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On α-solvable fundamental groups

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Abstract

We introduce a specific kind of equivalence relation ${\xi^*}_n^\alpha$ on a fuzzy hypergroup S such that the quotient $S/{\xi^*}_n^\alpha$, the set of all equivalence classes, is an α-solvable group. This helps us to introduce the α-solvable fundamental relation $\xi^{*\alpha}$. In particular, we obtain an equivalent condition with transitivity of ξ^α.

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1 Introduction

A solvable group with respect to an automorphism α is called an α-solvable group. An α-solvable group is a group that α-derived series terminates in the trivial subgroups. In [2], α-solvable groups, as a generalization of solvable groups, were introduced and some properties of α-solvabel groups were discussed. Clearly, every solvable group is an α-solvable group, where α is the identity automorphism.

In 1965, Zadeh [14] proposed the concept of fuzzy sets. In 1971, Rosenfeld [12], applied fuzzy sets in group theory to introduce fuzzy subgroups of a group. Fuzzy hypergroups as a new approach on fuzzy sets, introduced by Corsini and Tofan [5]. The basic idea is that a fuzzy hyperoperation assigns to every pair of elements a fuzzy set. Some researchers extended the concepts of abstract algebra to fuzzy sets (see [3], [5], [9], [8], [7], [10], [13]). The study of fuzzy hyperstructures is an interesting topic on fuzzy sets theory. One way for connecting fuzzy hypergroups and groups is the fundamental relation. A fundamental relation of a hypergroup is the smallest equivalence

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relation such that a quotient is a group. The fundamental relation, β, as a vital concept on hyperstructures, is studied by many scholars [4]. This relation plays an important role in the theory of hyperstructures. Also, the relation γ^* is the least equivalence relation on a hypergroup H such that a quotient is an abelian group [6]. Moreover, γ^* is a commutative fundamental relation. It is known that if R is a fuzzy strongly regular equivalence relation on a fuzzy hypergroup S, then we can define a binary operation $\otimes$ on the quotient set S/R, the set of all equivalence classes of S with respect to R, such that $(S/R, \otimes)$ is a group (see [13]). Ameri and Nozari [1], followed the results obtained by Sen et. all on fuzzy hypersemigroups to introduce the fundamental relation of fuzzy hypersemigroups. Now, we introduce an α-solvable fundamental group. In addition, we define a strongly regular relation $\xi^{*\alpha}$ on a fuzzy hypergroup S. Then we prove that $S/\xi^{*\alpha}$, the set of all equivalence classes of $\xi^{*\alpha}$ under usual operation, is an α-solvable group. Finally, by the notion of $\xi^{*\alpha}$-part of a fuzzy hypergroup, we try to get an equivalent condition to transitivity of $\xi^{*\alpha}$.

2 Preliminaries

Let G be any group and α be an automorphism of G. For two elements x and y of G the α-commutator of G is $[x, y]_\alpha = xyx^{-1}y^{-\alpha}$, where $y^{-\alpha}$ is used for $\alpha(y^{-1})$. For any $x_1, x_2, \ldots, x_n$ of G one can define inductively $[x_1, x_2, \ldots, x_n]_\alpha$, the α-commutator of weight n, as follows:

$$[x_1, x_2, \ldots, x_n]_\alpha = [x_1, [x_2, \ldots, x_n]_\alpha]_\alpha.$$

For any non-empty subsets X_1 and X_2 of G the *α-commutator subgroup* of G, denoted by $[X_1, X_2]_\alpha$ is defined as the subgroup of G generated by the set $\{[x_1, x_2]_\alpha | x_1 \in X_1, x_2 \in X_2\}$. It is clear that $[X_1, X_2]_\alpha$ is not equal to $[X_2, X_1]_\alpha$ in general. Let N be a normal subgroup of G and $N^\alpha = N$. For the isomorphism $\overline{\alpha} : G/N \longrightarrow G/N$ given by $\overline{x}^{\overline{\alpha}} = x^\alpha N$ we have $[\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}]_{\overline{\alpha}} = [x_1, x_2, \ldots, x_n]_\alpha N$ (see [2]).

The *α-derived subgroup* of a group G with respect to an automorphism α is defined by $D^\alpha(G) = \langle [x, y]_\alpha | x, y \in G\rangle$. Also, $D^\alpha_0(G) = G$, $D^\alpha_1(G) = D^\alpha(G)$ and $D^\alpha_i(G) = D^\alpha(D^\alpha_{i-1}(G))$. A group G is α-solvable if and only if for some integer r, $D^\alpha_r(G) = \{1\}$, where 1 is the identity element. The smallest such r is called *length* of G (see [2]).

A *hypergroupoid* is a nonempty set H with a hyperoperation $\triangleright$ defined on H, that is, a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, then its image under $\triangleright$ is denoted by $x \triangleright y$. If A, B are non-empty subsets of H, then $A \triangleright B$ is given by $A \triangleright B = \bigcup\{x \triangleright y | x \in A, y \in B\}$. Thus $x \triangleright A$ is used for $\{x\} \triangleright A$ and $A \triangleright x$ for $A \triangleright \{x\}$. Generally, the singleton a is identified with its member a. The structure $(H, \triangleright)$ is called a *semihypergroup* if $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright c$ for any $a, b, c \in H$, and a semihypergroup $(H, \triangleright)$ is called a *hypergroup* in the sense of Marty if $x \triangleright H = H \triangleright x = H$, for any $x \in H$. This axiom means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in x \triangleright u$ and $y \in v \triangleright x$.

Let S be a non-empty set and $F^*(S)$ be the set of all non-zero fuzzy subsets of S. We denote by 0 the zero fuzzy set. Then $\circ : S \times S \to F^*(S)$ is a *fuzzy hyperoperation* on S and the couple $(S, \circ)$ is called a *fuzzy hypergroupoid*.

