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Hyper BZ-algebras and semihypergroups

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"This paper is dedicated to Professor Young Bae Jun on the occasion of his 70th birthday."

Abstract

In this paper, we introduce the new concept of a hyper BZ-algebra which is a generalization of BZ-algebra and hyper BCI-algebra, and give some examples and basic properties. We discuss the relationships among hyper BZ-algebras, hyper BCC-algebras and hyper BCI-algebra. Moreover, we propose the concepts of anti-grouped hyper BZ-algebras and generalized anti-grouped hyper BZ-algebras, and prove that the following important results:

(1) Every anti-grouped hyper BZ-algebra is an anti-grouped BZ-algebra;

(2) Every generalized anti-grouped hyper BZ-algebra corresponds to a semihypergroup.

Finally, we present a method to construct a new hyper BZ-algebra by using a hyper BCC-algebra and a standard generalized anti-grouped hyper BZ-algebra.

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1 Introduction

BCK-algebra and BCI-algebra are two kinds of algebraic structures closely related to non-classical logics (such as combinatorial logic, fuzzy logic, etc.), which have been studied extensively and deeply (see [28]-[34]). BCC-algebra and BZ-algebra are the extension of BCK-algebra and BCI-algebra, respectively, among which BZ-algebra was first proposed by Ye, and BZ-algebra is also called weak BCC-algebra (see [26]-[29]).

The notion of hyperstructrue (called also multialgebra) was introduced in 1934 by F. Marty at the 8th congress of Scandinavian Mathematicians, and hyperstructrues have many applications to several sectors of both pure and applied sciences (see [6]-[30]). Naturally, the idea of hyperstructrue is also applied to the study of non-classical logic algebras. In 2000, Jun et al. introduced

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the concept of hyper BCK-algebra, and investigated hyper BCK-ideals and some related hyper algebras such as hyper K-algebra and hyper MV-algebra (see [17]-[24]). In 2006, Jun and Borzooei et al. independently proposed the new concept of hyper BCC-algebra; also in 2006, Xin introduced the concept of hyper BCI-algebra, and since then a large number of research papers on hyper logical algebras have emerged (see [2]-[20]). However, as a common generalization of $BCK/BCI/BCC$-algebra, BZ-algebra has no corresponding hyper algebraic structure, and this paper will fill this gap.

In the study of non-classical logic algebras, an important direction is to explore the relationships between logic algebras and classical abstract algebraic structures (such as groups, semigroups, rings). For examples, the semigroup structure induced by a BCI-algebra (see [12]), the mixed structure of BCI-algebra and semigroup (see [23]). In 1995, Zhang and Ye first revealed the internal relation between BZ-algebras and general groups which can be non-commutative (see [37]); In 2013, Zhang investigated the close connection between BCC-algebras and residuated partially-ordered groupoids (see [33]); In 2014, Zhang and Jun proposed the notion of anti-grouped pseudo-BCI algebra and analyzed the relationship between group and pseudo-BCI algebra. Therefore, for the study of hyper BZ-algebra in this paper, we naturally pay attention to the relation between it and semihypergroup, which is a new direction in the study of hyper logical algebras.

The rest of the paper is arranged as follows. In Section 2, we give some basic concepts and properties of non-classical logic algebras and related hyper algebraic structures. In Section 3, we introduce the concept of hyper BZ-algebra, analyze its basic properties and internal relations with other related hyper logic algebras, and study the special properties of standard hyper BZ-algebra and transitive hyper BZ-algebra. In Section 4, we construct an adjoint semigroup from any hyper BZ-algebra, propose the new notions of anti-grouped hyper BZ-algebra and generalized anti-grouped hyper BZ-algebra, obtain some new properties and results. Specially, we investigate the relationship between generalized anti-grouped hyper BZ-algebra and semihypergoup. Finally, we present a construction method of hyper BZ-algebra from a hyper BCC-algebra and standard generalized anti-grouped hyper BZ-algebra.

2 Preliminaries

Definition 2.1. [13, 28] *An algebraic structure* $< X; *, 0 >$ *of type (2,0) is said to be a BCI-algebra if it satisfies: for all* $x, y, z \in X$,
1) $((x * z) * (y * z)) * (x * y) = 0$;
2) $(x * (x * y)) * y = 0$;
3) $x * 0 = x$;
4) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

If a BCI-algebra satisfies the following condition: for all $x \in X$,
5) $0 * x = 0$,
then we call it a BCK-algebra.

Definition 2.2. [26] *An algebraic structure* $< X; *, 0 >$ *of type (2,0) is said to be a BCC-algebra if it satisfies: for all* $x, y, z \in X$,
1) $((x * z) * (y * z)) * (x * y) = 0$;
2) $x * 0 = x$;
3) $x * x = 0$;
4) $0 * x = 0$;
5) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

Definition 2.3. [32] *An algebraic structure* $< X; *, 0 >$ *of type (2,0) is said to be a BZ-algebra if it satisfies: for all* $x, y, z \in X$,
1) $((x * z) * (y * z)) * (x * y) = 0$;
2) $x * 0 = x$;
3) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

Definition 2.4. [37] *A BZ-algebra* $< X; *, 0 >$ *is said to be an anti-grouped BZ-algebra if for any* $x \in X$, *satisfying* $0 * (0 * x) = x$.

Theorem 2.5. [37] *A BZ-algebra* $< X; *, 0 >$ *is an anti-grouped if and only if it satisfies*

$$(x * y) * (z * y) = x * z, \forall x, y, z \in X.$$

Theorem 2.6. [37] *Let* $< X; *, 0 >$ *be an anti-grouped BZ-algebra. Define "+" :*

$$x + y = x * (0 * y), \forall x, y \in X.$$

Then $< X; +, 0 >$ *is a group.*

Theorem 2.7. [37] *Let* $< G; \circ, e >$ *be a group. Define "∗", for all* $x, y \in G$, $x * y = x \circ y^{-1}$. *Then* $< G; *, e >$ *is an anti-grouped BZ-algebra.*

Let H be a non-empty set and $P^*(H)$ the set of all non-empty subsets of H. A map $\circ$: $H \times H \to P^*(H)$ is called (binary) hyperoperation (or hypercomposition), and $(H, \circ)$ is called a hypergroupoid. If $A, B \in P^*(H)$ and $x \in H$, then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b,\ A \circ x = A \circ \{x\},\ x \circ B = \{x\} \circ B.$$

Definition 2.8. [30] *Let* $(H, \circ)$ *be a hypergroupoid. If for all* $x, y, z \in H$ *we have* $(x \circ y) \circ z = x \circ (y \circ z)$, *then* $(H, \circ)$ *is called a semihypergroup. That is,* $\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v$.

Note that, if $(H, \circ)$ *is a semihypergroup, then* $(A \circ B) \circ C = A \circ (B \circ C)$, *for all* $A, B, C \in P^*(H)$.

