. Thus by (i), we have $x \in \bigcap \{P \mid P \in \text{Spec}(\mathcal{H})\} = \{0\}$, and so $x = 0$, which is a contradiction. \square

Theorem 3.4. Let $I \in \text{Id}(\mathcal{H})$ and $P_i \in \text{Spec}(\mathcal{H})$, for $1 \leq i \leq n$ such that $I \subseteq \bigcup_{i=1}^n P_i$. Then there exists P_i , for $1 \leq i \leq n$, such that $I \subseteq P_i$.

Proof. We prove it by induction on n . If $n = 2$, then $I \subseteq P_1 \cup P_2$. We show that $I \subseteq P_1$ or $I \subseteq P_2$. Suppose $I \not\subseteq P_1$ and $I \not\subseteq P_2$. Then there exist $x_2 \in I \setminus P_1$ and $x_1 \in I \setminus P_2$. Since $I \subseteq P_1 \cup P_2$, we have $x_1 \in P_1$ and $x_2 \in P_2$. Moreover, from $I \in \mathcal{Id}(\mathcal{H})$, we get $x_1 \odot x_2 \in I \subseteq P_1 \cup P_2$. We have the following cases:

Case 1. If $x_1 \odot x_2 \in P_1$, since $x_2 \leq x_1 \odot x_2$, we get $x_2 \in P_1$, which is a contradiction.

Case 2. If $x_1 \odot x_2 \in P_2$, since $x_1 \leq x_1 \odot x_2$, we get $x_1 \in P_2$, which is a contradiction.

Hence, $x_1 \odot x_2 \notin P_1 \cup P_2$, that is a contradiction. Therefore, $I \subseteq P_1$ or $I \subseteq P_2$. Suppose that the theorem holds for $n = k$. We prove that it holds for $n = k + 1$. For this, let $I \subseteq \bigcup_{i=1}^{k+1} P_i$. If for any $1 \leq i \leq k + 1$, we have $I \not\subseteq P_i$, then there exists $x_j \in I \setminus \bigcup_{i=1, i \neq j}^{k+1} P_i$ such that $x_j \in P_j$ and $x_j \notin \bigcap_{i=1, i \neq j}^{k+1} P_i$. Thus $x_1 \wedge x_2 \wedge \dots \wedge x_k \notin P_{k+1}$, and so $x_1 \wedge x_2 \wedge \dots \wedge x_k \in \bigcap_{i=1}^k P_i \setminus P_{k+1}$ and $x_{k+1} \in P_{k+1} \setminus \bigcup_{i=1}^k P_i$. Now, let $y = (x_1 \wedge x_2 \wedge \dots \wedge x_k) \odot x_{k+1}$. From $x_j \in I \setminus \bigcup_{i=1, i \neq j}^{k+1} P_i$ and $I \in \mathcal{Id}(\mathcal{H})$, we have $y \in I$. If $y \in P_{k+1}$, then since $x_1 \wedge x_2 \wedge \dots \wedge x_k \leq y$, we get $x_1 \wedge x_2 \wedge \dots \wedge x_k \in P_{k+1}$, which is a contradiction, and so $y \notin P_{k+1}$. Since $I \subseteq \bigcup_{i=1}^{k+1} P_i$ and $y \notin P_{k+1}$, we obtain $y \in \bigcup_{i=1}^k P_i$. Then by assumption of induction, there exists $1 \leq i \leq k$ such that $I \subseteq P_i$. This complete the proof. \square

Lemma 3.5. *Suppose \mathcal{H} has (DNP). Then for any $x, y, z \in \mathcal{H}$ we have $x \wedge (y \odot z) \leq (x \wedge y) \odot (x \wedge z)$.*

Proof. Let $x, y, z \in \mathcal{H}$. Then

$$\begin{aligned}
& [x \wedge (y \odot z)] \rightarrow [(x \wedge y) \odot (x \wedge z)] \\
&= [x \wedge (y' \rightarrow z)] \rightarrow [(x \wedge y)' \rightarrow (x \wedge z)] \quad \text{by (HP3)} \\
&= (x \wedge y)' \rightarrow [(x \wedge (y' \rightarrow z)) \rightarrow (x \wedge z)] \quad \text{by Proposition 2.1(vii)} \\
&= (x \wedge y)' \rightarrow [((x \wedge (y' \rightarrow z)) \rightarrow x) \wedge ((x \wedge (y' \rightarrow z)) \rightarrow z)] \\
&= (x \wedge y)' \rightarrow [((x \wedge (y' \rightarrow z)) \rightarrow z)] \quad \text{by (HP3)} \\
&= ((x \wedge (y' \rightarrow z)) \rightarrow [(x \wedge y)' \rightarrow z]) \quad \text{by (DNP)} \\
&= (x \wedge (y' \rightarrow z)) \rightarrow [(x \wedge y)' \rightarrow z''] \quad \text{by (HP3)} \\
&= (x \wedge (y' \rightarrow z)) \rightarrow [z' \rightarrow (x \wedge y)''] \quad \text{by (DNP)} \\
&= (x \wedge (y' \rightarrow z)) \rightarrow [z' \rightarrow (x \wedge y)] \quad \text{by Proposition 2.1(vii)} \\
&= (x \wedge (y' \rightarrow z)) \rightarrow [(z' \rightarrow x) \wedge (z' \rightarrow y)] \quad \text{by Proposition 2.1(vii)} \\
&= [(x \wedge (y' \rightarrow z)) \rightarrow (z' \rightarrow x)] \wedge [(x \wedge (y' \rightarrow z)) \rightarrow (z' \rightarrow y)] \quad \text{by (HP3)} \\
&= [z' \rightarrow ((x \wedge (y' \rightarrow z)) \rightarrow x)] \wedge [(x \wedge (y' \rightarrow z)) \rightarrow (z' \rightarrow y'')] \quad \text{by (DNP)} \\
&= (x \wedge (y' \rightarrow z)) \rightarrow (y' \rightarrow z) \\
&= 1.
\end{aligned}$$