Let μ, ν be two fuzzy subsets of a fuzzy hypergroupoid $(S, \circ)$. In [13] for any $a, r \in S$ we have the following statements:

(i) $(\mu \odot \nu)(r) = \bigvee_{p,q \in S} (\mu(p) \wedge (p \circ q)(r) \wedge \nu(q))$,

(ii) $(a \bullet \mu)(r) = \begin{cases} \bigvee_{t \in S} ((a \circ t)(r) \wedge \mu(t)), & \mu \neq 0 \\ 0. & \mu = 0, \end{cases}$ $\quad (\mu \bullet a)(r) = \begin{cases} \bigvee_{t \in S} (\mu(t) \wedge (t \circ a)(r)), & \mu \neq 0 \\ 0. & \mu = 0 \end{cases}$

Definition 2.1. [13]
(i) A fuzzy hypergroupoid $(S, \circ)$ is a fuzzy semihypergroup if for any $x, y, z \in S$ we have $(x \circ y) \bullet z = x \bullet (y \circ z)$.
(iii) A fuzzy semihypergroup is a fuzzy hypergroup (FHG) if $x \circ S = \chi_S = S \circ x$. A fuzzy subhypergroup $(K, \circ)$ of an FHG $(S, \circ)$ is a non-empty subset $K \subseteq S$ such that for any $k \in K$, $k \circ K = K \circ k = \chi_K$. Let $(S_1, \circ_1)$ and $(S_2, \circ_2)$ be two FHG. A map $f : S_1 \to S_2$ is called a fuzzy hypergroup homomorphism if, for any $x, y \in S_1, f(x \circ_1 y) = f(x) \circ_2 f(y)$.

Definition 2.2. [13] *Let ρ be an equivalence relation on a fuzzy semihypergroup $(S, \circ)$ and μ, ν be two fuzzy subsets of $(S, \circ)$. We say that $\mu \overline{\overline{\rho}} \nu$ if for all $x, y \in S$ such that $\mu(x) > 0$ and $\nu(y) > 0$, then $x \rho y$.*

Definition 2.3. [13] *An equivalence relation ρ on a fuzzy semihypergroup $(S, \circ)$ is said to be a fuzzy strongly regular relation if $a \rho b$ and $a' \rho b'$ imply $a \circ a' \ \overline{\overline{\rho}} \ b \circ b'$.*

Theorem 2.4. [13] *Let $(S, \circ)$ be a fuzzy semihypergroup and ρ be an equivalence relation on S. For any $\rho_a, \rho_b \in S/\rho$ consider the operation $\oplus$ as follows:*

$$\rho_a \oplus \rho_b = \{\rho_c | (a' \circ b')(c) > 0, \ a \rho a', \ b \rho b'\}.$$

Then ρ is a fuzzy strongly regular relation on $(S, \circ)$ iff $(S/\rho, \oplus)$ is a semigroup.

Definition 2.5. [1, 11] *Let $(S, \circ)$ be a fuzzy semihypergroup and $\mathbb{S}_n$ be the symmetric group on n letters ($n \in \mathbb{N}$). We define the relations λ and ϵ_n on S in the following way:*
(i)

$$a \lambda b \Leftrightarrow \exists x_1, ..., x_n \in S, \text{ such that } (x_1 \circ ... \circ x_n)(a) > 0 \text{ and } (x_1 \circ ... \circ x_n)(b) > 0.$$

(ii) $\epsilon = \bigcup_{n \geq 1} \epsilon_n$, *where* $\epsilon_1 = \{(s, s) | s \in S\}$ *and for any* $n \geq 2$;

$$a \epsilon_n b \Leftrightarrow \exists x_1, ..., x_n \in S, \exists \sigma \in \mathbb{S}_n \text{ such that } (x_1 \circ ... \circ x_n)(a) > 0 \text{ and } (x_{\sigma_1} \circ ... \circ x_{\sigma_n})(b) > 0.$$

One can see that λ and ϵ are symmetric and reflexive. Let ϵ^* and λ^* be the transitive closure of ϵ and λ, respectively. Then ϵ^* and λ^* are equivalence relations.

Definition 2.6. [13] *Let $(S, \circ)$ be a fuzzy semihypergroup. The smallest equivalence relation ρ on S is called the fundamental relation if the quotient structure $(S/\rho, \oplus)$ is a semigroup.*

Theorem 2.7. [11] *The relation ϵ^* is the abelian fundamental relation on fuzzy semihypergroup $(S, \circ)$.*

3 Characterization of α-solvable groups via a fuzzy strongly regular relation

We introduce a new fuzzy strongly regular relation on an FHG such that the quotient group is an α-solvable group.

Note: Let $(S, \circ)$ be an FHG and $m \in \mathbb{N}$. From now on for simplify we use the following notations:

(1)For any $x, y \in S$ we use xy instead of $x \circ y$.
(2) For any fuzzy strongly regular relation ρ on S and any $x \in X$ we use $\overline{x}$ for ρ_x.
(3) For any $z_1, \ldots, z_m$ of S we denote $z_1 \circ z_2, \cdots \circ z_m$ by $\prod_{i=1}^{m} z_i$.
(4) Let $Aut(S)$ denote the set of all one to one and onto fuzzy homomorphisms on an FHG.

Definition 3.1. *Let $(S, \circ)$ be an FHG and $\alpha \in Aut(S)$. Suppose $A_0^\alpha(S) = S$ and for any $k \geq 0$,*

$$A_{k+1}^\alpha(S) = \{t \in S | \exists r \in S \text{ such that } (xy)(r) > 0 \ \text{ and } (t \bullet y^\alpha x)(r) > 0 \text{ for some } \ x, y \in A_k^\alpha(S)\}.$$

For integers $n \geq 1$ and $m > 1$, consider $\xi_{1,n}^\alpha$ is the diagonal relation on S. We define the relation $\xi_{m,n}^\alpha$ as follows:

$x \xi_{m,n}^\alpha y \Leftrightarrow \exists (z_1, ..., z_m) \in S^m, \exists \sigma \in \mathbb{S}_m$with $\sigma(i) = i$ if $z_i \notin A_n^\alpha(S)$ such that

$$(\prod_{i=1}^{m} z_i)(x) > 0 \text{ and } (\prod_{i=1}^{m} z_{\sigma(i)})(y) > 0.$$

