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On fuzzy implicative ideals in BL-algebras

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Abstract

In this paper, the concept of fuzzy implicative ideal in BL-algebras is introduced and several properties of it are stated. Using the concept of level subsets, some characterizations of fuzzy implicative ideals are proved. Also, it is proved that the concepts of fuzzy implicative ideal and fuzzy Boolean ideal in BL-algebras are coincide. Moreover, it is shown that a BL-algebra L is a Boolean algebra if and only if any fuzzy ideal of L is a fuzzy implicative ideal. Finally, it is proved that the homomorphic image and preimage of fuzzy implicative ideals are fuzzy implicative ideal.

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1 Introduction

BL-algebras are the algebraic structure for Hájek's basic logic [3] in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Godel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general"many-valued logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0, 1]. Most familiar example of a BL-algebra is the unit interval [0,1] endowed with the structure induced by a continuous t-norm. In 1958, Chang [1] introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. Turunen [8] introduced the notion of an implicative filter and a Boolean filter and proved that these notions are equivalent in BL-algebras. Boolean filters are an important class

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of filters, because the quotient BL-algebra induced by these filters are Boolean algebras. The notion of (fuzzy) ideal has been introduced in many algebraic structures such as lattices, rings, MV-algebras. Ideal theory is very effective tool for studying various algebraic and logical systems. In the theory of MV-algebras, as various algebraic structures, the notion of ideal is at the center, while in BL-algebras, the focus has been on deductive systems also filters. The study of BL-algebras has experienced a tremendous growth over resent years and the main focus has been on filters. In the meantime, several authors have claimed in recent works that the notion of ideals is missing in BL-algebras. Zhang et al. [10] studied the notion of fuzzy ideals in BL-algebras and in 2013, Lele [4], introduced the notion of Boolean ideals and analyzed the relationship between ideals and filters by using the set of complement elements. Yhang et al.[9] introduced the notion of implicative ideals in BL-algebras and they studied some characterizations of implicative ideals by pseudo implication operation showed. This motivates us to study the fuzzy structure for implicative ideals in BL-algebras, so the notions of fuzzy implicative ideals are introduced in BL-algebras.
Now, in this paper, we introduce the concept of fuzzy implicative ideal in BL-algebras and we state and prove several properties of it. Using the concept of level subset of a fuzzy set in a BL-algebra, we give characterization of fuzzy implicative ideals. Also, we prove that the concepts of fuzzy implicative ideal and fuzzy Boolean ideal in BL-algebras are coincide. Moreover, we show that a BL-algebra L is a Boolean algebra if and only if any fuzzy ideal of L is a fuzzy implicative ideal. Finally, we prove that the homomorphic image and preimage of fuzzy implicative ideals are fuzzy implicative ideal.

2 Preliminaries

In this section, we give some fundamental definitions and results from the literature. For more details, we refer to the references.

Definition 2.1. [3] *A BL-algebra is an algebra $(L, \vee, \wedge, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that*
$(BL1)$ $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice,*
$(BL2)$ $(L, \odot, 1)$ *is a commutative monoid,*
$(BL3)$ $z \leq x \to y$ *if and only if* $x \odot z \leq y$*,*
$(BL4)$ $x \wedge y = x \odot (x \to y)$*,*
$(BL5)$ $(x \to y) \vee (y \to x) = 1$*.*
for all $x, y, z \in L$*,*

Proposition 2.2. [2] *In any BL-algebra the following hold:*
$(BL6)$ $x \leq y$ *if and only if* $x \to y = 1$*,*
$(BL7)$ $x \odot y \leq x, y$ *and* $x^{n+1} \leq x^n$*,* $\forall n \in \mathbb{N}$*, where* $x^n = x \odot ... \odot x$*(n-times),*
$(BL8)$ $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$*,*
$(BL9)$ $0 \leq x$*,* $0 \odot x = 0$ *and* $x \odot x^- = 0$*, where* $x^- = x \to 0$*.*
$(BL10)$ $1 \to x = x$ *and* $x \to 1 = 1$*,*
for all $x, y, z \in L$*.*

Lele and Nganou adopted the pseudo addition operation $\oslash$ in a BL-algebra L as $x \oslash y := x^- \to y$, for any $x, y \in L$ and the pseudo implication operation $\rightharpoonup$, which is defined by Yang and Xin [9], $x \rightharpoonup y := x \odot y^-$, for any $x, y \in L$.

Lemma 2.3. [9] *Let L be a BL-algebra. Then for any $x, y, z \in L$,*
(i) $x \leq y$*, implies* $z \rightharpoonup y \leq z \rightharpoonup x$ *and* $x \rightharpoonup z \leq y \rightharpoonup z$*,*

(ii) $(x \rightharpoonup y) \rightharpoonup z = (x \rightharpoonup z) \rightharpoonup y = x \rightharpoonup (y \oslash z)$,
(iii) $x \rightharpoonup 0 = x, 0 \rightharpoonup x = 0, x \rightharpoonup x = 0$,
(iv) $(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \leq x \rightharpoonup y$,
(v) $x \rightharpoonup y \leq x$,
(vi) $x \leq y$ *implies* $x \rightharpoonup y = 0$,
(vii) $x \rightharpoonup y = 0$ *implies* $y^- \leq x^-$ *and* $x \leq y^{--}$,
$(viii)$ $(y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y)) = 0$,
(ix) $(x \rightharpoonup z) \leq (y \rightharpoonup z) \oslash (x \rightharpoonup y)$,

Definition 2.4. [4] *Let L be a BL-algebra and I be a non-empty subset of L. Then*
(i) *I is called an ideal of L if $x \oslash y = x^- \rightarrow y \in I$, for any $x, y \in I$ and if $y \in I$ and $x \leq y$, then $x \in I$, for all $x, y \in L$.*
(ii) *An ideal I of L is called a Boolean ideal if $x \wedge x^- \in I$, for all $x \in L$.*
(iii) *I is called an implicative ideal if $0 \in I$ and $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$ imply $x \rightharpoonup z \in I$, for any $x, y, z \in L$.*

