Ideals in pseudo-hoop algebras

F. Xie¹ and H. Liu²

¹,²School of Mathematics and Statistics, Shandong Normal University, 250014, Jinan, P. R. China
850938132@qq.com, lhxshanda@163.com

Abstract

Pseudo-hoop algebras are non-commutative generalizations of hoop-algebras, originally introduced by Bosbach. In this paper, we study ideals in pseudo-hoop algebras. We define congruences induced by ideals and construct the quotient structure. We show that there is a one-to-one correspondence between the set of all normal ideals of a pseudo-hoop algebra \( A \) with condition (pDN) and the set of all congruences on \( A \). Also, we prove that if \( A \) is a good pseudo-hoop algebra with pre-linear condition, then a normal ideal \( P \) of \( A \) is prime if and only if \( A/P \) is a pseudo-hoop chain. Furthermore, we analyse the relationship between ideals and filters of \( A \).

Article Information

Corresponding Author: H. Liu;
Received: July 2020;
Revised: September 2020;
Accepted: September 2020,
Paper type: Original.

Keywords:
Pseudo-hoop algebra, ideal, congruence, filter.

1 Introduction

Hoop algebras were presented by Bosbach in [4, 5]. Then Büchi and Owens investigated this algebraic structure in an unpublished paper. Pseudo-hoop algebras were presented as non-commutative generalizations of hoop algebras by Georgescu, Leuştean and Preoteasa in [13], following after the notions of pseudo-MV algebras in [12] and pseudo-BL algebras ([10]). Pseudo-hoop algebras are weaker structures. Pseudo-MV algebras and pseudo-BL algebras are particular cases of pseudo-hoop algebras. In recent years, the study of hoop algebras and pseudo-hoop algebras has made great progress. And the main focus has been on filters in [2, 6, 9, 15].

Ideal theory plays a fundamental role in many algebraic structures, such as lattices, rings and pseudo-MV algebras. Georgescu and Iorgulescu in [12] introduced the notion of ideals in pseudo-MV algebras, which was shown effective in studying structure properties of pseudo-MV algebras. In addition, Dvurečenskij in [11] studied states on pseudo-MV algebras by exploiting ideals. In recent years, the notion of ideals has been introduced as a dual notion of filters in some algebraic structures using multiplication operations. Lele and Nganou in [14] presented the notion of ideals.
in BL-algebras and defined quotient algebraic structures by ideals. Using ideals, they proved that an ideal of a BL-algebra is prime if and only if the quotient algebraic structure is a linear MV-algebra. Also, Rachůnek and Šalounová in [16] introduced ideals of general residuated lattices. It was proved that a congruence can be defined by an ideal in some cases, and the corresponding quotient structure is involutive. In [1], Kologani and Borzooei introduced the notions of ideals, implicative (maximal, prime) ideals of hoop algebras and studied the relationships between these ideals.

In (pseudo-) MV-algebras, filters and ideals are dual. However, in pseudo-hoop algebras, we mainly study filters. As pseudo-hoop algebras may not have lattice structures, not all pseudo-hoop algebras are general residuated lattices. Since pseudo-MV algebras are particular cases of general residuated lattices, the notion of ideals in pseudo-hoop algebras can not be similar to that in pseudo-MV algebras. Therefore, we want to introduce the notion of ideals in pseudo-hoop algebras, as a dual notion of filters in \([\text{pseudo-MV algebras}].\) Another inspiration is the notion of ideals in hoop algebras defined in [2]. Since pseudo-hoop algebras are non-commutative generalizations of hoop algebras, we shall generalize the notion of ideals in hoop algebras to the case of pseudo-hoop algebras. Also, by Theorem 1.5 and Theorem 6.3, it is noticeable that ideals and filters behave differently in pseudo-hoop algebras. Therefore, it is meaningful to investigate ideals in pseudo-hoop algebras.

The paper is constructed as follows. In Section 2, we recall some definitions and results on pseudo-hoop algebras which are useful. In Section 3, we define the notions of left, right and both-sided ideals of pseudo-hoop algebras. In Section 4, we analyse congruences induced by ideals and construct the quotient pseudo-hoop algebras via ideals. In addition, we get an isomorphism theorem. In Section 5, we introduce the notion of prime ideals in pseudo-hoop algebras and give some equivalent conditions of prime ideals. In Section 6, we analyse the relationship between ideals and filters. Also, we introduce the notion of \(\circ\)-prime ideals in pseudo-hoop algebras. The relationship between \(\circ\)-prime ideals and maximal filters is discussed.

2 Preliminaries

In this section, we recall some definitions and results to be used in this paper.

**Definition 2.1.** [13] A pseudo-hoop algebra is an algebra \((A, \circ, \rightarrow, \rightsquigarrow, 1)\) of the type \((2, 2, 2, 0)\) that for all \(u, v, w \in A\), it is satisfying in the following conditions:

1. (ph1) \(u \circ 1 = 1 \circ u = u\);
2. (ph2) \(u \rightarrow u = u \rightsquigarrow u = 1\);
3. (ph3) \((u \circ v) \rightarrow w = u \rightarrow (v \rightarrow w)\);
4. (ph4) \((u \circ v) \rightsquigarrow w = v \rightsquigarrow (u \rightsquigarrow w)\);
5. (ph5) \((u \rightarrow v) \circ u = (v \rightarrow u) \circ v = u \circ (u \rightsquigarrow v) = v \circ (v \rightsquigarrow u)\).

We define \(u^0 = 1\) and \(u^n = u^{n-1} \circ u\) for any \(n \in \mathbb{N}_+\) on \(A\). The relation \(\leq\) defined by \(u \leq v \iff u \rightarrow v = 1 \iff u \rightsquigarrow v = 1\) is a partial order on \(A\). If \(\circ\) is commutative or equivalently \(\rightarrow = \rightsquigarrow\), \(A\) is called to be a hoop algebra. Also, \(A\) is called bounded if \(u \geq 0\) for any \(u \in A\). In this case, we define \(u^0 = v \rightarrow 0\) and \(u^\sim = u \rightsquigarrow 0\) on \(A\). If \(u^\sim\) = \(u^\sim\) for all \(u \in A\), then the bounded pseudo-hoop algebra is called good (see [8]). In a bounded pseudo-hoop algebra \(A\), if \(u^\sim = u^\sim\) for all \(u \in A\), then \(A\) is called satisfying the \((pDN)\) condition (see [8]). A good pseudo-hoop algebra \(A\) is called normal if it satisfies \((u \circ v)^\sim = u^\sim \circ v^\sim\) for all \(u, v \in A\).

We summarize some properties of pseudo-hoop algebras that we will use later. For more details, see [8] and [13].
Proposition 2.2. \[13\] Let \((A, \odot, \rightarrow, \rightsquigarrow, 1)\) be a pseudo-hoop algebra. Then for all \(u, v, w \in A\), the following conditions hold:

1. \(u \odot v \leq w\) iff \(u \leq v \rightarrow w\) iff \(v \leq u \rightsquigarrow w\);
2. \((A, \odot, 1)\) is a monoid;
3. if \(u \leq v\), then \(u \odot w \leq v \odot w\) and \(w \odot u \leq w \odot v\);
4. \(u \odot v = (u \rightarrow v) \odot u = (v \rightarrow u) \odot v = u \odot (u \rightsquigarrow v) \odot (v \rightsquigarrow u)\);
5. if \(u \leq v\), then \(v \rightarrow w \leq u \rightarrow w\) and \(v \rightsquigarrow w \leq u \rightsquigarrow w\);
6. if \(u \leq v\), then \(w \rightarrow u \leq w \rightarrow v\) and \(w \rightsquigarrow u \leq w \rightsquigarrow v\);
7. \((v \rightarrow w) \odot (u \rightarrow v) \leq u \rightarrow w\), \((u \rightsquigarrow v) \odot (v \rightsquigarrow w) \leq u \rightsquigarrow w\).