Definition 2.9. [30] *Assume that* $(H, \circ)$ *is a semihypergroup.*
(1) *If* $a \in H$ *satisfies for all* $x \in H$, $|a \circ x| = |x \circ a| = 1$, *then* a *is called scalar.*
(2) *If* $e \in H$ *satisfies for all* $x \in H$, $x \circ e = e \circ x = \{x\}$, *then* e *is called scalar identity.*
(3) *If* $e \in H$ *satisfies for all* $x \in H$, $x \in e \circ x \cap x \circ e$, *then* e *is called identity.*
(4) *Let* $a, b \in H$. *If there exists an identity* $e \in H$ *satisfies* $e \in a \circ b \cap b \circ a$, *then* b *is called an inverse of* a.
(5) *If* $0 \in H$ *satisfies for all* $x \in H$, $x \circ 0 = 0 \circ x = \{0\}$, *then* 0 *is called a zero element.*

Definition 2.10. [24] *By a hyper BCK-algebra we mean a hyper groupoid* $(H, \circ)$ *that contains a constant 0 and satisfies the following axioms: for all* $x, y, z \in H$,
$(HK1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$,
$(HK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,
$(HK3)$ $x \circ H \ll x$,
$(HK4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$.

Definition 2.11. [20] *By a hyper BCC-algebra (Jun's definition) we mean a hyper groupoid* $(H, \circ)$ *that contains a constant 0 and satisfies the following axioms: for all* $x, y, z \in H$,
$(HC1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$,
$(HC2)$ $x \ll x$,
$(HC3)$ $x \circ y \ll x$,
$(HC4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$.

Definition 2.12. [3] *By a hyper BCC-algebra (Borzooei's definition) we mean a hyper groupoid* $(H, \circ)$ *that contains a constant 0 and satisfies the following axioms: for all* $x, y, z \in H$,
$(HC1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$,
$(HC2)$ $0 \circ x = \{0\}$,
$(HC3)$ $x \circ 0 = \{x\}$,
$(HC4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$.

Jun and Borzooei in [20] and [3] gave different definitions of hyper BCC-algebra respectively. But in this paper, we mainly use Definition 2.12 about hyper BCC-algebra.

Definition 2.13. [31] *By a hyper BCI-algebra we mean a hyper groupoid* $(H, \circ)$ *that contains a constant 0 and satisfies the following axioms: for all* $x, y, z \in H$,
$(HK1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$,
$(HK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,
$(HI3)$ $x \ll x$,
$(HK4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$,
$(HI5)$ $0 \circ (0 \circ x) \ll x$.

Proposition 2.14. [31] *In any hyper BCI-algebra* $(H, \circ)$, *the following hold: for all* $x, y, z \in H$ *and for all non-empty subsets* A *and* B *of* H,
(1) $x \ll 0$ *implies* $x = 0$,
(2) $0 \in x \circ (x \circ 0)$,
(3) $x \ll x \circ 0$,
(4) $0 \circ (x \circ y) \ll y \circ x$,
(5) $A \ll A$,
(6) $A \subseteq B$*implies* $A \ll B$,
(7) $A \ll 0$ *implies* $A = 0$,
(8) $x \circ 0 \ll \{y\}$ *implies* $x \ll y$,
(9) $y \ll z$ *implies* $x \circ z \ll x \circ y$,
(10) $x \circ y = 0$ *implies* $(x \circ z) \circ (y \circ z) = 0$ *and* $x \circ z \ll y \circ z$,
(11) $A \circ 0 = 0$ *implies* $A = 0$.

3 Hyper BZ-algebras and basic properties

In this section, we introduce the concept of hyper BZ-algebras and investigate some related properties. In the following, we replace singleton set $\{x\}$ with x.

Definition 3.1. *By a hyper BZ-algebra we mean a hyper groupoid* $(H, \circ)$ *that contains a constant 0 and satisfies the following axioms: for all* $x, y, z \in H$,
$(HZ1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$,
$(HZ2)$ $x \ll x$,
$(HZ3)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$,
$(HZ4)$ $0 \circ (0 \circ x) \ll x$,
$(HZ5)$ $x \ll x \circ 0$.

Also we define $x \ll y$ by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ means that for all $a \in A$ there is $b \in B$ such that $a \ll b$.

Remark 3.2. (1) *Every BZ-algebra is a hyper BZ-algebra.*
(2) *Every hyper BCK/BCI/BCC-algebra is a hyper BZ-algebra.*

Now, we give some examples about hyper BZ-algebras and some examples shows that every hyper BZ-algebra is not a hyper $BCK/BCI/BCC$-algebra.

Example 3.3. (1) *Let* $H = \{0, 1, 2, 3, 4\}$. *Define an operation* $*$ *on* H *as follows,*

$*$	0	1	2	3	4
0	0	0	2	2	0
1	1	0	3	2	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	4	3	3	0

Then $(H, *, 0)$ *is a* BZ*-algebra, and it is a hyper* BZ*-algebra. But it is not a (hyper)* BCI*-algebra, since* $(4 * (4 * 2)) * 2 = 1 \neq 0$.
(2) *Let* $H = \{0, 1, 2, 3\}$. *Define the operation* $\circ$ *on* H *as follows,*

$\circ$	0	1	2	3
0	0	0	0	0
1	$\{1,2\}$	$\{0,2\}$	0	0
2	2	2	0	0
3	3	3	2	0

Then $(H, \circ, 0)$ *is a hyper* BZ*-algebra, but it isn't a hyper* BCK*-algebra since* $1 \circ H = \{0, 1, 2\}$ *and* $0 \notin 2 \circ 1$(*i.e.*$2 \ll 1$ *is not true), this means that the condition (HK3) in Definition 2.10 doesn't hold.*
(3) *Let* $H = \{0, 1, 2, 3\}$. *Define the operation* $\circ$ *on* H *as follows,*

$\circ$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	$\{0,1\}$	2
3	3	$\{1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$

Then $(H, \circ, 0)$ *is a hyper* BZ*-algebra, but it isn't a hyper* BCI*-algebra since* $(2 \circ 1) \circ 2 = \{0, 1\} \neq 0 = (2 \circ 2) \circ 1$.
(4) *Let* $H = \{0, 1, 2, 3\}$. *Define the operation* $\circ$ *on* H *as follows,*

$\circ$	0	1	2	3
0	0	0	2	2
1	1	$\{0,1\}$	3	3
2	2	2	0	0
3	3	3	1	$\{0,1\}$

Then $(H, \circ, 0)$ *is a hyper* BZ*-algebra, but it isn't a hyper* BCC*-algebra since* $0 \circ 3 = 2 \neq 0$.

Proposition 3.4. *In any hyper BZ-algebra* $(H, \circ)$*, the followings hold: for all* $x, y, z \in H$ *and for all non-empty subsets* A *and* B *of* H*,*
(1) $x \ll 0$ *implies* $x = 0$,
(2) $0 \circ (x \circ y) \ll y \circ x$,
(3) $A \ll A$,

(4) $A \subseteq B$*implies* $A \ll B$,
(5) $A \ll 0$ *implies* $A = 0$,
(6) $0 \circ 0 = 0$,
(7) $(0 \circ x) \circ (0 \circ x) = 0$,
(8) $0 \circ x$ *is a singleton set,*
(9) $x \circ y = 0$ *implies* $(x \circ z) \circ (y \circ z) = 0$ *and* $x \circ z \ll y \circ z$,
(10) $A \circ 0 = 0$ *implies* $A = 0$,
(11) $x \ll y$ *implies* $0 \ll y \circ x$,
(12) $0 \circ (0 \circ (0 \circ x)) \ll 0 \circ x$,
(13) $x \circ x = 0$ *implies* $|y \circ z| = 1$.

