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Fuzzy n-fold obstinate and maximal (pre)filters of EQ-algebras

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Abstract

In this paper, we defined the concepts of fuzzy n-fold obstinate (pre)filter and maximal fuzzy (pre)filter of EQ-algebras and discussed the properties of them. We show that every maximal fuzzy (pre)filter of $\mathcal{E}$ is normalized and takes only the values $\{0, 1\}$. Also we show that in good EQ-algebra, if μ is a normalized fuzzy (pre)filter of $\mathcal{E}$, then μ is a fuzzy n-fold obstinate (pre)filter of $\mathcal{E}$ if and only if every normalized fuzzy (pre)filter of quotient algebra $\mathcal{E}/\mu$ is a fuzzy n-fold obstinate (pre)filter of $\mathcal{E}/\mu$. Also, we verify relation between fuzzy obstinate n-fold (pre)filters and other fuzzy (pre)filters of EQ-algebras.

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1 Introduction

Recently, a special algebra called EQ-algebra has been introduced by Novák in [16]. [These algebras are intended to become algebras of truth values for a higher-order fuzzy logic (a fuzzy type theory, FTT)]. An EQ-algebra has three basic binary operations (meet, multiplication and a fuzzy equality) and a top element. The implication is defined from the fuzzy equality "$\sim$" by the formula $a \to b =(a \wedge b) \sim a$. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes residuated lattice. From the point of view of potential application, it seems very interesting that unlike Hájek [9], we can have non-commutativity without the necessity to introduce, two kinds of implication.

Novák and De Baets in [17] introduced various kinds of EQ-algebras. El-Zekey in [6], proved that the class of EQ-algebras is a variety. El-Zekey in [6] introduced prelinear good EQ-algebras and proved that a prelinear good EQ-algebra is a distributive lattice. Novák and De Baets in

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[17] defined the concept of prefilter on EQ-algebras which is the same as filter of other algebraic structures such as residuated lattices, MTL-algebras, and etc. But the binary relation has been introduced by prefilters is not a congruence relation. For solving this problem, they added another condition to the definition of prefilter so filter of EQ-algebras is defined. To read more about EQ-algebras, the reader can read articles [1, 3, 12, 20, 22, 24].

Filter theory plays an important role in studying logical algebras. From a logic point of view, various filters have a natural interpretation as various sets of provable formulas. Up to now, some types of n-fold filters on BCK-algebra, BL-algebra and etc., are studied (see [4, 11, 15]).

The concept of fuzzy set and various operations on it were first introduced by Zadeh in [23]. Since then, fuzzy sets have been applied to diverse field. The study of fuzzy subsets and their application to mathematical contexts has reached to what is now commonly called fuzzy mathematics. Fuzzy algebra is an important branch of fuzzy mathematics. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy sub-groups in 1971 by Rosenfeld [19]. Since then these ideas have been applied to other algebraic structures such as semigroups, rings, ideals, modules and vector spaces. In this paper, we defined the concepts of fuzzy n-fold obstinate (pre)filter and maximal fuzzy (pre)filter of EQ-algebras and discussed the properties of them. We show that every maximal fuzzy (pre)filter of $\mathcal{E}$ is normalized and takes only the values $\{0, 1\}$ and in good EQ-algebra, if μ is a normalized fuzzy (pre)filter of $\mathcal{E}$, then μ is a fuzzy n-fold obstinate (pre)filter of $\mathcal{E}$ if and only if every normalized fuzzy (pre)filter of quotient algebra $\mathcal{E}/\mu$ is a fuzzy n-fold obstinate (pre)filter of $\mathcal{E}/\mu$. Also, we verify relation between fuzzy obstinate n-fold (pre)filters and other fuzzy (pre)filters of EQ-algebras.

2 Preliminaries

In this section, we recollect some definitions and results which will be used in the next sections.

Definition 2.1. [6] *An EQ-algebra is an algebraic structure $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type (2, 2, 2, 0) such that the following conditions hold, for all $x, y, z, t \in E$:*
(E1) $\langle E, \wedge, 1\rangle$ is a commutative idempotent monoid (i.e. $\wedge$-semilattice with top element 1);
(E2) $\langle E, \otimes, 1\rangle$ is a commutative monoid and $\otimes$ is isotone w.r.t. $''\leq''$, where $x \leq y$ is defined as $x \wedge y = x$;
(E3) $x \sim x = 1$; (reflexivity axiom)
(E4) $((x \wedge y) \sim z) \otimes (t \sim x) \leq z \sim (t \wedge y)$; (substitution axiom)
(E5) $(x \sim y) \otimes (z \sim t) \leq (x \sim z) \sim (y \sim t)$; (congruence axiom)
(E6) $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$; (monotonicity axiom)
(E7) $x \otimes y \leq x \sim y$. (boundedness axiom)

Note. From now one $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra.

Proposition 2.2. [17] *Define $x \to y := (x \wedge y) \sim x$ and $\bar{x} := x \sim 1$. Then the following properties hold, for all $x, y, z, t \in E$:*
(i) $x \otimes y \leq x, y$ and $x \otimes y \leq x \wedge y$;
(ii) $x \to y \leq (z \to x) \to (z \to y)$ and $x \to y \leq (y \to z) \to (x \to z)$;
(iii) if $x \leq y$, then $x \sim y = y \to x$, $z \to x \leq z \to y$ and $y \to z \leq x \to z$;

Definition 2.3. [17] *The algebraic structure $\mathcal{E}$ is called:*
(i) separated if $x \sim y = 1$, then $x = y$, for all $x, y \in E$, (in other words $x \sim y = 1$ if and only if $x = y$);

(ii) *good if $x \sim 1 = x = 1 \sim x$, for all $x \in E$;*
(iii) *residuated if $x \leq y \to z$ if and only if $x \otimes y \leq z$, for all $x, y, z \in E$;*
(iv) *lattice EQ-algebra(ℓEQ-algebra) if it is a lattice and for all $x, y, z, t \in E$ we have,*

$$((x \vee y) \sim z) \otimes (t \sim x) \leq z \sim (t \vee y);$$

(v) *lattice ordered if $\mathcal{E}$ is a lattice.*

Proposition 2.4. [6] *The following statements are equivalent, for all $x, y \in E$:*
(i) *$\mathcal{E}$ is good;*
(ii) $x \otimes (x \sim y) \leq y$;
(iii) $x \otimes (x \to y) \leq y$;
(iv) $1 \to x = x$.

Proposition 2.5. [6] *The following statements are equivalent, for all $x, y, z \in E$:*
(i) *$\mathcal{E}$ is residuated;*
(ii) *$\mathcal{E}$ is good and $x \to y \leq (x \otimes z) \to (y \otimes z)$;*
(iii) *$\mathcal{E}$ is good and $x \leq y \to (x \otimes y)$;*
(iv) *$\mathcal{E}$ is separated and $(x \otimes y) \to z = x \to (y \to z)$.*

Definition 2.6. [17] *A nonempty subset $F \subseteq E$ is called a prefilter of $\mathcal{E}$, if for all $x, y \in E$,*
$(F1)$ $1 \in F$;
$(F2)$ *If $x, x \to y \in F$, then $y \in F$.*
A prefilter F is said to be a filter of $\mathcal{E}$.
$(F3)$ *if $x \to y \in F$ implies $(x \otimes z) \to (y \otimes z) \in F$, for all $x, y, z \in E$.*

Let F be a filter of $\mathcal{E}$. Then a binary relation $\equiv_F$ on E is defined as follows:

$$x \equiv_F y \text{ if and only if } x \sim y \in F.$$

Then $\equiv_F$ is a congruence relation on E. Denote $E/F := \{[x]_F \mid x \in E\}$ and $[x]_F = \{y \in E \mid x \equiv_F y\}$ and define the operations $\wedge_F$, $\otimes_F$, $\sim_F$ and relation $\leq_F$ on E/F as follows:

$$[x]_F \wedge_F [y]_F = [x \wedge y]_F, \quad [x]_F \otimes_F [y]_F = [x \otimes y]_F, \quad [x]_F \sim_F [y]_F = [x \sim y]_F,$$

$$[x]_F \leq_F [y]_F \text{ if and only if } x \to y \in F \text{ if and only if } [x]_F \to_F [y]_F = [1]_F.$$

Theorem 2.7. [6] *Let F be a filter of an (ℓ)EQ-algebra $\mathcal{E}$. Then the quotient algebra $\mathcal{E}/F = (E/F, \wedge_F, \otimes_F, \sim_F, F)$ is a separated (ℓ)EQ-algebra.*

