



## Cyclicity in some classes of $H_v$ -groups

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### Abstract

The study of cyclicity in hyperstructures was started very early, almost from the beginning of the introduction of a hypergroup by F. Marty in 1934. New concepts appeared in hyperstructures the main of which are *the period of a generator* and *the single power cyclicity*. These terms have no meaning in the classical structures as groups. We study the cyclicity in special large classes of  $H_v$ -groups.

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## 1 Introduction

In the hyperstructure theory, especially in  $H_v$ -structures or in  $h/v$ -structures, a lot of new concepts appeared which are not possible to be seen in the classical theory. Defining the powers and cyclicity in  $H_v$ -structures, we discover the cyclic hyperoperation, the period of a generator, the single power cyclicity, and so on. The fundamental relations relate the hyperstructures with the corresponding classical structures. Therefore, the cyclicity is connected with the fundamental classes, as well. In this paper, we present some of the theory of cyclicity and we define the concept of cyclicity with respect to the fundamental classes. We give examples on the concept.

## 2 Preliminaries

The class of hyperstructures called  $H_v$ -structures, introduced in 1990 [7], [8] by Vougiouklis, satisfy the *weak axioms*, where the non-empty intersection replaces equality.

**Definition 2.1.** An algebraic hyperstructure  $(H, \cdot)$  is called a set  $H$  equipped with a hyperoperation (abbreviated hope)

$$\cdot : H \times H \rightarrow \mathcal{P}(H) - \{\emptyset\}.$$

We abbreviate by WASS the weak associativity:

$$(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$$

and by COW the weak commutativity:

$$xy \cap yx \neq \emptyset, \forall x, y \in H.$$

The algebraic structure  $(H, \cdot)$  is an  $H_v$ -semigroup if it is WASS, and it is called an  $H_v$ -group if it is a reproductive  $H_v$ -semigroup, i.e.

$$xH = Hx = H, \forall x \in H.$$

The algebraic structure  $(R, +, \cdot)$  is called an  $H_v$ -ring if  $(+)$  and  $(\cdot)$  are WASS;  $(+)$  is reproductive and  $(\cdot)$  is weak distributive to  $(+)$ :

$$x(y + z) \cap (xy + xz) \neq \emptyset \text{ and } (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R.$$

Let  $(R, +, \cdot)$  be an  $H_v$ -ring, a COW  $H_v$ -group  $(M, +)$  is called an  $H_v$ -module over  $R$ , if there is an external hope

$$\cdot : R \times M \rightarrow \mathcal{P}(M) : (a, x) \rightarrow ax$$

such that for any  $a, b \in R$  and for any  $x, y \in M$ , we have

$$a(x + y) \cap (ax + ay) \neq \emptyset, (a + b)x \cap (ax + bx) \neq \emptyset \text{ and } (ab)x \cap a(bx) \neq \emptyset.$$

For more definitions and applications on  $H_v$ -structures one can see in books and papers as [1], [3], [4], [8], [12], [13], [14].

**Definition 2.2.** Let  $(H, \cdot)$  and  $(H, \star)$  be two  $H_v$ -semigroups. Then the hope  $(\cdot)$  is smaller than  $(\star)$ , and  $(\star)$  greater than  $(\cdot)$ , iff there exists an automorphism

$$f \in \text{Aut}(H, \star) \text{ such that } xy \subset f(x \star y), \forall x, y \in H.$$

We write  $\cdot \leq \star$  and say  $(H, \star)$  contains  $(H, \cdot)$ . If  $(H, \cdot)$  is a structure, then it is basic structure and  $(H, \star)$  is an  $H_b$ -structure.

The following little theorem leads to a partial order on  $H_v$ -structures and to posets.

**The Little Theorem.** Greater hopes than the ones which are WASS or COW, are WASS or COW, respectively.

**Example 2.3.** In quaternions  $Q = \{1, -1, i, -i, j, -j, k, -k\}$ , with  $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$ , denote  $\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}$ , define hopes  $(\star)$  by enlarging only few products. For example,  $(-1) \star k = \underline{k}, k \star i = \underline{j}$  and  $i \star j = \underline{k}$ . Then  $(Q, \star)$  is an  $H_v$ -group.