Proof. (1) Let $x \ll 0$. Then $0 \in x \circ 0$ and by (HZ1) and (HZ2), $0 \in 0 \circ 0 \subseteq 0 \circ (x \circ 0) \subseteq (0 \circ 0) \circ (x \circ 0) \ll 0 \circ x$, that is $0 \ll 0 \circ x$. By (HZ4), $0 \in 0 \circ (0 \circ x) \ll x$. Then $0 \ll x$. Combining $x \ll 0$, we get $x = 0$.
(2) By (HZ1) and (HZ2), $0 \circ (x \circ y) \subseteq (y \circ y) \circ (x \circ y) \ll y \circ x$. That is $0 \circ (x \circ y) \ll y \circ x$.
(3) By (HZ2), for any $x \in A, x \ll x$, that is $0 \in x \circ x$. Then $A \ll A$.
(4) Let $a \in A$. Then $a \in B$. By (HZ2), $x \ll x$ and $0 \in x \circ x$. Then $A \ll B$.
(5) Let $a \in A$. Then $a \ll 0$ and so $a = 0$. Then $A = \{0\}$.
(6) By (HZ4), $0 \circ (0 \circ 0) \ll 0$, then $0 \circ (0 \circ 0) = 0$. By (HZ2), $0 \in 0 \circ 0 \subseteq 0 \circ (0 \circ 0) = 0$, so $0 \circ 0 = 0$.
(7) By (HZ1), $(0 \circ x) \circ (0 \circ x) \ll 0 \circ 0 = 0$, then $(0 \circ x) \circ (0 \circ x) = 0$.
(8) For any $a, b \in 0 \circ x$, and $a \neq b$. $a \circ b \subseteq (0 \circ x) \circ (0 \circ x) = 0$, $b \circ a \subseteq (0 \circ x) \circ (0 \circ x) = 0$, by (HZ3), $a \ll b, b \ll a$, then $a = b$. So $0 \circ x$ is a singleton set.
(9) By (HZ1), $(x \circ z) \circ (y \circ z) \ll x \circ y = \{0\}$. By (5), $(x \circ z) \circ (y \circ z) = \{0\}$. So $x \circ z \ll y \circ z$.
(10) Assume that $A \circ 0 = 0$, then $A \ll 0$. So $A = 0$.
(11) Assume that $x \ll y$. Then $0 \in x \circ y$, and so $0 \in 0 \circ 0 \subseteq (y \circ y) \circ (x \circ y) \ll y \circ x$. Hence $0 \ll y \circ x$.
(12) By (HZ4), $0 \circ (0 \circ x) \ll x$. Let $a = 0 \circ x$, $0 \circ (0 \circ a) \ll a$, so $0 \circ (0 \circ (0 \circ x)) \ll 0 \circ x$.
(13) For any $x \in H$, let $x \circ x = \{0\}$. Assume that $|y \circ z| > 1$, let $a, b \in y \circ z$, and $a \neq b$. Then

$$a \circ b \subseteq (y \circ z) \circ (y \circ z) \ll y \circ y = 0 \text{ and } b \circ a \subseteq (y \circ z) \circ (y \circ z) \ll y \circ y = 0,$$

thus $a \circ b \ll 0$, $b \circ a \ll 0$and $a \ll b, b \ll a$,. Hence $a = b$ and so $|y \circ z| = 1$. □

In some hyper BZ-algebras $(H, \circ)$, for all $x \in H$, $x \circ 0 = x$, but the others don't satisfy this condition. So we give the concept of standard hyper BZ-algebras and study their properties.

Definition 3.5. *A hyper BZ-algebra* $(H, \circ)$ *is said to be a standard hyper BZ-algebra if for any* $x \in H$, *satisfying* $x \circ 0 = x$.

Proposition 3.6. *In any standard hyper BZ-algebra* $(H, \circ)$, *the followings hold: for all* $x, y, z \in H$ *and for all non-empty subset* A *of* H,
$(SHZ1)$ $A \circ 0 = A$,
$(SHZ2)$ $x \ll y$ *implies* $z \circ y \ll z \circ x$.

Proof. (SHZ1) For any $a \in A$, $a \circ 0 = a$. So $A \circ 0 = A$.
(SHZ2) Assume that $x \ll y$. Then $0 \in x \circ y$. By (HZ1), $(z \circ y) \circ (x \circ y) \ll z \circ x$. For any $m \in z \circ y$, according to the definition of standard hyper BZ-algebra, $m = m \circ 0 \subseteq (z \circ y) \circ (x \circ y) \ll z \circ x$. So $z \circ y \ll z \circ x$. □

Remark 3.7. *Every hyper BCC-algebra is a standard hyper BZ-algebra.*

But every standard hyper BZ-algebra is not a BCC-algebra, see Example 3.3(4). Example 3.8 shows that hyper BCI-algebra is not necessarily a standard hyper BZ-algebra.

Example 3.8. *Let $H = \{0, 1, 2, 3\}$. Define the operation $\circ$ on H as follows,*

$\circ$	0	1	2	3
0	0	0	0	3
1	$\{1, 2\}$	$\{0, 2\}$	0	3
2	2	2	0	3
3	3	3	3	0

Then $(H, \circ, 0)$ is a hyper BCI-algebra, but it isn't a standard hyper BZ-algebra.

Definition 3.9. *A hyper BZ-algebra $(H, \circ)$ is said to be a transitive hyper BZ-algebra if for any $x, y, z \in H$, satisfying $x \ll y$ and $y \ll z$ imply $x \ll z$.*

Proposition 3.10. *In any transitive hyper BZ-algebra $(H, \circ)$, the following conditions hold: for all $x, y, z, u \in H$ and for all non-empty subsets A, B and C of H,*
$(THZ1)$ $A \ll B$ and $B \ll C$ imply $A \ll C$,
$(THZ2)$ $x \circ y \ll z$ implies $(x \circ u) \circ (y \circ u) \ll z$.

Proof. (THZ1) Let $a \in A$. There exists $b \in B$ such that $a \ll b$. Also, for any $b \in B$, there exists $c \in C$ such that $b \ll c$. So $a \ll c$. Then for any $a \in A$, there exists $c \in C$ such that $a \ll c$, that is $A \ll C$.

(THZ2) By (HZ1), $(x \circ u) \circ (y \circ u) \ll x \circ y$, and $x \circ y \ll z$. So $(x \circ u) \circ (y \circ u) \ll z$. □

Example 3.11. (1) *Let $H = \{0, 1, 2, 3\}$. Define the operation $\circ$ on H as follows,*

$\circ$	0	1	2	3
0	0	0	0	0
1	1	$\{0, 1\}$	0	0
2	2	2	$\{0, 2\}$	$\{0, 2\}$
3	3	2	$\{1, 2\}$	$\{0, 1, 2\}$

Then $(H, \circ)$ is a transitive hyper BZ-algebra.
(2) *Let $H = \{0, 1, 2, 3\}$. Define the operation $\circ$ on H as follows,*

$\circ$	0	1	2	3
0	0	0	0	0
1	$\{1, 2\}$	$\{0, 2\}$	0	0
2	2	2	0	0
3	3	3	2	0

Then $(H, \circ)$ is a transitive hyper BZ-algebra, but it isn't standard.