Proposition 2.3. \[8\] Let \(A\) be a bounded pseudo-hoop algebra. Then for all \(u, v, w \in A\) the following statements hold:

1. \(u \odot 0 = 0 = 0 \odot u\);
2. \(u \odot \rightsquigarrow u = 0\);
3. \(u \odot v = 0\) iff \(u \leq v \rightsquigarrow v \leq u \rightsquigarrow v\);
4. \(u \leq u \rightsquigarrow v\), \(u \leq u \rightsquigarrow v\);
5. \(u \rightsquigarrow v = u \rightsquigarrow u = u \rightsquigarrow v\);
6. if \(A\) is good, then \((u \rightarrow v) \rightsquigarrow v = u \rightsquigarrow v \rightarrow v \rightsquigarrow v \equiv (u \rightsquigarrow v) \rightsquigarrow v \equiv u \rightsquigarrow v \rightsquigarrow v \equiv v \rightsquigarrow v\);
7. if \(A\) is good, then \(v \rightarrow u = u \rightarrow u \rightarrow u \rightarrow u \rightarrow u \rightarrow v \rightarrow v \rightarrow v\).

A pseudo-hoop algebra \(A\) is said to satisfy the pre-linear condition if we have \((x \rightarrow y) \odot (y \rightarrow x) = (x \rightsquigarrow y) \odot (y \rightsquigarrow x) = 1\) for any \(x, y \in A\). By \[8\], Proposition 3.4, \((A, \odot, \rightarrow, \rightsquigarrow, 0, 1)\) is a bounded pseudo-hoop algebra with pre-linear condition if and only if \((A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)\) is a pseudo-\(\mathcal{BL}\) algebra.

A filter \(F\) of a pseudo-hoop algebra \(A\) is a nonempty subset of \(A\) which satisfies \((F1): u, v \in F\) implies \(u \odot v \in F\) and \((F2): \) for any \(u, v \in A\), if \(u \leq v\) and \(u \in F\), then \(v \in F\) (see \[13\]). In a pseudo-hoop algebra, \(A\), filters are coincided with deductive systems. A filter \(F\) of \(A\) satisfying \(F \neq A\) is called proper. If \(F\) is a proper filter of \(A\) and there is no proper filter containing \(F\), \(F\) is called maximal. A filter \(F\) of \(A\) is normal if \(u \rightarrow v \in F\) iff \(u \rightsquigarrow v \in F\) for any \(u, v \in A\). Let \(X\) be a subset of \(A\). We use \([X]\) to denote the filter of \(A\) generated by \(X\).

Proposition 2.4. \[13\] Let \(A\) be a pseudo-hoop algebra, \(W\) a normal filter of \(A\) and \(u \in A\). Then

\[
(W \cup \{u\}) = \{a \in A | w \odot u^n \leq a, \text{ for some } n \in \mathbb{N}, w \in W\} = \{a \in A | w^n \odot u \leq a, \text{ for some } n \in \mathbb{N}, w \in W\}.
\]

Let \(A_1\) and \(A_2\) be pseudo-hoop algebras. In \[8\], a map \(f : A_1 \rightarrow A_2\) is called a pseudo-hoop homomorphism if \(f\) preserves the operations \(\odot, \rightarrow\) and \(\rightsquigarrow\). The pseudo-hoop homomorphism \(f : A_1 \rightarrow A_2\) is called a bounded pseudo-hoop homomorphism if \(A_1, A_2\) are bounded and \(f(0) = 0\).

3 Ideals

In this section, we shall introduce two kinds of binary operations (left and right additions) and the notion of ideals in pseudo-hoop algebras. We give some equivalent characterizations of ideals of good pseudo-hoop algebras.

Definition 3.1. Let \((A, \odot, \rightarrow, \rightsquigarrow, 1)\) be a bounded pseudo-hoop algebra. We define left addition \(\odot\) and right addition \(\odot\) as follows: for any \(x, y \in A\),

\[
x \odot y = y \rightsquigarrow x \quad \text{and} \quad x \odot y = x \rightarrow y.
\]
Example 3.2. Let $A = \{0, a, b, c, d, 1\}$. Define the operations $\rightarrow$, $\rightsquigarrow$ and $\odot$ on $A$ as follows:

<table>
<thead>
<tr>
<th>$\rightarrow \rightsquigarrow$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>c</td>
<td>1</td>
<td>b</td>
<td>c</td>
<td>b</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>d</td>
<td>a</td>
<td>1</td>
<td>b</td>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>a</td>
<td>1</td>
<td>1</td>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>d</td>
<td>b</td>
<td>1</td>
<td>1</td>
<td>b</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\odot$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>d</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>d</td>
<td>c</td>
<td>c</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>c</td>
<td>c</td>
<td>0</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>d</td>
<td>0</td>
<td>d</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>1</td>
</tr>
</tbody>
</table>

Then $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ is a bounded hoop algebra. It is easy to see that $b \odot c = c^{-} \rightsquigarrow b = a \rightsquigarrow b = b$ and $c \odot a = c^{-} \rightarrow a = a \rightarrow a = 1$.

Proposition 3.3. Let $A$ be a pseudo-hoop algebra. For all $x, y, m, n \in A$, if $x \leq y$ and $m \leq n$, then $x \odot m \leq y \odot n$ and $x \odot m \leq y \odot n$.

Proof. If $x \leq y$ and $m \leq n$, then $y^{-} \leq x^{-}$, $n^{-} \leq m^{-}$. By Proposition (5) and (6), we have $x \odot m = m^{-} \rightsquigarrow x \leq n^{-} \rightsquigarrow y = y \odot n$. Similarly, we have $x \odot m \leq y \odot n$.

Proposition 3.4. Let $A$ be a pseudo-hoop algebra. If $A$ is normal, then left addition $\odot$ and right addition $\odot$ are associative.

Proof. For all $x, y, z \in A$, we obtain

\[
\begin{align*}
x \odot (y \odot z) &= x^{-} \rightarrow (y^{-} \rightarrow z) \\
&= (x^{-} \odot y^{-}) \rightarrow z \\
&= (x^{-} \odot y^{-})^{-} \rightarrow z \\
&= (x^{-} \rightarrow y^{-})^{-} \rightarrow z \\
&= (x^{-} \rightarrow y^{-})^{-} \rightarrow z \\
&= (x^{-} \rightarrow y^{-})^{-} \rightarrow z \\
&= (x^{-} \rightarrow y^{-})^{-} \rightarrow z \\
&= (x^{-} \rightarrow y^{-})^{-} \rightarrow z \\
&= (x^{-} \rightarrow y^{-})^{-} \rightarrow z \\
&= (x \odot y) \odot z.
\end{align*}
\]

Similarly, we can prove $(x \odot y) \odot z = x \odot (y \odot z)$.

Definition 3.5. Let $I$ be a nonempty subset of a bounded pseudo-hoop algebra $A$. Then $I$ is called a left ideal of $A$ if it satisfies:

(LI1) $x, y \in I$ implies $x \odot y \in I$;
(LI2) for any $x, y \in A$, $x \leq y$ and $y \in I$ imply $x \in I$.

Similarly, $I$ is called a right ideal of $A$ if it satisfies:

(RI1) $x, y \in I$ implies $x \odot y \in I$;
(RI2) for any $x, y \in A$, $x \leq y$ and $y \in I$ imply $x \in I$.

If $I$ is both a left ideal and a right ideal of $A$, we call $I$ to be an ideal of $A$.

For any ideal $I$ of $A$, we have $0 \in I$. For all $x \in A$, we have $x \in I$ iff $x^{-} \in I$ iff $x^{-} \in I$. An ideal $I$ of $A$ is called proper if $I \neq A$. An ideal $I$ of $A$ is called normal if $x^{-} \odot y \in I$ iff $y \odot x^{-} \in I$ for all $x, y \in A$. The intersection of any family of ideals of a bounded pseudo-hoop algebra $A$ is also an ideal of $A$. For any subset $H \subseteq A$, the smallest ideal of $A$ containing $H$ is said to be the ideal generated by $H$, and it is denoted by $\langle H \rangle$. 

42 F. Xie, H. Liu
Example 3.6. Let $A$ be a pseudo-hoop algebra as in Example 2. Then $I_{1} = \{0\}, I_{2} = \{0, c\}, I_{3} = \{0, a, d\}$ and $I_{4} = A$ are all ideals of $A$.