Therefore, $x \wedge (y \odot z) \leq (x \wedge y) \odot (x \wedge z)$. \square

Proposition 3.6. *Let \mathcal{H} has (DNP) and $P \in \text{Spec}(\mathcal{H})$. Then*

$$I_P = \{x \in \mathcal{H} \mid \text{there exists } y \notin P \text{ such that } x \wedge y = 0\},$$

is a proper ideal of \mathcal{H} and $I_P \subseteq P$.

Proof. For any $y \notin P$, $0 \wedge y = 0$, and so $0 \in I_P$. Suppose $a \in \mathcal{H}$ and $b \in I_P$ such that $a \leq b$. Then there is $y \notin P$ such that $b \wedge y = 0$. Since $a \leq b$, we have $a \wedge y \leq b \wedge y = 0$, and so $a \wedge y = 0$.

Hence, $a \in I_P$. Now, suppose $a, b \in I_P$. Then there are $x, y \notin P$ such that $a \wedge y = b \wedge x = 0$. Since $P \in \text{Spec}(\mathcal{H})$, it is clear that $x \wedge y \notin P$. Thus

$$\begin{aligned} (a \ominus b) \wedge (y \wedge x) &= ((a \ominus b) \wedge y) \wedge x && \text{by Lemma 3.5} \\ &\leq ((a \wedge y) \ominus (b \wedge y)) \wedge x \\ &= (0 \ominus (b \wedge y)) \wedge x = (b \wedge y) \wedge x = 0. \end{aligned}$$

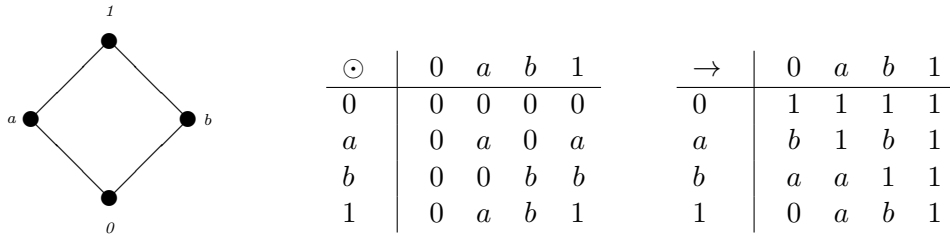
Hence, $(a \ominus b) \wedge (y \wedge x) = 0$, and so $a \ominus b \in I_P$. Hence, $I_P \in \mathcal{Id}(\mathcal{H})$. In addition, obviously $1 \notin I_P$, and so I_P is a proper ideal of \mathcal{H} . Let $x \in I_P$. Then there exists $y \notin P$ such that $x \wedge y = 0$. Since $P \in \text{Spec}(\mathcal{H})$, $y \notin P$ and $x \wedge y = 0 \in P$, we get $x \in P$. Therefore, $I_P \subseteq P$. \square

Definition 3.7. An ideal P of \mathcal{H} is called a minimal prime ideal of \mathcal{H} if (MP_1) $P \in \text{Spec}(\mathcal{H})$.

(MP_2) If there exists $Q \in \text{Spec}(\mathcal{H})$ such that $Q \subseteq P$, then $Q = P$.

The set of all minimal prime ideals of \mathcal{H} is denoted by $\text{Min}(\mathcal{H})$.

Example 3.8. Let $\mathcal{H} = \{0, a, b, 1\}$ be a set by following Hasse diagram. Define the operations \odot and \rightarrow on \mathcal{H} as follows:



Then $(\mathcal{H}, \odot, \rightarrow, 0, 1)$ is a bounded hoop. Clearly, $\mathcal{Id}(\mathcal{H}) = \{\{0\}, \{0, a\}, \{0, b\}, \mathcal{H}\}$ and $\text{Min}(\mathcal{H}) = \{\{0, a\}, \{0, b\}\}$. Obviously, $\{0\} \notin \text{Spec}(\mathcal{H})$.

Note. Every minimal prime ideal of \mathcal{H} is prime.

Corollary 3.9. Let \mathcal{H} be a \sqcup -hoop with (DNP). Then

(i) $\bigcap \{P \mid P \in \text{Min}(\mathcal{H})\} = \{0\}$.

(ii) If $0 \neq x \in \mathcal{H}$, then there exists $P \in \text{Min}(\mathcal{H})$ such that $x \notin P$.

Proof. Since every minimal prime ideal of \mathcal{H} is prime ideal, by Proposition 3.3 the proof is clear. \square

Theorem 3.10. Let I be a proper ideal of \mathcal{H} and $P \in \text{Spec}(\mathcal{H})$ such that $I \subseteq P$. Then there is $Q \in \text{Min}(\mathcal{H})$ such that $Q \subseteq P$.