Definition 2.8. [21] *A (pre)filter F of $\mathcal{E}$ is called maximal if and only if it is proper and no (pre)filter of $\mathcal{E}$ strictly contains F that is, for each (pre)filter G of $\mathcal{E}$, if $F \subsetneq G$, then $G = E$.*

Definition 2.9. [14] *Let X be a nonempty subset of E. Then the smallest (pre)filter of $\mathcal{E}$ which contains X, i.e. $\bigcap\{F \mid F$ is a (pre)filter of E such that $X \subseteq F\}$ is said to be a (pre)filter of $\mathcal{E}$ generated by X and is denoted by $\langle X \rangle$. If $a \in E$ and $X = \{a\}$, then we denote by $\langle a \rangle$ the (pre)filter generated by $\{a\}$ ($\langle a \rangle$ is called principal). For (pre)filter F and $a \in E$, we denote by $F(a) = \langle F \cup \{a\} \rangle$. It is clear that $a \in F$ implies $F(a) = F$. We can prove*

$$\langle X \rangle = \{a \in E \mid x_1 \to (x_2 \to (x_3 \to ...(x_n \to a)...)) = 1, \textit{ for some } x_i \in X \textit{ and } n \in \mathbb{N}\}.$$

Definition 2.10. [10] *A prefilter F of an EQ-algebra $\mathcal{E}$ is called*
(i) an implicative prefilter of $\mathcal{E}$, if for all $x, y, z \in E$,
$(F4)$ *$z \to ((x \to y) \to x) \in F$ and $z \in F$ imply $x \in F$.*
(ii) a positive implicative prefilter of $\mathcal{E}$, if for all $x, y, z \in E$,
$(F5)$ *$x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$.*
(iii) an obstinate prefilter of $\mathcal{E}$,
$(F6)$ *$x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$.*
A prefilter F is said to be a (positive implicative, implicative, obstinate) filter of $\mathcal{E}$, if F satisfies in $(F3)$.

Corollary 2.11. [21] *Let $\mathcal{E}$ be good and F be a nonempty subset of $\mathcal{E}$. Then F is an implicative (pre)filter if and only if F is both a positive implicative (pre)filter and a fantastic (pre)filter of $\mathcal{E}$.*

Definition 2.12. [7] *A nonempty subset $F \subseteq E$ such that $1 \in F$ is called*
(i) an n-fold prefilter of $\mathcal{E}$, if for all $x, y \in E$ if $x^n, x^n \to y \in F$, then $y \in F$.
(ii) an n-fold positive implicative prefilter of $\mathcal{E}$ if for all $x, y, z \in E$, $x^n \to (y \to z) \in F$ and $x^n \to y \in F$ imply $x^n \to z \in F$.
(iii) an n-fold implicative prefilter of $\mathcal{E}$ if for all $x, y, z \in E$, $z \to ((x^n \to y) \to x) \in F$ and $z \in F$ imply $x \in F$.
(iv) an n-fold obstinate (pre)filter of $\mathcal{E}$ if $x, y \notin F$ imply $x^n \to y \in F$ and $y^n \to x \in F$.
(v) an n-fold fantastic prefilter of $\mathcal{E}$ if for all $x, y \in E$, $z \to (y \to x) \in F$ and $z \in F$ imply $((x^n \to y) \to y) \to x \in F$.

Note. An n-fold prefilter F is said to be an n-fold (positive implicative, implicative, obstinate, fantastic) filter of $\mathcal{E}$ if F satisfies in $(F3)$.

Definition 2.13. [23] *Let E be a set. A fuzzy set μ in E is a function $\mu : E \to [0, 1]$. The set of all fuzzy sets on $\mathcal{E}$ is denoted by $\mathcal{F}(\mathcal{E})$.*

Note.Let $\mu \in \mathcal{F}(\mathcal{E})$. For all $t \in [0, 1]$, the set $\mu_t = \{x \in E \mid \mu(x) \geq t\}$ is called a level subset of μ. Suppose that F is a nonempty subset of E. Then we denote the characteristic function of F by χ_F. For convenience, for any $a, b \in [0, 1]$, we denote $\sup\{a, b\}$ and $\inf\{a, b\}$ by $a \vee b$ and $a \wedge b$, respectively. For $\mu, \nu \in \mathcal{F}(\mathcal{E})$, we define $\mu \leq \nu$ if and only if for any $x \in E$, $\mu(x) \leq \nu(x)$.

Definition 2.14. [21] *Let $\mu \in \mathcal{F}(\mathcal{E})$. Then μ is called a fuzzy prefilter of $\mathcal{E}$ if for all $x, y \in E$, it satisfies in the following conditions:*
(FF1) $\mu(x) \leq \mu(1)$,
(FF2) $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$.
A fuzzy prefilter μ is called a fuzzy filter of $\mathcal{E}$ if for all $x, y \in E$, we have
(FF3) $\mu(x \to y) \leq \mu((x \otimes z) \to (y \otimes z))$.

Note. The set of all fuzzy (pre)filters on $\mathcal{E}$ is denoted by $\mathcal{FF}(\mathcal{E})(\mathcal{FPF}(\mathcal{E}))$.

Theorem 2.15. [21] *The following statements hold:*
(i) if $\mu \in \mathcal{F}(\mathcal{E})$, then μ is a fuzzy (pre)filter of $\mathcal{E}$ if and only if for all $t \in [0, 1]$, μ_t is either empty or a (pre)filter of E.
(ii) if $\emptyset \neq F \subseteq E$, then F is a (pre)filter of $\mathcal{E}$ if and only if $\chi_F \in \mathcal{FF}(\mathcal{E})$.

Proposition 2.16. [21] *Let $\mu \in \mathcal{FF}(\mathcal{E})$. Then for any $x, y, z \in E$, we have:*
(i) $\mu(x \otimes y) = \mu(x) \wedge \mu(y)$.
(ii) $\mu(x \to y) \wedge \mu(y \to z) \leq \mu(x \to z)$.
(iii) If $x \leq y$, then $\mu(x) \leq \mu(y)$; that is μ is order preserving.

Let $\mu \in \mathcal{FF}(\mathcal{E})$. For any $x, y \in E$, define a fuzzy relation $\equiv_\mu$ on $\mathcal{E}$ as follows:

$$x \equiv_\mu y \text{ if and only if } \mu(x \sim y) = \mu(1).$$

Theorem 2.17. [8] *The relation $\equiv_\mu$ is a congruence relation on $\mathcal{E}$ and $\mathcal{E}/\mu = (E/\mu, \otimes_\mu, \sim_\mu, \wedge_\mu, [1]_\mu)$ is a separated EQ-algebra with operations $\otimes_\mu$, $\sim_\mu$ and $\wedge_\mu$ on E/μ which are defined as follows:*

$$[x]_\mu \otimes_\mu [y]_\mu = [x \otimes y]_\mu,\ [x]_\mu \sim_\mu [y]_\mu = [x \sim y]_\mu \text{ and } [x]_\mu \wedge_\mu [y]_\mu = [x \wedge y]_\mu.$$

We define a binary relation on $\mathcal{E}/\mu$ by $[x]_\mu \leq_\mu [y]_\mu$ if and only if $\mu(x \to y) = \mu(1)$, for any $x, y \in E$. Obviously, $(E/\mu, \leq_\mu)$ is a poset.

Definition 2.18. [8] *Let $\mu \in \mathcal{F}(\mathcal{E})$. Then μ is called*
(i) a fuzzy n-fold prefilter of $\mathcal{E}$ if for all $x, y \in E$, $\mu(x) \leq \mu(1)$ and $\mu(x^n) \wedge \mu(x^n \to y) \leq \mu(y)$.
(ii) a fuzzy n-fold positive implicative prefilter of $\mathcal{E}$ if for any $x, y \in E$, $\mu(x) \leq \mu(1)$ and

$$\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z).$$

(iii) a fuzzy n-fold implicative prefilter of $\mathcal{E}$ if for any $x, y \in E$, $\mu(x) \leq \mu(1)$ and

$$\mu(z \to ((x^n \to y) \to x)) \wedge \mu(z) \leq \mu(x).$$

A fuzzy n-fold prefilter μ of $\mathcal{E}$ is called a fuzzy n-fold filter of $\mathcal{E}$ if for all $x, y, z \in E$, μ satisfies in

$$\mu(x \to y) \leq \mu((x \otimes z) \to (y \otimes z)).$$

Note. The set of all fuzzy n-fold (pre)filter on $\mathcal{E}$ is denoted by $\mathcal{FF}_n(\mathcal{E})(\mathcal{FPF}_n(\mathcal{E}))$.