Proposition 3.12. *In any transitive standard hyper BZ-algebra $(H, \circ)$, the followings hold: for all $x, y, z, u \in H$,*
$(TSHZ1)$ $x \circ y \ll z$ implies $(u \circ y) \circ z \ll u \circ x$,
$(TSHZ2)$ $(0 \circ x) \circ (y \circ x) = 0 \circ y$,
$(TSHZ3)$ $(0 \circ (x \circ y)) \circ (0 \circ x) = 0 \circ (0 \circ y)$,
$(TSHZ4)$ $0 \circ (0 \circ (x \circ y)) = (0 \circ (0 \circ x)) \circ (0 \circ (0 \circ y))$.

Proof. (TSHZ1) Assume that $x\circ y \ll z$, by (HZ1), $((u\circ y)\circ z)\circ((x\circ y)\circ z) \ll (u\circ y)\circ(x\circ y) \ll u\circ x$. By (THZ1), $((u\circ y)\circ z)\circ((x\circ y)\circ z) \ll u\circ x$. Since $x\circ y \ll z$, that is $0 \in (x\circ y)\circ z$. For any $m \in (u\circ y)\circ z$, there exists $p \in u\circ x$ such that $0 \in (m\circ 0)\circ p = m\circ p$. So $(u\circ y)\circ z \ll u\circ x$.

(TSHZ2) Since $(y\circ x)\circ(0\circ x) \ll y\circ 0$, by (TSHZ1), $((0\circ x)\circ(0\circ x))\circ(y\circ 0) \ll (0\circ x)\circ(y\circ x)$, that is $0\circ y \ll (0\circ x)\circ(y\circ x)$. According to (HZ1), $(0\circ x)\circ(y\circ x) \ll 0\circ y$, then $(0\circ x)\circ(y\circ x) = 0\circ y$.

(TSHZ3) According to (TSHZ2), $(0\circ(x\circ y))\circ((0\circ y)\circ(x\circ y)) = 0\circ(0\circ y)$ and $(0\circ y)\circ(x\circ y) = 0\circ x$, so $(0\circ(x\circ y))\circ(0\circ x) = 0\circ(0\circ y)$.

(TSHZ4) By (TSHZ2) and (TSHZ3), $(0\circ(0\circ x))\circ((0\circ(x\circ y))\circ(0\circ x)) = 0\circ(0\circ(x\circ y))$, and $(0\circ(x\circ y))\circ(0\circ x) = 0\circ(0\circ y)$, so $0\circ(0\circ(x\circ y)) = (0\circ(0\circ x))\circ(0\circ(0\circ y))$. □

In the following, we investigate some kinds of hyper BZ-subalgebra.

Definition 3.13. *Let $(H,\circ)$ be a hyper BZ-algebra and S be a subset of H contains 0. If S is a hyper BZ-algebra with respect to the hyper operation "$\circ$" on H, we say that S is a hyper subalgebra of H.*

Proposition 3.14. *Let S be a non-empty subset of a hyper BZ-algebra $(H,\circ)$. If $x\circ y \subseteq S$ for all $x,y \in S$, then $0 \in S$.*

Proof. Assume that $x\circ y \subseteq S$ for all $x,y \in S$ and let $a \in S$. Since $a \ll a$, we have $0 \in a\circ a \subseteq S$. □

Theorem 3.15. *Let S be a non-empty subset of a hyper BZ-algebra $(H,\circ)$. Then S is a hyper subalgebra of H if and only if $x\circ y \subseteq S$ for all $x,y \in S$.*

Proof. ($\Rightarrow$) Clearly.

($\Leftarrow$) Assume that $x\circ y \subseteq S$ for all $x,y \in S$. Then $0 \in S$ by Proposition 3.14. For any $x,y,z \in S$, we have $x\circ z \subseteq S$, $y\circ z \subseteq S$, and $x\circ y \subseteq S$. Hence,

$$(x\circ z)\circ(y\circ z) = \bigcup_{a\in x\circ z, b\in y\circ z} a\circ b \subseteq S.$$

Because $S \subseteq H$, $(x\circ z)\circ(y\circ z) \ll x\circ y \subseteq S$. Therefore (HZ1) holds in S. Similarly we can prove that (HZ2), (HZ3), (HZ4) and (HZ5) are true in S. Therefore, S is a hyper subalgebra of H. □

Theorem 3.16. *Let $(H,\circ)$ be a hyper BZ-algebra. Then the set*

$$S_I := \{x \in H | x\circ x = 0\}$$

is a hyper subalgebra of H and for any $x,y \in S_I$, $x\circ y$ is a singleton set.

Proof. Let $x,y \in S_I$ and $a \in x\circ y$. Then $(x\circ y)\circ(x\circ y) \ll x\circ x = 0$ and hence $(x\circ y)\circ(x\circ y) = 0$ and $a\circ a \subseteq (x\circ y)\circ(x\circ y) = 0$. Therefore, $x\circ y \subseteq S_I$. By Theorem 3.15, S_I is a hyper subalgebra of H.

According to Proposition 3.4(13), for any $x,y \in S_I$, $x\circ y$ is a singleton set. □

However, S_I isn't necessarily a BZ-algebra, see the following example:

Example 3.17. *Let $H = \{0,1,2,3\}$. Define the operation $\circ$ on H as follows,*

$\circ$	0	1	2	3
0	0	0	0	0
1	$\{1,2\}$	$\{0,2\}$	0	0
2	2	2	0	2
3	2	2	0	0

Then $(H, \circ)$ is a hyper BZ-algebra, and $S_I = \{0, 2, 3\}$ is a hyper BZ-subalgebra. But it isn't a BZ-algebra, since $3 \circ 0 = 2 \neq 3$.

Theorem 3.18. *Let $(H, \circ)$ be a standard hyper BZ-algebra. Then the set*

$$BCC(H) := \{x \in H | 0 \circ x = 0\},$$

is a hyper BCC-subalgebra of H.

Proof. Let $x, y \in BCC(H)$ and $a \in x \circ y$. Then $0 \circ (x \circ y) = (0 \circ y) \circ (x \circ y) \ll 0 \circ x = 0$ and hence $0 \circ (x \circ y) = 0$. Therefore $x \circ y \subseteq BCC(H)$. By Theorem 3.15, $BCC(H)$ is a hyper subalgebra of H. Since for any $x \in BCC(H)$, $x \circ 0 = x$, we get $BCC(H)$ is a hyper BCC-algebra. □

4 (Generalized) Anti-grouped hyper BZ-algebras and semihypergroups

In this section, we firstly discuss the relation between hyper BZ-algebra and semigroup, and then explore the relationships between hyper BZ-algebras and semihypergroups by introducing the concepts of anti-grouped hyper BZ-algebra and generalized anti-grouped hyper BZ-algebra.