Proof. Let $\Sigma = \{U \in \text{Spec}(\mathcal{H}) \mid I \subseteq U \subseteq P\}$. Since $P \in \Sigma$, we have $\Sigma \neq \emptyset$. Define \ll on Σ by $U_1 \ll U_2$ if and only if $U_2 \subseteq U_1$. Clearly, (Σ, \ll) is a poset. Suppose $\{U_i\}_{i \in I}$ is a family of prime ideals of \mathcal{H} such that for any $i \in I$, $U_i \in \Sigma$. Set $Q = \bigcap_{i \in I} U_i$. Obviously, Q is a proper ideal

of \mathcal{H} which is true in terms of the set Σ . It is enough to prove that Q is prime. For this suppose $x \wedge y \in Q$ such that $x \notin Q$. Then there exists $i \in I$ such that $x \notin U_i$. Suppose $U_j \in \Sigma$ such that $j \neq i$. We have the following cases:

Case 1. If $U_j \subseteq U_i$, then from $x \notin U_i$, we have $x \notin U_j$ where $x \wedge y \in Q \subseteq U_j$. Thus $y \in U_j$, and so for any $j \in I$, $y \in U_j$. Hence, $y \in Q$, and so $Q \in \text{Spec}(\mathcal{H})$.

Case 2. If $U_i \subseteq U_j$, then since $x \notin U_i$ and $U \in \text{Spec}(\mathcal{H})$ such that $x \wedge y \in Q \subseteq U_j$, we have $y \in U_i$. Thus for any $j \in I$, $y \in U_j$ and so $y \in Q$. Hence, $Q \in \text{Spec}(\mathcal{H})$.

Thus, Q is an upper bound of \sum and by Zorn's Lemma, \sum has a maximal element such as Q^* . Now, we prove that $Q^* \in \text{Min}(\mathcal{H})$. For this, suppose there is $\hat{P} \in \text{Spec}(\mathcal{H})$ such that $I \subseteq \hat{P} \subseteq Q^*$. Then $\hat{P} \in \sum$ and $Q^* \ll \hat{P}$, which is a contradiction. Therefore, $Q^* \in \text{Min}(\mathcal{H})$ such that $Q^* \subseteq P$. \square

Theorem 3.11. *Let \mathcal{H} be a \sqcup -hoop with (DNP) and $P \in \text{Spec}(\mathcal{H})$. Then $P \in \text{Min}(\mathcal{H})$ if and only if for each $x \in P$, there exists $r \in \mathcal{H} \setminus P$ such that $x \wedge r = 0$.*

Proof. (\Rightarrow) Suppose there exists $x \in P$ such that for any $r \in \mathcal{H} \setminus P$, we have $r \wedge x \neq 0$. Define $S = \{x \wedge r \mid r \in \mathcal{H} \setminus P\}$. Let $x \wedge r_1, x \wedge r_2 \in S$, for $r_1, r_2 \in \mathcal{H} \setminus P$. Then $(x \wedge r_1) \wedge (x \wedge r_2) = x \wedge (r_1 \wedge r_2)$. If $r_1 \wedge r_2 \in P$, then since $P \in \text{Spec}(\mathcal{H})$, we get $r_1 \in P$ or $r_2 \in P$, which is a contradiction. Thus, $r_1 \wedge r_2 \in \mathcal{H} \setminus P$, and so $(x \wedge r_1) \wedge (x \wedge r_2) \in S$. Hence, S is \wedge -closed. By Theorem 2.6, there exists $Q \in \text{Spec}(\mathcal{H})$ such that $S \cap Q = \emptyset$. Now, we have two following cases:

Case 1. If $Q \subseteq P$, then by assumption, since $P \in \text{Min}(\mathcal{H})$, we get $Q = P$ and so $P \cap S = \emptyset$. But by hypothesis, $x \in P$, $1 \in \mathcal{H} \setminus P$, and $x = 1 \wedge x \in S$, and so $P \cap S \neq \emptyset$, which is a contradiction.

Case 2. If $Q \not\subseteq P$, then there exists $r \in Q \setminus P$, and so $r \in \mathcal{H} \setminus P$. Thus $x \wedge r \in S$. Also, since $Q \in \text{Spec}(\mathcal{H})$ and $x \wedge r \leq r$, we get $x \wedge r \in Q$. Hence, $x \wedge r \in Q \cap S \neq \emptyset$, which is a contradiction. Since in both cases we have a contradiction, we consequence that for each $x \in P$, there exists $r \in \mathcal{H} \setminus P$ such that $x \wedge r = 0$.

(\Leftarrow) Suppose that there exists $Q \in \text{Spec}(\mathcal{H})$ such that $Q \subseteq P$. If $Q \neq P$, then there is $x \in P \setminus Q$. By assumption, there exists $r \in \mathcal{H} \setminus P$ such that $x \wedge r = 0$. Since $Q \in \text{Spec}(\mathcal{H})$, $x \wedge r = 0 \in Q$ and $x \notin Q$, we have $r \in Q$. Also, from $Q \subseteq P$ we get $r \in P$, which is a contradiction. Therefore, $Q = P$, and so $P \in \text{Min}(\mathcal{H})$. \square

Corollary 3.12. *Let \mathcal{H} be a \sqcup -hoop with (DNP). If $P \in \text{Min}(\mathcal{H})$, then for any $x \in P$ there is $y \in \mathcal{H} \setminus P$ and $k \in \mathbb{N}$ such that $y \wedge kn = 0$.*

