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On S-prime hyperideals in multiplicative hyperrings

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Abstract

Let R be a multiplicative hyperring and $S \subseteq R$ be a multiplicatively closed subset of R. In this paper, we introduce and study the concept of S-prime hyperideals which is a generalization of prime hyperideals. Some properties of S-prime hyperideals in multiplicative hyperring are presented. Then we investigate the behaviour of S-prime hyperideals under homomorphism hyperrings, in factor hyperrings, Cartesian products of hyperrings, and the fundamental relation in the context of multiplicative hyperring.

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1 Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Hyperstructures have many applications to several sectors of both pure and applied mathematics (see [5, 8]). The hypergroup notion was introduced in 1934 by a French mathematician Marty [11], at the 8th Congress of Scandinavian Mathematicians. Contrary to classical algebra, in hyperstructure theory, there are various kinds of hyperrings and studied by many authors. The notion of hyperrings was introduced by M. Krasner in 1983, where the addition is a hyperoperation, while the multiplication is an operation [10]. The notion of multiplicative hyperrings are an important class of algebraic hyperstructures that generalize rings, initiated the study by Rota in 1982, where the multiplication is a hyperoperation, while the addition is an operation [14]. The principal notions of algebraic hyperstructure theory can be found in [4, 6, 8]. Procesi and Rota introduced and studied, in brief, the prime hyperideals

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of multiplicative hyperrings [12, 13] and this idea is further generalized in a paper by U. Dasgupta in [6, 7]. Ameri *et al.* in [1] described multiplicative hyperring of fractions and coprime hyperideals. Later on, many types of research have observed that generalizations of prime hyperideals in multiplicative hyperrings. Ghiasvand in [9] has introduced and studied the concept of 2-absorbing hyperideals of a multiplicative hyperring as a generalization of prime hyperideals. Also, Anbarloei has studied 2-absorbing and 2-absorbing primary hyperideals of a multiplicative hyperring in [2, 3]. In this paper, we introduce and study the concept of S-prime hyperideal in a multiplicative hyperring which is also a generalization of prime hyperideals, and obtain their basic properties. For example, we show that every prime hyperideal is an S-prime hyperideal, but the converse is not true in general (see Example 3.4). After we investigate the behaviour of S-prime hyperideals under homomorphism hyperrings, in factor hyperrings, and Cartesian products of hyperrings. Also, we show that the hyperideal P is S-prime if and only if P/γ^* is an S/γ^*-prime ideal of R/γ^*.

2 Preliminaries

Definition 2.1. [8] *Let H be a non-empty set. By $P^*(H)$, we mean the set of all non-empty subsets of H. A hyperoperation on H is a map $\circ : H \times H \to P^*(H)$. Then $(H, \circ)$ is called a hypergroupoid. A hypergroup is a hypergroupoid $(H, \circ)$ that satisfies the associative and the reproductive law, i.e.,*
(1) $x \circ (y \circ z) = (x \circ y) \circ z, \ \forall x, y, z \in H$, *(associative law)*
(2) $x \circ H = H \circ x = H, \ \forall x \in H$. *(reproductive law)*

A hypergroupoid $(H, \circ)$ is called a semihypergroup if only the associative law holds. Here, we mean a semihypergroup by a non-empty set H with an associative hyperoperation $\circ$, i. e.,

$$a \circ (b \circ c) = \bigcup_{t \in (b \circ c)} a \circ t = \bigcup_{s \in (a \circ b)} s \circ c = (a \circ b) \circ c$$

for all $a, b, c \in H$.

For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ x = A \circ \{x\}.$$

Definition 2.2. [8] *A non-empty subset A of a hypergroup $(H, \circ)$ is called a sub-hypergroup of H if $(A, \circ)$ is itself a hypergroup.*

Definition 2.3. [8] *A triple $(R, +, \circ)$ is called a multiplicative hyperring, if it has the following properties:*
(i) $(R, +)$ *is an Abelian group;*
(ii) $(R, \circ)$ *is a semihypergroup;*
(iii) *For all $a, b, c \in R$, $a \circ (b + c) \subseteq a \circ b + a \circ c$ and $(b + c) \circ a \subseteq b \circ a + c \circ a$;*
(iv) $a \circ (-b) = (-a) \circ b = -(a \circ b)$.

If in (iii) we have equalities instead of inclusions, then we say that the multiplicative hyperring is strongly distributive.