Let $(H, \circ)$ be a hyper BZ-algebra. For any $a, x \in H$, denote a map:

$$\rho_a : H \to P^*(H); x \mapsto x \circ a.$$

For any $a, b \in H$, for any $x \in H$, denote $\rho_a \circ \rho_b$:

$$(\rho_a \circ \rho_b)(x) = \bigcup_{\forall y \in \rho_b(x)} \rho_a(y),$$

where $\circ$ represents the composition operation of mappings.

Theorem 4.1. *Denote $M(X)$ is a set which is all compositional results of finite mappings which are for all $a \in H$, we have ρ_a. Then $M(X)$ is a semigroup.*

Proof. For any $x \in H$, $a, b, c \in H$, for any $s \in ((\rho_a \circ \rho_b) \circ \rho_c)(x)$, there exists $y \in \rho_c(x)$ such that $s \in (\rho_a \circ \rho_b)(y)$. Then there exists $u \in \rho_b(y)$ such that $u \in \rho_b(\rho_c(x)) = \rho_b \circ \rho_c(x)$ and $s \in \rho_a(u)$. Then $s \in (\rho_a \circ (\rho_b \circ \rho_c))(x)$ and $((\rho_a \circ \rho_b) \circ \rho_c)(x) \subseteq (\rho_a \circ (\rho_b \circ \rho_c))(x)$.

For any $t \in (\rho_a \circ (\rho_b \circ \rho_c))(x)$, there exists $m \in \rho_b \circ \rho_c(x)$ such that $t \in \rho_a(m)$. Then there exists $n \in \rho_c(x)$ such that $m \in \rho_b(n)$ and $t \in \rho_a(\rho_b(n)) = \rho_a \circ \rho_b(n)$. Then $t \in ((\rho_a \circ \rho_b) \circ \rho_c)(x)$ and $(\rho_a \circ (\rho_b \circ \rho_c))(x) \subseteq ((\rho_a \circ \rho_b) \circ \rho_c)(x)$.

So $(\rho_a \circ (\rho_b \circ \rho_c))(x) = ((\rho_a \circ \rho_b) \circ \rho_c)(x)$. Then $M(X)$ satisfies associative law. □

Proposition 4.2. *Let $(H, \circ)$ be a standard hyper BZ-algebra. Then $(M(X), \circ)$ is a semigroup with the identity ρ_0.*

Proof. For any $x \in H$, $x \circ 0 = x$. Then for any $a \in H$,

$$\rho_0 \circ \rho_a(x) = \rho_0(x \circ a) = (x \circ a) \circ 0 = x \circ a = \rho_a(x),$$

$$\rho_a \circ \rho_0(x) = \rho_a(x \circ 0) = (x \circ 0) \circ a = x \circ a = \rho_a(x).$$

So ρ_0 is the identity. □

Example 4.3. *Let $H = \{0, 1, 2, 3\}$. Define the operation $\circ$ on H as follows,*

$\circ$	0	1	2	3
0	0	0	2	2
1	1	$\{0,1\}$	2	2
2	2	2	0	0
3	3	3	1	$\{0,1\}$

Then $(H, \circ)$ is a hyper BZ-algebra and $M(X) = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_2^2, \rho_3^3\}$, where $\rho_2^2 = \rho_2 \circ \rho_2, \rho_3^3 = \rho_3 \circ \rho_3 \circ \rho_3$.

We can verify the following:
$\rho_0 \circ \rho_0 = \rho_0, \rho_0 \circ \rho_1 = \rho_1, \rho_0 \circ \rho_2 = \rho_2, \rho_0 \circ \rho_3 = \rho_3, \rho_0 \circ \rho_2^2 = \rho_2^2, \rho_0 \circ \rho_3^3 = \rho_3^3$;
$\rho_1 \circ \rho_0 = \rho_1, \rho_1 \circ \rho_1 = \rho_1, \rho_1 \circ \rho_2 = \rho_3, \rho_1 \circ \rho_3 = \rho_3, \rho_1 \circ \rho_2^2 = \rho_2^2, \rho_1 \circ \rho_3^3 = \rho_3^3$;
$\rho_2 \circ \rho_0 = \rho_2, \rho_2 \circ \rho_1 = \rho_2, \rho_2 \circ \rho_2 = \rho_2^2, \rho_2 \circ \rho_3 = \rho_2^2, \rho_2 \circ \rho_2^2 = \rho_3^3, \rho_2 \circ \rho_3^3 = \rho_2^2$;
$\rho_3 \circ \rho_0 = \rho_3, \rho_3 \circ \rho_1 = \rho_3, \rho_3 \circ \rho_2 = \rho_2^2, \rho_3 \circ \rho_3 = \rho_2^2, \rho_3 \circ \rho_2^2 = \rho_3^3, \rho_3 \circ \rho_3^3 = \rho_2^2$;
$\rho_2^2 \circ \rho_0 = \rho_2^2, \rho_2^2 \circ \rho_1 = \rho_2^2, \rho_2^2 \circ \rho_2 = \rho_3^3, \rho_2^2 \circ \rho_3 = \rho_3^3, \rho_2^2 \circ \rho_2^2 = \rho_2^2, \rho_2^2 \circ \rho_3^3 = \rho_3^3$;
$\rho_3^3 \circ \rho_0 = \rho_3^3, \rho_3^3 \circ \rho_1 = \rho_3^3, \rho_3^3 \circ \rho_2 = \rho_2^2, \rho_3^3 \circ \rho_3 = \rho_2^2, \rho_3^3 \circ \rho_2^2 = \rho_3^3, \rho_3^3 \circ \rho_3^3 = \rho_2^2$.

Then $(M(X), \circ)$ is a semigroup, but it isn't commutative, since $\rho_1 \circ \rho_2 = \rho_3 \neq \rho_2 = \rho_2 \circ \rho_1$.

$\circ$	ρ_0	ρ_1	ρ_2	ρ_3	ρ_2^2	ρ_3^3
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	ρ_2^2	ρ_3^3
ρ_1	ρ_1	ρ_1	ρ_3	ρ_3	ρ_2^2	ρ_3^3
ρ_2	ρ_2	ρ_2	ρ_2^2	ρ_2^2	ρ_3^3	ρ_2^2
ρ_3	ρ_3	ρ_3	ρ_2^2	ρ_2^2	ρ_3^3	ρ_2^2
ρ_2^2	ρ_2^2	ρ_2^2	ρ_3^3	ρ_3^3	ρ_2^2	ρ_3^3
ρ_3^3	ρ_3^3	ρ_3^3	ρ_2^2	ρ_2^2	ρ_3^3	ρ_2^2

Definition 4.4. *A hyper BZ-algebra $(H, \circ)$ is said to be an anti-grouped hyper BZ-algebra if it is standard and for any $x, y, z \in H$, satisfying $(x \circ z) \circ (y \circ z) = x \circ y$.*

Proposition 4.5. *Let $(H, \circ)$ be an anti-grouped hyper BZ-algebra. Then for any $x \in H$, $0 \circ (0 \circ x) = x$.*

Proof. For any $x \in H$, $(x \circ x) \circ (0 \circ x) = x \circ 0 = x$, and $0 \in x \circ x$. So $0 \circ (0 \circ x) \subseteq (x \circ x) \circ (0 \circ x) = x \circ 0 = x$. That is, $0 \circ (0 \circ x) = x$. Hence $0 \circ (0 \circ x) = x$. □

Proposition 4.6. *In any hyper BZ-algebra $(H, \circ)$ satisfying $0 \circ (0 \circ x) = x$, for all $x \in H$, the following conditions hold: for all $x, y \in H$,*
(1) $x \circ x = 0$,
(2) $x \circ 0 = x$,
(3) $|x \circ y| = 1$.