Consider $\xi_n^\alpha = \bigcup_{m \geq 1} \xi_{m,n}^\alpha$. Then $\xi_n^{*\alpha}$, the transitive closure of ξ_n^α, is an equivalence relation on S, since ξ_n^α is symmetric. For this let $a \xi_n^\alpha b$. Then there exists an integer $m \geq 1$ such that $a \xi_{m,n}^\alpha b$. It follows that

$$\exists (z_1, ..., z_m) \in S^m, \exists \delta \in \mathbb{S}_m \text{with } \delta(i) = i \text{ if } z_i \notin A_n^\alpha(S) \text{ such that } (\prod_{i=1}^{m} z_i)(a) > 0 \text{ and } (\prod_{i=1}^{m} z_{\delta(i)})(b) > 0.$$

Put $I = \delta(i)$. Now, for $(z_1, ..., z_m) \in S^m$ and $\delta^{-1} \in \mathbb{S}_m$ with $\delta^{-1}(I) = i$ if $z_I \notin A_n^\alpha(S)$, then $(\prod_{I=1}^{m} z_I)(a) > 0$ and $(\prod_{I=1}^{m} z_{\delta(I)})(b) > 0$. Therefore, $b \xi_n^\alpha a$ and so ξ_n^α is symmetric. Also, ξ_n^α is reflexive. Since for any $a \in S$ we have $a(a) = (\chi_a)(a) = 1$.

Example 3.2. *Let $S = \mathbb{Z}_2$ and α be the identity isomorphism. For any $x, y \in \mathbb{Z}_2$ we define a fuzzy hyper operation $\circ$ on $\mathbb{Z}_2$ by $x \circ y = \chi_{\{x,y\}}$. Clearly, $(\mathbb{Z}_2, \circ)$ is an FHG and $A_0^\alpha(\mathbb{Z}_2) = \mathbb{Z}_2$. Also,*

$$A_1^\alpha(\mathbb{Z}_2) = \{t \in \mathbb{Z}_2 | \exists r \in \mathbb{Z}_2; (x \circ y)(r) > 0 \ \text{and } (t \bullet (y^\alpha \circ x))(r) > 0, \text{for some } x, y \in \mathbb{Z}_2\}.$$

Let $r = 0, x = 0$ and $y = 1$. Then $(x \circ y)(r) = \chi_{\{x,y\}}(r) = \chi_{\{0,1\}}(0) > 0$ and

$$\begin{aligned}
(t \bullet (y^\alpha \circ x))(r) &= \bigvee_{s \in \mathbb{Z}_2} (t \circ s)(r) \wedge (y^\alpha \circ x)(s) \\
&= \bigvee_{s \in \mathbb{Z}_2} \chi_{\{t,s\}}(0) \wedge \chi_{\{0,1^\alpha\}}(s) \\
&= (\chi_{\{t,0\}}(0) \wedge \chi_{\{1,0\}}(0)) \vee (\chi_{\{t,1\}}(0) \wedge \chi_{\{1,0\}}(1)) \\
&= 1.
\end{aligned}$$

Therefore, $A_1^\alpha(\mathbb{Z}_2) = \mathbb{Z}_2$.

Example 3.3. *Let α be the identity isomorphism and $S=\{a,b,c\}$. We define the fuzzy hyperoperation "$\circ$" on S as follows:*
$(a\circ a)(a)=(b\circ b)(a)=(c\circ c)(a)=0.5,\ (a\circ b)(b)=(b\circ a)(b)=(b\circ c)(b)=(c\circ b)(b)=0.1,\ (a\circ c)(c)=(b\circ b)(c)=(c\circ a)(c)=0.7,$ *and* $(a\circ a)(b)=(a\circ a)(c)=(a\circ b)(a)=(a\circ b)(c)=(a\circ c)(a)=(a\circ c)(b)=(b\circ a)(a)=(b\circ a)(c)=(b\circ b)(b)=(b\circ c)(a)=(b\circ c)(c)=(c\circ a)(a)=(c\circ a)(b)=(c\circ b)(a)=(c\circ b)(c)=(c\circ c)(b)=(c\circ c)(c)=0.$
Let $\rho=\{(a,a),(b,b),(c,c),(a,c),(c,a)\}$. Then $A_0^\alpha(S)=S$ and $A_1^\alpha(S)=\{a\}$.

Theorem 3.4. *The relation $\xi_n^{*\alpha}$ is a fuzzy strongly regular relation.*

Proof. It is clear that $\xi_{m,n}^{*\alpha}$ is an equivalence relation. First, we show that for any $x,y,z\in S$

$$x\xi_n^\alpha y\Rightarrow xz\overline{\overline{\xi_n^\alpha}}yz \text{ and } zx\overline{\overline{\xi_n^\alpha}}zy \qquad (*).$$

If $x\xi_n^\alpha y$, then there exists an integer m such that $x\xi_{m,n}^\alpha y$, and so there exist $(z_1,\ldots,z_m)\in S^m$ and $\sigma\in\mathbb{S}_m$ with $\sigma(i)=i$ if $z_i\notin A_n^\alpha(S)$ such that $(\prod_{i=1}^m z_i)(x)>0$ and $(\prod_{i=1}^m z_{\sigma(i)})(y)>0$.

Let $z\in S$ such that for any r,s we have $(xz)(r)>0$ and $(yz)(s)>0$. Let $p=x$ and $q=y$. Then

$$((\prod_{i=1}^m z_i)\bullet z)(r)=\bigvee_p\{(\prod_{i=1}^m z_i)(p)\wedge(pz)(r)\}>0$$

and

$$((\prod_{i=1}^m z_{\sigma(i)})\bullet z)(s)=\bigvee_q\{(\prod_{i=1}^m z_{\sigma(i)})(q)\wedge(qz)(s)\}>0.$$

Now, suppose that $z_{m+1}=z$. We define σ' as follows:

$$\sigma'(i)=\begin{cases}\sigma(i), & \forall i\in\{1,2,\ldots,m\}\\ m+1, & i=m+1.\end{cases}$$

It is clear that σ' is one to one and onto. Thus for any $r,s\in S$

$$(\prod_{i=1}^{m+1} z_i)(r)>0 \text{ and } (\prod_{i=1}^{m+1} z_{\sigma'(i)})(s)>0.$$