Theorem 3.13. *Let \mathcal{H} be a \sqcup -hoop with (DNP), $P \in \text{Min}(\mathcal{H})$ and I be a finitly generated ideal of \mathcal{H} . Then $I \subseteq P$ if and only if $I^\perp \not\subseteq P$.*

Proof. (\Rightarrow) By assumption, I is a finitly generated ideal of \mathcal{H} . Then there exist $a_1, a_2, \dots, a_n \in \mathcal{H}$ such that $I = (a_1, a_2, \dots, a_n]$. Thus, for any $x \in I$, we have $x \leq a_1 \ominus (a_2 \ominus (\dots \ominus (a_{n-1} \ominus a_n) \dots))$. Since $I \subseteq P$, for any $1 \leq i \leq n$, $a_i \in P$ and by Theorem 3.11, there exist $u_i \in \mathcal{H} \setminus P$ such that $a_i \wedge u_i = 0$. Set $u = \bigwedge_{i=1}^n u_i$. Clearly $u \notin P$. Then by Lemma 3.5, we have

$$u \wedge x \leq u \wedge [a_1 \ominus (a_2 \ominus (\dots \ominus (a_{n-1} \ominus a_n) \dots))] \leq (u \wedge a_1) \ominus (u \wedge a_2) \ominus \dots \ominus (u \wedge a_n) = 0.$$

Thus $u \wedge x = 0$, and so $u \in I^\perp$. Hence, there is an element $u \in I^\perp$ such that $u \notin P$. Therefore, $I^\perp \not\subseteq P$.

(\Leftarrow) Suppose $I^\perp \not\subseteq P$. Then there exists $x \in I^\perp$ such that $x \notin P$. Since $x \in I^\perp$, for any $y \in I$, $x \wedge y = 0$. From $P \in \text{Min}(\mathcal{H})$, $x \wedge y = 0 \in P$ and $x \notin P$, we have $y \in P$. Hence, $I \subseteq P$. \square

Proposition 3.14. *Let \mathcal{H} be a \sqcup -hoop with (DNP) and $P \in \text{Spec}(\mathcal{H})$. If $I_P = P$, then $P \in \text{Min}(\mathcal{H})$.*

Proof. By Proposition 3.6 and Theorem 3.11, the proof is clear. \square

Definition 3.15. *Let X be a non-empty subset of \mathcal{H} . The set of all zero divisors of X is denoted by $\mathcal{Z}_X(\mathcal{H})$ and defined as follows:*

$$\mathcal{Z}_X(\mathcal{H}) = \{a \in \mathcal{H} \mid \text{there exists } 0 \neq x \in X \text{ such that } x \wedge a = 0\}.$$

The set of all zero divisors of \mathcal{H} is denoted by $\mathcal{Z}_{\mathcal{H}}$.

Example 3.16. (i) The zero element of \mathcal{H} is a trivial zero divisor of \mathcal{H} .

(ii) Let \mathcal{H} be the hoop as Example 3.8. Suppose $X_1 = \{a\}$ and $X_2 = \{a, b\}$. Then $\mathcal{Z}_{X_1}(\mathcal{H}) = \{0, b\}$ and $\mathcal{Z}_{X_2}(\mathcal{H}) = \{0, a, b\}$.

Theorem 3.17. Let \mathcal{H} be a \sqcup -hoop with (DNP). Then the set of all zero divisors of \mathcal{H} contains at least one prime ideal of \mathcal{H} .

Proof. For proving this theorem, we prove that for $0 \neq x \in \mathcal{H} \setminus \mathcal{Z}_{\mathcal{H}}$, there exists $P \in \text{Spec}(\mathcal{H})$ such that $x \in P$ and $P \subseteq \mathcal{Z}_{\mathcal{H}}$. Let $x, y \notin \mathcal{Z}_{\mathcal{H}}$. Then for any $a, b \in \mathcal{H} \setminus \{0\}$, $x \wedge a \neq 0$ and $y \wedge b \neq 0$. If $x \wedge y \in \mathcal{Z}_{\mathcal{H}}$, then there is $u \in \mathcal{H}$ such that $(x \wedge y) \wedge u = 0$, and so $x \wedge (y \wedge u) = 0$ or $y \wedge (x \wedge u) = 0$. Thus $x \in \mathcal{Z}_{\mathcal{H}}$ or $y \in \mathcal{Z}_{\mathcal{H}}$, which is a contradiction. Thus $x \wedge y \notin \mathcal{Z}_{\mathcal{H}}$, and so $\mathcal{H} \setminus \mathcal{Z}_{\mathcal{H}}$ is \wedge -closed. Let $x \in \mathcal{Z}_{\mathcal{H}}$. Then set $I = [x]$. We prove $I \subseteq \mathcal{Z}_{\mathcal{H}}$. Let $y \in I \setminus \mathcal{Z}_{\mathcal{H}}$. Then there is $n \in \mathbb{N}$ such that $y \leq nx$ and $y \notin \mathcal{Z}_{\mathcal{H}}$. Thus for any $a \in \mathcal{H} \setminus \{0\}$, $y \wedge a \neq 0$. Since $x \in \mathcal{Z}_{\mathcal{H}}$, by Definition 3.15, there is $0 \neq b \in \mathcal{H}$ such that $x \wedge b = 0$. By Lemma 3.5, $y \wedge b \leq (nx) \wedge b \leq n(x \wedge b) = 0$, and so $y \wedge b = 0$, which is a contradiction. Thus $I \subseteq \mathcal{Z}_{\mathcal{H}}$. Since $\mathcal{H} \setminus \mathcal{Z}_{\mathcal{H}}$ is \wedge -closed and $I \cap (\mathcal{H} \setminus \mathcal{Z}_{\mathcal{H}}) = \emptyset$, by Theorem 2.6, there exists $P \in \text{Spec}(\mathcal{H})$ such that $[x] = I \subseteq P$ and $P \cap (\mathcal{H} \setminus \mathcal{Z}_{\mathcal{H}}) = \emptyset$ which means that $P \subseteq \mathcal{Z}_{\mathcal{H}}$. \square