Example 3.7. Let $u$ be an element of an arbitrary $\ell$-group $G = (G, +, -, 0, \lor, \land)$ and $u \geq 0$. Define the operations $\to$, $\sim$ and $\circ$ on $G[u] = [0, u]$ as follows:

$$x \circ y = (x - u + y) \lor 0, \quad x \to y = (y - x + u) \land u, \quad \text{and} \quad x \sim y = (u - x + y) \land u.$$ 

By Proposition 5.1, $G[u]$ is a bounded pseudo-hoop algebra. Let $W$ be a normal convex $\ell$-subgroup of $G$ and $F = \{x \in G[u] : u - x \in W\}$. We define $I_{0} = \{x \in G[u] : x \notin F\}$ and $I'_{0} = \{x \in G[u] : x^- \notin F\}$. Then $I_{0}$ and $I'_{0}$ are ideals of $G[u]$.

We shall show that $I_{0}$ is an ideal of $G[u]$. Let $x, y \in G[u]$. Then $x \to 0 = (0 - x + u) \land u = -x + u, x \sim 0 = (u - x + 0) \land u = u - x,$

$$x \circ y = x^- \to y = (y - (u - x) + u) \land u = (y + x - u + u) \land u = (y + x) \land u,$$

and $x \circ y = y^- \sim x = (u - (y + u) + x) \land u = (y + x) \land u$. Also, we have $x \circ y = x \circ y$.

By Proposition 5.2, $F$ is a normal filter of $G[u]$. Suppose $x, y \in G[u]$ such that $x \leq y$ and $y \in I_{0}$. Then $y^- \leq x^-$ and $y^- \in F$. Using (F2), we obtain $x^- \in F$, i.e. $x \in I_{0}$. Suppose $x, y \in I_{0}$, i.e. $x^-, y^- \in F$. We have $x^- \circ y^- \in F$, by (F1). Since

$$x^- \circ y^- = (x^- - u + y^-) \lor 0 = [(-x + u) - u + (-y + u)] \lor 0 = (-x - y + u) \lor 0,$$

and

$$(x \circ y)^- = ((y + x) \land u)^- = -((y + x) \land u) + u = (-x - y + u) \lor (-u + u) = (-x - y + u) \lor 0,$$

we obtain $(x \circ y)^- = (x \circ y)^- = x^- \circ y^- \in F$. Hence, $x \circ y, x \circ y \in I_{0}$. Thus, $I_{0}$ is an ideal of $G[u]$.

Similarly, we can show that $I'_{0}$ is an ideal of $G[u]$.

Theorem 3.8. Let $I$ be a nonempty subset of a good pseudo-hoop algebra $A$ containing 0. The following conditions are equivalent:

1. $I$ is an ideal of $A$;
2. for any $x, y \in A$, $x^- \circ y \in I$ and $x \in I$ imply $y \in I$;
3. for any $x, y \in A$, $y \circ x^- \in I$ and $x \in I$ imply $y \in I$.

Proof. $(1) \Rightarrow (2)$ Suppose $I$ is an ideal of $A$. If $x, y \in A$ such that $x, x^- \circ y \in I$, then $(x^- \circ y) \circ x \in I$. Since $x^- \circ y \leq x^- \circ y$, we obtain $y \leq x^- \Rightarrow (x^- \circ y) = (x^- \circ y) \circ x$ by Proposition 2.2(1). Using (2), we have $y \in I$.

$(2) \Rightarrow (1)$ Let $x, y \in A$ such that $y \in I$ and $x \leq y$. Then $y^- \leq x^-$. Thus, $y^- \circ x \leq x^- \circ x = 0$. So $y^- \circ x = 0 \in I$. By (2), we obtain $x \in I$. Therefore, condition (12) holds. Let $x, y \in I$. Since $y^- \circ (x \circ y) = y^- \circ (y^- \sim x) \leq x \in I$, we have $y^- \circ (x \circ y) \in I$. Therefore, $x \circ y \in I$. In addition, we have $x \in I$ and $x^- \circ x^- = 0 \in I$. It follows that $x^- \in I$. Since $x^- = x^-$, we have $y^- \circ (x \circ y) = y^- \circ (x^- \sim y) \leq x^- \in I$ by Proposition 2.2(7). Using (12), we obtain $y^- \circ (x \circ y) \in I$. Thus, $x \circ y \in I$. Therefore, $I$ is an ideal of $A$.

This proves that $(1) \iff (2)$. Similarly, we can prove that $(2) \iff (3)$. \qed

Remark 3.9. Let $I$ be a nonempty subset of a bounded pseudo-hoop algebra $A$ containing 0, where $A$ does not have to be good. By the previous proof, if $I$ is an ideal of $A$, then conditions (2) and (3) hold. Also, $I$ is a left (right) ideal of $A$ if and only if condition (2) ((3)) holds.
Theorem 3.10. Let $I$ be a nonempty subset of a good pseudo-hoop algebra $A$ containing $0$. The following conditions are equivalent:
(1) $I$ is an ideal of $A$;
(2) for $x, y \in A$, $(x^- \to y^-)^\sim \in I$ and $x \in I$ imply $y \in I$;
(3) for $x, y \in A$, $(x^- \leadsto y^-)^\sim \in I$ and $x \in I$ imply $y \in I$.

Proof. $(1) \Rightarrow (2)$ Suppose $I$ is an ideal of $A$. Let $x, y \in A$ such that $(x^- \to y^-)^\sim \in I$ and $x \in I$. Then $x^- \circ y^- \leq (x^- \circ y^-)^\sim = (x^- \to y^-)^\sim = (x^- \to y^-)^\sim \in I$. Using $(2)$, we obtain $x^- \circ y^- \in I$. Thus $y^- \in I$ by Theorem 3.8. Since $y \leq y^\sim$, we obtain $y \in I$.

$(2) \Rightarrow (1)$ Suppose that the condition $(2)$ holds. Let $x \in I$. Then $(x^- \to x^-)^\sim = (x^- \to x^-)^\sim = 0 \in I$. It follows that $x^\sim \in I$ by $(2)$. Hence, we show that $x \in I$ implies $x^\sim \in I$. Let $x^\circ y, x \in I$. Then $(x^\circ y)^\sim \in I$, and so $(x^- \to y^-)^\sim \in I$. Thus, $y \in I$ by $(2)$. Therefore, $I$ is an ideal of $A$ by Theorem 3.8.

This proves that $(1) \Leftrightarrow (2)$. Similarly, we can prove that $(1) \Leftrightarrow (3)$. \qed

Proposition 3.11. Let $H$ be a subset of a bounded pseudo-hoop algebra $A$.
(1) If $H$ is empty, then $\langle H \rangle = \{0\}$.
(2) If $H$ is not empty and $A$ is normal, then
\[
\langle H \rangle = \{h \in A : h \leq x_1 \circ x_2 \circ x_3 \circ \ldots \circ x_n, \text{ for some } x_1, x_2, \ldots, x_n \in H\}
\]
\[
= \{h \in A : h \leq x_1 \circ x_2 \circ x_3 \circ \ldots \circ x_n, \text{ for some } x_1, x_2, \ldots, x_n \in H\}.
\]

Proof. $(1)$ It is obvious.

$(2)$ If $A$ is normal, $\circ$ and $\circ$ are associative. Let
\[
B = \{h \in A : h \leq x_1 \circ x_2 \circ x_3 \circ \ldots \circ x_n, \text{ for some } x_1, x_2, \ldots, x_n \in H\}.
\]

Let $a, b \in A$ such that $a \in B$ and $a^- \circ b \in B$. We obtain $a \leq x_1 \circ x_2 \circ x_3 \circ \ldots \circ x_n$ and $a^- \circ b \leq y_1 \circ y_2 \circ y_3 \circ \ldots \circ y_m$, for some $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in H$. Since $b \leq a^- \leadsto (a^- \circ b) = (a^- \circ b) \circ a \leq y_1 \circ y_2 \circ y_3 \circ \ldots \circ y_m \circ x_1 \circ x_2 \circ x_3 \circ \ldots \circ x_n,$

we have $b \in B$. By the notion of normal pseudo-hoop algebras, we know that $A$ is good. Thus, $B$ is an ideal of $A$ by Theorem 3.8.

Suppose $D$ is an ideal of $A$ containing $H$. For any $b \in B$, we have $b \leq x_1 \circ x_2 \circ x_3 \circ \ldots \circ x_n$ for some $x_1, x_2, \ldots, x_n \in H$. Since $H \subseteq D$, we obtain $x_1 \circ x_2 \circ x_3 \circ \ldots \circ x_n \in D$. Then $b \in D$.