Hence σ' is a permutation of $\mathbb{S}^{m+1}$ such that $\sigma'(i)=i$ if $z_i\notin A_n^\alpha(S)$. Therefore, $xz\overline{\overline{\xi_n^\alpha}}yz$. Now, if $x\xi_n^{*\alpha}y$, then there exists $k\in\mathbb{N}$ and $u_0=x,u_1,\ldots,u_k=y\in S$ such that $u_0=x\xi_n^\alpha u_1\xi_n^\alpha u_2\xi_n^\alpha\ldots\xi_n^\alpha u_k=y$. By the above result we have $u_0z=xz\overline{\overline{\xi_n^\alpha}}u_1z\overline{\overline{\xi_n^\alpha}}u_2z\overline{\overline{\xi_n^\alpha}}\ldots\overline{\overline{\xi_n^\alpha}}u_kz=yz$ and so $xz\overline{\overline{\xi_n^\alpha}}yz$. By the similar way, we can show that $zx\overline{\overline{\xi_n^\alpha}}zy$. Therefore, $\xi_n^{*\alpha}$ is a fuzzy strongly regular relation on S. □

Proposition 3.5. *For any integer n we have $\xi_{n+1}^{*\alpha}\subseteq\xi_n^{*\alpha}$.*

Proof. Let $x\xi_{n+1}^\alpha y$. Then there exist $m\in\mathbb{N}$, $(z_1,...,z_m)\in S^m$ and $\delta\in\mathbb{S}_m$ with $\delta(i)=i$ if $z_i\notin A_{n+1}^\alpha(S)$ such that $(\prod_{i=1}^m z_i)(x)>0$ and $(\prod_{i=1}^m z_{\delta(i)})(y)>0$. Now, for $(z_1,...,z_m)\in S^m$ and $\delta\in\mathbb{S}_m$ with $\delta(i)=i$ if $z_i\notin A_n^\alpha(S)$ we have $z_i\notin A_{n+1}^\alpha(S)$ (since $A_{n+1}^\alpha(S)\subseteq A_n^\alpha(S)$) and so $(\prod_{i=1}^m z_i)(x)>0$ and $(\prod_{i=1}^m z_{\delta(i)})(y)>0$. Therefore, $x\xi_n^\alpha y$. □

Proposition 3.6. *For any integer n we have $\lambda^* \subseteq \xi_n^{*\alpha} \subseteq \epsilon^*$. In particular, if S is a commutative FHG, then $\epsilon^* = \xi_n^{*\alpha} = \lambda^*$.*

Proof. It is clear that $\lambda^* \subseteq \xi_n^{*\alpha} \subseteq \epsilon^*$. It is enough to show that if S is commutative, then $\lambda^* = \xi_n^{*\alpha} = \epsilon^*$. For this, let $a\xi_n^{*\alpha}b$. Then there exists an integer m , $(x_1, x_2, \ldots, x_m) \in S^m$ and $\varrho \in \mathbb{S}^m$ with $\varrho(i) = i$ if $x_i \notin A_n^\alpha(S)$ such that $(x_1 \circ \ldots \circ x_m)(a) > 0$ and$(x_{\sigma_1} \circ \ldots \circ x_{\sigma_m})(b) > 0$. For any i since S is commutative, we conclude that each element $x_{\varrho(i)}$ can commute with others and so $\lambda^* = \xi_n^{*\alpha} = \epsilon^*$. □

Example 3.7. *Let S be an FHG as Example 3.3. Then it is routine to verify that ρ is a fuzzy strongly regular relation* [1].

Now, we are ready to state one of our main results of this section.

Theorem 3.8. *$S/\xi_n^{*\alpha}$ is an α-solvable group of length at most $n+1$.*

Proof. Let φ be a fuzzy strongly regular relation on S. Then we show that for any integer k

$$D_k^\alpha(S/\varphi)) = \langle \bar{t} | t \in A_k^\alpha(S) \rangle.$$

We proceed by induction on k. Put $G = S/\varphi$. Since $G = \langle \bar{t} | t \in S \rangle$ the case $k = 0$ is clear. Now, suppose that $\bar{a} \in \langle \bar{t} | t \in A_{k+1}^\alpha(S) \rangle$, then there exists $t \in A_{k+1}^\alpha(S)$ such that $\bar{a} = \bar{t}$. By Definition 3.1, there exist $r_1 \in S$ and $x, y \in A_k^\alpha(S)$ such that $(xy)(r_1) > 0$ and $(t \bullet y^\alpha x)(r_1) > 0$. It follows from Theorem 2.4 that $\bar{x} \oplus \bar{y} = \overline{r_1}$ and $\bar{t} \oplus \overline{y^\alpha} \oplus \bar{x} = \overline{r_1} = \bar{x} \oplus \bar{y}$. So $\bar{t} = [\bar{x}, \bar{y}]_{\bar{\alpha}}$. The hypotheses of induction implies that $\bar{a} = \bar{t} \in D_{k+1}^\alpha(G)$.

Conversely, let $\bar{a} \in D_{k+1}^\alpha(G)$. Then there exist $\bar{x}, \bar{y} \in D_k^\alpha(G)$ such that $\bar{a} = [\bar{x}, \bar{y}]_{\bar{\alpha}}$. So by hypotheses of induction we have $\bar{x} = \bar{u}$ and $\bar{y} = \bar{v}$, where $u, v \in A_k^\alpha(S)$. As uv is a non-zero fuzzy subset of S so there exists $c \in S$ such that $(uv)(c) > 0$. By Definition 2.1, we have $1 = \chi_S(c) = (Su)(c) = \bigvee_{r \in S}(ru)(c)$ and so there exists $r \in S$ such that $(ru)(c) > 0$. Moreover, $1 = \chi_S(r) = (Sv^\alpha)(r) = \bigvee_{t \in S}(tv^\alpha)(r)$. Hence, by Definition 2.1 we have:

$$(t \bullet v^\alpha u)(c) = (tv^\alpha \bullet u)(c) = \bigvee_p ((tv^\alpha)(p) \wedge (pu)(c)) \geq (tv^\alpha)(r) \wedge (ru)(c) > 0.$$