Lemma 2.5. [4] *Let I be a non-empty subset of a BL-algebra L. Then I is an ideal of L if and only if it satisfies:*
(i) $0 \in I$,
(ii) *for any $x, y \in L$, if $x \rightharpoonup y \in I$ and $y \in I$, then $x \in I$.*

Proposition 2.6. [9] *Let I and J be ideals of a BL-algebra L such that $I \subseteq J$. If I is an implicative ideal, then so is J.*

Theorem 2.7. [9] *Let I be an ideal of a BL-algebra L. Then the following conditions are equivalent:*
(i) *I is an implicative ideal of L,*
(ii) *for any $a \in L$, the set $I_a := \{x \in L \mid x \rightharpoonup a \in I\}$ is an ideal of L.*

Definition 2.8. [3] *Let L_1 and L_2 be two BL-algebras. Then the map $f : L_1 \to L_2$ is called a BL-homomorphism if and only if it satisfies the following conditions, for every $x, y \in L_1$:*
(i) $f(0) = 0$,
(ii) $f(x \odot y) = f(x) \odot f(y)$,
(iii) $f(x \rightarrow y) = f(x) \rightarrow f(y)$.
If f is a bijective, then f is called BL-isomorphism. In this case we write $L_1 \cong L_2$.
Note that for every $x, y \in L_1$, $f(x) \rightharpoonup f(y) = f(x) \odot f(y)^- = f(x \odot y^-) = f(x \rightharpoonup y)$.

Definition 2.9. [3, 5] *Let L_1 and L_2 be two BL-algebras, μ a fuzzy subset of L_1, η a fuzzy subset of L_2 and $f : L_1 \to L_2$ a BL-homomorphism. The image of μ under f denoted by $f(\mu)$ is a fuzzy set of L_2 defined by:*

$$f(\mu)(y) = \sup_{x \in f^{-1}(y)} \mu(x) \text{ if } f^{-1}(y) \neq \emptyset \text{ and } f(\mu)(y) = 0 \text{ if } f^{-1}(y) = \emptyset, \text{ for all } y \in L_2.$$

The preimage of η under f denoted by $f^{-1}(\eta)$ is a fuzzy set of L_1 defined by, $f^{-1}(\eta)(x) = \eta(f(x))$, for all $x \in L_1$.

Definition 2.10. [5, 6] *Let L be a BL-algebra and $\mu : L \to [0, 1]$ be a fuzzy set of L. Then*
(i) *μ is called a fuzzy ideal of L if $\mu(x) \leq \mu(0)$ and $\mu((x^- \rightarrow y^-)^-) \wedge \mu(x) \leq \mu(y)$, for all $x, y \in L$.*
(ii) *A fuzzy ideal μ is called a fuzzy Boolean ideal of L, if $\mu(x \wedge x^-) = \mu(0)$, for all $x \in L$.*

Theorem 2.11. [5] *Let L be a BL-algebra and $\mu : L \to [0,1]$ be a fuzzy set of L. Then μ is a fuzzy ideal of L if and only if the following conditions hold:*
$(FI1)$: *For every $x, y \in L$, $\mu(x) \wedge \mu(y) \leq \mu(x \oslash y)$,*
$(FI2)$: *For every $x, y \in L$, if $x \leq y$, then $\mu(y) \leq \mu(x)$.*
It is easy to see that for any fuzzy ideal μ, $\mu(x^{--}) = \mu(x)$.

Lemma 2.12. [5, 9] *Let L be a BL-algebra and μ be a fuzzy ideal of L. Then the following properties hold:*
(i) *if $x \leq y$, then $\mu(y) \leq \mu(x)$, for all $x, y \in L$.*
(ii) $\mu(x \rightharpoonup (x \rightharpoonup y)) = \mu(y \rightharpoonup (y \rightharpoonup x))$.

Theorem 2.13. [5] *A fuzzy subset μ of a BL-algebra L is a fuzzy ideal if and only if the following conditions hold:*
(i) *For every x, $\mu(x) \leq \mu(0)$,*
(ii) *For every $x, y \in L$, $\mu(x \rightharpoonup y) \wedge \mu(y) \leq \mu(x)$.*

Lemma 2.14. [6] *Let $f : L_1 \to L_2$ be a BL-homomorphism, μ a fuzzy ideal of L_2 and η a fuzzy ideal of L_1. Then*
(i) *$f^{-1}(\mu)$ is a fuzzy ideal of L_1,*
(ii) *If f is a BL-isomorphism, then $f(\eta)$ is a fuzzy ideal of L_2.*

Note. From now on, in this paper, L will denote a BL-algebra, unless otherwise stated.

3 Fuzzy implicative ideals in BL-algebras

In this section, we introduce the notion of fuzzy implicative ideals in BL-algebras and investigate some of their properties.

Definition 3.1. *Let μ be a fuzzy set of L. Then μ is called a fuzzy implicative ideal if for all $x, y, z \in L$, it satisfies:*
(i) $\mu(x) \leq \mu(0)$,
(ii) $\mu((x \rightharpoonup y) \rightharpoonup z) \wedge \mu(y \rightharpoonup z) \leq \mu(x \rightharpoonup z)$.

The following example shows that fuzzy implicative ideals exist.

Example 3.2. [?] *Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Suppose $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\to$ are defined as the following tables:*

Table. 3

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table. 4

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra. Now, let the fuzzy set μ of L is defined by

$$\mu(1) = 0.5, \ \mu(a) = \mu(b) = \mu(0) = 0.7.$$

Thus μ is a fuzzy implicative ideal of L.