Theorem 2.19. [8] *Let $\mu \in \mathcal{F}(\mathcal{E})$. Then μ is a fuzzy n-fold (pre)filter of $\mathcal{E}$ if and only if for any $t \in [0, 1]$, μ_t is an empty set or an n-fold (pre)filter of $\mathcal{E}$.*

3 Maximal fuzzy n-fold filter of EQ-algebra

In this section we introduce maximal fuzzy n-fold (pre)filter and investigate some properties about them. Then we show that each maximal fuzzy (n-fold) (pre)filter takes only values $\{0, 1\}$.
By definition of a fuzzy subset of EQ-algebra, it is clear that $\mu(1)$ is the greatest value of μ.

Definition 3.1. *Let $\mu \in \mathcal{FF}(\mathcal{E})(\mathcal{FF}_n(\mathcal{E}))$. Then μ is called normalized if $\mu(1) = 1$.*

Example 3.2. *Let $E = \{0, a, b, c, 1\}$ be a chain such that $0 \leq a \leq b \leq c \leq 1$. Define the operations $\wedge, \otimes$ and $\sim$ on E as follows:*

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	0	c
1	0	a	b	c	1

$\sim$	0	a	b	c	1
0	1	a	0	0	0
a	a	1	a	a	a
b	0	a	1	b	b
c	0	a	b	1	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	a	1	1	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$x \wedge y = \min\{x, y\}$.

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Define the fuzzy set μ on E as follows:

$$\mu(0) = 0.4,\ \mu(a) = 0.4,\ \mu(b) = 0.5,\ \mu(c) = 0.6 \text{ and } \mu(1) = 1.$$

Then μ is a normalized fuzzy (pre)filter of $\mathcal{E}$.

Definition 3.3. *Let $\mu \in \mathcal{F}(\mathcal{E})$. The normalization μ is a fuzzy subset $\bar{\mu} : E \to [0,1]$ given by $\bar{\mu}(x) = \mu(x) + 1 - \mu(1)$, for any $x \in E$.*

Note. In what follows we prove properties for filters instead of (pre)filters, unless other wise state.

Proposition 3.4. *Let $\mu \in \mathcal{F}(\mathcal{E})$. Then fuzzy subset $\bar{\mu}$ is a normalized fuzzy filter of $\mathcal{E}$.*

Proof. Obviously, $\bar{\mu}(x) \leq 1 = \mu(1) + 1 - \mu(1) = \bar{\mu}(1)$. Since $\mu \in \mathcal{FF}(\mathcal{E})$, for any $x, y \in E$, we have

$$\begin{aligned}\bar{\mu}(x) \wedge \bar{\mu}(x \to y) &= (\mu(x) + 1 - \mu(1)) \wedge (\mu(x \to y) + 1 - \mu(1)) \\ &\leq (\mu(x) \wedge \mu(x \to y)) + 1 - \mu(1) \leq \mu(y) + 1 - \mu(1)) \\ &= \bar{\mu}(y).\end{aligned}$$

Also,

$$\bar{\mu}(x \to y) = \mu(x \to y) + 1 - \mu(1) \leq \mu((x \otimes z) \to (y \otimes z)) + 1 - \mu(1) = \bar{\mu}((x \otimes z) \to (y \otimes z)).$$

Clearly, $\bar{\mu}(1) = 1$. Therefore, $\bar{\mu}$ is a normalized fuzzy filter of $\mathcal{E}$. $\square$

Corollary 3.5. *Let μ be a fuzzy n-fold filter of $\mathcal{E}$. Then $\bar{\mu}$ is a normalized fuzzy n-fold filter of $\mathcal{E}$.*

Note. We denote the set of all normalized fuzzy filters of $\mathcal{E}$ by $\mathcal{NFF}(\mathcal{E})$ and the set of all normalized fuzzy n-fold filters of $\mathcal{E}$ by $\mathcal{NFF}_n(\mathcal{E})$

Proposition 3.6. *Let $\mu, \nu \in \mathcal{F}(\mathcal{E})$. Then a mapping $\mu \wedge \nu : E \to [0,1]$ is defined by $(\mu \wedge \nu)(x) = \mu(x) \wedge \nu(x)$, for any $x, y \in E$. If $\mu, \nu \in \mathcal{NFF}_n(\mathcal{E})$, then $\mu \wedge \nu \in \mathcal{NFF}_n(\mathcal{E})$.*

Proof. Clearly, $(\mu \wedge \nu)(x) = \mu(x) \wedge \nu(x) \leq \mu(1) \wedge \nu(1) = (\mu \wedge \nu)(1)$. In addition, for any $x, y \in E$, we have

$$(\mu \wedge \nu)(x^n) \wedge (\mu \wedge \nu)(x^n \to y) = \mu(x^n) \wedge \nu(x^n) \wedge \mu(x^n \to y) \wedge \nu(x^n \to y) \leq \mu(y) \wedge \nu(y) = (\mu \wedge \nu)(y).$$

Also, since $\mu, \nu \in \mathcal{NFF}(\mathcal{E})$, we have $(\mu \wedge \nu)(1) = \mu(1) \wedge \nu(1) = 1$. Hence $\mu \wedge \nu \in \mathcal{NFF}_n(\mathcal{E})$. $\square$

Proposition 3.7. *Let $\mu, \nu \in \mathcal{NFF}_n(\mathcal{E})$. Then:*
(i) $\mu \leq \bar{\mu}$;
(ii) If $\mu \in \mathcal{NFF}_n(\mathcal{E})$, then $\bar{\mu} = \mu$;
(iii) If $E_\mu = \{x \in E \mid \mu(x) = \mu(1)\}$, then $E_\mu = E_{\bar{\mu}}$;
(iv) $(\mathcal{NFF}_n(\mathcal{E}), \leq)$ is a $\wedge$-semilattice ($\chi_{\{[0]\}}$ is the smallest element of $\mathcal{NFF}_n(\mathcal{E})$ and $1(x) = 1$, for all $x \in E$ is the greatest element of $\mathcal{NFF}_n(\mathcal{E})$);
(v) If $\bar{\mu}(x) = 0$, for some $x \in E$, then $\mu(x) = 0$;
(vi) If $\mu, \nu \in \mathcal{FF}_n(\mathcal{E})$ such that $\bar{\mu} \in \mathcal{NFF}_n(\mathcal{E})$ and $\bar{\mu} \leq \nu$, then $\nu = \bar{\nu}$;
(vii) If F and G are two filters of $\mathcal{E}$, then $\chi_{F \cap G} = \chi_F \wedge \chi_G$;
$(viii)$ If $\mu, \nu \in \mathcal{F}(\mathcal{E})$, then $E_{\mu \wedge \nu} = E_\mu \cap E_\nu$.

Proof. The proof of (i), (ii) and (iv) is clear.
(iii): $E_{\bar{\mu}} = \{x \in E \mid \bar{\mu}(x) = \bar{\mu}(1)\} = \{x \in E \mid \mu(x) + 1 - \mu(1) = 1\} = E_\mu$.
(v): Since $\bar{\mu}(x) = 0$, for some $x \in E$ we get $\mu(x) + 1 - \mu(1) = 0$ and so $\mu(x) = \mu(1) - 1 \leq 0$. Thus $\mu(x) = 0$.
(vi): By assumption $1 = \bar{\mu}(1) \leq \nu(1)$. Hence $\nu(1) = 1$ and by (ii), $\nu = \bar{\nu}$.
(vii): Let $x \in F \cap G$. Then $\chi_F = \chi_G = \chi_{F \cap G} = 1$. If $x \notin F \cap G$, then $x \notin F$ or $x \notin G$. Thus

$\chi_F \wedge \chi_G = 0$ and $\chi_{F\cap G} = 0$. Hence, $\chi_{F\cap G} = \chi_F \wedge \chi_G$.
$(viii)$:

$$\begin{aligned} E_{\mu\wedge\nu} &= \{x \in E \mid (\mu \wedge \nu)(x) = \mu(x) \wedge \nu(x) = \mu(1) \wedge \nu(1) = 1\} \\ &= \{x \in E \mid \mu(x) = \mu(1) = 1 \text{ and } \nu(x) = \nu(1) = 1\} \\ &= E_\mu \cap E_\nu. \end{aligned}$$

□

Definition 3.8. *Let $\mu \in \mathcal{FF}(\mathcal{E})$. We call μ is a maximal fuzzy filter of $\mathcal{E}$, if it is non-constant and $\bar{\mu}$ is a maximal element of $(\mathcal{NFF}(\mathcal{E}), \leq)$.*

Definition 3.9. *Let $\mu \in \mathcal{FF}_n(\mathcal{E})$. We call μ is a maximal fuzzy n-fold filter of $\mathcal{E}$, if it is non-constant and $\bar{\mu}$ is a maximal element of $(\mathcal{NFF}_n(\mathcal{E}), \leq)$.*

Example 3.10. *Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2. Define the fuzzy set μ on E as follows:*

$$\mu(0) = 0 \text{ and } \mu(a) = \mu(b) = \mu(c) = \mu(1) = 1.$$

Then μ is a maximal fuzzy n-fold prefilter of $\mathcal{E}$.