Thus $(uv)(c) > 0$ and $(t \bullet v^\alpha u)(c) > 0$. So $t \in A_{k+1}^\alpha(S)$. It follows from Theorem 2.4, that $\bar{u} \oplus \bar{v} = \bar{c} = \bar{t} \oplus \overline{v^\alpha} \oplus \bar{u}$, and so $\bar{t} = [\bar{u}, \bar{v}]_{\bar{\alpha}} = [\bar{x}, \bar{y}]_{\bar{\alpha}} = \bar{a}$. Therefore, $\bar{a} = \bar{t} \in \langle \bar{t}; t \in A_{k+1}^\alpha(S) \rangle$ i.e $D_{k+1}^\alpha(S/\varphi) = \langle \bar{t} | t \in A_{k+1}^\alpha(S) \rangle$. Consequently, $D_n^\alpha(S/\xi_n^{*\alpha})$ is an abelian group and $D_{n+1}^\alpha(S/\xi_n^{*\alpha}) = \{e\}$. □

In the following, we introduce the smallest fuzzy strongly regular relation $\xi^{*\alpha}$ on a finite FHG S such that $S/\xi^{*\alpha}$ is an α-solvable group.

Theorem 3.9. *The fuzzy relation $\xi^{*\alpha} = \bigcap_{n \geq 1} \xi_n^{*\alpha}$ is the smallest fuzzy strongly regular relation on a finite FHG S such that $S/\xi^{*\alpha}$ is an α-solvable. In particular, $\xi^{*\alpha}$ is an α-solvable fundamental relation.*

Proof. First, we show that $\xi^{*\alpha}$ is a fuzzy strongly regular relation on S such that $S/\xi^{*\alpha}$ is α-solvable. By $\xi^{*\alpha} = \bigcap_{n \geq 1} \xi_n^{*\alpha}$ and Theorem 3.4, it is easy to see that $\xi^{*\alpha}$ is a fuzzy strongly regular relation on S. Since S is finite, Proposition 3.5 implies that there exists an integer k such that $\xi^{*\alpha}_{k+1} = \xi^{*\alpha}_{k}$. Thus, for some m $\xi^{*\alpha} = \xi^{*\alpha}_{m}$ and so by Theorem 3.8, $S/\xi^{*\alpha}$ is α-solvable.

Now, we prove $\xi^{*\alpha}$ is the smallest relation with this property. Suppose ρ is a fuzzy strongly regular relation on S such that $K = S/\rho$ is α-solvable of class c. We show that $\xi^{*\alpha} \subseteq \rho$. For this, let $x, y \in S$ and $x\xi^{\alpha}y$, where $\xi^{\alpha} = \bigcap_{n\geq 1}\xi^{\alpha}_{n}$. Then there exists integers n and m such that $x\xi^{\alpha}_{m,n}y$ and so there exist $(z_1, \ldots, z_m) \in S^m$ and $\delta \in \mathbb{S}_m$ with $\delta(i) = i$ if $z_i \notin A^{\alpha}_{n}(S)$ such that $(\prod_{i=1}^{m} z_i)(x) > 0$ and $(\prod_{i=1}^{m} z_{\delta(i)})(y) > 0$. Thus by Theorem 2.4, we get

$$\overline{x} = \prod_{i=1}^{m}\overline{z_i} \text{ and } \overline{y} = \prod_{i=1}^{m}\overline{z_{\delta(i)}}.$$

By the proof of Theorem 3.8, we have

$$D_c(S/\rho) = \langle \overline{t} | t \in A^{\alpha}_{c}(S)\rangle = \{\overline{e}\}.$$

And so for any $z_i \in A^{\alpha}_{c}(S)$ we get $\overline{z_i} = \overline{e}$. Hence, $\overline{x} = \overline{y}$. Therefore, $x\rho y$ as required. Now, $\xi^{*\alpha} \subseteq \rho$, because, let $z, t \in S$ and $z\xi^{*\alpha}t$. Then for some integer n, $z\xi^{*\alpha}_{n}t$ and so there exist $z_0, z_1, \ldots, z_k \in S$ $(k \in \mathbb{N})$ such that $(z = z_0)\xi^{*\alpha}_{n}z_1\xi^{*\alpha}_{n} \ldots \xi^{*\alpha}_{n}(z_k = t)$. So we have $(z = z_0)\rho z_1\rho \ldots \rho(z_k = t)$. Hence, $\xi^{*\alpha} \subseteq \rho$. Therefore, $\xi^{*\alpha}$ is the smallest relation such that $S/\xi^{*\alpha}$ is an α-solvable group. □

Example 3.10. *Let S be an FHG as Example 3.2. Then, by Proposition 3.6, we have $\epsilon^* = \xi^{*\alpha}_{n}$ and so $S/\xi^{*\alpha}_{n} = S/\epsilon* \cong S$. Therefore, it follows from Theorem 3.8 that S is an α-solvable group.*

Example 3.11. *Let α be the identity isomorphism and $S = \{a, b, c\}$. Consider fuzzy hyperoperation "$\circ$" on S as follows:*
$(a\circ a)(a) = (b\circ b)(a) = (c\circ c)(a) = 0.5$, $(a\circ b)(b) = (b\circ a)(b) = (b\circ c)(b) = (c\circ b)(b) = 0.1$, $(a\circ c)(c) = (b\circ b)(c) = (c\circ a)(c) = 0.7$, *and* $(a\circ a)(b) = (a\circ a)(c) = (a\circ b)(a) = (a\circ b)(c) = (a\circ c)(a) = (a\circ c)(b) = (b\circ a)(a) = (b\circ a)(c) = (b\circ b)(b) = (b\circ c)(a) = (b\circ c)(c) = (c\circ a)(a) = (c\circ a)(b) = (c\circ b)(a) = (c\circ b)(c) = (c\circ c)(b) = (c\circ c)(c) = 0$.
Let $\rho_1 = \{(a, a), (b, b), (c, c)\}$. It is clear that ρ_1 is the smallest fuzzy strongly regular relation.

4 ξ^{α}-part of an FHG

In this section, we use the concept of an ξ^{α}-part of an FHG to make a transitive fuzzy relation ξ^{α} on an FHG.