Example 3.3. [4] *Let $L = \{0, a, b, c, d, e, f, 1\}$ such that $0 < a < b < c < 1$, $0 < d < e < f < 1$, $a < e$ and $b < f$. Define $\odot$ and $\to$ as follows:*

Table. 3

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

Table. 4

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	d	1	1	1
f	0	b	c	c	d	f	1	1
1	0	a	b	c	d	e	f	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. Now, let the fuzzy set μ of L is defined by

$$\mu(d) = \mu(e) = \mu(f) = \mu(1) = 0.5,\ \mu(a) = \mu(b) = \mu(c) = 0.6,\ \mu(0) = 0.7.$$

Thus, μ is a fuzzy implicative ideal of L.

Theorem 3.4. *Let μ be a fuzzy implicative ideal of L. Then μ is a fuzzy ideal of L.*

Proof. Let μ be a fuzzy implicative ideal of L. Then $\mu(x) \le \mu(0)$, for any $x \in L$ and by Lemma 2.3(iii), for $x, y \in L$,

$$\mu(x \rightharpoonup y) \wedge \mu(y) = \mu((x \rightharpoonup y) \rightharpoonup 0) \wedge \mu(y \rightharpoonup 0) \le \mu(x \rightharpoonup 0) = \mu(x).$$

Thus, μ is a fuzzy ideal of L. □

Next example shows that the converse of Theorem 3.4 does not hold.

Example 3.5. *Let L be the BL-algebra in Example 3.2. Now, suppose the fuzzy set μ of L is defined by*

$$\mu(a) = \mu(b) = \mu(1) = 0.5,\ \mu(0) = 0.7.$$

Then μ is a fuzzy ideal. But, it is not a fuzzy implicative ideal of L, because $\mu((b \rightharpoonup a) \rightharpoonup a) \wedge \mu(a \rightharpoonup a) = 0.7$ and $\mu(b \rightharpoonup a) = 0.5$ and so $\mu((b \rightharpoonup a) \rightharpoonup a) \wedge \mu(a \rightharpoonup a) \nleq \mu(b \rightharpoonup a)$.

Theorem 3.6. *Let μ be a fuzzy set of L. Then μ is a fuzzy implicative ideal if and only if for each $t \in [0, 1]$, $\emptyset \neq \mu_t$ is an implicative ideal of L.*

Proof. Let μ be a fuzzy implicative ideal of L and $\emptyset \neq \mu_t$, for $t \in [0, 1]$. Then there exists $x \in L$ such that $x \in \mu_t$ and so $t \le \mu(x)$. Since $\mu(x) \le \mu(0)$, we get $t \le \mu(0)$ and so $0 \in \mu_t$. Now, let $(x \rightharpoonup y) \rightharpoonup z \in \mu_t$ and $y \rightharpoonup z \in \mu_t$, for $x, y, z \in L$. Then $t \le \mu((x \rightharpoonup y) \rightharpoonup z)$ and $t \le \mu(y \rightharpoonup z)$ and since μ is a fuzzy implicative ideal of L, we conclude

$$\mu((x \rightharpoonup y) \rightharpoonup z) \wedge \mu(y \rightharpoonup z) \le \mu(x \rightharpoonup z),$$

and so $t \le \mu(x \rightharpoonup z)$. Hence, $x \rightharpoonup z \in \mu_t$ and so $\emptyset \neq \mu_t$ is an implicative ideal of L.

Conversely, suppose μ is not a fuzzy implicative ideal of L. Then there exist $a, b, c \in L$ such that $\mu((a \rightharpoonup b) \rightharpoonup c) \wedge \mu(b \rightharpoonup c) > \mu(a \rightharpoonup c)$. Let

$$t_0 = 1/2\{\mu((a \rightharpoonup b) \rightharpoonup c) \wedge \mu(b \rightharpoonup c) + \mu(a \rightharpoonup c)\}.$$

Then

$$\mu(a \rightharpoonup c) < t_0 < \mu((a \rightharpoonup b) \rightharpoonup c) \wedge \mu(b \rightharpoonup c),$$

and so $t_0 < \mu((a \rightharpoonup b) \rightharpoonup c)$ and $t_0 < \mu(b \rightharpoonup c)$. Hence, $((a \rightharpoonup b) \rightharpoonup c) \in \mu_{t_0}$ and $b \rightharpoonup c \in \mu_{t_0}$ and since μ_{t_0} is an implicative ideal of L, we get $a \rightharpoonup c \in \mu_{t_0}$ and so $\mu(a \rightharpoonup c) \geq t_0$, which is a contradiction. Hence, for every $x, y, z \in L$,

$$\mu((x \rightharpoonup y) \rightharpoonup z) \wedge \mu(y \rightharpoonup z) \leq \mu(x \rightharpoonup z).$$

Therefore, μ is a fuzzy implicative ideal of L. $\square$

Proposition 3.7. *Let μ be a fuzzy implicative ideal of L. Then for any $a \in L$,*

$$\mu^a = \{x \in L \mid \mu(x \rightharpoonup a) = \mu(0)\},$$

is an ideal of L.

Proof. Let μ be a fuzzy implicative ideal of L and $a \in L$. Then by $(BL9)$, $\mu(0 \rightharpoonup a) = \mu(0 \odot a^-) = \mu(0)$. Hence, $0 \in \mu^a$. Let $x \rightharpoonup y \in \mu^a$ and $y \in \mu^a$, for $x, y \in L$. Then $\mu((x \rightharpoonup y) \rightharpoonup a) = \mu(0)$ and $\mu(y \rightharpoonup a) = \mu(0)$. Now, since μ is a fuzzy implicative ideal of L, we have

$$\mu(x \rightharpoonup a) \geq \mu((x \rightharpoonup y) \rightharpoonup a) \wedge \mu(y \rightharpoonup a) = \mu(0) \wedge \mu(0) = \mu(0).$$

Hence, $\mu(x \rightharpoonup a) = \mu(0)$ and so $x \in \mu^a$. Therefore, μ^a is an ideal of L. $\square$

Proposition 3.8. *Let μ be a fuzzy ideal of L. Then the following are equivalent:*
(i) μ is a fuzzy implicative ideal of L,
(ii) for each $t \in [0, 1]$ and for every $a \in L$, $\mu_{a,t} = \{x \in L \mid x \rightharpoonup a \in \mu_t\}$ is an ideal of L, when $\mu_t \neq \emptyset$.