Definition 2.4. [6] (a) *If $(R, \circ)$ is a multiplicative hyperring with $a \circ b = b \circ a$ for all $a, b \in R$, then $(R, \circ)$ is called a commutative multiplicative hyperring.*

(b) *A non-zero element a of a multiplicative hyperring R is said to be unit if $1 \in a \circ x$ and $1 \in x \circ a$ for some $x \in R$. The set of all unit elements of R is denoted by $U(R)$.*
(c) *A commutative hyperring R with identity 1 is called hyperfield if every non-zero element of R is unit.*

Example 2.5. *Let $(R, +, \circ)$ be a ring and I be an ideal of R. We define the following hyperoperation on R: For all $a, b \in R$, $a \circ b = a \cdot b + I$. Then $(R, +, \circ)$ is a multiplicative hyperring.*

Definition 2.6. [6] (a) *Let $(R, +, \circ)$ be a multiplicative hyperring and S be a non-empty subset of R. Then S is said to be a sub-hyperring of R if $(S, +, \circ)$ is itself a multiplicative hyperring.*
(b) *A non-empty subset I of a multiplicative hyperring R is a hyperideal of R if*
(i) *$a, b \in I$, then $a - b \in I$,*
(ii) *$a \in I$ and $r \in R$, then $r \circ a \subseteq I$.*

A hyperideal I of a commutative multiplicative hyperring R with identity 1 is finitely generated if $I = \langle r_1, \cdots, r_n \rangle$ for some $r_1, \cdots, r_n \in R$, i. e., for any $x \in I$, there exist $x_1, \cdots, x_n \in R$ such that $x \in r_1 \circ x_1 + \cdots + r_n \circ x_n$. A hyperideal I of R is called principal if $I = \langle x \rangle$ for some $x \in R$. Also, R is called principal hyperideal hyperring, if every hyperideal of R is principal.

Let I, J be two hyperideals of R. We define $(I :_R J) = \{a \in R \mid a \circ J \subseteq I\}$. It is clear that $(I :_R J)$ is a hyperideal of R.

Definition 2.7. [6] (a) *A proper hyperideal M of a multiplicative hyperring R is maximal in R if for any hyperideal I of R, $M \subset I \subseteq R$ implies $I = M$ or $I = R$.*
(b) *A proper hyperideal P of a multiplicative hyperring R is said to be a prime hyperideal of R if for any $a, b \in R$, $a \circ b \subseteq P$ implies $a \in P$ or $b \in P$.*
(c) *A proper hyperideal Q of a multiplicative hyperring R is said to be a primary hyperideal of R if for any $a, b \in R$, $a \circ b \subseteq Q$ implies $a \in Q$ or $b^n \subseteq Q$, for some $n \in \mathbb{N}$.*

Definition 2.8. [1] *Let I be a hyperideal of a multiplicative hyperring R and $R/I = \{r + I \mid r \in R\}$. Define the operations $+$ and $\circ$ on R/I by $(a + I) + (b + I) = a + b + I$ and $(a + I) \circ (b + I) = \cup\{c + I \mid c \in a \circ b\}$. Then $(R/I, +, \circ)$ is called a quotient hyperring.*

Definition 2.9. [6] *Let C be the class of all finite products of elements of a multiplicative hyperring R i. e. $C = \{r_1 \circ r_2 \circ \cdots \circ r_n : r_i \in R, n \in \mathbb{N}\} \subseteq P^*(R)$. A hyperideal I of R is said to be a C-ideal of R if for any $A \in C$, $A \cap I \neq \emptyset$, then $A \subseteq I$.*

Let I be a hyperideal of a multiplicative hyperring $(R, +, \circ)$. The intersection of all prime hyperideals of R containing I is called the radical of I, being denoted by $Rad(I)$. If the multiplicative hyperring R does not have any prime hyperideal containing I, we define $Rad(I) = R$. Also, the hyperideal $\{r \in R : r^n \subseteq I \text{ for some } n \in \mathbb{N}\}$ will be designated by $D(I)$ and note that the inclusion $D(I) \subseteq Rad(I)$ always holds. In addition, if I is a C-ideal of R, other inclusion holds by [6, Proposition 3.2].

Definition 2.10. [1] *Let R and S be hyperrings. A mapping $\phi : R \to S$ is said to be a hyperring good homomorphism if for all $a, b \in R$;*
(1) $\phi(a + b) = \phi(a) + \phi(b)$,
(2) $\phi(a \circ b) = \phi(a) \circ \phi(b)$.

Throughout this paper, R is a commutative multiplicative hyperring with scalar identity 1.

3 S-prime hyperideals

In this section, the basic properties of S-prime hyperideals are studied.

Definition 3.1. *A nonempty subset S of a multiplicative hyperring $(R, +, \circ)$ with identity 1 is called a multiplicatively closed subset of R if $S \subseteq R$ has the following properties:*
(i) $1 \in S$,
(ii) $s_1 \circ s_2 \cap S \neq \emptyset$ *for all* $s_1, s_2 \in S$.