Thus, $D \subseteq B$. Therefore, $B = \langle H \rangle$.

Similarly, $\langle H \rangle = \{h \in A : h \leq x_1 \circ x_2 \circ x_3 \circ \ldots \circ x_n, \text{ for some } x_1, x_2, \ldots, x_n \in H\}$. \qed

4 Ideals and congruences

In this section, we define congruences on pseudo-hoop algebras induced by ideals. We construct the quotient pseudo-hoop algebras via ideals and prove that there is a one-to-one correspondence between the set of all normal ideals of a pseudo-hoop algebra $A$ with condition (pDN) and the set of all congruences relation on $A$. Also, we obtain an isomorphism theorem.

Definition 4.1. Let $(A, \circ, \to, \leadsto)$ be a pseudo-hoop algebra and $\sim$ an equivalence relation on $A$.

The equivalence relation $\sim$ is called a left congruence relation if $x \sim y$ implies $(a \circ x) \sim (a \circ y)$, $(a \to x) \sim (a \to y)$ and $(a \leadsto x) \sim (a \leadsto y)$ for any $x, y, a \in A$. 


The equivalence relation $\sim$ is called a right congruence relation if $x \sim y$ implies $(x \circ a) \sim (y \circ a)$, $(x \rightarrow a) \sim (y \rightarrow a)$ and $(x \leftarrow a) \sim (y \leftarrow a)$ for any $x, y, a \in A$.

The equivalence relation $\sim$ is called a congruence relation if $x_1 \sim y_1$ and $x_2 \sim y_2$ imply $(x_1 \circ x_2) \sim (y_1 \circ y_2)$, $(x_1 \rightarrow x_2) \sim (y_1 \rightarrow y_2)$ and $(x_1 \leftarrow x_2) \sim (y_1 \leftarrow y_2)$.

Example 4.2. Let $A$ be a hoop algebra of Example 3.2. It is easy to check that

$$\rho = \{(0,0), (0,a), (0,d), (a,0), (a,a), (a,d), (d,0), (d,d), (b,b), (b,c), (b,1), (c,b), (c,c), (c,1), (1,b), (1,c), (1,1)\}$$

is a congruence relation on $A$.

Proposition 4.3. A relation on a pseudo-hoop algebra $(A, \odot, \rightarrow, \sim)$ is a congruence relation if and only if it is both a left and a right congruence relation.

Proof. The proof is obvious.\hfill \Box

If $I$ is an ideal of a bounded pseudo-hoop algebra $A$, then define $\sim_I$ on $A$ as follows:

$$\forall x, y \in A, \ x \sim_I y \text{ iff } x^- \odot y \in I, y^- \odot x \in I, x \odot y^- \in I, y \odot x^- \in I.$$  

Proposition 4.4. Let $A$ be a bounded pseudo-hoop algebra and $I$ an ideal of $A$. Then $\sim_I$ is an equivalence relation on $A$.

Proof. It is clear that $\sim_I$ is symmetric. And we know that $\sim_I$ is reflexive by Proposition 4.2(2). We only need to show that $\sim_I$ is transitive. If $x \sim_I y$ and $y \sim_I z$, then

$$(z^- \odot y)^- \odot (z^- \odot x) = ((z^- \rightarrow y^-) \odot z^-) \odot x \leq y^- \odot x \in I.$$ 

So $(z^- \odot y) \sim (z^- \odot x) \in I$. Since $z^- \odot y \in I$, we get $z^- \odot x \in I$ by Theorem 4.8 and Remark 4.9. Similarly, $x^- \odot z \in I$.

Since $(x \odot z^-) \odot (y \odot z^-) = x \odot (z^- \odot (z^- \sim y^-)) \leq x \odot y^- \in I$, we get $x \odot z^- \in I$. Similarly, $z \odot x^- \in I$. Therefore, $x \sim_I z$. \hfill \Box

Theorem 4.5. Let $A$ be a good pseudo-hoop algebra and $I$ a normal ideal of $A$. Then $\sim_I$ is a congruence relation on $A$.

Proof. Let $x, y \in A$. By Propositions 4.8 and 4.9, we only need to show that $x \sim_I y$ implies $(x \odot a) \sim_I (y \odot a)$, $(a \odot x) \sim_I (a \odot y)$, $(x \rightarrow a) \sim_I (y \rightarrow a)$, $(a \rightarrow x) \sim_I (a \rightarrow y)$, $(x \leftarrow a) \sim_I (y \leftarrow a)$ and $(a \leftarrow x) \sim_I (a \leftarrow y)$ for any $a \in A$.

Suppose $x \sim_I y$. Then $x^- \odot y \in I$, $y^- \odot x \in I$, $x \odot y^- \in I$ and $y \odot x^- \in I$. Since

$$(x \odot a) \odot (y \odot a)^- = x \odot (a \odot (a \sim y^-)) \leq x \odot y^- \in I,$$

we obtain $(x \odot a) \odot (y \odot a) \in I$. Since $I$ is normal, we have $(y \odot a) \in I$. Similarly, we have $(y \odot a) \odot (x \odot a) \in I$ and $(x \odot a) \odot (y \odot a) \in I$. So $(x \odot a) \sim_I (y \odot a)$.

Similarly, $x \sim_I y$ implies $(a \odot x) \sim_I (a \odot y)$ for any $a \in A$. Moreover, by

$$(x^- \odot y^-) \odot (x^- \odot y^-) = ((x^- \rightarrow y^-) \odot x^-) \odot y^- \leq y^- \odot y^- = 0 \in I,$$

we obtain $(x^- \odot y^-) \odot (x^- \odot y^-) \in I$. Thus, $x^- \odot y^- \in I$ by Theorem 4.8. Similarly, we have $y^- \odot x^- \in I$. Since $I$ is normal, we obtain $y^- \odot x^- \in I$ and $x^- \odot y^- \in I$. Hence, $x^- \sim_I y^-$. Similarly, $x \sim_I y$ implies $x^- \sim_I y^-$.\hfill \Box
If $x \sim_I y$, for any $a \in A$, then $(x \odot a^-)^- \sim_I (y \odot a^-)^-$, and so $(x \rightarrow a^-)^- \sim_I (y \rightarrow a^-)^-$. Since $A$ is good, we obtain $(x \rightarrow a^-) \sim_I (y \rightarrow a^-)$. For any $b \in A$, we have $b^- \odot b^- = 0 \in I$ and $b \odot b^- = b^- \sim_I b$. Thus, $x^- \sim_I x \sim_I y \sim_I y^-$. Then $(x^- \rightarrow a^-) \sim_I (y^- \rightarrow a^-)$ for any $a \in A$. By Proposition 4.6, we have $(x \rightarrow a^-) \sim_I (y \rightarrow a^-)$. Hence, $(x \rightarrow a) \sim_I (y \rightarrow a)$. Similarly, we can show $(x \sim a) \sim_I (y \sim a)$ for any $a \in A$.

If $x \sim_I y$, for any $a \in A$, then $(a \odot x^-)^- \sim_I (a \odot y^-)^-$, and so $(a \rightarrow x^-)^- \sim_I (a \rightarrow y^-)^-$. Since $a \sim_I a^-\sim_I a^-\sim_I a^-$, we obtain $(a \rightarrow x^-) \sim_I (a^- \rightarrow x^-)$ and $(a \rightarrow y^-) \sim_I (a^- \rightarrow y^-)$ by the above proof. Thus, $(a^- \rightarrow x^-) \sim_I (a^- \rightarrow y^-)$ by transitivity. Hence $(a \rightarrow x)^- \sim_I (a \rightarrow y)^-$ by Proposition 4.6. Therefore, $(a \rightarrow x) \sim_I (a \rightarrow y)$. Analogously, we have $(a \sim x) \sim_I (a \sim y)$. □

Let $A$ be a good pseudo-hoop algebra and $I$ a normal ideal of $A$. We define $A/I = \{ [a]: a \in A \}$ where $[a] = \{ x \in A : x \sim_I a \}$. For any $x, y \in A$, we define the operations $\odot, \rightarrow$ and $\sim$ on $A/I$ by:

$$[x] \odot [y] = [x \odot y], [x] \rightarrow [y] = [x \rightarrow y] \text{ and } [x] \sim [y] = [x \sim y].$$

It is easy to know that $(A/I, \odot, \rightarrow, \sim, [1])$ is a bounded pseudo-hoop algebra with condition (pDN).