Note. The set of all maximal fuzzy n-fold filter of $\mathcal{E}$ is denoted by $\mathcal{MFF}_n(\mathcal{E})$.

Proposition 3.11. *Let μ be non-constant. If μ is a maximal element of $(\mathcal{NFF}_n(\mathcal{E}), \leq)$, then it takes only the values $\{0, 1\}$.*

Proof. Let μ be non-constant. We claim that if $\mu(x) \neq 1$, then $\mu(x) = 0$, for all $x \in E$. Suppose that there exists $a \in E$ such that $\mu(a) \neq 1$ and $\mu(a) \neq 0$. Then $0 < \mu(a) < 1$. Define $\nu : E \to [0, 1]$ as follows,

$$\nu(x) = \begin{cases} \frac{1}{2}(1 + \mu(x)), & \mu(x) \geq \frac{1}{2} \\ \\ \frac{3}{2}\mu(x), & \mu(x) < \frac{1}{2} \end{cases}$$

Thus $\nu \in \mathcal{FF}(\mathcal{E})$. We have $\nu(1) = 1$ and if $\mu(x) = 0$, then $\nu(x) = 0$. Also, $\mu(x) \geq \frac{1}{2}$ if and only if $\nu(x) = \frac{1}{2}(1 + \mu(x)) \geq \frac{3}{4}$ and $\mu(x) < \frac{1}{2}$ if and only if $\nu(x) < \frac{3}{4}$. Now, let $t \in [0, 1]$. If $t \geq \frac{3}{4}$, then

$$\nu_t = \{x \in E \mid \nu(x) \geq t\} = \{x \in E \mid \mu(x) \geq \frac{1}{2}\} = \mu_{\frac{1}{2}}.$$

If $t < \frac{3}{4}$, then

$$\begin{aligned} \nu_t = \{x \in E \mid \nu(x) \geq t\} &= \{x \in E \mid \nu(x) \geq \frac{3}{4}\} \cup \{x \in E \mid t \leq \nu(x) < \frac{3}{4}\} \\ &= \{x \in E \mid \mu(x) \geq \frac{1}{2}\} \cup \{x \in E \mid t \leq \frac{3}{2}\mu(x) < \frac{3}{4}\} \\ &= \{x \in E \mid \mu(x) \geq \frac{1}{2}\} \cup \{x \in E \mid \frac{2t}{3} \leq \mu(x) < \frac{1}{2}\} \\ &= \{x \in E \mid \mu(x) \geq \frac{2t}{3}\} \\ &= \mu_{\frac{2t}{3}}. \end{aligned}$$

Hence for each $t \in [0,1]$, ν_t is either empty or is an n-fold filter of $\mathcal{E}$ and so $\nu \in \mathcal{FF}_n(\mathcal{E})$, that is $\nu \in \mathcal{NFF}_n(\mathcal{E})$. Since $\mu(x) \leq \nu(x)$, for all $x \in \mathcal{E}$ and $\mu(a) < \nu(a)$, which is a contradiction of maximality of μ. Therefore, μ takes only the values $\{0,1\}$. □

Theorem 3.12. *Every $\mu \in \mathcal{MFF}_n(\mathcal{E})$ is normalized and takes only the values $\{0,1\}$.*

Proof. Let $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Then $\bar{\mu}$ is a maximal element of $(\mathcal{NFF}_n(\mathcal{E}), \leq)$. Since μ is not constant, we get $\bar{\mu}$ is not, too. By Lemma 3.11, $\bar{\mu}$ takes only the values $\{0,1\}$, that is $\bar{\mu}(x) = \mu(x) + 1 - \mu(1)$, takes only the values $\{0,1\}$. Clearly, $\bar{\mu}(x) = 1$ if and only if $\mu(x) = \mu(1)$ and $\bar{\mu}(x) = 0$ if and only if $\mu(x) = \mu(1) - 1$ and by Proposition 3.7(v), we have $\mu(x) = 0$. Thus $\mu(1) = 1$. Hence $\mu \in \mathcal{NFF}_n(\mathcal{E})$ and so $\mu = \bar{\mu}$. Therefore, μ takes only the values $\{0,1\}$. □

In general, obviously, if $\mu \in \mathcal{NFF}_n(\mathcal{E})$, then $\chi_{E_\mu} \leq \mu$. But, if $\mu \in \mathcal{MFF}_n(\mathcal{E})$, then the following theorem holds.

Theorem 3.13. *Let $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Then $\chi_{E_\mu} = \mu$.*

Proof. Clearly $\chi_{E_\mu} \leq \mu$ and χ_{E_μ} takes only values $\{0,1\}$. Since by Theorem 3.12, μ only takes values $\{0,1\}$, we have $\mu \leq \chi_{E_\mu}$. Therefore, $\chi_{E_\mu} = \mu$. □

Theorem 3.14. *Let $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Then E_μ is a maximal n-fold filter of $\mathcal{E}$.*

Proof. Since μ is non-constant, we can easily see that E_μ is a proper n-fold filter of $\mathcal{E}$. Let F be a filter of $\mathcal{E}$ such that $E_\mu \subsetneq F$. Then $\chi_{E_\mu} \leq \chi_F$. Hence by Theorem 3.13, $\mu \leq \chi_F$. Since $\mu, \chi_F \in \mathcal{FF}_n(\mathcal{E})$ and $\mu = \bar{\mu}$ is a maximal element of $\mathcal{NFF}_n(\mathcal{E})$, we have $\mu = \chi_F$ or $\chi_F = 1$. Thus, if $\chi_F = 1$, then $F = E$ and if $\mu = \chi_F$, then $E_\mu = E_{\chi_F}$. Therefore, E_μ is a maximal n-fold filter of $\mathcal{E}$. □

Definition 3.15. [17] *A proper (pre)filter F is called a prime (pre)filter of $\mathcal{E}$ if $x \to y \in F$ or $y \to x \in F$, for all $x, y \in E$.*

Definition 3.16. *Let $\mu \in \mathcal{F}(\mathcal{E})$. Then μ is a prime fuzzy filter of lattice ordered EQ-algebra $\mathcal{E}$ if μ is non-constant and $\mu(x \vee y) = \max\{\mu(x), \mu(y)\}$, for all $x, y \in E$.*

Definition 3.17. *A proper (pre)filter F is called a prime n-fold (pre)filter of $\mathcal{E}$ if $x^n \to y \in F$ or $y^n \to x \in F$, for all $x, y \in E$.*

Remark 3.18. *We can see in lattice ordered EQ-algebra, each maximal n-fold (pre)filter of $\mathcal{E}$ is a prime n-fold (pre)filter of $\mathcal{E}$.*

Definition 3.19. *Let $\mu \in \mathcal{F}(\mathcal{E})$. Then μ is a prime fuzzy n-fold filter of lattice ordered EQ-algebra $\mathcal{E}$ if μ is non-constant and $\mu(x^n \vee y^n) = \max\{\mu(x^n), \mu(y^n)\}$, for all $x, y \in E$.*

Example 3.20. *According to Example 3.2, μ is a prime fuzzy 1-fold filter of $\mathcal{E}$. Since*

$$0.5 = \mu(b) = \mu(a \vee b) = \max\{\mu(a^n), \mu(b^n)\} = \max\{\mu(0), \mu(0)\} = \mu(0) = 0.4,$$

then μ is not prime fuzzy n-fold filter of $\mathcal{E}$, for all $n \geqslant 2$.