Proof. (1) Since $0 \circ x$ is a singleton set, we get $x \circ x = (0 \circ (0 \circ x)) \circ (0 \circ (0 \circ x)) \ll 0 \circ 0 = 0$. So $x \circ x = 0$.
(2) By (HZ5), $x \ll x \circ 0$. Then $x \circ 0 = (0 \circ (0 \circ x)) \circ ((0 \circ x) \circ (0 \circ x)) \ll 0 \circ (0 \circ x) = x$. So $x = x \circ 0$.
(3) According to Proposition 3.4(13), for any $x, y \in H$, $|x \circ y| = 1$. □

Theorem 4.7. *A hyper BZ-algebra $(H, \circ)$ satisfies $0 \circ (0 \circ x) = x$, for all $x \in H$, if and only if it satisfies $x \ll y$ implies $x = y$, for all $x, y \in H$.*

Proof. (⇐) Let $m = 0 \circ (0 \circ x)$. Since $m \ll x$, we have $m = x$. That is, $0 \circ (0 \circ x) = x$.

(⇒) Assume that $x \ll y$. Then $0 = 0 \circ 0 \subseteq 0 \circ (x \circ y) \ll y \circ x$. For $0 \in x \circ y$, there exists $m \in y \circ x$ such that $0 \ll m$. If $m \neq 0$, $|0 \circ m| = 1$ and $0 = 0 \circ m$, then $0 \circ (0 \circ m) = 0 \circ 0 = 0 \neq m$. Hence $m = 0$, and so $y \ll x$. therefore, by (HZ3), $x = y$. □

Theorem 4.8. *In any hyper BZ-algebra $(H, \circ)$, the following conditions are equivalent: for any $x, y, z \in H$,*
(1) $(H, \circ)$ is a hyper BZ-algebra satisfying $0 \circ (0 \circ x) = x$;
(2) $x \circ x = 0$ and $(x \circ z) \circ (y \circ z) = x \circ y$;
(3) $(H, \circ)$ is an anti-grouped BZ-algebra.

Proof. (1)⇒(2) By (HZ1), $(x \circ z) \circ (y \circ z) \ll x \circ y$. By (AGHZ3), for any $x, y \in H, |x \circ y| = 1$. By Theorem 4.7, $(x \circ z) \circ (y \circ z) = x \circ y$.

(2)⇒(3) For any $x, y \in H$, assume $|x \circ y| > 1$. There exist $a, b \in x \circ y$ such that $a \neq b$. Then

$$a \circ b \subseteq (x \circ y) \circ (x \circ y) \ll x \circ x = 0 \text{ and } b \circ a \subseteq (x \circ y) \circ (x \circ y) \ll x \circ x = 0.$$

Thus $a \circ b \ll 0$ and $b \circ a \ll 0$ and so $a \ll b$ and $b \ll a$. By (HZ4) $a = b$. So $|x \circ y| = 1$. Let $x \circ 0 = y$ and $y \neq x$. By (HZ5), $x \ll x \circ 0 = y$ and $y \circ x = (y \circ y) \circ (x \circ y) = 0 \circ 0 = 0$, so $y \ll x$. Hence $x = y$ and $0 \circ (0 \circ x) = (x \circ x) \circ (0 \circ x) = x \circ 0 = x$. Therefore, $(H, \circ)$ is an anti-grouped BZ-algebra.

(3)⇒(1) Obviously. □

According to Proposition 4.5 and Theorem 4.8, we know every anti-grouped hyper BZ-algebra is a anti-grouped BZ-algebra.

Definition 4.9. *A hyper BZ-algebra $(H, \circ)$ is said to be a generalized anti-grouped hyper BZ-algebra if for any $x, y, z \in H$, satisfying $(x \circ (0 \circ y)) \circ (0 \circ z) = x \circ (0 \circ (y \circ (0 \circ z)))$.*

Example 4.10. *Let $H = \{0, 1, 2, 3\}$. Define the operation $\circ$ on H as follows,*

$\circ$	0	1	2	3
0	0	0	2	2
1	1	$\{0,1\}$	3	3
2	2	2	0	0
3	3	3	1	$\{0,1\}$

Then $(H, \circ)$ is a generalized anti-grouped hyper BZ-algebra. But it is neither a hyper BCC-algebra, since $0 \circ 3 = 2 \neq 0$, or a hyper BCI-algebra, since $(1 \circ 1) \circ 2 = \{2, 3\} \neq 3 = (1 \circ 2) \circ 1$.

Proposition 4.11. *In any generalized anti-grouped hyper BZ-algebra $(H, \circ)$, for any $x, y \in H$ define "$\oplus$" :*

$$x \oplus y = x \circ (0 \circ y).$$

Then $(H, \oplus, 0)$ is a semihypergroup.

Proof. For any $x, y, z \in H$, $(x \oplus y) \oplus z = (x \circ (0 \circ y)) \circ (0 \circ z) = x \circ (0 \circ (y \circ (0 \circ z))) = x \oplus (y \oplus z)$. Then $(H, \oplus, 0)$ is a semihypergroup. □

Proposition 4.12. *Let $(H, \circ)$ be a hyper BZ-algebra such that for all $x \in H$, $0 \circ x = 0$ and $(x \circ 0) \circ 0 = x \circ 0$. For any $x, y \in H$, define "$\oplus$":*

$$x \oplus y = x \circ (0 \circ y).$$

Then $(H, \oplus, 0)$ is a semihypergroup.

Proof. For any $x, y, z \in H$, $(x \oplus y) \oplus z = (x \circ (0 \circ y)) \circ (0 \circ z) = (x \circ 0) \circ 0 = x \circ 0$, and $x \oplus (y \oplus z) = x \oplus (y \circ (0 \circ z)) = x \circ (0 \circ (y \circ (0 \circ z))) = x \circ 0$. Then $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, and so $(H, \oplus, 0)$ is a semihypergroup. □

Proposition 4.13. *In any hyper BCC-algebra $(H, \circ)$, for any $x, y \in H$ define "$\oplus$":*

$$x \oplus y = x \circ (0 \circ y).$$

Then $(H, \oplus, 0)$ is a semigroup, and every element is a right identity in H.