Example 3.2. *Let $R = (\mathbb{Z}_5, +, \cdot)$. For all $a, b \in R$ we define the hyperoperation $a * b = \{a \cdot b, 2a \cdot b, 3a \cdot b\}$. Then $(R, +, *)$ is a multiplicative hyperring, which is not strongly distributive. Now, let $S = \{\overline{1}, \overline{3}\}$. Then S is a multiplicatively closed subset of $(R, +, *)$.*

Definition 3.3. *Let R be a multiplicative hyperring, $S \subseteq R$ be a multiplicatively closed subset of R and P be a hyperideal of R such that $P \cap S = \emptyset$. We say that P is S-prime if there exists an $s \in S$ such that for all $a, b \in R$ with $a \circ b \subseteq P$, we have $s \circ a \subseteq P$ or $s \circ b \subseteq P$.*

Example 3.4. *Let $(R, +, \cdot)$ be a ring. Then corresponding to every subset $A \in P^*(R)(|A| \geq 2)$, there exists a multiplicative hyperring with absorbing zero $(R_A, +, \circ)$, where $R_A = R$ and for any $\alpha, \beta \in R_A$, $\alpha \circ \beta = \{\alpha \cdot a \cdot \beta : a \in A\}$. Let $(R_A, +, \circ)$ be a commutative multiplicative hyperring and element x indeterminate over R_A. Consider the polynomial multiplicative hyperring $T = (R_A[x], +, *)$, where operation $+$ and hyperoperation $*$ defined on T as follows:*
for all $f(x) = \sum_{k=0}^{n} a_k x^k$ and $g(x) = \sum_{k=0}^{m} b_k x^k$ of T, we consider

$$f(x) + g(x) = \sum_{k=0} (a_k + b_k)x^k, \quad f(x) * g(x) = \left\{ \sum_{k=0}^{n+m} c_k x^k \mid c_k \in \sum_{i+j=k} a_i \circ b_j \right\}.$$

Let $R_A = (\mathbb{Z}, +, \circ)$ with $A = \{1, 2, 4\}$ be the multiplicative hyperring. Consider the multiplicative polynomial hyperring $T = (\mathbb{Z}_A[x], +, *)$ and $S = \{2^n \mid n \in \mathbb{N}\}$. Then S is a multiplicatively closed subset of T. Let $P = \langle 4x \rangle$. It is easy to see that $P \cap S = \emptyset$. Put $s = 4$. Thus P is an S-prime hyperideal of T, but P is not a prime hyperideal of T because $2 \circ 2x = \{4x, 8x, 16x\} \subseteq P$ but $2 \notin P$ and $2x \notin P$.

Proposition 3.5. *Let R be a multiplicative hyperring and $S \subseteq R$ be a multiplicatively closed subset of R. Then:*
(i) Every prime hyperideal P of R such that $P \cap S = \emptyset$ is also an S-prime hyperideal of R.
(ii) If S consists of units of R, then a hyperideal P of R is prime if and only if P is S-prime.
(iii) Let $S_1 \subseteq S_2$ be a multiplicatively closed subset of R. If P is an S_1-prime hyperideal of R such that $P \cap S_2 = \emptyset$, then P is an S_2-prime hyperideal of R.

Proof. The proof is straightforward. □

Let I be a hyperideal of a multiplicative hyperring R and $x \in I$. We note $(I : x) = \{r \in R \mid r \circ x \subseteq I\}$. Then for all $x \in R$, $(I : x)$ is a hyperideal of R.

Proposition 3.6. *Let R be a multiplicative hyperring, $S \subseteq R$ be a multiplicatively closed subset of R and P be a hyperideal of R such that $P \cap S = \emptyset$. Then P is an S-prime hyperideal of R if and only if $(P : s)$ is a prime hyperideal of R for some $s \in S$.*

Proof. ($\Rightarrow$) Since P is an S-prime hyperideal of R, there exists an $s \in S$ such that for all $a, b \in R$ with $a \circ b \subseteq P$, then $s \circ a \subseteq P$ or $s \circ b \subseteq P$. Let $a \circ b \subseteq (P : s)$ where $a, b \in R$. Hence $(s \circ a) \circ b \subseteq P$. Thus for any $t \in s \circ a$, $t \circ b \subseteq P$. Since P is an S-prime hyperideal, then $s_1 \circ t \subseteq P$ or $s_1 \circ b \subseteq P$. Since for all $t \in s \circ a$, we have $s_1 \circ t \subseteq P$, so $(s_1 \circ s) \circ a \subseteq P$. Since $s, s_1 \in S$ and S is a subalgebra, we get $(s_1 \circ s) \cap S \neq \emptyset$. So there exists $u \in (s_1 \circ s) \cap S$ such that $u \circ a \subseteq P$. Thus $s \circ u \subseteq P$ or $s \circ a \subseteq P$ because P is an S-prime hyperideal. If $s \circ u \subseteq P$, then $(s \circ u) \cap S \subseteq P \cap S = \emptyset$ which is a contradiction. Thus $s \circ a \subseteq P$ or $s \circ b \subseteq P$. Therefore $a \in (P : s)$ or $b \in (P : s)$, and so P is a prime hyperideal of R.
($\Leftarrow$) Let $(P : s)$ is a prime hyperideal. Assume that $a \circ b \subseteq P$ for some $a, b \in R$. It is clear that $P \subseteq (P : s)$ since P is a hyperideal. Thus $a \circ b \subseteq (P : s)$, so since $(P : s)$ is prime, $a \in (P : s)$ or $b \in (P : s)$. Therefore $s \circ a \subseteq P$ or $s \circ b \subseteq P$, as required. □