Remark 3.21. *Let $\mathcal{E}$ be a chain. Then every fuzzy n-fold filter of $\mathcal{E}$ is a prime fuzzy n-fold filter of $\mathcal{E}$.*

Theorem 3.22. *Let $\mathcal{E}$ be a lattice ordered EQ-algebra and $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Then μ is a prime fuzzy n-fold filter of $\mathcal{E}$.*

Proof. Since $x^n, y^n \leq x^n \vee y^n$, we get $\mu(x^n), \mu(y^n) \leq \mu(x^n \vee y^n)$ and so $\max\{\mu(x^n), \mu(y^n)\} \leq \mu(x^n \vee y^n)$. By Theorem 3.12, $\mu \in \mathcal{NFF}_n(\mathcal{E})$ and takes only values $\{0, 1\}$. If $\mu(x^n \vee y^n) = 1 = \mu(1)$, then $x^n \vee y^n \in E_\mu$. By Theorem 3.14, E_μ is a maximal n-fold filter of $\mathcal{E}$. Thus E_μ is a prime n-fold filter of $\mathcal{E}$ and so $x^n \in E_\mu$ or $y^n \in E_\mu$, that is $\mu(x^n) = 1 = \mu(1)$ or $\mu(y^n) = 1 = \mu(1)$. Hence $\max\{\mu(x^n), \mu(y^n)\} = 1 = \mu(x^n \vee y^n)$. Therefore, $\mu(x^n) \vee \mu(y^n) = \max\{\mu(x^n), \mu(y^n)\}$. Similarly, if $\mu(x^n \vee y^n) = 0$, we have $\max\{\mu(x^n), \mu(y^n)\} = \mu(x^n \vee y^n)$. Therefore, μ is a prime fuzzy n-fold filter of $\mathcal{E}$. $\square$

Definition 3.23. *Let $\mathcal{E}$ be bounded and μ be a non-constant fuzzy filter (n-fold filter) of $\mathcal{E}$. Then μ is called completely normalized if $\mu(1) = 1$ and $\mu(0) = 0$*

Example 3.24. *According to Example 3.10, μ is a completely normalized fuzzy n-fold filters of $\mathcal{E}$.*

Definition 3.25. *Let $\mathcal{E}$ be bounded and μ be a non-constant fuzzy filter (n-fold filter) of $\mathcal{E}$. Then the complete normalization of μ is $\bar{\bar{\mu}} : E \to [0, 1]$ given by $\bar{\bar{\mu}}(x) = \dfrac{\mu(x) - \mu(0)}{\mu(1) - \mu(0)}$.*

Proposition 3.26. *Let $\mathcal{E}$ be bounded and μ be a non-constant fuzzy n-fold filter of $\mathcal{E}$. Then $\bar{\bar{\mu}}$ is a completely normalized fuzzy n-fold filter of $\mathcal{E}$. If μ is already completely normalized, then $\bar{\bar{\mu}} = \mu$.*

Proof. Let $\mu \in \mathcal{FF}_n(\mathcal{E})$. Since $\mu(x) \leq \mu(1)$, we get $\bar{\bar{\mu}}(x) \leq \bar{\bar{\mu}}(1)$ and we have

$$\bar{\bar{\mu}}(x^n \to y) \wedge \bar{\bar{\mu}}(x^n) = \frac{1}{\mu(1) - \mu(0)}((\mu(x^n \to y) \wedge \mu(x^n)) - \mu(0)) \leq \frac{1}{\mu(1) - \mu(0)}(\mu(y) - \mu(0)) = \bar{\bar{\mu}}(y).$$

Hence $\bar{\bar{\mu}} \in \mathcal{FF}_n(\mathcal{E})$.
By definition $\bar{\bar{\mu}}(1) = 1$ and $\bar{\bar{\mu}}(0) = 0$, we get $\bar{\bar{\mu}}$ is a completely normalized. If μ is completely normalized, then $\mu(1) = 1$ and $\mu(0) = 0$. Thus $\bar{\bar{\mu}}(x) = \dfrac{\mu(x) - 0}{1 - 0} = \mu(x)$ and so $\bar{\bar{\mu}} = \mu$. $\square$

Theorem 3.27. *Let $\mathcal{E}$ be bounded. Then $\mathcal{NFF}_n(\mathcal{E})$ is the set of all completely normalized fuzzy n-fold filters of $\mathcal{E}$.*

Proof. Let $\mu \in \mathcal{NFF}_n(\mathcal{E})$. Then there exists $x_0 \in E$ such that $\mu(x_0) = 0$. Hence

$$\mu(x_0) \geq \mu(0^n \to x_0) \wedge \mu(0^n) = \mu(1) \wedge \mu(0) = \mu(0),$$

and so $\mu(x_0) = \mu(0) = 0$. Obviously $\mu(1) = 1$. $\square$

Remark 3.28. *If $\mu \in \mathcal{NFF}_n(\mathcal{E})$, then by Theorem 3.27, $\bar{\bar{\mu}} = \mu$ and $E_\mu = E_{\bar{\bar{\mu}}}$.*

Definition 3.29. *Let $\mu \in \mathcal{F}(\mathcal{E})$ and non-constant. Then μ is maximal fuzzy completely normalized n-fold filter of $\mathcal{E}$ if $\bar{\bar{\mu}}$ is a maximal element of $(\mathcal{NFF}_n(\mathcal{E}), \leq)$.*

Example 3.30. *Let $\mathcal{E}$ be the EQ-algebra as Example 3.2, and $\mu(0) = 0.4, \mu(a) = \mu(b) = m(c) = \mu(1) = 0.5$. Then μ is a maximal fuzzy completely normalized n-fold filter of $\mathcal{E}$.*

Theorem 3.31. *Let $\mathcal{E}$ be bounded. Then every maximal fuzzy n-fold filter of $\mathcal{E}$ is completely normalized and takes only the values $\{0, 1\}$.*

Proof. Let $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Then by Theorem 3.12, μ is non-constant, $\mu = \bar{\mu}$ and μ takes only the values $\{0, 1\}$. Hence by Theorem 3.27, $\mu(0) = 0$. Since $\mu \in \mathcal{NFF}_n(\mathcal{E})$, then μ is completely normalized. $\square$

Let $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Set $supp(\mu) = \{x \in E \mid \mu(x) > 0\}$.

Proposition 3.32. *Let $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Then set $supp(\mu)$ is empty or an n-fold filter of $\mathcal{E}$*

Proof. Let $supp(\mu) \neq \emptyset$. Then there exists $x \in supp(\mu)$ such that $\mu(x) > 0$. Thus $0 < \mu(x) \leq \mu(1)$ and so $1 \in supp(\mu)$. Now, let $x^n, x^n \to y \in supp(\mu)$. Then $\mu(x^n) > 0$ and $\mu(x^n \to y) > 0$. Since $0 < \mu(x^n) \wedge \mu(x^n \to y) \leq \mu(y)$, we get $y \in supp(\mu)$. Hence $supp(\mu)$ is an n-fold prefilter of $\mathcal{E}$. Suppose that, $x \to y \in supp(\mu)$. Since $\mu(x \to y) \leq \mu((x \otimes z) \to (y \otimes z))$, we have $supp(\mu)$ is an n-fold filter of $\mathcal{E}$. □

Proposition 3.33. *Let $\mathcal{E}$ be bounded and $\mu \in \mathcal{NFF}_n(\mathcal{E})$. Then:*
(i) $\mu \leq \chi_{supp(\mu)}$.
(ii) If μ is a maximal element of $\mathcal{NFF}_n(\mathcal{E})$, we have $\mu = \chi_{supp(\mu)}$ and $E_\mu = supp(\mu)$.
(iii) Every fuzzy maximal completely normalized n-fold filter μ of $\mathcal{E}$ takes only the values $\mu(0)$ and $\mu(1)$.
(iv) If μ is a maximal element of $\mathcal{NFF}_n(\mathcal{E})$, then E_μ is a maximal n-fold filter of $\mathcal{E}$.