Proof. $(i) \Rightarrow (ii)$: Let μ be a fuzzy implicative ideal of L. Then for each $t \in [0, 1]$, $\emptyset \neq \mu_t$ is an implicative ideal of L. Since $a \rightharpoonup a = 0 \in \mu_t$, we conclude $a \in \mu_{a,t}$ and so $\mu_{a,t} \neq \emptyset$, when $\mu_t \neq \emptyset$. Hence, by Theorem 2.7, for any $a \in L$, $\mu_{a,t} = \{x \in L \mid x \rightharpoonup a \in \mu_t\}$ is an ideal of L, when $\mu_t \neq \emptyset$.
$(ii) \Rightarrow (i)$: Let for each $t \in [0, 1]$ and for every $a \in L$, $\mu_{a,t} = \{x \in L \mid x \rightharpoonup a \in \mu_t\}$ be an ideal of L. Then $\mu_{a,t} \neq \emptyset$ and so $\mu_t \neq \emptyset$ and since μ is a fuzzy ideal of L, we get $\mu_t \neq \emptyset$ is an ideal of L, for each $t \in [0, 1]$. Hence, by Theorem 2.7, μ_t is an implicative ideal of L, for each $t \in [0, 1]$. Therefore, by Theorem 3.6, μ is a fuzzy implicative ideal of L. $\square$

Proposition 3.9. *Let μ be a fuzzy implicative ideal of L. Then for each $t \in [0, 1]$ and for every $a \in L$, $\mu_{a,t}$ is the least ideal of L containing a and μ_t.*

Proof. Let μ be a fuzzy implicative ideal of L. Then by Proposition 3.8, for each $t \in [0, 1]$ and for every $a \in L$, $\mu_{a,t}$ is an ideal of L, when $\mu_t \neq \emptyset$. By $(BL7)$, $x \rightharpoonup a = x \odot a^- \leq x$ and since μ is a fuzzy ideal of L, by Lemma 2.12 (i), $\mu(x) \leq \mu(x \rightharpoonup a)$. Now, if $x \in \mu_t$, then $t \leq \mu(x)$, and so $t \leq \mu(x \rightharpoonup a)$. Hence, $x \rightharpoonup a \in \mu_t$, and so $x \in \mu_{a,t}$. Therefore, $\mu_t \subseteq \mu_{a,t}$. Moreover, since μ_t is an ideal of L and $a \rightharpoonup a = 0 \in \mu_t$, we obtain $a \in \mu_{a,t}$. Now, let J be an ideal of L containing a and μ_t, for $t \in [0, 1]$ and $a \in L$, $z \in \mu_{a,t}$. Then $z \rightharpoonup a \in \mu_t \subseteq J$ and since $a \in J$, by Lemma 2.5, we conclude $z \in J$. Therefore, $\mu_{a,t} \subseteq J$ and so $\mu_{a,t}$ is the least ideal of L containing a and μ_t. $\square$

Corollary 3.10. *If μ is a fuzzy implicative ideal of L, then for each $t \in [0, 1]$ and for every $a \in L$, $\mu_{a,t}$ is an implicative ideal of L.*

Proof. Let μ be a fuzzy implicative ideal of L. Then by Theorem 3.6, for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is an implicative ideal of L, and since by Proposition 3.9, $\mu_t \subseteq \mu_{a,t}$, for each $t \in [0,1]$ and for every $a \in L$, by Proposition 2.6, we get $\mu_{a,t}$ is an implicative ideal of L. □

Proposition 3.11. *(Extension property for fuzzy implicative ideals) Let μ and η be two fuzzy ideals of L such that $\mu \subseteq \eta$ and μ be a fuzzy implicative ideal. Then η is a fuzzy implicative ideal.*

Proof. Let μ be a fuzzy implicative ideal of L. Then by Theorem 3.6, for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is an implicative ideal of L and since $\mu \subseteq \eta$, we get $\mu_t \subseteq \eta_t$. Now, by Proposition 2.6, for each $t \in [0,1]$, $\emptyset \neq \eta_t$ is an implicative ideal of L and so by Theorem 3.6, η is a fuzzy implicative ideal of L. □

Proposition 3.12. *Let μ and ν be two fuzzy ideals of L. Then the followings hold for any $a, b \in L$ and for each $t \in [0,1]$:*
(i) $\mu_t = \mu_{a,t}$ if and only if $a \in \mu_t$.
(ii) $a \leq b$ implies $\mu_{a,t} \subseteq \mu_{b,t}$.
(iii) $\mu \subseteq \nu$ implies $\mu_{a,t} \subseteq \nu_{a,t}$.
(iv) $(\mu \cup \nu)_{a,t} = \mu_{a,t} \cup \nu_{a,t}$.
(v) $(\mu \cap \nu)_{a,t} = \mu_{a,t} \cap \nu_{a,t}$.
(vi) If μ is a fuzzy implicative ideal, then $\mu_{a \oslash b,t} = (\mu_{a,t})_b = (\mu_{b,t})_a$.