Corollary 3.18. Let \mathcal{H} be a \sqcup -hoop with (DNP). Then every element of a minimal prime ideal of \mathcal{H} is zero divisor.

Proof. By Theorem 3.17 and Corollary 3.12, the proof is clear. \square

In the following example we show that the converse of Theorem 3.17 does not hold.

Example 3.19. Let \mathcal{H} be the hoop as in Example 3.8. Then $\mathcal{Z}_{\mathcal{H}} = \{0, a, b, 1\} = \mathcal{H}$ and clearly there is not any prime ideal containing \mathcal{H} .

Proposition 3.20. Let \mathcal{H} be a \sqcup -hoop with (DNP). Then the set $\mathcal{Z}_{\mathcal{H}}$ is a union of all minimal prime ideals of \mathcal{H} .

Proof. We prove $\mathcal{Z}_{\mathcal{H}} = \bigcup_{P \in \text{Min}(\mathcal{H})} P$. Let $x \in \bigcup_{P \in \text{Min}(\mathcal{H})} P$. Then there is $P \in \text{Min}(\mathcal{H})$ such that $x \in P$. By Corollary 3.18, $x \in \mathcal{Z}_{\mathcal{H}}$ and so $\bigcup_{P \in \text{Min}(\mathcal{H})} P \subseteq \mathcal{Z}_{\mathcal{H}}$. Conversely, suppose $x \in \mathcal{Z}_{\mathcal{H}}$. Then there is $0 \neq y \in \mathcal{H}$ such that $x \wedge y = 0$. Thus by Corollary 3.9(ii), there is $P \in \text{Min}(\mathcal{H})$ such that $y \notin P$. Since $P \in \text{Spec}(\mathcal{H})$ and $x \wedge y = 0 \in P$, we have $x \in P$. Hence, $\mathcal{Z}_{\mathcal{H}} \subseteq \bigcup_{P \in \text{Min}(\mathcal{H})} P$. Therefore, $\mathcal{Z}_{\mathcal{H}}$ is a union of all minimal prime ideals of \mathcal{H} . \square

Note. Define $P_b = \bigcap \{P \mid P \in \text{Min}(\mathcal{H}) \text{ and } b \in P\}$ and $M_b = \bigcap \{M \mid M \in \text{Max}(\mathcal{H}) \text{ and } b \in M\}$.

Theorem 3.21. Let \mathcal{H} be a \sqcup -hoop with (DNP). If $x, y \in \mathcal{H}$ such that $y \in (x)^\perp$, then $(x)^\perp \subseteq P_y$ if and only if $x \odot y \notin \mathcal{Z}_{\mathcal{H}}$.

Proof. (\Rightarrow) Let $x, y \in \mathcal{H}$ such that $x \odot y \in \mathcal{Z}_{\mathcal{H}}$. Then by Proposition 3.20, there exists $P \in \text{Min}(\mathcal{H})$ such that $x \odot y \in P$. Since $y \leq x \odot y$, we get $y \in P$. Thus $P_y \subseteq P$. In addition, $x \leq x \odot y$, and so $x \in P$. By assumption, $(x)^\perp \subseteq P$ and by Theorem 3.13, $(x)^\perp \not\subseteq P$ and so $x \notin P$, which is a contradiction. Hence, $x \odot y \notin \mathcal{Z}_{\mathcal{H}}$.

(\Leftarrow) Suppose $(x)^\perp \not\subseteq P_y$. Then there is $a \in (x)^\perp$ such that $a \notin P_y$. Thus there is $P \in \text{Min}(\mathcal{H})$ such that $y \in P$ and $a \notin P$. Also, by Theorem 3.13, since $(x)^\perp \not\subseteq P_y$, we have $(x)^\perp \subseteq P$. Thus $x, y \in P$ and since P is ideal, we have $x \odot y \in P$. By Corollary 3.18, since $P \subseteq \mathcal{Z}_{\mathcal{H}}$, we get $x \odot y \in \mathcal{Z}_{\mathcal{H}}$ which is a contradiction. Hence $(x)^\perp \subseteq P_y$. \square

Theorem 3.22. *Let \mathcal{H} be a \sqcup -hoop with (DNP) and $x \in \mathcal{H}$. Then $P_x = \{y \in \mathcal{H} \mid x^\perp \subseteq y^\perp\}$.*