Proof. Obviously, $(H, \oplus, 0)$ is a semigroup. For any $x, y \in H$, $x \oplus y = x \circ (0 \circ y) = x \circ 0 = x$, that is, y is a right identity of x. This means that every element is a right identity in H. □

Proposition 4.14. *All anti-grouped (hyper) BZ-algebras are generalized anti-grouped hyper BZ-algebra.*

Proof. It follows from Theorem 2.6. □

Example 4.15. (1) *Let $H = \{0, 1, 2, 3\}$. Define the operation $\circ$ on H as follows,*

$\circ$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	$\{0,1\}$	2
3	3	$\{1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$

Then $(H, \circ)$ is a generalized anti-grouped hyper BZ-algebra. According to Proposition 4.11, we get a semigroup $(H, \oplus)$ and the operation $\oplus$ on it is as follows,

$\oplus$	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

(2) *Let $H = \{0, 1, 2, 3\}$. Define the operation $\circ$ on H as follows,*

$\circ$	0	1	2	3
0	0	0	0	0
1	$\{1,2\}$	$\{0,2\}$	0	2
2	2	2	0	2
3	3	3	0	0

Then $(H, \circ)$ is a generalized anti-grouped hyper BZ-algebra. According to Proposition 4.11, we get a semihypergroup $(H, \oplus)$ and the operation $\oplus$ on it is as follows.

$\oplus$	0	1	2	3
0	0	0	0	0
1	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$
2	2	2	2	2
3	3	3	3	3

In the follwing, we present a constrcution method of hyper BZ-algebra by a hyper BCC-algebra and a standard generalized anti-grouped hyper BZ-algebra.

Theorem 4.16. *Let $(H, \circ, 0)$ be a hyper BCC-algebra, $(G, *, 0)$ a standard generalized anti-grouped hyper BZ-algebra, and $H \cap G = 0$. Denote $X = H \cup G$, and define all $x, y \in X$*

$$x \cdot y = \begin{cases} x \circ y, & x, y \in H \\ x * y, & x, y \in G \\ 0 * y, & x \in H, y \in G - \{0\} \\ x, & x \in G - \{0\}, y \in H \end{cases}$$

Then $(X, \cdot, 0)$ is a hyper BZ-algebra.

Proof. For $x, y, z \in X$,
(1) **Case 1:** If $x, y, z \in H$, then obviously, $(x \cdot z) \cdot (y \cdot z) = (x \circ z) \circ (y \circ z) \ll x \circ y = x \cdot y$;
Case 2: If $x, y, z \in G$, then obviously, $(x \cdot z) \cdot (y \cdot z) = (x * z) * (y * z) \ll x * y = x \cdot y$;
Case 3: If $x, y \in H$ and $z \in G - \{0\}$, then

$$((x \cdot z) \cdot (y \cdot z)) \cdot (x \cdot y) = ((0 * z) * (0 * z)) \cdot (x \cdot y) = 0 \cdot (x \cdot y) = 0 \circ (x \circ y) = 0,$$

and so $(x \cdot z) \cdot (y \cdot z) \ll x \cdot y$;
Case 4: If $x, y \in G - \{0\}$ and $z \in H$, then

$$(x \cdot z) \cdot (y \cdot z) = x \cdot y = x * y \ll x * y = x \cdot y.$$

Thus $(x \cdot z) \cdot (y \cdot z) \ll x \cdot y$;
Case 5: If $x, z \in H$ and $y \in G - \{0\}$, then

$$((x \cdot z) \cdot (y \cdot z)) \cdot (x \cdot y) = ((x \cdot z) \cdot y) \cdot (0 * y) = (0 * y) * (0 * y) = 0,$$

and so $(x \cdot z) \cdot (y \cdot z) \ll x \cdot y$;
Case 6: If $x, z \in G - \{0\}$ and $y \in H$, then

$$(x \cdot z) \cdot (y \cdot z) = (x \cdot z) \cdot (0 * z) = (x * z) * (0 * z) \ll x * 0 = x.$$

Since $x \cdot y = x$. Hence $(x \cdot z) \cdot (y \cdot z) \ll x \cdot y$;
Case 7: If $x \in H$ and $y, z \in G - \{0\}$, then

$$(x \cdot z) \cdot (y \cdot z) = (0 * z) \cdot (y * z) = (0 * z) * (y * z) \ll 0 * y.$$

On the other hand, $x \cdot y = 0 * y$. Thus $(x \cdot z) \cdot (y \cdot z) \ll x \cdot y$;
Case 8: If $x \in G - \{0\}$ and $y, z \in H$, then

$$(x \cdot z) \cdot (y \cdot z) = x \cdot (y \circ z) = x,$$

and $x \cdot y = x$. Since $x \ll x$, then $(x \cdot z) \cdot (y \cdot z) \ll x \cdot y$.
Therefore, (HZ1) holds.

(2) For any $x \in H$, $x \ll x$; for any $x \in G, x \ll x$. So for any $x \in X, x \ll x$. Then (HZ2) holds.

(3) For any $x, y \in H, x \ll y, y \ll x$, then $x = y$; for any $x, y \in G, x \ll y, y \ll x$, then $x = y$; for any $x \in H, y \in G, 0 \in x \cdot y = 0 * y$, because $|0 * y| = 1, 0 * y = 0, y \cdot x = y = 0$. Then $x \ll 0, 0 \ll x$. And $x = 0, x = y$. Then (HZ3) holds.

(4) For any $x \in H$, obviously, $0 \cdot (0 \cdot x) \ll x$; For any $x \in G$, $0 \cdot (0 \cdot x) \ll x$. Then (HZ4) holds.

(5) For any $x \in H$, obviously, $x \ll x \cdot 0$; For any $x \in G, x \ll x \cdot 0$. Then (HZ5) holds.

So $(X, \cdot, 0)$ is a hyper BZ-algebra. □

Example 4.17. *Let $H = \{0,1,2\}$ be a hyper BCC-algebra, and $G = \{0,3,4,5\}$ a standard generalized anti-grouped hyper BZ-algebra. The operation $\circ$ on H is defined as follows:*

$\circ$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	$\{0,1\}$

The operation $*$ on G is defined as follows:

$*$	0	3	4	5
0	0	0	4	4
3	3	$\{0,3\}$	5	5
4	4	4	0	0
5	5	5	3	$\{0,3\}$

Denote $X = H \cup G$, according to the construction method of Theorem 4.16, we get a hyper BZ-algebra $(X, \cdot, 0)$ and the operation "$\cdot$" on X is defined as follows:

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	4	4
1	1	0	0	0	4	4
2	2	2	$\{0,1\}$	0	4	4
3	3	3	3	$\{0,3\}$	5	5
4	4	4	4	4	0	0
5	5	5	5	5	3	$\{0,3\}$

5 Conclusions

In this paper, we investigated hyper BZ-algebras and some related hyper algebraic structures. We proposed some new concepts of standard (transitive) hyper BZ-algebra, anti-grouped hyper BZ-algebra, generalized anti-grouped BZ-algebra, and proved some important results: (1) every hyper BZ-algebra has a hyper BZ sub-algebra ; every standard hyper BZ-algebra has a hyper subalgebra which is a hyper BCC-algebra; (2) every hyper BZ-algebra can leads to a semigroup which is made up of some mapping defined on it; (3) every anti-grouped hyper BZ-algebra is an anti-grouped BZ-algebra; (4) every generalized anti-grouped hyper BZ-algebra corresponds to a special semihypergroup; (5) one can obtain a new hyper BZ-algebra by using a hyper BCC-algebra and a standard generalized anti-grouped hyper BZ-algebra.