Proof. (i) Let $a \in \mu_t$, for $a \in L$. Since μ is a fuzzy ideal of L, we get $\emptyset \neq \mu_t$ is an ideal of L, for each $t \in [0,1]$, so by Proposition 3.9, $\mu_t \subseteq \mu_{a,t}$. Now, let $x \in \mu_{a,t}$. Then $x \rightharpoonup a \in \mu_t$ and since $a \in \mu_t$, by Lemma 2.5, we get $x \in \mu_t$. Hence, $\mu_{a,t} \subseteq \mu_t$ and so $\mu_t = \mu_{a,t}$. Conversely, let $\mu_t = \mu_{a,t}$. Then by Proposition 3.9, $a \in \mu_{a,t}$ and so $a \in \mu_t$.
(ii) Let $a \leq b$ and $x \in \mu_{a,t}$, for $x \in L$. Then $x \rightharpoonup a \in \mu_t$. By Lemma 2.3$(i)$, $x \rightharpoonup b \leq x \rightharpoonup a$ and since μ_t is an ideal of L, we have $x \rightharpoonup b \in \mu_t$ and so $x \in \mu_{b,t}$. Thus, $\mu_{a,t} \subseteq \mu_{b,t}$.
(iii) Let μ and ν be two fuzzy ideals of L such that $\mu \subseteq \nu$ and $x \in \mu_{a,t}$, for $x \in L$. Then $x \rightharpoonup a \in \mu_t$ and since $\mu \subseteq \nu$, we have $\mu_t \subseteq \nu_t$, for each $t \in [0,1]$. Hence, $x \rightharpoonup a \in \nu_t$ and so $x \in \nu_{a,t}$. Therefore, $\mu_{a,t} \subseteq \nu_{a,t}$.
(iv) Since $\mu, \nu \subseteq \mu \cup \nu$, by (iii) we get $\mu_{a,t}, \nu_{a,t} \subseteq (\mu \cup \nu)_{a,t}$ and so $\mu_{a,t} \cup \nu_{a,t} \subseteq (\mu \cup \nu)_{a,t}$. Now, let $x \in (\mu \cup \nu)_{a,t}$, for $x \in L$. Then $x \rightharpoonup a \in (\mu \cup \nu)_t$ and so $(\mu \cup \nu)(x \rightharpoonup a) \geq t$. Hence, $\max\{\mu(x \rightharpoonup a), \nu(x \rightharpoonup a)\} \geq t$ and so $\mu(x \rightharpoonup a) \geq t$ or $\nu(x \rightharpoonup a) \geq t$. Thus, $x \rightharpoonup a \in \mu_t$ or $x \rightharpoonup a \in \nu_t$ and so $x \in \mu_{a,t}$ or $x \in \nu_{a,t}$. Therefore, $x \in \mu_{a,t} \cup \nu_{a,t}$ and so $(\mu \cup \nu)_{a,t} \subseteq \mu_{a,t} \cup \nu_{a,t}$.
(v) Since $\mu \cap \nu \subseteq \mu, \nu$, by (iii) we get $(\mu \cap \nu)_{a,t} \subseteq \mu_{a,t}, \nu_{a,t}$ and so $(\mu \cap \nu)_{a,t} \subseteq \mu_{a,t} \cap \nu_{a,t}$. Now, let $x \in \mu_{a,t} \cap \nu_{a,t}$, for $x \in L$. Then $x \rightharpoonup a \in \mu_t$ and $x \rightharpoonup a \in \nu_t$ and so $\mu(x \rightharpoonup a) \geq t$ and $\nu(x \rightharpoonup a) \geq t$. Hence, $(\mu \cap \nu)(x \rightharpoonup a) = \min\{\mu(x \rightharpoonup a), \nu(x \rightharpoonup a)\} \geq t$ and so $x \rightharpoonup a \in (\mu \cap \nu)_t$. Therefore, $x \in (\mu \cap \nu)_{a,t}$ and so $\mu_{a,t} \cap \nu_{a,t} \subseteq (\mu \cap \nu)_{a,t}$.
(vi) Let μ be a fuzzy implicative ideal of L. Then by Corollary 3.10, for each $t \in [0,1]$ and for every $a \in L$, $\mu_{a,t}$ is an implicative ideal of L. Now, by Lemma 2.3(ii), $x \in \mu_{a \oslash b,t}$ if and only if $x \rightharpoonup (a \oslash b) \in \mu_t$ if and only if $(x \rightharpoonup b) \rightharpoonup a \in \mu_t$ if and only if $x \rightharpoonup b \in \mu_{a,t}$ if and only if $x \in (\mu_{a,t})_b$, for every $x \in L$. Hence, $\mu_{a \oslash b,t} = (\mu_{a,t})_b$. By similar way, $\mu_{a \oslash b,t} = (\mu_{b,t})_a$. Therefore,

$$\mu_{a \oslash b,t} = (\mu_{a,t})_b = (\mu_{b,t})_a.$$

□

Theorem 3.13. *Let μ be a fuzzy set of L. Then the following conditions are equivalent:*
(i) μ is a fuzzy implicative ideal of L,
(ii) μ is a fuzzy ideal and for any $x, y \in L$, $\mu((x \rightharpoonup y) \rightharpoonup y) \leq \mu(x \rightharpoonup y)$,

(iii) μ *is a fuzzy ideal and for any* $x, y, z \in L$, $\mu((x \rightharpoonup y) \rightharpoonup z) \leq \mu((x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z))$,
(iv) $\mu(x) \leq \mu(0)$ *and for any* $x, y, z \in L$, $\mu(((x \rightharpoonup y) \rightharpoonup y) \rightharpoonup z) \wedge \mu(z) \leq \mu(x \rightharpoonup y)$.
(v) $\mu(x) \leq \mu(0)$ *and for any* $x, y, z \in L$, $\mu((x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z) \wedge \mu(z) \leq \mu(x)$,
(vi) μ *is a fuzzy ideal and for any* $x, y \in L$, $\mu(x \rightharpoonup (y \rightharpoonup x)) \leq \mu(x)$.