Proposition 3.7. *Let R be a multiplicative hyperring, $S \subseteq R$ be a multiplicatively closed subset of R and P be a hyperideal of R disjoint with S. Then*
(i) If J is a hyperideal of R such that $J \cap S \neq \emptyset$ and P is an S-prime hyperideal of R, then JP is an S-prime hyperideal of R.
(ii) Let $R \subseteq R'$ be an extension of R. If Q is an S-prime hyperideal of R', then $Q \cap R$ is an S-prime hyperideal of R.

Proof. (*i*) Let $t \in J \cap S$. Let $a \circ b \subseteq JP \subseteq P$ where $a, b \in R$. Hence there exists an $s \in S$ such that $s \circ a \subseteq P$ or $s \circ b \subseteq P$. Thus $t \circ s \circ a \subseteq JP$ or $t \circ s \circ b \subseteq JP$. Since $(t \circ s) \cap S \neq \emptyset$, so there is $u \in (t \circ s) \cap S$. Thus $u \circ a \subseteq JP$ or $u \circ b \subseteq JP$. So JP is an S-prime hyperideal of R.
(*ii*) Let $a \circ b \subseteq Q \cap R$ where $a, b \in R$. Since Q is an S-prime hyperideal of R', there exists $s \in S$ such that $s \circ a \subseteq Q$ or $s \circ b \subseteq Q$. Therefore, $s \circ a \subseteq Q \cap R$ or $s \circ b \subseteq Q \cap R$. □

Theorem 3.8. *Let R be a multiplicative hyperring and $S \subseteq R$ be a multiplicatively closed subset of R. Suppose $f : R \to R'$ is a good homomorphism of hyperrings such that $f(S)$ does not contain zero. If Q is an $f(S)$-prime hyperideal of R', then $f^{-1}(Q)$ is an S-prime hyperideal of R.*

Proof. It is easy to see that $f(S)$ is a multiplicatively closed subset of R'. Let Q be an $f(S)$-prime hyperideal of R'. Hence there exists an $s \in S$ such that for all $x, y \in R'$ if $x \circ y \subseteq Q$, then $f(s) \circ x \subseteq Q$ or $f(s) \circ y \subseteq Q$. Let $P = f^{-1}(Q)$. Hence we have $P \cap S = \emptyset$ since $Q \cap f(S) = \emptyset$. Let $a \circ b \subseteq P$ where $a, b \in R$. Thus $f(a \circ b) = f(a) \circ f(b) \subseteq Q$ which implies that $f(s) \circ f(a) \subseteq Q$ or $f(s) \circ f(b) \subseteq Q$. Therefore $f(s \circ a) \subseteq Q$ or $f(s \circ b) \subseteq Q$ since f is a good homomorphism. Hence $s \circ a \subseteq f^{-1}(Q) = P$ or $s \circ b \subseteq f^{-1}(Q) = P$, as needed. □

Let R be a multiplicative hyperring, $S \subseteq R$be a multiplicatively closed subset of R and I be a hyperideal of R such that $I \cap S = \emptyset$. Let $s \in S$, we denote by $\bar{s} = s + I$ the equivalence class of s in the quotient hyperring R/I. Let $\bar{S} = \{s + I \mid s \in S\}$. Then $\bar{S}$ is a multiplicatively closed subset of R/I, because if $s + I, t + I \in \bar{S}$, then $(s + I) \circ (t + I) \cap \bar{S} \neq \emptyset$ because $s \circ t \cap S \neq \emptyset$.

Proposition 3.9. *Let R be a multiplicative hyperring, $S \subseteq R$ be a multiplicatively closed subset of R and I be a hyperideal of R such that $I \cap S = \emptyset$. Let P be a proper hyperideal of R containing I such that $(P/I) \cap \bar{S} = \emptyset$. Then P is an S-prime hyperideal of R if and only if P/I is an $\bar{S}$-prime hyperideal of R/I.*

Proof. Let P be an S-prime hyperideal. Then there exists an $s \in S$ such that for all $a, b \in R$ with $a \circ b \subseteq P$, then $s \circ a \subseteq P$ or $s \circ b \subseteq P$. Let $(a + I) \circ (b + I) \subseteq P/I$ where $a + I, b + I \in R/I$. Hence $a \circ b \subseteq P$, and so $s \circ a \subseteq P$ or $s \circ b \subseteq P$. Thus $(s + I) \circ (a + I) \subseteq P/I$ or $(s + I) \circ (b + I) \subseteq P/I$. Therefore, P/I is an $\bar{S}$-prime hyperideal of R/I.