**Proposition 4.6.** Let $A$ be a good pseudo-hoop algebra.

1. If $\sim$ is a congruence relation on $A$, then $B = \{ x \in A : x \sim 0 \}$ is a normal ideal of $A$. Also, $\sim_B$ is a congruence relation on $A$. If $A$ satisfies the condition (pDN), then $\sim_B$ coincides with $\sim$.

2. If $I$ is a normal ideal of $A$, then $\sim_I$ is a congruence relation on $A$. Also, $[0] = \{ x \in A : x \sim_I 0 \}$ is a normal ideal of $A$ and coincides with $I$.

3. If $A$ satisfies the condition (pDN), then there is a one-to-one correspondence between the set of congruence relations on $A$ and the set of normal ideals of $A$.

**Proof.**

1. By reflexivity, we have $0 \in B$. So $B \neq \emptyset$. Let $x, y \in B$. Then $(y^- \sim x) \sim (0^- \sim x)$, i.e. $(x \odot y) \sim x$. Since $x \sim 0$, we obtain $x \odot y \in B$. Similarly, $x \odot y \in B$. Suppose $x, y \in A$ such that $x \leq y$ and $y \in B$. Then $(x \odot y^-) \sim (x \odot 0^-) = x$. Since $x \leq y \leq y^-$, we have $x \odot y^- = 0$ by Proposition 4.6. Thus, $x \sim 0$. Hence, $B$ is an ideal of $A$.

Suppose $x, y \in A$ such that $x^- \odot y \in B$. Then $y \sim x^- = (x^- \odot y)^- \sim (y \odot 1)$, and so $(y \wedge x^-) \sim y$. Therefore, $(y \odot x^-) \sim ((y \wedge x^-) \odot x^-)$. Since $A$ is good, we obtain

$$(y \wedge x^-) \odot x^- = (x^- \rightarrow y) \odot x^- \sim x^- \sim (x^- \rightarrow y) \odot (x^- \rightarrow x^-) = 0.$$  

Then $y \odot x^- \in B$. Similarly, $y \odot x^- \in B$ implies $x^- \odot y \in B$. Therefore, $B$ is normal.

By Theorem 1.8, $\sim_B$ is a congruence on $A$. Suppose $A$ satisfies condition (pDN). If $x \sim y$, we have $(x^- \odot y^-) \sim (y^- \odot y^-) = 0$, $(y^- \odot x^- \sim (x^- \odot x^-) = 0$, $(y \odot x^-) \sim (y \odot y^-) = 0$ and $(x \odot y^-) \sim (x \odot x^-) = 0$. So $x \sim_B y$. Conversely, if $x \sim_B y$, then $(y \odot x^-) \sim 0$. Thus $((y \odot x^-) \odot y) \sim (0^- \odot y)$, and so $(y \wedge x^-) \sim y$. Using condition (pDN), we have $(y \wedge x) \sim y$. Similarly, $(y \wedge x) \sim x$. Hence, $x \sim y$. Therefore $\sim_B$ coincides with $\sim$.

2. By Theorem 1.8, $\sim_I$ is a congruence relation on $A$. Then $[0]$ is a normal ideal of $A$ by (1). So we only need to show that $[0]$ coincides with $I$. For any $x \in I$, we have $x^- \odot 0 = 0 \in I$, $0^- \odot x = x \in I$, $x \odot x^- = e \in I$ and $0 \odot x^- = 0 \in I$. So $x \sim_I 0$, i.e. $x \in [0]$. Therefore, $I \subseteq [0]$. Conversely, if $x \in [0]$, then $x \odot 0^- \in I$. Thus $x = x \odot 0^- \in I$. Hence, $I = [0]$.

3. It is obvious by (1) and (2). □

**Proposition 4.7.** Let $X, Y$ be two bounded pseudo-hoop algebras and $f : X \rightarrow Y$ a bounded pseudo-hoop homomorphism. We have the following results:
(1) If $I$ is an (normal) ideal of $Y$, then $f^{-1}(I)$ is an (normal) ideal of $X$.
(2) If $f : X \to Y$ is a bounded pseudo-hoop isomorphism and $J$ is an (normal) ideal of $X$, then $f(J)$ is a (normal) ideal of $Y$.

Proof. (1) Let $I$ be an ideal of $Y$. Since $0 \notin f^{-1}(I)$, we have $f^{-1}(I) \neq \emptyset$. Let $x, y \in X$ such that $x \leq y$ and $y \in f^{-1}(I)$. Then $f(y) \in I$ and $f(x) \to f(y) = f(x) \to y = f(1) = 1$, i.e. $f(x) \leq f(y)$. Using (I2), we have $f(x) \in I$, i.e. $x \in f^{-1}(I)$. Suppose $x, y \in f^{-1}(I)$. Since $f(x \otimes y) = f(x) \otimes f(y)$ and $f(x), f(y) \in I$, we obtain $f(x \otimes y) \in I$, i.e. $x \otimes y \in f^{-1}(I)$. Similarly, $x \otimes y \in f^{-1}(I)$. Hence, $f^{-1}(I)$ is an ideal of $X$.

Let $I$ be a normal ideal of $Y$. Then $x^{-} \otimes y \in f^{-1}(I)$ iff $f(x^{-} \otimes y) \in I$ iff $f(y) \circ f(x^{-}) \in I$ iff $y \circ x^{-} \in f^{-1}(I)$ for any $x, y \in X$. Therefore, $f^{-1}(I)$ is a normal ideal of $X$.

(2) Let $J$ be an ideal of $X$. Suppose $x, y \in Y$ such that $x \leq y$ and $y \in f(J)$. Then there is $v \in J$ such that $f(v) = y$. Since $f$ is surjective, there is $u \in X$ such that $f(u) = x$. Since $f(u \to v) = x \to y = f(1)$ and $f$ is injective, we have $u \to v = 1$, i.e. $u \leq v \in J$. Thus, $u \in J$. So $x \in f(J)$. Let $x, y \in f(J)$. Then there exist $u, v \in J$ such that $f(u) = x$ and $f(v) = y$. Since $u \otimes v, u \otimes v \in J$, we have $f(u) \circ f(v) = f(u \otimes v) \in f(J)$ and $f(u) \otimes f(v) = f(u \otimes v) \in f(J)$. Therefore, $f(J)$ is an ideal of $Y$.

Let $J$ be a normal ideal of $X$. Then $f(u^{-} \circ f(v) \in f(J)$ iff $u^{-} \circ v \in J$ iff $u \circ u^{-} \in J$ iff $f(v) \circ f(u^{-}) \in f(J)$ for any $u, v \in X$. Thus, $f(J)$ is a normal ideal of $Y$. □

Let $f : X \to Y$ be a bounded pseudo-hoop homomorphism. Denote $\{x \in X : f(x) = 0\} = f^{-1}(0)$ by $\ker f$. Then $\ker f$ is an ideal of $X$.

**Proposition 4.8.** Let $X, Y$ be two bounded pseudo-hoop algebras and $f : X \to Y$ a bounded pseudo-hoop homomorphism. If $Y$ is good, then $\{0\}$ is a normal ideal of $Y$ and $\ker f$ is a normal ideal of $X$.

Proof. It is clear that $\{0\}$ is an ideal of $Y$. Since $Y$ is good, we obtain $x^{-} \otimes y = 0$ iff $y \leq x^{-}$ iff $y \leq x^{-}$ iff $y \circ x^{-} = 0$ for any $x, y \in Y$. Therefore, $\{0\}$ is normal. Hence, $\ker f$ is a normal ideal of $X$ by Proposition 4.7(1). □

Let $W$ be a nonempty subset of a bounded pseudo-hoop algebra $X$. We define

$$W^{-} = \{x^{-} : x \in W\} \quad \text{and} \quad W^{\sim} = \{x^{\sim} : x \in W\}.$$ 

Let $X, Y$ be two good pseudo-hoop algebras and $f : X \to Y$ a bounded pseudo-hoop homomorphism. Since $X$ is good and $\ker f$ is a normal ideal of $X$, we know that $X/\ker f$ is a bounded pseudo-hoop algebra. Then we have the following result.