Proof. $(i) \Rightarrow (ii)$: Let μ be a fuzzy implicative ideal of L and $x, y \in L$. Then by Theorem 3.4, μ is a fuzzy ideal of L. By Lemma 2.3(iii)

$$\begin{aligned} \mu((x \rightharpoonup y) \rightharpoonup y) &= \mu((x \rightharpoonup y) \rightharpoonup y) \wedge \mu(0) \\ &= \mu((x \rightharpoonup y) \rightharpoonup y) \wedge \mu(y \rightharpoonup y) \\ &\leq \mu(x \rightharpoonup y). \end{aligned}$$

$(ii) \Rightarrow (iii)$: By Lemma 2.3(i), (ii) and (iv), for any $x, y, z \in L$,

$$((x \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z) \rightharpoonup z = ((x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z \leq (x \rightharpoonup y) \rightharpoonup z.$$

Now, since μ is a fuzzy ideal, by (ii) and Lemma 2.12(i) we get

$$\begin{aligned} \mu((x \rightharpoonup y) \rightharpoonup z) &\leq \mu(((x \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z) \rightharpoonup z) \\ &\leq \mu((x \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z) \\ &= \mu((x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)). \end{aligned}$$

$(iii) \Rightarrow (iv)$: Since μ is a fuzzy ideal of L, we have $\mu(x) \leq \mu(0)$, for any $x \in L$. By (iii) and Theorem 2.13, for any $x, y, z \in L$,

$$\begin{aligned} \mu(((x \rightharpoonup y) \rightharpoonup y) \rightharpoonup z) \wedge \mu(z) &\leq \mu((x \rightharpoonup y) \rightharpoonup y) \\ &\leq \mu((x \rightharpoonup y) \rightharpoonup (y \rightharpoonup y)) \\ &= \mu((x \rightharpoonup y) \rightharpoonup 0) \\ &= \mu(x \rightharpoonup y). \end{aligned}$$

$(iv) \Rightarrow (i)$: Firstly, we prove μ is a fuzzy ideal of L. Let $y = 0, z = y$, in (iv). Then

$$\mu(((x \rightharpoonup 0) \rightharpoonup 0) \rightharpoonup y) \wedge \mu(y) \leq \mu(x \rightharpoonup 0),$$

and so for any $x, y \in L$,

$$\mu(x \rightharpoonup y) \wedge \mu(y) \leq \mu(x).$$

Therefore, by Theorem 2.13, μ is a fuzzy ideal of L. Now, by Lemma 2.3(ii) and (iv), for any $x, y, z \in L$,

$$(x \rightharpoonup z) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \leq (x \rightharpoonup z) \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup z.$$

By Lemma 2.12(i), we get

$$\mu((x \rightharpoonup y) \rightharpoonup z) \leq \mu(((x \rightharpoonup z) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)),$$

and so by (iv),

$$\mu((x \rightharpoonup y) \rightharpoonup z) \wedge \mu(y \rightharpoonup z) \leq \mu((x \rightharpoonup z) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)) \wedge \mu(y \rightharpoonup z) \leq \mu(x \rightharpoonup z).$$

Therefore, μ is a fuzzy implicative ideal of L.
$(i) \Rightarrow (v)$: Let μ be a fuzzy implicative ideal of L. Then $\mu(x) \leq \mu(0)$, for any $x \in L$ and for $x, y, z \in L$, $\mu((x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z) \wedge \mu(z) \leq \mu(x \rightharpoonup (y \rightharpoonup x))$ and since

$$\mu((x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z) \wedge \mu(z) \leq \mu(z),$$

we get

$$\mu((x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z) \wedge \mu(z) \leq \mu(x \rightharpoonup (y \rightharpoonup x)) \wedge \mu(z).$$

By Lemma 2.3(ii), (iii) and (iv), we have

$$\begin{aligned} ((y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x)) &= ((y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x))) \rightharpoonup (y \rightharpoonup x) \\ &\leq (y \rightharpoonup x) \rightharpoonup (y \rightharpoonup x) \\ &= 0. \end{aligned}$$

Hence, by By Lemma 2.3 (vi), we obtain $((y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (y \rightharpoonup x)) \leq ((x \rightharpoonup (y \rightharpoonup x)))^{--}$. So by Lemma 2.12 ($i$) and Theorem 2.11, we conclude

$$\mu(x \rightharpoonup (y \rightharpoonup x)) = \mu(((x \rightharpoonup (y \rightharpoonup x)))^{--}) \leq \mu((y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (y \rightharpoonup x)).$$

Since μ is a fuzzy implicative ideal of L, by Lemma 2.12 (ii), we get

$$\mu((y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (y \rightharpoonup x)) \leq \mu(y \rightharpoonup (y \rightharpoonup x)) = \mu(x \rightharpoonup (x \rightharpoonup y)).$$

Now, by Lemma 2.3(ii),

$$((x \rightharpoonup y) \rightharpoonup z) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x))) = ((x \rightharpoonup y) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x))) \rightharpoonup z,$$

and by Lemma 2.3(i) and (v), $y \rightharpoonup x \leq y$ implies $x \rightharpoonup y \leq x \rightharpoonup (y \rightharpoonup x)$, thus $(x \rightharpoonup y) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x)) = 0$, and so

$$((x \rightharpoonup y) \rightharpoonup z) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x))) = ((x \rightharpoonup y) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x))) \rightharpoonup z = 0.$$

Hence, by Lemma 2.3(vii), $((x \rightharpoonup y) \rightharpoonup z) \leq (x \rightharpoonup (y \rightharpoonup x)))^{--}$ and so by Lemma 2.12(i),

$$\mu(x \rightharpoonup (y \rightharpoonup x)) \leq \mu((x \rightharpoonup y) \rightharpoonup z).$$