Proof. (i) If $\chi_{supp(\mu)}(x) = 1$, then $x \in supp(\mu)$ and so $\mu(x) \leq \chi_{supp(\mu)}(x)$. Let $\chi_{supp(\mu)}(x) = 0$. Then $x \notin supp(\mu)$ and so $\mu(x) = 0$. Hence, $\mu \leq \chi_{supp(\mu)}$.
(ii) Since $\mu(1) = 1 \leq \chi_{supp(\mu)}(1)$, we get $\chi_{supp(\mu)}(1) = 1$ and so $\chi_{supp(\mu)} \in \mathcal{NFF}_n(\mathcal{E})$. Thus $\mu = \chi_{supp(\mu)}$. Let $x \in E_\mu$. Then $\mu(x) = \mu(1) = 1 > 0$ and so $x \in supp(\mu)$. If $x \in supp(\mu)$, then $\mu(x) > 0$ and so $\mu(x) = 1 = \mu(1)$, i. e. $x \in E_\mu$, Hence, $E_\mu = supp(\mu)$.
(iii) Since $\bar{\bar{\mu}}$ only takes the values $\{0, 1\}$ and $\bar{\bar{\mu}} = \dfrac{\mu(x) - \mu(1)}{\mu(0) - \mu(1)}$, we get $\mu(x) = \mu(1)$ or $\mu(x) = \mu(0)$.
(iv) By (ii), we have $\mu = \chi_{supp(\mu)}$. Let F be a proper n-fold filter of $\mathcal{E}$ such that $E_\mu \subsetneq F$. Then $\mu = \chi_{E_\mu} \leq \chi_F$ and so $\mu = \chi_F$. Thus $E_\mu = E_{\chi_F} = F$. Therefore, E_μ is a maximal n-fold filter of $\mathcal{E}$. □

Now, we try to gain one to one corresponding between maximal fuzzy n-fold filters and maximal n-fold filters.

Proposition 3.34. *Let $\mathcal{E}$ be bounded and F be a maximal n-fold filter of $\mathcal{E}$. Then χ_F is a maximal element of $\mathcal{NFF}_n(\mathcal{E})$.*

Proof. Let $\mu \in \mathcal{NFF}_n(\mathcal{E})$ and non-constant such that $\chi_F \leq \mu$. Then by Proposition 3.33(i), $\chi_F \leq \mu \leq \chi_{supp(\mu)}$. If $y \in F$, then $\chi_F(y) = 1$ and so $\chi_{supp(\mu)}(y) = 1$. Hence $y \in supp(\mu)$, that is $F \subseteq supp(\mu)$. Since $\mu \in \mathcal{NFF}_n(\mathcal{E})$, we have $supp(\mu)$ is a proper n-fold filter of $\mathcal{E}$. Thus $F = supp(\mu)$ and so $\mu = \chi_F$. Therefore, χ_F is a maximal element of $\mathcal{NFF}_n(\mathcal{E})$. □

Corollary 3.35. *Let $\mathcal{E}$ be bounded and $\mu \in \mathcal{NFF}_n(\mathcal{E})$ such that E_μ be a maximal n-fold filter of $\mathcal{E}$. Then $\mu = \chi_{E_\mu}$.*

Proof. By Proposition 3.34, χ_{E_μ} is a maximal element of $\mathcal{NFF}_n(\mathcal{E})$. Also, $\chi_{E_\mu} \leq \mu$. Hence $\chi_{E_\mu} = \mu$. □

Corollary 3.36. *Let $\mathcal{E}$ be bounded and $\mu \in \mathcal{NFF}_n(\mathcal{E})$. Then E_μ is a maximal n-fold filter of $\mathcal{E}$ if and only if μ is a maximal element of $\mathcal{NFF}_n(\mathcal{E})$. Similarly, F is a maximal n-fold filter of $\mathcal{E}$ if and only if χ_F is a maximal element of $\mathcal{NFF}_n(\mathcal{E})$.*

Proof. By Proposition 3.34 and Corollary 3.35, the proof is clear. □

4 Fuzzy n-fold obstinate filter of EQ-algebra

In this section, we introduce the notion of fuzzy n-fold obstinate (pre)filter of EQ-algebra and some related properties of them are investigated.

Definition 4.1. *Let $\mu \in \mathcal{FF}(\mathcal{E})$. Then μ is called a fuzzy n-fold obstinate prefilter of $\mathcal{E}$ if for all $x, y \in E$, $(1-\mu(x)) \wedge (1-\mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x)$. A fuzzy n-fold obstinate prefilter of $\mathcal{E}$ is called a fuzzy n-fold obstinate filter of $\mathcal{E}$ if μ satisfies in (FF3).*

Note. The set of all n-fold obstinate filter of $\mathcal{E}$ is denoted by $\mathcal{OF}_n(\mathcal{E})$ and the set of all fuzzy n-fold obstinate prefilter of $\mathcal{E}$ is denoted by $\mathcal{FOF}_n(\mathcal{E})$.

Example 4.2. *By Example 3.2, define the fuzzy set μ on E as follows:*

$$\mu(0) = 0.4,\ \mu(a) = 0.4,\ \mu(b) = 0.5,\ \mu(c) = 0.6 \text{ and } \mu(1) = 0.8.$$

Then $\mu \in \mathcal{FOF}_n(\mathcal{E})$, for all $n \geq 2$. But it is not a fuzzy 1-fold obstinate prefilter of $\mathcal{E}$ because

$$0.5 = 0.5 \wedge 0.6 = (1-\mu(a)) \wedge (1-\mu(b)) \nleq \mu(a \to b) \wedge \mu(b \to a) = \mu(1) \wedge \mu(a) = \mu(a) = 0.4.$$

Theorem 4.3. *Let $\emptyset \neq F \subseteq E$. Then $F \in \mathcal{OF}_n(\mathcal{E})$ if and only if $\chi_F \in \mathcal{FOF}_n(\mathcal{E})$.*

Proof. Let $F \in \mathcal{OF}_n(\mathcal{E})$. It is enough to prove that $(1-\chi_F(x)) \wedge (1-\chi_F(y)) \leq \chi_F(x^n \to y) \wedge \chi_F(y^n \to x)$. For this, suppose $x \in F$ or $y \in F$. Then $1-\chi_F(x) = 0$ or $1-\chi_F(y) = 0$. Also, if we suppose $x \notin F$ and $y \notin F$, then $1-\chi_F(x) = 1$ and $1-\chi_F(y) = 1$. On the other side, since F is an n-fold obstinate filter of $\mathcal{E}$, we have $x^n \to y \in F$ and $y^n \to x \in F$, and so $\chi_F(x^n \to y) = 1$ and $\chi_F(y^n \to x) = 1$. Hence in both cases we have $(1-\chi_F(x)) \wedge (1-\chi_F(y)) \leq \chi_F(x^n \to y) \wedge \chi_F(y^n \to x)$. Therefore, $\chi_F \in \mathcal{FOF}_n(\mathcal{E})$.
Conversely, let $\chi_F \in \mathcal{FOF}_n(\mathcal{E})$ and $x, y \notin F$. Then $1-\chi_F(x) = 1$ and $1-\chi_F(y) = 1$, and so

$$1 = (1-\chi_F(x)) \wedge (1-\chi_F(y)) \leq \chi_F(x^n \to y) \wedge \chi_F(y^n \to x) \leq \chi_F(x^n \to y), \chi_F(y^n \to x) \leq 1.$$

Hence, $\chi_F(x^n \to y) = \chi_F(y^n \to x) = 1$, and so $x^n \to y \in F$ and $y^n \to x \in F$. Therefore, $F \in \mathcal{OF}_n(\mathcal{E})$. □

Theorem 4.4. *Let $\mu \in \mathcal{FF}(\mathcal{E})$. Then the level subset $\mu_t = \{x \in E \mid \mu(x) \geq t\}$ is an n-fold obstinate filter of $\mathcal{E}$, for all $t \in [0, \frac{1}{2}]$ if $\mu \in \mathcal{FOF}_n(\mathcal{E})$.*

Proof. Let $\mu \in \mathcal{FOF}_n(\mathcal{E})$. If $t = 0$, then $\mu_t = E$, and so $\mu_t \in \mathcal{OF}_n(\mathcal{E})$. Suppose $0 \leq t \leq \frac{1}{2}$ and $x, y \notin \mu_t$. Then $\mu(x) < t$ and $\mu(y) < t$, and so

$$1-t \leq (1-\mu(x)) \wedge (1-\mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x) \leq \mu(x^n \to y), \mu(y^n \to x).$$

Thus $t \leq 1-t \leq \mu(x^n \to y), \mu(y^n \to x)$, and so $x^n \to y, y^n \to x \in \mu_t$. Therefore, $\mu_t \in \mathcal{FOF}_n(\mathcal{E})$. □

Proposition 4.5. *If $\mu \in \mathcal{FOF}_n(\mathcal{E})$, then $\mu \in \mathcal{FOF}_{n+1}(\mathcal{E})$.*

Proof. Let $\mu \in \mathcal{FOF}_n(\mathcal{E})$ and $x, y \in E$. Then by Proposition 2.2(iii), $x^n \to y \leq x^{n+1} \to y$ and $y^n \to x \leq y^{n+1} \to x$. Thus we have $\mu(x^n \to y) \leq \mu(x^{n+1} \to y)$ and $\mu(y^n \to x) \leq \mu(y^{n+1} \to x)$. Hence,

$$(1-\mu(x)) \wedge (1-\mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x) \leq \mu(x^{n+1} \to y) \wedge \mu(y^{n+1} \to x).$$

Therefore, $\mu \in \mathcal{FOF}_{n+1}(\mathcal{E})$. □

In the next example we show that the converse of Proposition 4.5 does not hold, in general.