Definition 4.1. *Let X be a non-empty subset of S. Then X is called an ξ^{α}-part of S if for any $m \in \mathbb{N}$, $(z_1, ..., z_m) \in S^m$ and $\sigma \in \mathbb{S}_m$ with $\sigma(i) = i$ if $z_i \notin \bigcup_{n\geq 1} A^{\alpha}_{n}(S)$, then*

there exists $x \in X$ such that $(\prod_{i=1}^{m} z_i)(x) > 0$ implies for all $y \in S\backslash X$, $(\prod_{i=1}^{m} z_{\sigma(i)})(y) = 0$.

Theorem 4.2. *Let X be a non-empty subset of S. Then for any $x, y \in S$ the following conditions are equivalent:*

(i) X is an ξ^{α}-part of S,
(ii) If $x \in X$ and $x\xi^{\alpha}y$, then $y \in X$,
(iii) If $x \in X$ and $x\xi^{*\alpha}y$, then $y \in X$.

Proof. $(i) \Rightarrow (ii)$ For $x, y \in S$ if $x \in X$ and $x\xi^{\alpha}y$, then there exist $n, m \in \mathbb{N}$ such that $x\xi^{\alpha}_{m,n}y$ and so there exist $(z_1, ..., z_m) \in S^m$ and $\sigma \in \mathbb{S}_m$ with $\sigma(i) = i$ if $z_i \notin \bigcup_{n\geq 1} A^{\alpha}_n(S)$ such that $(\prod_{i=1}^{m} z_i)(x) > 0$ and $(\prod_{i=1}^{m} z_{\sigma(i)})(y) > 0$. As X is an ξ^{α}-part of S and $(\prod_{i=1}^{m} z_i)(x) > 0$ if $y \notin X$ we have $(\prod_{i=1}^{m} z_{\sigma(i)})(y) = 0$, a contradiction. Therefore, $y \in X$.
$(ii) \Rightarrow (iii)$ Let $x, y \in S$, $x \in X$, and $x\xi^{*\alpha}y$. Then there is an integer m and $(z_0, ..., z_m) \in S^m$ such that $x = z_0\xi^{\alpha}z_1\xi^{\alpha} \dots \xi^{\alpha}z_m = y$. Applying (ii) m times, we have $y \in X$.
$(iii) \Rightarrow (i)$ For $(z_1, ..., z_m) \in \mathbb{S}^m$ and $\sigma \in S_m$ with $\sigma(i) = i$ if $z_i \notin \bigcup_{n\geq 1} A^{\alpha}_n(S)$, let $x \in X$ and $(\prod_{i=1}^{m} z_i)(x) > 0$. If $y \notin X$, then $(\prod_{i=1}^{m} z_{\sigma(i)})(y) > 0$. It follows that $x\xi_n y$ and so $x\xi y$. Hence, (iii) implies that $y \in X$, a contradiction and so $(\prod_{i=1} z_{\sigma(i)})(y) = 0$, i.e X is an ξ^{α}-part of S. □ □

Example 4.3. *Let $X = \{a, c\}$ be as Example 3.7. Then by Theorem 4.2 and Proposition 3.6, X is an ξ^{α}-part of S.*

Theorem 4.4. *For any $a \in S$, $\xi^{\alpha}(a)$ is an ξ^{α}-part of S if and only if ξ^{α} is transitive.*

Proof. $(\Leftarrow)$ Let $x, y \in S$, $z \in \xi^{\alpha}(x)$ and $z\xi^{\alpha}y$. Since ξ^{α} is transitive, we have $y \in \xi^{\alpha}(x)$. So, by Theorem 4.2, $\xi^{\alpha}(x)$ is an ξ^{α}-part of S.
$(\Rightarrow)$ Suppose that $x\xi^{*\alpha}y$. Then there exists an integer k and $(z_1, \dots, z_k) \in S^k$ such that

$$x = z_0\xi^{\alpha}z_1\xi^{\alpha} \dots \xi^{\alpha}z_k = y$$

thus, $z_i \in \xi^{\alpha}(z_{i-1})$. Since $\xi^{\alpha}(z_i)$ is an ξ^{α}-part $(0 \leq i \leq k)$ it follows that $y \in \xi^{\alpha}(x)$ by Theorem 4.2, i.e $x\xi^{\alpha}y$ and so $\xi^{*\alpha} = \xi^{\alpha}$.

□

Definition 4.5. *Let A be a non-empty subset of S. We define $K(A)$ and $W(A)$ as follows:*
1) $K(A) = \bigcap\{B : A \subseteq B$ and B is an $\xi^{\alpha} -$ part of $S\}$. We use $K(a)$ for $K(\{a\})$,
2) $W(A) = \bigcup_{n\geq 1} W_n(A)$, where $W_1(A) = A$ and for $n \geq 1$,

$$W_{n+1}(A) = \{x \in S | \exists m \in \mathbb{N} \text{ and } \exists (z_1, \dots, z_m) \in S^m \text{ such that for some } a \in W_n(A) \text{ we have } (\prod_{i=1}^{m} z_i)(x) > 0 \text{ and } \exists \sigma \in \mathbb{S}_m \text{ with } \sigma(i) = i \text{ if } z_i \notin \bigcup_{s\geq 1} A^{\alpha}_s(S) \text{ such that } (\prod_{i=1}^{m} z_{\sigma(i)})(a) > 0\}.$$

Example 4.6. *Let $A = \{a, c\}$ be as Example 3.7. Since X is an ξ^{α}-part of S we have $K(A) = A$.*

Theorem 4.7. *The following statements hold:*
(1) $W(A) = K(A)$,
(2) $K(A) = \bigcup_{a\in A} K(a)$,
(3) $W_n(W_2(z)) = W_{n+1}(z)$, *for* $n \geq 2$ *and* $z \in S$.

Proof. (1) We show that $W(A)$ is an ξ^α-part. Let $a \in W(A)$, $(\prod_{i=1}^{m} z_i)(a) > 0$ and $\sigma \in \mathbb{S}_m$ with $\sigma(i) = i$, if $z_i \notin \bigcup_{s\geq 1} A_s^\alpha(S)$. Then there exists an integer n such that $a \in W_n(A)$. If $t \notin W(A)$ and $(\prod_{i=1} z_{\sigma(i)})(t) > 0$, then $t \in W_{n+1}(A)$ and so $t \in W(A)$, a contradiction. Therefore, $(\prod_{i=1}^{m} z_{\sigma(i)})(t) = 0$ and $W(A)$ is an ξ^α-part.