Proof. Suppose $B = \{y \in \mathcal{H} \mid x^\perp \subseteq y^\perp\}$. Let $z \in P_x$ such that $z \notin B$. Since $z \notin B$, we get $x^\perp \not\subseteq z^\perp$. Then there exists $u \in x^\perp$ such that $u \notin z^\perp$. Thus $u \wedge z \neq 0$. By Corollary 3.9, there exists $Q \in \text{Min}(\mathcal{H})$ such that $u \wedge z \notin Q$, and so $u \notin Q$ and $z \notin Q$. In addition, since $u \in x^\perp$, we have $u \wedge x = 0 \in Q$, and so $x \in Q$. Hence, we find $Q \in \text{Min}(\mathcal{H})$ such that $x \in Q$ and $z \notin Q$. Thus $z \notin P_x$, which is a contradiction. Therefore, $P_x \subseteq B$. Conversely, suppose $y \in B$ such that $y \notin P_x$. Since $y \in B$, we have $x^\perp \subseteq y^\perp$. From $y \notin P_x$ we get that there is $Q \in \text{Min}(\mathcal{H})$ such that $x \in Q$ and $y \notin Q$. Thus $(y) \not\subseteq Q$. By Theorem 3.13, since (y) is finite generated, $y^\perp \subseteq Q$. Also, from $x \in Q$, we have $(x) \subseteq Q$ and by Theorem 3.13, $x^\perp \not\subseteq Q$. Thus there exists $u \in x^\perp$ such that $u \notin Q$. Since $y^\perp \subseteq Q$, we get $u \notin y^\perp$, and so $x^\perp \not\subseteq y^\perp$, which is a contradiction. Hence, $B \subseteq P_x$. Therefore, $P_x = \{y \in \mathcal{H} \mid x^\perp \subseteq y^\perp\}$. \square

Definition 3.23. (i) *Let I be a proper ideal of \mathcal{H} . Then I is called a p -ideal of \mathcal{H} if $P_x \subseteq I$, for any $x \in I$.*

(ii) *Let I be a proper ideal of \mathcal{H} . Then I is called an m -ideal of \mathcal{H} if $M_x \subseteq I$, for any $x \in I$.*

Example 3.24. (i) *Let \mathcal{H} be the hoop as in Example 3.8. Then $\{0\}$ is a p -ideal of \mathcal{H} .*

(ii) *Let $\mathcal{H} = \{0, a, b, c, d, e, f, 1\}$. Define two operations \odot and \rightarrow on \mathcal{H} as follows,*

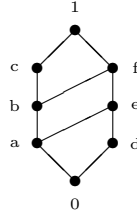


Figure 1: The Hasse diagram of \mathcal{H}

\rightarrow	0	a	b	c	d	e	f	1	\odot	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
a	d	1	1	1	d	1	1	1	a	0	a	a	a	0	a	a	a
b	d	f	1	1	d	f	1	1	b	0	a	a	b	0	a	a	b
c	d	e	f	1	d	e	f	1	c	0	a	b	c	0	a	b	c
d	c	c	c	c	1	1	1	1	d	0	0	0	0	d	d	d	d
e	0	c	c	c	d	1	1	1	e	0	a	a	a	d	e	e	e
f	0	b	c	c	d	f	1	1	f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1	1	0	a	b	c	d	e	f	1

Then $(\mathcal{H}, \odot, \rightarrow, 0, 1)$ is a bounded hoop. Then $\mathcal{I}d(\mathcal{H}) = \{\{0\}, \{0, d\}, \{0, a, b, c\}, \mathcal{H}\}$ and $\text{Min}(\mathcal{H}) = \{\{0, d\}, \{0, a, b, c\}\}$. Clearly, $\{0, d\}$ and $\{0, a, b, c\}$ are p -ideal and m -ideal of \mathcal{H} . Also, $P_a = \{0, a, b, c\}$ and $P_e = \emptyset$.

Clearly, any arbitrary intersection of p -ideals of \mathcal{H} is a p -ideal of \mathcal{H} .

Theorem 3.25. *Let \mathcal{H} be a \sqcup -hoop with (DNP). If I is a proper ideal of \mathcal{H} , then I^\perp is a p -ideal of \mathcal{H} .*

Proof. Let $x \in I^\perp$ and $u \in P_x$. Then by Theorem 3.22, $u \in \{y \in \mathcal{H} \mid x^\perp \subseteq y^\perp\}$. Thus $x^\perp \subseteq u^\perp$. If $u \notin I^\perp$, then $u \wedge I \neq 0$, and so $I \not\subseteq u^\perp$. Since $x^\perp \subseteq u^\perp$, we get $I \not\subseteq x^\perp$. Hence, $x \notin I^\perp$, which is a contradiction. Thus, $P_x \subseteq I^\perp$, for any $x \in I^\perp$. Therefore, I^\perp is a p -ideal of \mathcal{H} . \square

Corollary 3.26. *Let \mathcal{H} be a \sqcup -hoop with (DNP). If $I \in \mathcal{I}d(\mathcal{H})$, then $I^{\perp\perp}$ is a p -ideal of \mathcal{H} .*

Proof. By Proposition 2.10, I^\perp is a proper ideal of \mathcal{H} . Then by Theorem 3.25, the proof is clear. \square

Theorem 3.27. *Let \mathcal{H} be a \sqcup -hoop with (DNP). If I is a p -ideal of \mathcal{H} , then every element of I is zero divisor.*

Proof. Suppose $x \in I$. By assumption, $P_x \subseteq I$. Since $P_x = \bigcap\{P \mid P \in \mathcal{M}in(\mathcal{H}) \text{ such that } x \in P\}$, there is $P \in \mathcal{M}in(\mathcal{H})$ such that $x \in P$. By Corollary 3.18, we get x is zero divisor. \square