Example 4.6. *Let $\mathcal{E}$ be an EQ-algebra as in Example 4.2. Then μ is a fuzzy 2-fold obstinate prefilter of $\mathcal{E}$. Since*

$$0.5 = 0.5 \wedge 0.6 = (1-\mu(a)) \wedge (1-\mu(b)) \nleq \mu(a \to b) \wedge \mu(b \to a) = \mu(1) \wedge \mu(a) = \mu(a) = 0.4,$$

then μ is not a fuzzy 1-fold obstinate prefilter of $\mathcal{E}$.

Proposition 4.7. *Let μ and ν be two fuzzy filters of $\mathcal{E}$ such that $\mu \leq \nu$. If $\mu \in \mathcal{FOF}_n(\mathcal{E})$, then ν is too.*

Proof. Since $\mu, \nu \in \mathcal{FF}(\mathcal{E})$ such that $\mu \leq \nu$, for any $x \in E$ we have $\mu(x) \leq \nu(x)$ and so $1-\nu(x) \leq 1-\mu(x)$. Moreover, from $\mu \in \mathcal{FOF}_n(\mathcal{E})$, we have

$$(1-\nu(x)) \wedge (1-\nu(y)) \leq (1-\mu(x)) \wedge (1-\mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x) \leq \nu(x^n \to y) \wedge \nu(y \to x^n).$$

Therefore, $\nu \in \mathcal{FF}_n(\mathcal{E})$. □

Now, we investigate relation between maximal fuzzy n-fold filter and fuzzy n-fold (positive) implication filter and fuzzy obstinate n-fold filters of $\mathcal{E}$.

Theorem 4.8. *Let $\mathcal{E}$ be bounded and $\mu \in \mathcal{NFF}_n(\mathcal{E})$ such that μ takes only the values $\{0,1\}$. Then $\mu \in \mathcal{FOF}_n(\mathcal{E})$ if and only if for any $x \in E$, $\mu(x) = \mu(1) = 1$ or there exists $m \in \mathbb{N}$ such that $\mu((\neg x^n)^m) = \mu(1) = 1$.*

Proof. Let $\mu \in \mathcal{FOF}_n(\mathcal{E})$ and there exists $x \in E$ such that $\mu(x) \neq \mu(1) = 1$. Then by assumption $\mu(x) = 0$. Since $(1-\mu(x)) \wedge (1-\mu(0)) \leq \mu(x^n \to 0) \wedge \mu(0^n \to x)$ and $1-\mu(x) \leq 1-\mu(0)$, we get $\mu(1) = 1 = 1-\mu(x) \leq \mu(x^n \to 0) = \mu(\neg x^n)$ and so $\mu(\neg x^n) = 1 = \mu(1)$.

Conversely, for all $x, y \in E$, we have $\mu(x) = 1$ or there exists $m \in \mathbb{N}$ such that $\mu((\neg x^n)^m) = 1$ and $\mu(y) = 1$ or there exists $k \in \mathbb{N}$ such that $\mu((\neg y^n)^k) = 1$. Since $(\neg x^n)^m \leq \neg x^n$ and $(\neg y^n)^k \leq \neg y^n$, we have $1 = \mu((\neg x^n)^m) \leq \mu(\neg x^n)$ and $1 = \mu((\neg y^n)^k) \leq \mu(\neg y^n)$, and so $\mu(\neg x^n) = \mu(\neg y^n) = 1$. Then by Proposition 2.2(iii), $\neg x^n = x^n \to 0 \leq x^n \to y$ and $\neg y^n = y^n \to 0 \leq y^n \to x$. Thus

$$1 = \mu(\neg x^n) = \mu(x^n \to 0) \leq \mu(x^n \to y),$$

and

$$1 = \mu(\neg y^n) = \mu(y^n \to 0) \leq \mu(y^n \to x).$$

Hence, $\mu(x^n \to y) = \mu(y^n \to x) = 1$ and so

$$(1-\mu(x)) \wedge (1-\mu(y)) \leq 1 = \mu(1) = \mu(x^n \to y) \wedge \mu(y^n \to x).$$

Therefore, $\mu \in \mathcal{FOF}_n(\mathcal{E})$. □

Corollary 4.9. *Let $\mathcal{E}$ be a bounded and $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Then for every $\mu \in \mathcal{FOF}_n(\mathcal{E})$, we have E_μ is a maximal n-fold filter of $\mathcal{E}$, for all $n \in \mathbb{N}$.*

Proof. Let $\mu \in \mathcal{FOF}_n(\mathcal{E})$ and G be a proper filter of E such that $E_\mu \subsetneq G$. Then there exists $x \in G$ such that $x \notin E_\mu$, and so $\mu(x) \neq \mu(1)$. Thus by Theorem 4.8, there exists $m \in \mathbb{N}$ such that $\mu((\neg x^n)^m) = \mu(1)$, and so $(\neg x^n)^m \in E_\mu \subsetneq G$. Since $(\neg x^n)^m \leq \neg x^n$, $x \in G$ and G is a filter of E, we have $0 \in G$, which is a contradiction. Therefore, E_μ is a maximal n-fold filter of $\mathcal{E}$, for all $n \in \mathbb{N}$. □

Theorem 4.10. *Let $\mathcal{E}$ be bounded and $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Then every $\mu \in \mathcal{FOF}_n(\mathcal{E})$ is a fuzzy n-fold implicative filter of $\mathcal{E}$.*

Proof. Let $\mu \in \mathcal{FOF}_n(\mathcal{E})$, but not a fuzzy n-fold implicative filter of $\mathcal{E}$. Then there exist $x, y \in E$ such that $\mu(z \to ((x^n \to y) \to x)) \wedge \mu(z) \nleq \mu(x)$. Clearly, $\mu(x) \neq \mu(1)$. Then by Theorem 4.8, there exists $l \in \mathbb{N}$ such that $\mu(\neg(x^n)^l) = \mu(1)$. Since $(\neg x^n)^l \leq \neg(x^n)$, we have $\mu(\neg x^n) = \mu(1)$. By Proposition 2.2(iii), we have $\mu(x^n \to 0) \leq \mu(x^n \to y)$ and so $\mu(x^n \to y) = 1$. Now, by assumption,

$$\mu(z \to ((x^n \to y) \to x)) \wedge \mu(z) \leq \mu((x^n \to y) \to x) \leq \mu((x^n \to y) \to x) \wedge \mu(x^n \to y) \leq \mu(x),$$

which is a contradiction. Therefore, μ is a fuzzy n-fold implicative filter of $\mathcal{E}$. $\square$

Next example shows that the converse of Theorem 4.10, does not hold, in general.