The results obtained in this paper show that hyper BZ-algebras are not only more extensive than hyper $BCK/BCC/BCI$-algebras, but also retain many important properties, and are closely related to some algebraic structures such as semigroups and semihypergroups, which can be used for reference for further study of the properties of various hyper logical algebras. The hyper ideals, quotient hyper algebras, hyper atoms and hyper lattice structures of hyper BZ-algebras will become interesting topics in the future.

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References

[1] A. Ahadpanah, A. Borumand Saeid, *Smarandache hyper BCC-algebra*, Computers and Mathematics with Applications, 61(9) (2011), 2490–2497.

[2] R.A. Borzooei, M. Aaly Kologani, *An overview of hyper logical algebras*, Journal of Algebraic Hyperstructrues and Logical Algebras, 1(3) (2020), 31–50.

[3] R.A. Borzooei, W.A. Dudek, N. Koohestanki, *On hyper BCC-algebras*, International Journal of Mathematics and Mathematical Sciences, 2006 (2006), DOI: 10.1155/IJMMS/2006/49703.

[4] R.A. Borzooei, B. Ganji Safar, R. Ameri, *On hyper EQ-algebras*, Italian Journal of Pure and Applied Mathematics, 31 (2013), 77–96.

[5] P. Corsini, V. Leoreanu, *Applications of hyperstructure theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[6] B. Davvaz, *Semihypergroup theory*, Elsevier: Amsterdam, 2016.

[7] B. Davvaz, A. Dehghan Nezhad, M.M. Heidari, *Inheritance examples of algebraic hyperstructures*, Information Sciences, 224 (2013), 180–187.

[8] W.A. Dudek, *Solid weak BCC-algebras*, International Journal of Computer Mathematics, 88(14) (2011), 2915–2925.

[9] W.A. Dudek, J. Thomys, *On some generalizations of BCC-algebras*, International Journal of Computer Mathematics, 89(12) (2012), 1596–1616.

[10] W.A. Dudek, X.H. Zhang, *On ideals and congruence in BCC-algebras*, Czechoslovak Mathematical Journal, 48(123) (1998), 21–29.

[11] W.A. Dudek, X.H. Zhang, Y.Q. Wang, *Ideals and atoms of BZ-algebras*, Mathematica Slovaca, 59 (2009), 387–404.

[12] W. Huang, *On BCI-algebras and semigroups*, Mathematica Japonicae, 42 (1995), 59–64.

[13] Y.S. Huang, *BCI-algebra*, Science Press, Beijing, 2006.

[14] A. Iorgulescu, *Implicative-groups vs. groups and generalizations*, Bucuresti: Matrix Rom, 2018.

[15] Y.B. Jun, *Multipolar fuzzy hyper BCK-ideals of hyper BCK-algebras*, Journal of Algebraic Hyperstructures and Logical Algebras, 1(1) (2020), 37–47.

[16] Y.B. Jun, M.S. Kang, H.S. Kim, *Hyper MV-deductive systems of hyper MV-algebras*, Communications of the Korean Mathematical Society, 25(4) (2010), 537–545.

[17] Y.B. Jun, M.S. Kang, H.S. Kim, *Bipolar fuzzy hyper BCK-ideals in hyper BCK-algebras*, Iranian Journal of Fuzzy Systems, 8(2) (2011), 105–120.

[18] Y.B. Jun, K.J. Lee, M.A. Öztürk, *Soft BCC-algebras*, Journal of Applied Mathematics and Informatics, 27(5-6) (2009), 1293–1305.

[19] Y.B. Jun, E.H. Roh, *Fuzzy (weak) implicative hyper K-ideals*, Bulletin of the Korean Mathematical Society, 43(43) (2006), 141–148.

[20] Y.B. Jun, E.H. Roh, H. Harizavi, *Hyper BCC-algebras*, Honam Mathematical Journal, 28(1) (2006), 57–67.

[21] Y.B. Jun, S.Z. Song, W.H. Shim, *On implicative hyper K-ideals of hyper K-algebras*, Scientiae Mathematicae Japonicae, 59(3) (2004), 443–450.

[22] Y.B. Jun, X.L. Xin, *Scalar elements and hyperatoms of hyper BCK-algebras*, Scientiae Mathematicae, 2(3) (1999), 303–309.

[23] Y.B. Jun, X.L. Xin, E.H. Roh, *A class of algebras related to BCI-algebras and semigroups*, Soochow Journal of Mathematics, 24(4) (1998), 309–321.

[24] Y.B. Jun, M.M. Zahedi, X.L. Xin, R.A. Borzooei, *On hyper BCK-algebras*, Italian Journal of Pure and Applied Mathematics, 8 (2000), 127–136.

[25] Y.B. Jun, X.H. Zhang, *General forms of BZ-ideals and T-ideals in BZ-algebras*, Honam Mathematical Journal, 30(2) (2008), 379–390.

[26] Y. Komori, *The class of BCC-algebras is not a variety*, Mathematica Japonica, 29(3) (1984), 391–394.

[27] X.Y. Mao, H.J. Zhou, *Classification of proper hyper BCI-algebras of order 3*, Applied Mathematics and Information Sciences, 9(1) (2015), 387–393.

[28] J. Meng, Y.B. Jun, *BCK-algebras*, Kyung Moon Sa Co. Seoul, Korea, 1994.

[29] J. Thomys, X.H. Zhang, *On weak-BCC-algebras*, The Scientific World Journal, 2013, Article ID 935097, 10 pages, http://dx.doi.org/10.1155/2013/935097.

[30] H.S. Wall, *Hypergroups*, American Journal of Mathematics, 59 (1937), 77–98.

[31] X.L. Xin, *Hyper BCI-algebras*, Discussiones Mathematicae, General Algebra and Applications, 26 (2006), 5–19.

[32] R.F. Ye, *BZ-algebras, selected paper on BCI, BCK-algebra and computer logics (in Chinese)*, Shanghai Jiaotong University Press, 1991, 25–27.

[33] X.H. Zhang, *BCC-algebras and residuated partially-ordered groupoid*, Mathematica Slovaca, 63(3) (2013), 397–410.

[34] X.H. Zhang, *A survey of algebraic structures derived from non-classical logics*, Journal of Sichuan Normal University (Natural Science), 42(1) (2019), 1–14.

[35] X.H. Zhang, Y.B. Jun, *Anti-grouped pseudo-BCI algebras and anti-grouped filters*, Fuzzy Systems and Mathematics, 28(2) (2014), 21–33.

[36] X.H. Zhang, W.H. Li, *On pseudo-BL algebras and BCC-algebras*, Soft Computing, 10(10) (2006), 941–952.

[37] X.H. Zhang, R.F. Ye, *BZ-algebra and group*, Journal of Mathematical and Physical Sciences, 29(5) (1995), 223–233.