Also, from μ is a fuzzy ideal of L, we have

$$\mu(x \rightharpoonup (y \rightharpoonup x)) \wedge \mu(z) \leq \mu((x \rightharpoonup y) \rightharpoonup z) \wedge \mu(z) \leq \mu(x \rightharpoonup y).$$

Thus,

$$\mu((x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z) \wedge \mu(z) \leq \mu(x \rightharpoonup (y \rightharpoonup x)) \wedge \mu(z) \leq \mu(x \rightharpoonup y).$$

Moreover,

$$\mu((x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z) \wedge \mu(z) \leq \mu(x \rightharpoonup (y \rightharpoonup x)) \leq \mu(x \rightharpoonup (x \rightharpoonup y)).$$

Therefore,

$$\mu((x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z) \wedge \mu(z) \leq \mu(x \rightharpoonup (x \rightharpoonup y)) \wedge \mu(x \rightharpoonup y) \leq \mu(x).$$

(v) $\Rightarrow$ (vi): For $x, y \in L$, by (ii) and Lemma 2.3(iii),

$$\mu(x \rightharpoonup y) \wedge \mu(y) = \mu((x \rightharpoonup (x \rightharpoonup x)) \rightharpoonup y) \wedge \mu(y) \leq \mu(x).$$

Hence, μ is a fuzzy ideal of L. Now, by (ii), for $x, y \in L$, we have

$$\mu(x \rightharpoonup (y \rightharpoonup x)) = \mu((x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup 0) \wedge \mu(0) \leq \mu(x).$$

Therefore, $\mu(x \rightharpoonup (y \rightharpoonup x)) \leq \mu(x)$.
$(vi) \Rightarrow (i)$: Assume (iii) holds. For $x, y \in L$, by Lemma 2.3(ii), we have

$$\begin{aligned}((x \rightharpoonup y) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y)) \rightharpoonup ((x \rightharpoonup y) \rightharpoonup y) &= ((x \rightharpoonup y) \rightharpoonup ((x \rightharpoonup y) \rightharpoonup y)) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y)) \\ &\leq (x \rightharpoonup (x \rightharpoonup y)) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y)) \\ &= 0.\end{aligned}$$

Hence, by Lemma 2.3(vii), we conclude $((x \rightharpoonup y) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y)) \leq (((x \rightharpoonup y) \rightharpoonup y))^{--}$. Now, by (iii), Lemma 2.12(i) and Theorem 2.11, we conclude

$$\mu((x \rightharpoonup y) \rightharpoonup y) = \mu(((x \rightharpoonup y) \rightharpoonup y)^{--}) \leq \mu((x \rightharpoonup y) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y)) \leq \mu(x \rightharpoonup y).$$

Therefore, by Theorem 3.13, μ is a fuzzy implicative ideal of L. □

Theorem 3.14. *Let μ be a fuzzy ideal of L. Then μ is a fuzzy implicative ideal of L if and only if for any $x \in L$, $\mu((x \oslash x) \rightharpoonup x) = \mu(0)$.*

Proof. Let μ be a fuzzy implicative ideal of L and $x \in L$. Then by Lemma 2.3(ii) and Theorem 3.13, we have

$$\mu(0) = \mu((x \oslash x) \rightharpoonup (x \oslash x)) = \mu(((x \oslash x) \rightharpoonup x) \rightharpoonup x) \leq \mu((x \oslash x) \rightharpoonup x) \leq \mu(0).$$

Therefore, $\mu((x \oslash x) \rightharpoonup x) = \mu(0)$.

Conversely, let $x \in L$ and $\mu((x \oslash x) \rightharpoonup x) = \mu(0)$. Then by Lemma 2.3$(ii)$ and Theorem 2.11, for every $x, y \in L$,

$$\begin{aligned}\mu((x \rightharpoonup y) \rightharpoonup y) &= \mu(0) \wedge \mu((x \rightharpoonup y) \rightharpoonup y) \\ &= \mu((y \oslash y) \rightharpoonup y) \wedge \mu(x \rightharpoonup (y \oslash y)) \\ &\leq \mu(((y \oslash y) \rightharpoonup y) \oslash (x \rightharpoonup (y \oslash y))).\end{aligned}$$

Now, by Lemma 2.3(ix), $x \rightharpoonup y \leq ((y \oslash y) \rightharpoonup y) \oslash (x \rightharpoonup (y \oslash y))$ and so by Lemma 2.12(i), $\mu(((y \oslash y) \rightharpoonup y) \oslash (x \rightharpoonup (y \oslash y))) \leq \mu(x \rightharpoonup y)$. Hence,

$$\mu((x \rightharpoonup y) \rightharpoonup y) \leq \mu(x \rightharpoonup y).$$

Therefore, by Theorem 3.13, μ is a fuzzy implicative ideal of L. □

Corollary 3.15. *Let μ be a fuzzy ideal of L. Then μ is a fuzzy implicative ideal of L if and only if μ is a fuzzy Boolean ideal of L.*

Proof. Since for every $x \in L$, we have $(x \oslash x) \rightharpoonup x = (x^- \to x) \odot x^- = x^- \wedge x = x \wedge x^-$, by Theorem 3.14, the proof is clear. □

By the following example, we give a fuzzy implicative ideal of infinite BL-algebra, as an application of the Corollary 3.15.

Example 3.16. [7] *Let $L = [0, 1]$, $x \odot y = \min\{x, y\}$ and*

$$x \to y = \begin{cases} 1, & if\ x \leq y, \\ y, & if\ y < x, \end{cases}$$

Then $([0, 1], \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra. Now, suppose the fuzzy set μ of L is defined by

$$\mu(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

It is easy to check μ is a fuzzy ideal on infinite BL-algebra L. Moreover, since $x \wedge x^- = 0$, for all $x \in L$, we conclude $\mu(x \wedge x^-) = \mu(0)$, for all $x \in L$. Hence, μ is a fuzzy Boolean ideal of L and so by Corollary 3.15, μ is a fuzzy implicative ideal of infinite BL-algebra L.