Conversely, since $(P/I) \cap \bar{S} = \emptyset$, we can easily prove that $P \cap S = \emptyset$. Let $a \circ b \subseteq P$ where $a, b \in R$. Thus $(a+I)\circ(b+I) \subseteq P/I$. There exists an $\bar{s} = s+I \in \bar{S}$ such that $(s+I)\circ(a+I) \subseteq P/I$ or $(s+I)\circ(b+I) \subseteq P/I$. Hence we conclude $s \circ a \subseteq P$ or $s \circ b \subseteq P$, as required. □

Theorem 3.10. *Let R be a multiplicative hyperring, $S \subseteq R$be a multiplicatively closed subset of R and P be a hyperideal of R such that $P \cap S = \emptyset$. Then P is an S-prime hyperideal of R if and only if there exists an $s \in S$ such that for all hyperideals I, J of R, if $IJ \subseteq P$, then $s \circ I \subseteq P$ or $s \circ J \subseteq P$.*

Proof. ($\Leftarrow$) Let $a \circ b \subseteq P$ where $a, b \in R$. Thus $\langle a \circ b \rangle \subseteq P$. By [6, Proposition 2.15], we have $\langle a \rangle \circ \langle b \rangle \subseteq \langle a \circ b \rangle$, and so $\langle a \rangle \circ \langle b \rangle \subseteq P$. Thus there exists an $s \in S$ such that $s \circ \langle a \rangle \subseteq P$ or $s \circ \langle b \rangle \subseteq P$ by hypothesis. Therefore $s \circ a \subseteq P$ or $s \circ b \subseteq P$, hence P is S-prime.
($\Rightarrow$) Since P is S-prime, there exists an $s \in S$ such that $a \circ b \subseteq P$, implies $s \circ a \subseteq P$ or $s \circ b \subseteq P$ for any $a, b \in R$. Let for all $t \in S$, there exist hyperideals A, B of R with $AB \subseteq P$, $t \circ A \nsubseteq P$ and $t \circ B \nsubseteq P$. Since $s \in S$, there exists hyperideals I, J of R with $IJ \subseteq P$, $t \circ I \nsubseteq P$ and $t \circ J \nsubseteq P$. Thus there exist $a \in I$ and $b \in J$ such that $s \circ a \nsubseteq P$ and $s \circ b \nsubseteq P$ with $a \circ b \subseteq IJ \subseteq P$ that it contradicts with hypothesis. □

Corollary 3.11. *Let R be a multiplicative hyperring, $S \subseteq R$ be a multiplicatively closed subset of R and P be a hyperideal of R such that $P \cap S = \emptyset$. Then P is an S-prime hyperideal of R if and only if there exists an $s \in S$ such that for all hyperideals $I_1, I_2, \cdots, I_n$ of R, if $I_1 I_2 \cdots I_n \subseteq P$, then $s \circ I_i \subseteq P$ for some $i \in \{1, 2, \ldots, n\}$.*

Proof. ($\Leftarrow$) It follows from Theorem 3.10.
($\Rightarrow$) Since P is S-prime, there exists an $s \in S$ so that $a \circ b \subseteq P$, implies $s \circ a \subseteq P$ or $s \circ b \subseteq P$ for any $a, b \in R$. We will proceed by induction on n. For $n = 2$, the result is true by Theorem 3.10. Let $n \geq 3$. Suppose that the property holds up to the order $n-1$ and let $I_1, I_2, \cdots, I_n$ be hyperideals of R such that $I_1 I_2 \cdots I_n \subseteq P$. Thus $(I_1 I_2 \cdots I_{n-1}) I_n \subseteq P$. Hence by Theorem 3.10, $s \circ (I_1 I_2 \cdots I_{n-1}) \subseteq P$ or $s \circ I_n \subseteq P$. If $s(I_1 I_2 \cdots I_{n-1}) \subseteq P$, then $(s \circ s) \circ I_1 \subseteq P$ or $s \circ I_i \subseteq P$ for some $i \in \{2, \cdots, n-1\}$. If $(s \circ s) \circ I_1 \subseteq P$, then $t \circ I_1 \subseteq P$ for some $t \in s \circ s \cap S$. Hence $s \circ t \subseteq P$ or $s \circ I_1 \subseteq P$, but if $s \circ t \subseteq P$, then $(s \circ t) \cap S \subseteq P \cap S = \emptyset$, a contradiction. Therefore, $s \circ I_i \subseteq P$ for some $i \in \{1, \cdots, n\}$. □

Proposition 3.12. *Let R be a multiplicative hyperring, $S \subseteq R$ be a multiplicatively closed subset of R and P be a hyperideal of R such that $P \cap S = \emptyset$. Then P is an S-prime hyperideal of R if and only if there exists an $s \in S$ such that for all $a_1, a_2, \cdots, a_n$ of R, if $a_1 \circ a_2 \circ \cdots \circ a_n \subseteq P$, then $s \circ a_i \subseteq P$ for some $i \in \{1, 2, \ldots, n\}$.*