Example 4.11. *Let $E = \{0, a, b, 1\}$ be a chain such that $0 \leq a \leq b \leq 1$. Define the operations $\wedge, \otimes$ and $\sim$ on E as follows:*

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	1
1	0	a	b	1

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	1	1

$$x \wedge y = \min\{x, y\}.$$

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. Define the fuzzy set μ on E as follows:

$$\mu(0) = 0.3 \text{ and } \mu(a) = \mu(b) = \mu(1) = 0.7.$$

Thus μ is a fuzzy n-fold implicative filter of $\mathcal{E}$, for all $n \in \mathbb{N}$. But $\mu \notin \mathcal{FF}_n\mathcal{OF}(\mathcal{E})$, for all $n \in \mathbb{N}$. Because

$$0.3 = 0.7 \wedge 0.3 = (1 - \mu(0)) \wedge (1 - \mu(a)) \nleq \mu(0^n \to a) \wedge \mu(a^n \to 0) = \mu(1) \wedge \mu(0) = 0.2.$$

Theorem 4.12. [8] *Let $\mathcal{E}$ be a residuated EQ-algebra. Then every fuzzy n-fold implicative filter of $\mathcal{E}$ is a fuzzy n-fold positive implicative filter of $\mathcal{E}$.*

Theorem 4.13. *Let $\mathcal{E}$ be a residuated EQ-algebra and and $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Then every $\mu \in \mathcal{FOF}_n(\mathcal{E})$ is a fuzzy n-fold positive implicative filter of $\mathcal{E}$.*

Proof. By Theorem 4.10 and Theorem 4.12, the proof is clear. $\square$

Theorem 4.14. *Let $\mathcal{E}$ be a bounded EQ-algebra and $\mu \in \mathcal{FF}(\mathcal{E})$. Then the following statements are equivalent:*
(i) E_μ is a maximal n-fold filter of $\mathcal{E}$,
(ii) for any $x \in E$, if $\mu(x) \neq \mu(1)$, then there exists $n \in \mathbb{N}$ such that $\mu(\neg x^n) = \mu(1)$.

Proof. $(i) \Rightarrow (ii)$ Let $x \in E$ such that $\mu(x) \neq \mu(1)$. Then $\langle E_\mu \cup \{x\} \rangle = E$, and so $0 \in \langle E_\mu \cup \{x\} \rangle$. Thus there exist $x_1, x_2, \cdots, x_t \in E_\mu$ such that $x_1 \to (x_2 \to (\cdots \to (x^n \to 0) \cdots)) = 1$. Since $\mu \in \mathcal{FF}(\mathcal{E})$, we have

$$\mu(1) = \mu(1) \wedge \mu(1) = \mu(x_1) \wedge \mu(x_1 \to (x_2 \to (\cdots \to (x^n \to 0) \cdots))) \leq \mu(x_2 \to (\cdots \to (x^n \to 0) \cdots)).$$

Thus $\mu(x_2 \to (\cdots \to (x^n \to 0) \cdots)) = \mu(1)$. By repeating this method we get $\mu(x \to (\cdots \to (x^n \to 0) \cdots)) = \mu(1)$, that is $\mu(\neg x^n) = \mu(1)$, for some $n \in \mathbb{N}$.
$(ii) \Rightarrow (i)$ Let G be a proper filter of $\mathcal{E}$ such that $E_\mu \subsetneq G$. Then there exists $x \in G$ $x \notin E_\mu$, and so $\mu(x) \neq \mu(1)$. Thus there exists $n \in \mathbb{N}$ such that $\mu((\neg x^n) = \mu(1)$, and so $\neg x^n \in G$. Since G is a filter of $\mathcal{E}$ and $x \in G$, we have $0 \in G$, which is a contradiction. Therefore, E_μ is a maximal n-fold filter of $\mathcal{E}$. $\square$

If $\mathcal{E}$ is an EQ-algebra and μ be a fuzzy 1-fold positive implicative filter of $\mathcal{E}$, then $\mathcal{E}/\mu$ is a good EQ-algebra and if μ is a fuzzy filter of $\mathcal{E}$, then μ is a fuzzy n-fold implicative (pre)filter of $\mathcal{E}$ if and only if every fuzzy n-fold filter is a fuzzy n-fold implicative (pre)filter of $\mathcal{E}$, see [8]. Now, we try to obtain the same result about fuzzy obstinate n-fold filter of $\mathcal{E}$.

Proposition 4.15. *Let $\mathcal{E}$ be a good EQ-algebra. Then $\chi_{\{1\}} \in \mathcal{FOF}_n(\mathcal{E})$ if and only if every normalized fuzzy filter of $\mathcal{E}$ is a fuzzy n-fold obstinate filter of $\mathcal{E}$.*

Proof. Let μ be a fuzzy filter of $\mathcal{E}$ and $x, y \in E$. If $x, y \neq 1$, then $1 = (1 - \chi_{\{1\}}(x)) \wedge (1 - \chi_{\{1\}}(y)) \leq \chi_{\{1\}}(x^n \to y) \wedge \chi_{\{1\}}(y^n \to x)$, and so $\chi_{\{1\}}(x^n \to y) = \chi_{\{1\}}(y^n \to x) = 1$, that is $x^n \to y = y^n \to x = 1$. Thus

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x) = \mu(1) = 1.$$

It means that $\mu \in \mathcal{FF}_{\setminus}\mathcal{OF}(\mathcal{E})$. If $x = 1$ or $y = 1$, then $\mu(x) = \mu(1)$ or $\mu(y) = \mu(1)$. Hence,

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x).$$

Therefore, $\mu \in \mathcal{FOF}_n(\mathcal{E})$.

Since $\chi_{\{1\}} \in \mathcal{NFF}_n(\mathcal{E})$, the proof of the converse is clear. □

Theorem 4.16. *Let $\mathcal{E}$ be a good EQ-algebra and $\mu \in \mathcal{MFF}_n(\mathcal{E})$. Then $\mu \in \mathcal{FOF}_n(\mathcal{E})$ if and only if every normalized fuzzy filter of quotient algebra $\mathcal{E}/\mu$ is a fuzzy n-fold obstinate filter of $\mathcal{E}/\mu$.*

Proof. Let $\mu \in \mathcal{FOF}_n(\mathcal{E})$. It suffices to show that $\chi_{\{[1]\}}$ is a fuzzy n-fold obstinate filter of $\mathcal{E}/\mu$. Let $[x] = [1]$ or $[y] = [1]$. Then $(1-\chi_{\{[1]\}}[x]) \wedge (1-\chi_{\{[1]\}}[y]) \leq \chi_{\{[1]\}}([x^n] \to [y]) \wedge \chi_{\{[1]\}}([y^n] \to [x])$. If $[x] \neq [1]$ and $[y] \neq [1]$, then $\mu(x) \neq \mu(1) = 1$ and $\mu(y) \neq \mu(1) = 1$ and so there exists $m, k \in \mathbb{N}$ such that $\mu(\neg x^n)^m = \mu(1) = 1$ and $\mu(\neg y^n)^k = \mu(1) = 1$. Thus $\mu(x^n \to y) = \mu(1) = \mu(y^n \to x)$. Hence $\chi_{\{[1]\}}([x]^n \to [y]) = [1] = \chi_{\{[1]\}}([y]^n \to [x])$. Therefore, $(1 - \chi_{\{[1]\}}[x]) \wedge (1 - \chi_{\{[1]\}}[y]) \leq \chi_{\{[1]\}}([x^n] \to [y]) \wedge \chi_{\{[1]\}}([y^n] \to [x])$.
Conversely, suppose that $\chi_{\{[1]\}}$ is a fuzzy n-fold obstinate filter of $\mathcal{E}/\mu$, then $(1 - \chi_{\{[1]\}}[x]) \wedge (1 - \chi_{\{[1]\}}[y]) \leq \chi_{\{[1]\}}([x^n] \to [y]) \wedge \chi_{\{[1]\}}([y^n] \to [x])$. If $[x] \neq [1]$ and $[y] \neq [1]$ (for $[x] = [1] = [y]$ it is clear), we get $[x^n \to y] = [1] = [y^n \to x]$ and so $\mu(x^n \to y) \wedge \mu(y^n \to x) = 1$, i.e.

$$(1 - \mu(x)) \wedge (1 - \mu(y)) \leq \mu(x^n \to y) \wedge \mu(y^n \to x).$$

Therefore, μ is a fuzzy n-fold obstinate filter of $(\mathcal{E})$. □

5 Conclusion

In this paper, we defined the concepts of fuzzy n-fold obstinate filter and maximal fuzzy filter of EQ-algebras are defined and discussed some properties about them. After defining normalized filters, we show that every maximal fuzzy (pre)filter of $\mathcal{E}$ is normalized and takes only the values $\{0, 1\}$. Then it was prove that if $\mathcal{E}$ is a good EQ-algebra and μ is a normalized fuzzy filter of $\mathcal{E}$, then μ is a fuzzy n-fold obstinate (pre)filter of $\mathcal{E}$ if and only if every normalized fuzzy filter of quotient algebra $\mathcal{E}/\mu$ is a fuzzy n-fold obstinate (pre)filter of $\mathcal{E}/\mu$. Also, the relation between fuzzy obstinate n-fold (pre)filters and other fuzzy (pre)filters of EQ-algebras is investigated. Most of the results proved about filters, obviously they are true for prefilters.

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