Proposition 3.28. *Let \mathcal{H} be a \sqcup -hoop with (DNP). Then for any $x \in \mathcal{H}$, $P_x = ((x^\perp)^\perp)$.*

Proof. Suppose $y \in ((x^\perp)^\perp)$ such that $y \notin P_x$. Thus there is $P \in \mathcal{M}in(\mathcal{H})$ such that $x \in P$ and $y \notin P$. Since $x \in P$, we have $(x) \subseteq P$ and by Theorem 3.13, $(x)^\perp \not\subseteq P$. Also, from $y \notin P$, we get $(y) \not\subseteq P$ and by Theorem 3.13, $(y)^\perp \subseteq P$. Moreover, since $y \in ((x^\perp)^\perp)$, we get $y \wedge (x)^\perp = 0 \in P$. Since $P \in \mathcal{M}in(\mathcal{H})$, we get $y \in P$ or $(x)^\perp \subseteq P$, which is contradiction in both cases. Hence, $((x^\perp)^\perp) \subseteq P_x$. Conversely, suppose $y \in P_x$ such that $y \notin ((x^\perp)^\perp)$. Since $y \in P_x$, we have for any $P \in \mathcal{M}in(\mathcal{H})$ such that $x \in P$, we have $y \in P$. From $y \notin ((x^\perp)^\perp)$, we have $y \wedge (x)^\perp \neq 0$. By Corollary 3.9(ii), there exists $P \in \mathcal{M}in(\mathcal{H})$ such that $y \wedge (x)^\perp \notin P$. Thus $y \notin P$ and $(x)^\perp \not\subseteq P$. Since (x) is finite generated, by Theorem 3.13 we have $(x) \subseteq P$, and so $x \in P$. Hence, we find $P \in \mathcal{M}in(\mathcal{H})$ such that $x \in P$ and $y \notin P$, which is a contradiction. Thus $P \subseteq ((x^\perp)^\perp)$. Therefore, $P_x = ((x^\perp)^\perp)$. \square

Proposition 3.29. *Let \mathcal{H} be a \sqcup -hoop with (DNP). We have:*

- (i) *If I is a p -ideal of \mathcal{H} , then every element of I is a zero divisor.*
- (ii) *$\{0\}$ is a p -ideal of \mathcal{H} .*
- (iii) *Every $P \in \mathcal{M}in(\mathcal{H})$ is a p -ideal of \mathcal{H} .*

Proof. (i) Since I is a p -ideal of \mathcal{H} , for any $x \in I$, we have $P_x \subseteq I$. Also, by definition of P_x , $x \in P_x$ and there is $P \in \mathcal{M}in(\mathcal{H})$ such that $x \in P$. Also, by Corollary 3.18, x is a zero divisor. Hence, every element of I is a zero divisor.

(ii) By Corollary 3.9, the proof is straightforward.

(iii) Since $P \in \mathcal{M}in(\mathcal{H})$, we know $P \in \mathcal{S}pec(\mathcal{H})$. Let $x \in P$. By definition of P_x we have $P_x \subseteq P$. Hence, P is a p -ideal of \mathcal{H} . \square

Proposition 3.30. *Let \mathcal{H} be a \sqcup -hoop with (DNP). Suppose $I \in \mathcal{I}d(\mathcal{H})$ such that every element of I is a zero divisor. Then there is a p -ideal of \mathcal{H} containing I .*

Proof. Let $x \in I$. Then by assumption, x is a zero divisor element. Thus by Proposition 3.20, $x \in \bigcup_{P_i \in \mathcal{M}in(\mathcal{H})} P_i$. So, there exists $P_i \in \mathcal{M}in(\mathcal{H})$ such that $x \in P_i$. Thus, for any $x \in I$, there exists $P_i \in \mathcal{M}in(\mathcal{H})$ such that $x \in P_i$. Hence, $I \subseteq \bigcup_{P_i \in \mathcal{M}in(\mathcal{H})} P_i$. By Theorem 3.4, there exists $P_i \in \mathcal{M}in(\mathcal{H})$ such that $I \subseteq P_i$. Also, by Proposition 3.29(iii), P_i is a p -ideal of \mathcal{H} . Therefore, there is a p -ideal of \mathcal{H} containing I . \square

Lemma 3.31. *For any $x, y \in \mathcal{H}$, $P_x \cup P_y \subseteq P_{x \wedge y}$.*

Proof. Let $\alpha \in P_x \cup P_y$. Then $\alpha \in P_x$ or $\alpha \in P_y$. If $\alpha \in P_x$, then for any $P \in \text{Min}(\mathcal{H})$ such that $x \in P$ we have $\alpha \in P$. Since $P \in \text{Min}(\mathcal{H})$, for any $y \in \mathcal{H}$, $x \wedge y \in P$. Thus for any $P \in \text{Min}(\mathcal{H})$, $x \wedge y \in P$ and so $\alpha \in P_{x \wedge y}$. The proof of other case is similar. Therefore, $P_x \cup P_y \subseteq P_{x \wedge y}$. \square

Theorem 3.32. *Let \mathcal{H} be a \sqcup -hoop with (DNP). Suppose I is a p -ideal of \mathcal{H} and S be a non-empty set of \mathcal{H} such that $S \not\subseteq I$. Then $\{x \in \mathcal{H} \mid \text{for any } s \in S, x \wedge s \in I\}$ is a p -ideal of \mathcal{H} .*