Proof. ($\Leftarrow$) Take $n = 2$.
($\Rightarrow$) Let P be an S-prime hyperideal of R. Then there exists an $s \in S$ so that $a \circ b \subseteq P$, implies $s \circ a \subseteq P$ or $s \circ b \subseteq P$ for any $a, b \in R$. Let $a_1 \circ a_2 \circ \cdots \circ a_n \subseteq P$ where $a_1, \cdots, a_n \in R$. Hence $\langle a_1 \circ a_2 \circ \cdots \circ a_n \rangle \subseteq P$, so by [6, Proposition 2.15] and induction, we get $\langle a_1 \rangle \circ \cdots \circ \langle a_n \rangle \subseteq \langle a_1 \circ a_2 \circ \cdots \circ a_n \rangle \subseteq P$. Thus by Corollary 3.11, $s \circ \langle a_i \rangle \subseteq P$ for some $i \in \{1, 2, \ldots, n\}$. □

Corollary 3.13. *Let R be a multiplicative hyperring, $S \subseteq R$ be a multiplicatively closed subset of R and P be a hyperideal of R such that $P \cap S = \emptyset$. Then P is prime hyperideal of R if and only if for all hyperideals $I_1, I_2, \cdots, I_n$ of R, if $I_1 I_2 \cdots I_n \subseteq P$, then $I_i \subseteq P$ for some $i \in \{1, 2, \ldots, n\}$.*

Proof. Take $S = \{1\}$ in Corollary 3.11. □

Proposition 3.14. *Let R be a multiplicative hyperring, $S \subseteq R$ be a multiplicatively closed subset of R and P be a hyperideal of R such that $P \cap S = \emptyset$. Then if I is a C-hyperideal of R and P an S-prime hyperideal of R such that $I \subseteq P$, then there exists an $s \in S$ such that $s \circ (Rad(I)) \subseteq P$.*

Proof. Let $a \in Rad(I)$. Since I is a C-ideal, we have $Rad(I) = D(I) = \{r \in R \mid r^n \subseteq I \ for \ some \ n \in \mathbb{N}\}$. Hence $a^n \subseteq I \subseteq P$ for some $n \in \mathbb{N}$. Thus there exists an $s \in S$ such that $s \circ a \subseteq P$ by Proposition 3.12, and so $s \circ (Rad(I)) \subseteq P$. □

Theorem 3.15. *Let R be a multiplicative hyperring and $S \subseteq R$ be a multiplicatively closed subset of R. Let I be a hyperideal of R and $P_1, P_2, \cdots, P_n$ be S-prime C-hyperideals of R such that $I \subseteq \bigcup_{i=1}^{n} P_i$. Then there exists an $s \in S$ and $i \in \{1, 2, \cdots, n\}$ such that $s \circ I \subseteq P_i$.*

Proof. Let $P_1, P_2, \cdots, P_n$ be S-prime C-hyperideals of R. Let $I \subseteq \bigcup_{i=1}^{n} P_i$. By Proposition 3.6, for all i, $1 \leq i \leq n$, there exists $s_i \in S$ such that $(P_i : s_i)$ is a prime hyperideal of R. We have $I \subseteq \bigcup_{i=1}^{n} P_i \subseteq \bigcup_{i=1}^{n} (P_i : s_i)$. Thus by [6, Proposition 2.15], there exists $i \in \{1, \cdots, n\}$ such that $I \subseteq (P_i : s_i)$, and so $s_i \circ I \subseteq P_i$. □

Let $(R_1, +, \circ)$ and $(R_2, +, \circ)$ be two multiplicative hyperrings with identity 1. Then $(R = R_1 \times R_2, +, \circ)$ is a multiplicative hyperring with operation $+$ and the hyperoperation $\circ$ are defined, respectively, as $(x, y) + (z, t) = (x + z, y + t)$ and $(x, y) \circ (z, t) = \{(a, b) \in R \mid a \in x \circ z, b \in y \circ t\}$ for all $(x, y), (z, t) \in R$.

Theorem 3.16. *Let $R = R_1 \times R_2$ be a decomposable hyperring where R_1 and R_2 be multiplicative hyperrings with identity 1 and $S = S_1 \times S_2$ where S_i be a multiplicatively closed subset of R_i. Suppose $P = P_1 \times P_2$ is a hyperideal of R. Then the following are equivalent:*
(i) P is an S-prime hyperideal of R.
(ii) P_1 is an S_1-prime hyperideal of R_1 and $P_2 \cap S_2 \neq \emptyset$ or P_2 is an S_2-prime hyperideal of R_2 and $P_1 \cap S_1 \neq \emptyset$.