Now, it is enough to prove that if B is an ξ^α-part and $A \subseteq B$, then for any n, $W_n(A) \subseteq B$ i.e $W(A)$ is the smallest ξ^α-part of S which contains A. We use induction on n. Since $W_1(A) = A \subseteq B$, the case $n = 1$ is clear. Let $W_n(A) \subseteq B$ and $z \in W_{n+1}(A)$. Then there exists an integer m and $(z_1, \ldots, z_m) \in S^m$ and $\sigma \in \mathbb{S}_m$ with $\sigma(i) = i$ if $z_i \notin \bigcup_{s\geq 1} A_s^\alpha(S)$ and $t \in W_n(A)$ such that $(\prod_{i=1}^{m} z_{\sigma(i)})(t) > 0$ and $(\prod_{i=1}^{m} z_i)(z) > 0$. Since $W_n(A) \subseteq B$ we have $t \in B$. Moreover, if $z \notin B$ as B is ξ^α-part, then $(\prod_{i=1}^{m} z_i)(z) = 0$, a contradiction, and so $z \in B$ and the result holds.

(2) We know that for any $a \in A$, $K(a) \subseteq K(A)$. We use induction on n to prove that $W_n(A) = \bigcup_{a\in A} W_n(a)$. It follows from (1) that $K(A) = \bigcup_{n\geq 1} W_n(A)$ and $W_1(A) = A = \bigcup_{a\in A}\{a\}$. Suppose that it is true for n and $z \in W_{n+1}(A)$. Then there exists an integer m and $(z_1, \ldots, z_m) \in S^m$ such that $(\prod_{i=1}^{m} z_i)(z) > 0$ and there exists $\sigma \in \mathbb{S}_m$ with $\sigma(i) = i$ if $z_i \notin \bigcup_{s\geq 1} A_s^\alpha(S)$ such that for some $a \in W_n(A)$, $(\prod_{i=1}^{m} z_{\sigma(i)})(a) > 0$. By the hypotheses of induction we have $W_n(A) = \bigcup_{b\in A} W_n(b)$ and so $a \in \bigcup_{b\in A} W_n(b)$. Therefore, for some $b \in A$, $a \in W_n(b)$. Hence, $z \in W_{n+1}(b)$ i.e $W_{n+1}(A) \subseteq \bigcup_{b\in A} W_{n+1}(b)$. Since for any $a \in A, K(a) \subseteq K(A)$ we obtain $K(A) = \bigcup_{n} W_n(A) \subseteq \bigcup_{n}\bigcup_{a} W_n(a) = \bigcup_{a\in A} K(a) \subseteq K(A)$ Therefore, $K(A) = \bigcup_{a\in A} K(a)$.

(3) We proceed by induction on n. For $n = 2$ we have

$$W_2(W_2(x)) \;=\; \{z | \exists q \in \mathbb{N}, \exists (a_1, \ldots, a_q) \in S^q \text{ and } \sigma \in \mathbb{S}_q \text{ with } \sigma(i) = i \text{ if } z_i \notin \bigcup_{s\geq 1} A_s^\alpha(S) \text{ such that } (\prod_{i=1}^{q} a_i)(z) > 0 \text{ and for some } y \in W_2(x), (\prod_{i=1}^{q} a_{\sigma(i)})(y) > 0\} = W_3(x).$$

Suppose $W_n(W_2(x)) = W_{n+1}(x)$. Then

$$\begin{aligned} W_{n+1}(W_2(x)) &= \{z \in S | \exists q \in \mathbb{N}, (a_1, \ldots, a_q) \in S^q \text{ and } \sigma \in \mathbb{S}_q \text{ with } \sigma(i) = i \text{ if } z_i \notin \bigcup_{s \geq 1} A_s^{\alpha}(S), \\ & \quad t \in W_n(W_2(x)) \text{ such that } (\prod_{i=1}^{q} a_i)(z) > 0 \text{ and } \prod_{i=1}^{q} a_{\sigma(i)})(t) > 0\} \\ &= W_{n+2}(x). \end{aligned}$$

This completes the proof.

□

Theorem 4.8. *Let $x, y \in S$. Then the following relation is an equivalence relation on S:*

$$xWy \quad \textit{if and only if } x \in W(\{y\}).$$

Proof. The relation W is reflexive, since Theorem 4.7 and Definition 4.5, imply that $W\{x\} = K\{x\}$ and $x \in W\{x\}$ i.e xWx. Also, W is transitive, since for $x, y, z \in S$ let xWy and yWz. Therefore, Theorem 4.7, implies $x \in K(y)$ and $y \in K(z)$. For any P, ξ^{α}-part of S which contains z, we have $K(z) \subseteq P$ and so $y \in P$. Then $K(y) \subseteq P$ and so $x \in P$. Thus for any P we have $x \in P$ and $K(z)$ is an ξ^{α}-part of S which contains z, so $x \in K(z)$. Therefore, by Theorem 4.7, xWz and so W is transitive. W is symmetric. For this first by induction on n we prove that $x \in W_n(y)$ if and only if $y \in W_n(x)$. For $n = 2$ it is clear. Suppose $x \in W_{n+1}(y)$, then there exists an integer $q \geq 1, (a_1, \ldots, a_q) \in S^q$ and $\sigma \in \mathbb{S}_q$ with $\sigma(i) = i$ if $a_i \notin \bigcup_{s \geq 1} A_s^{\alpha}(S)$ and $t \in W_n(y)$ such that $(\prod_{i=1}^{q} a_i)(x) > 0$ and $(\prod_{i=1}^{q} a_{\sigma(i)})(t) > 0$. It follows that $t \in W_2(x)$. By hypotheses of induction we have $y \in W_n(t)$. Therefore, by Theorem 4.7(3), we have $y \in W_n(W_2(x)) = W_{n+1}(x)$.