Proof. Let I be a p -ideal of \mathcal{H} and $B = \{x \in \mathcal{H} \mid \text{for any } s \in S, x \wedge s \in I\}$. Since $S \not\subseteq I$, there exists $s^* \in S$ such that $s^* \notin I$. Suppose $y \in B$. Then for any $s \in S$, $y \wedge s \in I$. Thus, $y \wedge s^* \in I$. Since I is a p -ideal of \mathcal{H} , we have $P_{y \wedge s^*} \subseteq I$. By Lemma 3.31, clearly $P_{s^*} \cap P_y \subseteq P_{s^* \wedge y}$, and so $P_{s^*} \cap P_y \subseteq I$. Then by Theorem 2.8, $P_{s^*} \subseteq I$ or $P_y \subseteq I$. If $P_{s^*} \subseteq I$, since $s^* \in P_{s^*}$, we get $s^* \in I$, which is a contradiction. Thus $P_y \subseteq I$. Also, for any $z \in I$ and $s \in S$, we have $z \wedge s \leq z$ and so $z \wedge s \in I$. Hence, $I \subseteq B$, and so $P_y \subseteq B$. Therefore, $\{x \in \mathcal{H} \mid \text{for any } s \in S, x \wedge s \in I\}$ is p -ideal. \square

Theorem 3.33. (i) *If \mathcal{H} has just one maximal ideal, then \mathcal{H} has only one m -ideal.*

(ii) *Every chain hoop has only one m -ideal.*

Proof. Clearly, every chain hoop has just one maximal ideal, and by (i), the proof is clear. \square

Proposition 3.34. *Let \mathcal{H} be a \sqcup -hoop with (DNP). Then $\{0\}$ is an m -ideal of \mathcal{H} .*

Proof. By Theorem 2.7, $\text{Max}(\mathcal{H}) \subseteq \text{Spec}(\mathcal{H})$. Then by Proposition 3.3, we have

$$\bigcap \{M \mid M \in \text{Max}(\mathcal{H})\} \subseteq \bigcap \{P \mid P \in \text{Spec}(\mathcal{H})\} = \{0\}.$$

Thus $\bigcap \{M \mid M \in \text{Max}(\mathcal{H})\} = \{0\}$. Clearly, $M_0 \subseteq \{0\}$ and so $\{0\}$ is an m -ideal of \mathcal{H} . \square

Theorem 3.35. *Let \mathcal{H} be a \sqcup -hoop with (DNP). Suppose I is an m -ideal of \mathcal{H} and S be a non-empty set of \mathcal{H} such that $S \not\subseteq I$. Then $\{x \in \mathcal{H} \mid \text{for any } s \in S, x \wedge s \in I\}$ is m -ideal.*

Proof. The proof is similar to the proof of Theorem 3.32. \square

Definition 3.36. *Let \mathcal{H} is a hoop. Then \mathcal{H} is called a semi-simple hoop if the intersection of all maximal ideals of \mathcal{H} is $\{0\}$.*

Example 3.37. *Let \mathcal{H} be the hoop as in Example 3.8. Clearly, $\text{Max}(\mathcal{H}) = \{\{0, a\}, \{0, b\}\}$ and so $\{0, a\} \cap \{0, b\} = \{0\}$. Hence, \mathcal{H} is semi-simple.*

Theorem 3.38. *Let \mathcal{H} be a semisimple \sqcup -hoop with (DNP). Then $M_x \subseteq P_x$, for any $x \in \mathcal{H}$.*

Proof. Consider $y \in M_x$ such that $y \notin P_x$. Then there exists $P \in \text{Min}(\mathcal{H})$ such that $x \in P$ and $y \notin P$, and so $(x) \subseteq P$. By Theorem 3.13 we have $(x)^\perp \not\subseteq P$. Thus there is $a \in (x)^\perp$ such that $a \notin P$. So $x \wedge a = 0$. Also, from $y, a \notin P$, we get $y \wedge a \notin P$ which implies $y \wedge a \neq 0$. Since \mathcal{H} is semisimple, there is $M \in \text{Max}(\mathcal{H})$ such $y \wedge a \notin M$, and so $a, y \notin M$. Moreover, from $x \wedge a = 0$, we have $x \in M$, which is a contradiction by $y \in M_x$. Therefore, $M_x \subseteq P_x$, for any $x \in \mathcal{H}$. \square

Corollary 3.39. *Let \mathcal{H} be a semisimple \sqcup -hoop with (DNP). Then:*

(i) *If I is a p -ideal, then I is an m -ideal of \mathcal{H} .*

(ii) *Every p -ideal of \mathcal{H} is m -ideal.*

4 Conclusions and future works

In this paper, the notion of minimal prime ideals of hoops are defined and investigated some properties of them. Then by using the notion of annihilators, the relation between minimal prime ideals and annihilators is studied. Also, the notion of zero divisors elements of hoops are introduced and proved that the set of all zero divisors of hoops is a union of all minimal prime ideals of hoop. Finally, by using the notions of minimal prime ideals and maximal ideals of hoop, two new ideals as p -ideal and m -ideal are defined. Then some properties of them are proved and the relation between them is investigated and showed every p -ideal of semi-simple hoop is an m -ideal of it.

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