**Proposition 4.9.** Let $X, Y$ be two good pseudo-hoop algebras and $f : X \to Y$ a bounded pseudo-hoop homomorphism. If $X$ is normal, then $X/\ker f \cong (Im f)^{-}$ and $X/\ker f \cong (Im f)^{\sim}$.

Proof. Define $\varphi : X/\ker f \to (Im f)^{-}$ by $\varphi([x]) = f(x)^{\sim} = f(x)^{-}$ for all $x \in X$. Then $\varphi([x]) \in (Im f)^{-}$. Since $X$ is normal, for any $x, y \in X$ we have

$$f(x)^{-} \circ f(y)^{-} = f(x^{\sim} \circ y^{\sim}) = f((x^{-} \circ y^{-})^{\sim}) \in (Im f)^{-}.$$ 

By Proposition 4.3(6), for any $x, y \in X$ we obtain

$$f(x)^{-} \to f(y)^{-} = f(x^{\sim} \to y^{\sim}) = f((x^{-} \to y^{-})^{\sim}) \in (Im f)^{-}.$$
Similarly, \( f(x)^{-} \sim f(y)^{-} \in (Imf)^{-} \). Thus, the operations \( \odot, \to \) and \( \sim \) are closed on \( (Imf)^{-} \). Also, \( 1 = f(0)^{-} \in (Imf)^{-} \) and \( 0 = f(1)^{-} \in (Imf)^{-} \). Therefore, \( (Imf)^{-} \) is a bounded pseudo-hoop algebra. It is clear that \( \varphi([0]) = 0 \). Since \( X \) is good, for any \( x, y \in X \) we have

\[
\varphi([x] \to [y]) = \varphi([x \to y]) = f((x \to y)^{-}) = f(x^{-} \to y^{-}) = \varphi([x]) \to \varphi([y]).
\]

Similarly, we have \( \varphi([x] \sim [y]) = \varphi([x]) \sim \varphi([y]) \). Since \( X \) is normal, we obtain

\[
\varphi([x] \odot [y]) = \varphi([x \odot y]) = f((x \odot y)^{-}) = f(x^{-} \odot y^{-}) = f(x^{-} \odot f(y)^{-}) = \varphi([x]) \odot \varphi([y]).
\]

Therefore, \( \varphi \) is a bounded pseudo-hoop homomorphism.

Since \( kerf \) is normal, we get \( [x] = [y] \) if \( x \sim_{kerf} y \) iff \( f(x^{-} \odot y) = f(y^{-} \odot x) = 0 \) iff \( f(x)^{-} \odot f(y) = f(y)^{-} \odot f(x) = 0 \) iff \( f(x)^{-} \leq f(y)^{-} \) and \( f(y)^{-} \leq f(x)^{-} \) iff \( f(x)^{-} = f(y)^{-} \) iff \( \varphi([x]) = \varphi([y]) \) for any \( x, y \in X \). Thus, \( \varphi \) is injective. Since \( f(a)^{-} = f(a^{-}) = \varphi([a^{-}]) \) for any \( a \in X \), we have \( \varphi \) is surjective. Hence, \( \varphi \) is isomorphic. Therefore, \( X/kerf \cong (Imf)^{-} \). Similarly, \( X/kerf \cong (Imf)^{-} \).

\[\square\]

5 Prime ideals

In this section, we introduce the concept of prime ideals in pseudo-hoop algebras and obtain several equivalent conditions of prime ideals.

**Definition 5.1.** Let \((A, \odot, \to, \sim, 1)\) be a bounded pseudo-hoop algebra and \(P\) an ideal of \(A\). Then \(P\) is called a prime ideal if \(P \neq A\) and \(x \wedge y \in P\) implies \(x \in P\) or \(y \in P\) for any \(x, y \in A\).

**Example 5.2.** Let \((A, \odot, \to, \sim, 1)\) be a bounded hoop algebra as in Example 3.2. Then \(I_2 = \{0, c\}\) and \(I_3 = \{0, a, d\}\) are all prime ideals of \(A\). Since \(a \wedge c = 0\) and \(a, c \notin \{0\}\), \(I_1 = \{0\}\) is not prime.

**Proposition 5.3.** Let \(X, Y\) be two bounded pseudo-hoop algebras and \(f : X \to Y\) be a bounded pseudo-hoop homomorphism. Then the following statements hold:

1. If \(I\) is a prime ideal of \(Y\) and \(f^{-1}(I) \neq X\), then \(f^{-1}(I)\) is a prime ideal of \(X\).
2. If \(f : X \to Y\) is a bounded pseudo-hoop isomorphism and \(J\) is a prime ideal of \(X\), then \(f(J)\) is a prime ideal of \(Y\).

**Proof.** (1) It is obvious that \(f^{-1}(I)\) is a proper ideal of \(X\). For any \(x, y \in X\), if \(x \wedge y \in f^{-1}(I)\), then \(f(x) \wedge f(y) = f(x \wedge y) \in I\). Since \(I\) is prime, we obtain \(f(x) \in I\) or \(f(y) \in I\). Thus, \(x \in f^{-1}(I)\) or \(y \in f^{-1}(I)\). Hence, \(f^{-1}(I)\) is prime.

(2) By Proposition 3.2(2), \(f(J)\) is an ideal of \(Y\). Since \(J \neq X\) and \(f\) is bijective, we have \(f(J) \neq Y\). Let \(x, y \in Y\) such that \(x \wedge y \in f(J)\). Since \(f\) is surjective, there exist \(u, v \in X\) such that \(f(u) = x\) and \(f(v) = y\). Then \(f(u \wedge v) = f(u) \wedge f(v) = x \wedge y \in f(J)\). Thus, \(u \wedge v \in J\). Since \(J\) is prime, we have \(u \in J\) or \(v \in J\). Hence, \(x \in f(J)\) or \(y \in f(J)\). Therefore, \(f(J)\) is prime. \(\square\)

**Theorem 5.4.** Let \(A\) be a bounded pseudo-hoop algebra with the pre-linear condition and \(P\) be an ideal of \(A\). Then the following conditions are equivalent:

1. \(P\) is prime;
2. If \(x \wedge y = 0\), then \(x \in P\) or \(y \in P\);
3. For any \(x, y \in A\), \((x \to y)^{-} \in P\) or \((y \to x)^{-} \in P\);
4. For any \(x, y \in A\), \((x \sim y)^{-} \in P\) or \((y \sim x)^{-} \in P\).
Proof. (1) ⇒ (2) It is obvious by (1).
    (2) ⇒ (3) Since \( A \) is a lattice, for any \( x, y \in A \) we have
    \[
    (x \rightarrow y)^\sim \wedge (y \rightarrow x)^\sim = ((x \rightarrow y) \vee (y \rightarrow x))^\sim = 1^\sim = 0.
    \]
    It follows that \( (x \rightarrow y)^\sim \in \mathcal{P} \) or \( (y \rightarrow x)^\sim \in \mathcal{P} \) by (2).
    (3) ⇒ (1) Suppose \( x \wedge y \in \mathcal{P} \) and \( (x \rightarrow y)^\sim \in \mathcal{P} \). We obtain \( (x \wedge y) \circ (x \rightarrow y)^\sim \in \mathcal{P} \) by (RI1). Since \( (x \wedge y)^\sim = ((x \rightarrow y) \circ x)^\sim = x \rightarrow (x \rightarrow y)^\sim \), we get
    \[
    x \leq (x \wedge y)^\sim \rightarrow (x \rightarrow y)^\sim = (x \wedge y) \circ (x \rightarrow y)^\sim \in \mathcal{P}.
    \]
    So \( x \in \mathcal{P} \). Similarly, if \( x \wedge y \in \mathcal{P} \) and \( (y \rightarrow x)^\sim \in \mathcal{P} \), then \( y \in \mathcal{P} \).
    (2) ⇒ (4) The proof is similar to (2) ⇒ (3).
    (4) ⇒ (1) The proof is similar to (3) ⇒ (1). \( \square \)

Corollary 5.5. Let \( A \) be a bounded pseudo-hoop algebra with the pre-linear condition. If \( \mathcal{P} \) is a prime ideal of \( A \), then every proper ideal of \( A \) containing \( \mathcal{P} \) is also prime.