Theorem 3.17. *Let a fuzzy set μ of L is defined by*

$$\mu(x) = \begin{cases} 0, & x \neq 0, \\ \alpha, & x = 0, \end{cases}$$

where $\alpha \in (0, 1]$. Then the following conditions are equivalent:
(i) L is a Boolean algebra,
(ii) any fuzzy ideal of L is a fuzzy Boolean(implicative) ideal of L,
(iii) fuzzy ideal μ is a fuzzy Boolean(implicative) ideal of L.

Proof. $(i) \Rightarrow (ii)$: Let L be a Boolean algebra. Then for any $x \in L$, $x \wedge x^- = 0$. Hence, for any fuzzy ideal μ of L, $\mu(x \wedge x^-) = \mu(0)$. Thus, any fuzzy ideal of L is a fuzzy Boolean(implicative) ideal of L.
$(ii) \Rightarrow (iii)$: It is clear μ is a fuzzy ideal of L. Therefore, by (ii), μ is a fuzzy Boolean ideal of L.
$(iii) \Rightarrow (i)$: Let μ be a fuzzy Boolean(implicative) ideal of L. Then $\mu_\alpha = \mu_{\mu(0)} = \{x \in L \mid \mu(x) \geq \mu(0)\} = \{0\}$ is a Boolean ideal and for any $x \in L$, $x \wedge x^- \in \{0\}$. Hence, $x \wedge x^- = 0$, for any $x \in L$, and so L is a Boolean algebra. □

Proposition 3.18. *Let $f : L_1 \to L_2$ be a BL-homomorphism and μ be a fuzzy implicative ideal of L_2. Then $f^{-1}(\mu)$ is a fuzzy implicative ideal of L_1.*

Proof. Let μ be a fuzzy implicative ideal of L_2 and $x, y \in L_1$. Then by Lemma 2.14(i), $f^{-1}(\mu)$ is a fuzzy ideal of L_1. Now, let $x, y \in L_1$. Then by Definition 2.9, we have

$$\begin{aligned} f^{-1}(\mu)((x \rightharpoonup y) \rightharpoonup y)) &= \mu(f((x \rightharpoonup y) \rightharpoonup y)) \\ &= \mu((f(x) \rightharpoonup f(y))) \rightharpoonup f(y)) \\ &\leq \mu(f(x) \rightharpoonup f(y)) \\ &= \mu(f(x \rightharpoonup y)) \\ &= f^{-1}(\mu)(x \rightharpoonup y). \end{aligned}$$

Therefore, by Theorem 3.13, $f(\mu)$ is a fuzzy implicative ideal of L_2. □

Proposition 3.19. *Let $f : L_1 \to L_2$ be a BL-isomorphism and μ be a fuzzy implicative ideal of L_1. Then $f(\mu)$ is a fuzzy implicative ideal of L_2.*

Proof. Let μ be a fuzzy implicative ideal of L_1. Since f is a BL-isomorphism, by Lemma 2.14(ii), we conclude $f(\mu)$ is a fuzzy ideal of L_2. Suppose $y_1, y_2 \in L_2$. Since f is a BL-isomorphism, we get $f(x_1) = y_1$ and $f(x_2) = y_2$, for some $x_1, x_2 \in L_1$ and so by Definition 2.9,

$$f(\mu)((y_1 \rightharpoonup y_2) \rightharpoonup y_2) = \sup\{\mu(t) \mid t \in f^{-1}((y_1 \rightharpoonup y_2) \rightharpoonup y_2)\}.$$

Now, if $t \in f^{-1}((y_1 \rightharpoonup y_2) \rightharpoonup y_2)$, then

$$f(t) = (y_1 \rightharpoonup y_2) \rightharpoonup y_2 = (f(x_1) \rightharpoonup f(x_2)) \rightharpoonup f(x_2) = f((x_1 \rightharpoonup x_2) \rightharpoonup x_2).$$

Since f is a BL-isomorphism, we get $t = (x_1 \rightharpoonup x_2) \rightharpoonup x_2$. Hence

$$f(\mu)((y_1 \rightharpoonup y_2) \rightharpoonup y_2) = \sup\{\mu(t) \mid t = (x_1 \rightharpoonup x_2) \rightharpoonup x_2\} = \mu((x_1 \rightharpoonup x_2) \rightharpoonup x_2).$$

By similarly, $f(\mu)(y_1 \rightharpoonup y_2) = \mu(x_1 \rightharpoonup x_2)$ and so

$$f(\mu)((y_1 \rightharpoonup y_2) \rightharpoonup y_2) = \mu((x_1 \rightharpoonup x_2) \rightharpoonup x_2) \leq \mu(x_1 \rightharpoonup x_2) = f(\mu)(y_1 \rightharpoonup y_2).$$

Therefore, $f(\mu)$ is a fuzzy implicative ideal of L_2. □

4 Conclusions

The results of this paper are devoted to study fuzzy implicative and fuzzy Boolean ideal on BL-algebras. Using level subset of a fuzzy set in a BL-algebra, characterization of them are studied. Also, it is proved that the concepts of fuzzy implicative ideal and fuzzy Boolean ideal in BL-algebras are coincide. Moreover, it is shown that a BL-algebra L is a Boolean algebra if and only if any fuzzy ideal of L is a fuzzy implicative ideal. Finally, it is proved that the homomorphic image and preimage of fuzzy implicative ideals are fuzzy implicative ideal.

For future work, we will study the congruence relation that is made by a fuzzy implicative ideal of BL-algebras and the quotient structure induced by them.

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