Proof. $(i) \Rightarrow (ii)$ Since $(1, 0) \circ (0, 1) \subseteq P$, then there exists an $s = (s_1, s_2) \in S$ so that $s \circ (1, 0) \subseteq P$ or $s(0, 1) \subseteq P$. Thus $P_1 \cap S_1 \neq \emptyset$ or $P_2 \cap S_2 \neq \emptyset$. We may assume that $P_1 \cap S_1 \neq \emptyset$. As $P \cap S = \emptyset$, we have $P_2 \cap S_2 = \emptyset$. Let $a \circ b \subseteq P_2$ for some $a, b \in R_2$. Since $(0, a) \circ (0, b) \subseteq P$ and P is an S-prime hyperideal of R, $s \circ (0, a) \subseteq P$ or $s \circ (0, b) \subseteq P$. Hence we get $s_2 \circ a \subseteq P_2$ or $s_2 \circ b \subseteq P_2$. Thus P_2 is an S_2-prime hyperideal of R_2. In the other case, one can easily show that P_1 is an S_1-prime hyperideal of R_1.
$(ii) \Rightarrow (i)$ Assume that $P_1 \cap S_1 \neq \emptyset$ and P_2 is an S_2-prime hyperideal of R_2. Therefore there exists an $s_1 \in P_1 \cap S_1$. Let $(a, b) \circ (c, d) \subseteq P$ for some $a, c \in R_1$ and $b, d \in R_2$. This yields that $b \circ d \subseteq P_2$ and so there exists an $s_2 \in S_2$ so that $s_2 \circ b \subseteq P_2$ or $s_2 \circ d \subseteq P_2$. Put $s = (s_1, s_2) \in S$. Thus we get $s \circ (a, b) \subseteq P$ or $s \circ (c, d) \subseteq P$. Therefore, P is an S-prime hyperideal of R. In the other case, one can similarly prove that P is an S-prime hyperideal of R. □

Let $(R, +, \circ)$ be a hyperring with identity 1. We define the relation γ as follows: $a\gamma b$ if and only if $a, b \subseteq U$ where U is a finite sum of finite products of elements of R, i.e.,

$$a\gamma b \ \Leftrightarrow \ \exists z_1, z_2, \cdots, z_n \in R \text{ such that } \{a, b\} \subseteq \sum_{j \in J} \prod_{i \in I_j} z_i, \ I_j, \ J \subseteq \{1, 2, \cdots, n\}.$$

We denote the transitive closure of γ by γ^*. The relation γ^* is the smallest equivalence relation on a multiplicative hyperring $(R, +, \circ)$ such that the quotient R/γ^*, the set of all equivalence classes, is a fundamental ring. Let U be the set of all finite sums of products of elements of R. We can rewrite the definition of γ^* on R as follows:

$a\gamma b \Leftrightarrow \exists z_1, z_2, \cdots, z_n \in R$ with $z_1 = a$ and $z_{n+1} = b$ and $u_1, u_2, \cdots, u_n \in U$ such that $\{z_i, z_{i+1}\} \subseteq u_i$ for $i \in \{1, 2, \cdots, n\}$.
Suppose that $\gamma^*(a)$ is the equivalence class containing $a \in R$. Then, both the sum $\oplus$ and the product $\odot$ in R/γ^* are defined as follows: $\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c)$ for all $c \in \gamma^*(a) + \gamma^*(b)$ and $\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d)$ for all $d \in \gamma^*(a) \circ \gamma^*(b)$. Then R/γ^* is a ring, which is called a fundamental ring of R (see [8]).

Proposition 3.17. *Let R be a multiplicative hyperring with scaler 1. Then $S \subseteq R$ is a multiplicatively closed subset of R if and only if S/γ^* is a multiplicatively closed subset of the ring R/γ^*.*

Proof. Let $S \subseteq R$ be a multiplicatively closed subset of R. Suppose $x, y \in S/\gamma^*$. We show that $xy \in S/\gamma^*$. We have $x = \gamma^*(s)$ and $y = \gamma^*(t)$ for some $s, t \in S$. Hence $xy = \gamma^*(s)\gamma^*(t) = \gamma^*(s \circ t)$. Since $(s \circ t) \cap S \neq \emptyset$, there exists $u \in (s \circ t) \cap S$. Thus $xy = \gamma^*(s \circ t) = \gamma^*(u) \in S/\gamma^*$ since γ^* is a strongly regular relation of R.

Conversely, let S/γ^* be a multiplicatively closed subset of R/γ^*. Let $s_1, s_2 \in S$. We show that $(s_1 \circ s_2) \cap S \neq \emptyset$. We have $\gamma^*(s_1), \gamma^*(s_2) \in S/\gamma^*$. Thus $\gamma^*(s_1)\gamma^*(s_2) \in S/\gamma^*$ since R/γ^* is a multiplicatively closed subset of R/γ^*. Hence $\gamma^*(s_1 \circ s_2) = \gamma^*(s_1)\gamma^*(s_2) \in S/\gamma^*$. Therefore $s_1 \circ s_2 \subseteq S$, and so $(s_1 \circ s_2) \cap S \neq \emptyset$. Thus S is a multiplicatively closed subset of R. □