Proof. By Theorem 5.4(3) or (4). \( \square \)

Corollary 5.6. Let \( A \) be a bounded pseudo-hoop algebra with the pre-linear condition. Then every proper ideal of \( A \) is prime if and only if the ideal \( \{0\} \) of \( A \) is prime.

Proposition 5.7. Let \( A \) be a good pseudo-hoop algebra and \( \mathcal{P} \) be a normal ideal of \( A \). If \( A \) satisfies the pre-linear condition, then \( \mathcal{P} \) is prime if and only if \( \mathcal{P} / \mathcal{P} \mathcal{P} \) is a pseudo-hoop chain.

Proof. It is enough to prove \( [x] \leq [y] \Leftrightarrow (x \rightarrow y)^\sim \in \mathcal{P} \) for \( x, y \in A \). Suppose \( [x] \leq [y] \), then \( [x \rightarrow y] = [1] \), i.e. \( (x \rightarrow y)^\sim \circ [1] = [1] \). Therefore, \( 1 \circ (x \rightarrow y)^\sim = (x \rightarrow y)^\sim \in \mathcal{P} \). Conversely, suppose \( (x \rightarrow y)^\sim \in \mathcal{P} \). We have \( 1 \circ (x \rightarrow y)^\sim = (x \rightarrow y)^\sim \in \mathcal{P} \) and \( (x \rightarrow y) \circ 1^\sim = 0 \in \mathcal{P} \). Since \( \mathcal{P} \) is normal, we obtain \( (x \rightarrow y)^\sim \circ [1] = [1] \), i.e. \( [x] \leq [y] \). So \( \mathcal{P} \) is prime if and only if \( (x \rightarrow y)^\sim \in \mathcal{P} \) or \( (y \rightarrow x)^\sim \in \mathcal{P} \) for any \( x, y \in A \) if and only if \( [x] \leq [y] \) or \( [y] \leq [x] \) for any \( [x], [y] \in \mathcal{P} \) if and only if \( \mathcal{P} / \mathcal{P} \mathcal{P} \) is a pseudo-hoop chain. \( \square \)

6  Ideals and filters

In this section, we shall investigate the relationship between ideals and filters in pseudo-hoop algebras. First, some results are obtained by using the set of complement elements of pseudo-hoop algebras. In addition, the notion of \( \circ \)-prime ideals in pseudo-hoop algebras is given and the relationship between \( \circ \)-prime ideals and maximal filters is discussed.

Definition 6.1. Let \( (A, \circ, \rightarrow, 
\end{proof}
Example 6.2. Let $A = \{0, a, b, c, d, e, f, 1\}$. Define $\to, \sim$ and $\ominus$ as follows:

$$
\begin{array}{cccccccc}
\to & = & \sim & \ominus \\
0 & = & 0 & 0 & 0 & 0 & 0 & 0 \\
n & = & d & 1 & 1 & 1 & d & 1 & 1 \\
b & = & d & f & 1 & 1 & d & f & 1 \\
c & = & c & c & c & c & 1 & 1 & 1 \\
d & = & 0 & c & c & d & 1 & 1 & 1 \\
e & = & f & 0 & b & c & d & f & 1 \\
f & = & 0 & b & c & d & f & 1 & 1 \\
1 & = & 0 & a & b & c & d & e & f
\end{array}
$$

Then $(A, \ominus, \to, \sim, 1)$ is a bounded hoop algebra. Let $F_1 = \{d, e, f, 1\}$ and $F_2 = \{c, 1\}$. Then $M(F_1) = N(F_1) = \{0, a, b, c\}$ and $M(F_2) = N(F_2) = \{0, d\}$.

It is easy to check that $F_1$ and $F_2$ are filters of $A$. Also, $J_1 = \{0, a, b, c\}$ is an ideal of $A$. Since $b \leq c \in F_1^0$ and $b \notin F_1^0$, $F_1^0 = F_1^0 = \{c, 0\}$ is not an ideal of $A$. Since $e \geq d \in J_1^0$ and $e \notin J_1^0$, $J_1^0 = J_1^0 = \{1, d\}$ is not a filter of $A$.

The above example shows that ideals and filters are not dual under complement. Then we have the following results.

Theorem 6.3. Let $F$ be a filter of a good pseudo-hoop algebra $A$. Then $M(F)$ is an ideal generated by $F^\sim$ and $N(F)$ is an ideal generated by $F^\sim$.

Proof. Suppose $x, y \in A$ such that $x^\sim \ominus y \in M(F)$ and $x \in M(F)$. Then $(x^\sim \ominus y)^\sim = x^\sim \to y^\sim \in F$ and $x^\sim \in F$. Since $F$ is a filter of $A$, we have $y^\sim \in F$, and so $y \in M(F)$. Thus, $M(F)$ is an ideal of $A$ by Theorem 6.6. For any $x \in F^\sim$, there exists $y \in F$ such that $x = y^\sim$. Since $y \leq y^\sim = x^\sim$, we have $x^\sim \in F$, i.e. $x \in M(F)$. Hence, $F^\sim \subseteq M(F)$. Suppose $I$ is an ideal of $A$ containing $F^\sim$. If $x \in M(F)$, i.e. $x^\sim \in F$, then $x^\sim \in F^\sim \subseteq I$. Since $x \leq x^\sim$, we have $x \in I$. Thus, $M(F) \subseteq I$. Therefore, $M(F)$ is an ideal generated by $F^\sim$. Similarly, $N(F)$ is an ideal generated by $F^\sim$. □

Theorem 6.4. Let $A$ be a bounded pseudo-hoop algebra and $I$ an ideal of $A$. If $A$ is good, then $M(I)$ and $N(I)$ are filters of $A$ such that $I^\sim \subseteq M(I)$ and $I^\sim \subseteq N(I)$.

Proof. If $x \leq y$ and $x \in M(I)$, then $y^\sim \leq x^\sim$ and $x^\sim \in I$. Using (I2), we obtain $y^\sim \in I$, i.e. $y \in M(I)$. For any $x, y \in M(I)$, we have $x^\sim, y^\sim \in I$, and so by Proposition 6.4(7),

$$(x \ominus y)^\sim = x \to y^\sim = x^\sim \to y^\sim = x^\sim \ominus y^\sim \in I.$$

That is $x \ominus y \in M(I)$. Hence, $M(I)$ is a filter of $A$. Suppose $x \in I^\sim$. There exists $y \in I$ such that $x = y^\sim$. Since $y \in I \Rightarrow y^\sim \in I$, we have $x^\sim = y^\sim \in I$, i.e. $x \in M(I)$. Hence, $I^\sim \subseteq M(I)$.

Similarly, we can show that $N(I)$ is a filter of $A$ and $I^\sim \subseteq N(I)$. □

Theorem 6.5. If $I$ is an ideal of a bounded pseudo-hoop algebra $A$, then $I = M(N(I)) = N(M(I))$.

Proof. For any $x \in A$, we obtain $x \in I$ iff $x^\sim \in I$ iff $x^\sim \in N(I)$ iff $x \in M(N(I))$. So $I = M(N(I))$. Analogously, we can show $I = N(M(I))$. □

Theorem 6.6. If $F$ is a filter of a bounded pseudo-hoop algebra $A$, then $F \subseteq M(N(F))$ and $F \subseteq N(M(F))$. 

Proof. 

\textit{Case 1:} $F$ is an ideal of $A$. Then $F \subseteq M(N(F))$.

\textit{Case 2:} $F$ is a filter of $A$. Then $F \subseteq N(M(F))$.

Thus, $F \subseteq M(N(F)) \cap N(M(F))$. Therefore, $F \subseteq M(N(F))$ and $F \subseteq N(M(F))$. □
Proof. Let $x \in F$. Since $x \leq x^\sim$ and $F$ is a filter of $A$, we have $x^\sim \in F$. Then $x \in M(N(F))$. Thus, $F \subseteq M(N(F))$. Similarly, $F \subseteq N(M(F))$. \hfill \Box

Remark 6.7. In Theorem 6.13, we do not necessarily have $F = M(N(F))$ and $F = N(M(F))$. For instance, we have $M(N(F_1)) = \{d, e, f, 1\} = F_1$ and $M(N(F_2)) = \{a, b, c, e, f, 1\} \supseteq F_2$ in Example 6.2. Also, the converse of Theorem 6.13 is not true in general. Let $D = \{c\}$. Then $N(M(D)) = M(N(D)) = \{a, b, c\} \supseteq D$. But $D$ is not a filter of $A$.