□

Example 4.9. *Let $\rho = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$ and $\pi : S \to S/\rho$ defined by $\pi(x) = \overline{x}$ for all $x \in S$ be the canonical homomorphism. We know that ρ is a fuzzy strongly regular relation so by Theorem 2.6, S/ρ is a group. Moreover, $S/\rho = \{\overline{a}, \overline{b}\}$ and $\overline{a} = \{a, c\}$ is the identity element of S/ρ. Also,*

$$\omega_S = Ker(\pi) = \{x | \overline{x} = \overline{a}\} = \{a, c\}.$$

By Example 4.3, $\{a, c\}$ is a ρ-part of S i.e ω_S is a ρ-part of S.

Let M be a non-empty subset of S. We Know that $(M\omega_S)(r) = \bigvee_{x \in \omega_S, m \in M} (m \circ x)(r)$.

Lemma 4.10. *Assume that M is a non-empty subset of S. Then we have*
(i) $\pi^{-1}(\pi(M)) = \{x \in S : (\omega_S M)(x) > 0\} = \{x \in S : (M\omega_S)(x) > 0\}$;
(ii) If M is an ξ^{α}-part of S, then $\pi^{-1}(\pi(M)) = M$.

Proof. (i) Let $x \in S$, $t \in \omega_S$ and $y \in M$ such that $(ty)(x) > 0$. Then by Theorem 2.4, $\pi(x) = \pi(t) \oplus \pi(y) = 1_{S/\xi^{*\alpha}} \oplus \pi(y) = \pi(y)$ and so $x \in \pi^{-1}(\pi(y)) \subset \pi^{-1}(\pi(M))$.

Conversely, for any $x \in \pi^{-1}(\pi(M))$, there exists $b \in M$ such that $\pi(x) = \pi(b)$. For $a \in S$ we have $aS = \chi_S$ and so $(ab)(x) > 0$. Since by Theorem 2.4, $\pi(b) = \pi(x) = \pi(a) \oplus \pi(b)$ we have $\pi(a) = 1_{S/\xi^{*\alpha}}$. So $a \in \pi^{-1}(1_{S/\xi^{*\alpha}}) = \omega_S$. Therefore, $(\omega_S M)(x) > 0$.

By the similar way, we can prove that $\pi^{-1}(\pi(M)) = \{x \in S : (M\omega_S)(x) > 0\}$.
(ii) It is clear that $M \subseteq \pi^{-1}(\pi(M))$. If $x \in \pi^{-1}(\pi(M))$, then there exists $b \in M$ such that $\pi(x) = \pi(b)$ i.e $\xi^{*\alpha}(x) = \xi^{*\alpha}(b)$. Therefore, $x \in M$ by Theorem 4.2(iii) and M is ξ^{α}-part.

□

Theorem 4.11. *For all $a, b \in S$, aWb if and only if $a\xi^{*\alpha}b$.*

Proof. ($\Leftarrow$) Let $a\xi^{*\alpha}b$. Then there exist integer n, m such that $a\xi^{\alpha}_{m,n}b$. So for any $(z_1, ..., z_m) \in S^m$ and $\sigma \in \mathbb{S}_m$ with $\sigma(i) = i$ if $z_i \notin \bigcup_{n\geq 1} A^{\alpha}_n(S)$ we have $(\prod_{i=1}^{q} a_i)(a) > 0$ and $(\prod_{i=1}^{q} a_{\sigma(i)})(b) > 0$ and so $a \in W_2(b)$. Thus, by Definition 4.5, aWb and $\xi^{*\alpha} \subset W$.
($\Rightarrow$) If xWy, then there exists $n \in \mathbb{N}$ such that $x \in W_n(y)$. So for any integer m, $(z_1, ..., z_m) \in S^m$ and $\sigma \in \mathbb{S}_m$ with $\sigma(i) = i$ if $z_i \notin \bigcup_{n\geq 1} A^{\alpha}_n(S)$ we have $(\prod_{i=1}^{q} a_i)(x) > 0$ and for some $x_1 \in W_{n-1}(y)$ we have $(\prod_{i=1}^{q} a_{\sigma(i)})(x_1) > 0$. Thus, $x\zeta^{\alpha}_n x_1$. Continuing this method there exist $\exists x_2, \ldots, x_{n-1} \in S$ such that $x_i \in W_{n-i}(y)$ and $x_{i-1}\xi^{\alpha}_n x_i$. Then $(x = x_0)\xi^{\alpha}_n x_1 \xi^{\alpha}_n \ldots \xi^{\alpha}_n (x_{n-1} = y)$. Therefore, $W \subseteq \zeta^{*\alpha}$.

□

Theorem 4.12. *ω_S is a fuzzy subhypergroup of S which is also an ξ^{α}-part of S.*

Proof. It is clear that $\omega_S \subseteq S$ and so for any $a, b, c \in \omega_S$, $(ab) \bullet c = a \bullet (bc)$. Let $x, y \in \omega_S$. Then $Sy = \chi_S$ implies that there exists $u \in S$ such that $(uy)(x) > 0$. By Theorem 2.4, $\overline{u} \oplus \overline{y} = \overline{x}$ and so $\overline{u} = \overline{1}$. i.e $u \in \omega_S$. Therefore, $\omega_S y = \chi_{\omega_S}$ and ω_S is a fuzzy subhypergroup of S. Now we prove that

$$K(y) = \pi^{-1}(\pi(\{y\})) = \{x \in S : (\omega_S y)(x) > 0\} = \omega_S.$$

Let $y, z \in S$. Then

$$\begin{aligned} z \in \pi^{-1}(\pi(\{y\})) &\iff \pi(z) = \pi(y) \\ &\iff \xi^{*\alpha}(z) = \xi^{*\alpha}(y) \\ &\iff z\xi^{*\alpha}y \\ &\iff z \in \xi^{*\alpha}(y) = W(\{y\}) = K(y). \end{aligned}$$

Moreover, $y \in \omega_S$, we have $\{x \in S : (\omega_S y)(x) > 0\} = \{x \in S : (\chi_{\omega_S})(x) > 0\} = \omega_S$. Therefore, $K(y) = \omega_S$ and so ω_S is an ξ^{α}-part.

□

5 Conclusions

In this paper, we defined a new strongly regular relation on an FHG to get an α-solvable group. Also, we introduced the concept of ξ^{α}-part of a fuzzy hypergroup. Basically, we studied the relation between their fundamental relation and ξ^{α}-parts of a given FHG. In addition, we can extend this work on α-Engel groups (α-nilpotant groups).

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