Theorem 3.18. *Let R be a multiplicative hyperring with identity 1 and $S \subseteq R$ be a multiplicatively closed subset of R. Then the hyperideal P is S-prime if and only if P/γ^* is an S/γ^*-prime ideal of R/γ^*.*

Proof. Let P be an S-prime hyperideal of R. Let $xy \in P/\gamma^*$ where $x, y \in R/\gamma^*$. Thus there exist $a, b \in R$ such that $x = \gamma^*(a)$ and $y = \gamma^*(b)$. Hence $xy = \gamma^*(a \circ b) \in P/\gamma^*$, and so $a \circ b \subseteq P$. Then there exists an $s \in S$ such that $s \circ a \subseteq P$ or $s \circ b \subseteq P$ since P is an S-prime hyperideal of R. Therefore $\gamma^*(s)\gamma^*(a) = \gamma^*(s \circ a) \in P/\gamma^*$ or $\gamma^*(s)\gamma^*(b) = \gamma^*(s \circ b) \in P/\gamma^*$ since γ^* is a strongly regular relation of R.

Conversely, assume that P/γ^* is an S/γ^*-prime ideal of R/γ^*. Let $a \circ b \subseteq P$ where $a, b \in R$. Thus $\gamma^*(a \circ b) = \gamma^*(a)\gamma^*(b) \in P/\gamma^*$. Thus there exists $s \in S$ such that $\gamma^*(s)\gamma^*(a) \in P/\gamma^*$ or $\gamma^*(s)\gamma^*(b) \in P/\gamma^*$ since P/γ^* is an S/γ^*-prime ideal of R/γ^*. Therefore $s \circ a \subseteq P$ or $s \circ b \subseteq P$, as required. □

Example 3.19. *Let $R = \{0, 1, 2, 3\}$. Consider the multiplicative hyperring $(R, +, \circ)$, where operation $+$ and hyperoperation $\circ$ are defined on R as follows,*

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1, 3\}$	$\{2\}$	$\{1, 3\}$
2	$\{0\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{0\}$	$\{1, 3\}$	$\{2\}$	$\{1, 3\}$

Let $S = \{1, 3\}$ and $P = \{0, 2\}$. Then it is easy to verify that S is a multiplicatively closed subset of R and P is a prime hyperideal of R. Since $S \cap P = \emptyset$, then P is an S-prime hyperideal of R.

Also, we have $S/\gamma^* = \{\gamma^*(1), \gamma^*(3)\}$ which is a multiplicatively closed subset of R/γ^* and $P/\gamma^* = \{\gamma^*(0), \gamma^*(2)\}$ which is an S/γ^*-prime ideal of R/γ^*.

Let R be a multiplicative hyperring. Then $M_n(R)$ denotes the set of all hypermatrixes of R. Also, for all $A = (A_{ij})_{nn}$, $B = (B_{ij})_{nn} \in P^*(M_n(R))$, $A \subseteq B$ if and only if $A_{ij} \subseteq B_{ij}$.

Theorem 3.20. *Let R be a multiplicative hyperring with identity 1, $S \subseteq R$ be a multiplicatively closed subset of R and P be a hyperideal of R. If $M_n(P)$ is an $M_n(S) = \left\{ \begin{pmatrix} s & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \mid s \in S \right\}$-prime hyperideal of $M_n(R)$, then P is an S-prime hyperideal of R.*

Proof. It is clear that $M_n(S)$ is a multiplicatively closed subset of $M_n(R)$. Let $x \circ y \subseteq P$ where $x, y \in R$. Then

$$\begin{pmatrix} x \circ y & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \subseteq M_n(R).$$

We have

$$\begin{pmatrix} x \circ y & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} x & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} y & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Since $M_n(P)$ is an $M_n(S)$-prime hyperideal of $M_n(R)$, then

$$\begin{pmatrix} s & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \subseteq M_n(P)$$

or

$$\begin{pmatrix} s & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} y & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \subseteq M_n(P)$$

Hence

$$\begin{pmatrix} s \circ x & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \subseteq M_n(P) \text{ or } \begin{pmatrix} s \circ y & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \subseteq M_n(P)$$

Therefore, $s \circ x \subseteq P$ or $s \circ y \subseteq P$. Hence P is an S-prime hyperideal of R. □

4 Conclusions

The concepts of multiplicatively closed subset of a multiplicative hyperrings and S-prime hyperideals of a multiplicative hyperring have been studied and some results where established.

In fact, the notion of S-prime hyperideals differ with the notion of prime hyperideals and many of the results concerning of prime hyperideals are not hold for S-prime hyperideals. The notion of S-prime hyperideals was proposed and basic properties of S-prime hyperideals based on their formations were introduced. We also explored the behaviour of S-prime hyperideals under homomorphism hyperrings, in factor hyperrings, Cartesian products of hyperrings and the fundamental relation in the context of multiplicative hyperrings with some related results.

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