In order to further discuss the relationship between ideals and filters of a pseudo-hoop algebra, we introduce the notion of \textit{⊙}-prime ideals in pseudo-hoop algebras.

Definition 6.8. Let $(A, \odot, \to, \sim, 1)$ be a bounded pseudo-hoop algebra and $P$ an ideal of $A$. Then $P$ is called a $\odot$-prime ideal of $A$ if $P \neq A$ and $x \odot y \in P$ implies $x \in P$ or $y \in P$ for any $x, y \in A$.

Example 6.9. Let $A$ be the pseudo hoop algebra as in Example 6.2. Then it is easy to show that $I_3 = \{0, a, d\}$ is a $\odot$-prime ideal of $A$.

Proposition 6.10. Let $A$ be a bounded pseudo-hoop algebra. Then every $\odot$-prime ideal of $A$ is a prime ideal of $A$. The converse may not hold.

Proof. Let $P$ be a $\odot$-prime ideal of $A$. If $P$ is not prime, there exist $x, y \in A$ such that $x \wedge y \in P$, but $x, y \notin P$. We obtain $x \odot y \in P$ by $x \odot y \leq x \wedge y$. Then $x \in P$ or $y \in P$, which is a contradiction. Therefore, $P$ is a prime ideal of $A$.

In Example 6.2, $I_2 = \{0, c\}$ is a prime ideal of $A$. Since $b \odot d = 0 \in I_2$ and $b, d \notin I_2$, we get $I_2$ is not a $\odot$-prime ideal of $A$. Therefore, the converse may not hold. \hfill \Box

Proposition 6.11. Let $A$ be a bounded pseudo-hoop algebra and $P$ an ideal of $A$. Then $P$ is a $\odot$-prime ideal of $A$ if and only if $P$ is a prime ideal of $A$ and $x \odot y \in P$ implies $x \wedge y \in P$ for any $x, y \in P$.

Proof. Let $P$ be a $\odot$-prime ideal of $A$. Then $P$ is a prime ideal of $A$ by Proposition 6.10. Suppose $x \odot y \in P$. We obtain $x \in P$ or $y \in P$ by Definition 6.8. Since $x \wedge y \leq x, y$, we obtain $x \wedge y \in P$. Therefore, $x \odot y \in P$ implies $x \wedge y \in P$ for any $x, y \in P$.

Conversely, if $x \odot y \in P$, then $x \wedge y \in P$. By the notion of prime ideals, we know that $x \in P$ or $y \in P$. Therefore, $P$ is a $\odot$-prime ideal of $A$. \hfill \Box

Let $X$ be a subset of a pseudo-hoop algebra $A$. We denote $A - X$ by $\overline{X}$. The following results study the relationship between ideals and filters in pseudo-hoop algebras.

Theorem 6.12. Let $A$ be a bounded pseudo-hoop algebra and $P$ an ideal of $A$. If $P$ is a $\odot$-prime ideal of $A$, then $\overline{P}$ is a maximal filter of $A$.

Proof. Suppose $P$ is a $\odot$-prime ideal of $A$. Since $P \neq A$, we obtain $\overline{P} \neq \emptyset$. Since $0 \in P$, i.e. $0 \notin \overline{P}$, we have $\overline{P} \neq A$. Let $x, y \in \overline{P}$. If $x \odot y \in P$, then $x \in P$ or $y \in P$, which is a contradiction. Thus, $x \odot y \in \overline{P}$. Suppose $x, y \in A$ such that $x \leq y$ and $x \in \overline{P}$. It follows that $y \in \overline{P}$, i.e. $y \notin P$. If not, since $P$ is an ideal of $A$ and $x \leq y$, we have $x \in P$, which is a contradiction. Therefore, $\overline{P}$ is a filter of $A$.

Let $Q$ be a filter of $A$ strictly containing $\overline{P}$. Then there exists $a \in A$ such that $a \notin Q$ and $a \notin \overline{P}$. So $a \in P \cap Q$. It follows that $a^\sim, a^\sim \notin P$. If not, then $a^\sim \odot a = a^\sim \sim a^\sim = 1 \in P$ and $a \odot a^\sim = a^\sim \to a^\sim = 1 \in P$, which is a contradiction. So $a^\sim \notin \overline{P} \subseteq Q$. Using (F1), we have $0 = a \odot a^\sim \in Q$. Then $Q = A$. Hence, $\overline{P}$ is a maximal filter of $A$. \hfill \Box
Remark 6.13. By the previous proof, if \( P \) is a proper ideal of \( A \) and \( a \in P \), then \( a^- \not\in P \).

Theorem 6.14. Let \( A \) be a bounded pseudo-hoop algebra and \( P \) be an ideal of \( A \). If \( \overline{P} \) is a normal and maximal filter of \( A \), then \( P \) is a \( \circ \)-prime ideal of \( A \).

Proof. Let \( \overline{P} \) be a normal and maximal filter of \( A \). Then \( P \neq \emptyset \). Since \( 1 \in \overline{P} \), i.e. \( 1 \not\in P \), we have \( P \neq A \). Suppose \( x, y \in A \) such that \( x \circ y \in P \), i.e. \( x \circ y \not\in \overline{P} \). Therefore, \( \overline{P} \) is strictly contained in \( \{ P \cup \{ x \circ y \} \} \). So \( \overline{P} \cup \{ x \circ y \} = A \). By Proposition 2.4, there exists \( n \in \mathbb{N} \) and \( h \in \overline{P} \) such that \( h \circ (x \circ y)^n \leq 0 \). That is \( h \leq ((x \circ y)^n)^- \). So \( ((x \circ y)^n)^- \in P \). Suppose \( x, y \not\in P \). Since \( \overline{P} \) is a filter of \( A \), we obtain \( (x \circ y)^n \in \overline{P} \). It follows that \( 0 = ((x \circ y)^n)^- \circ (x \circ y)^n \in \overline{P} \). Using (F2), we have \( \overline{P} = A \), which is a contradiction. Therefore, \( x \circ y \in P \) implies \( x \in P \) or \( y \in P \). Thus, \( P \) is a \( \circ \)-prime ideal of \( A \). \( \square \)

7 Conclusions

We defined ideals in pseudo-hoop algebras using two kinds of addition operations. We gave some equivalent characterizations of ideals of good pseudo-hoop algebras. Also, the congruence relation on a pseudo-hoop algebra is induced by ideals are defined. Using ideals, we constructed the quotient pseudo-hoop algebras and got an isomorphism theorem. We proved that if a pseudo-hoop algebra \( A \) satisfies condition (pDN), then there is a one-to-one correspondence between the set of all congruence relation on \( A \) and the set of all normal ideals of \( A \). The notion of prime ideals in pseudo-hoop algebras is introduced. We showed that the normal ideal of a good pseudo-hoop algebra with the pre-linear condition is prime if and only if the corresponding quotient pseudo-hoop algebra is a pseudo-hoop chain. In addition, we discussed the relationship between ideals and filters in pseudo-hoop algebras. We found that ideals and filters behave differently in pseudo-hoop algebras. Also, we discussed the relationship between \( \circ \)-prime ideals and maximal filters.

For future works, we will study other types of ideals in pseudo-hoop algebras and discuss the relationships between these ideals. The notion of implicative ideals of hoop algebras was studied in [1]. We shall investigate the notion of implicative ideals in pseudo-hoop algebras. Similarly to the notion of nodal filters in hoop algebras in [15], we shall define the notion of nodal ideals in pseudo-hoop algebras. In this paper, we can observe that the operators \( M \) and \( N \) defined in Definition 6.11 transform filters into ideals and vice versa. We shall further study other properties of \( M \) and \( N \). In addition, stabilizers in hoop algebras were introduced in [3]. We shall study stabilizers in pseudo-hoop algebras. Furthermore, we shall discuss the relationship between ideals and stabilizers in pseudo-hoop algebras.

Acknowledgement

We are very grateful to the editor and reviewers for their valuable comments and suggestions for improvements in this paper.

References