Let  $(G, \cdot)$  be a hypergroupoid [8]. An  $n$ -ary hyperproduct can be defined, induced by  $(\cdot)$ , by inserting  $n - 2$  parentheses in the sequence of elements  $a_1, a_2, \dots, a_n$  in a standard position. Let us denote by  $p(a_1, \dots, a_n)$  such a pattern of  $n - 2$  parentheses and by  $P_n$  the set of all such patterns. For example, for  $n = 4$ , we have,

$$P_4(a_1, a_2, a_3, a_4) = \{((a_1 a_2) a_3) a_4, (a_1 (a_2 a_3)) a_4, (a_1 a_2) (a_3 a_4), a_1 ((a_2 a_3) a_4), a_1 (a_2 (a_3 a_4))\}.$$

If it is commutative, the cases are two:

$$P_4(a_1, a_2, a_3, a_4) = \{((a_1 a_2) a_3) a_4, (a_1 a_2) (a_3 a_4)\}$$

In an  $H_v$ -semigroup  $(H, \cdot)$ , we define  $(\circ)$  the  $n$ -ary circle hope, by the union of hyperproducts  $n$  times, with all possible patterns of parentheses on them [4], [5], [8]. Thus,  $\forall h \in H$ , we have

$$\begin{aligned} h^1 &= \{h\} \\ h^2 &= h \circ h = h \cdot h \\ h^3 &= h \circ h \circ h = [(h \cdot h) \cdot h] \cup [h \cdot (h \cdot h)] \\ h^4 &= h \circ h \circ h \circ h = [((h \cdot h) \cdot h) \cdot h] \cup [(h \cdot h) \cdot (h \cdot h)] \cup [h \cdot (h \cdot (h \cdot h))] \cup [h \cdot ((h \cdot h) \cdot h)] \cup [(h \cdot (h \cdot h)) \cdot h] \\ &\dots \end{aligned}$$

**Definition 2.4.** In an  $H_v$ -semigroup  $(H, \cdot)$ , the powers are defined as follows:

$$h^1 = \{h\}, h^2 = h \cdot h \text{ and } h^n = h \circ h \circ \dots \circ h,$$

where  $(\circ)$  is the  $n$ -ary circle hope. An  $(H, \cdot)$  is cyclic of period  $s$ , if there is a generator  $h \in H$  and the minimum integer  $s$ , such that

$$H = h^1 \cup h^2 \cup \dots \cup h^s.$$

If there are  $h \in H$  and  $s$ , the minimum one,  $H = h^s$ , then  $(H, \cdot)$  is a *single-power cyclic of period  $s$* .

Analogously, the cyclicity for infinite period, is defined [5], [8]. Thus, for example, an  $H_v$ -semigroup  $(H, \cdot)$  is called *single-power cyclic of infinite period* with generator  $h \in H$ , if every element of  $H$  belongs to a power of  $h$  and there exists  $s_o \geq 1$ , such that for every  $s \geq s_o$  we have

$$h^1 \cup h^2 \cup \dots \cup h^{s-1} \subset h^s.$$

Denote  $[h] = h^1 \cup h^2 \cup \dots \cup h^s \cup \dots$ , which is the set generating by the element  $h \in H$ .

The main tool in hyperstructures is the *fundamental relation* [7], [8], [11], [12], [4], [14].

**Definition 2.5.** The fundamental relations  $\beta^*$ ,  $\gamma^*$  and  $\epsilon^*$  are defined in  $H_v$ -groups,  $H_v$ -rings and  $H_v$ -vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.

Let  $(G, \cdot)$  be a group and  $R$  be a partition in  $G$ , then  $(G/R, \cdot)$  is an  $H_v$ -group. Therefore, the quotient  $(G/R, \cdot) / \beta^*$  is a group, the *fundamental one*.

**Theorem 2.6.** Let  $(H, \cdot)$  be an  $H_v$ -group and denote by  $U$  the set of all finite products of elements of  $H$ . Define the relation  $\beta$  in  $H$  by:  $x\beta y$  iff  $\{x, y\} \subset u$ , where  $u \in U$ . Then  $\beta^*$  is the transitive closure of  $\beta$ .

Analogous theorems are for  $H_v$ -rings,  $H_v$ -vector spaces and so on.

An element is called *single* if its fundamental class is singleton.

General structures can be defined using fundamental structures.

**Definition 2.7.** An  $H_v$ -ring  $(R, +, \cdot)$  is an  $H_v$ -field if  $R/\gamma^*$  is a field. An  $H_v$ -module over an  $H_v$ -field  $F$ , it is called an  $H_v$ -vector space.

### 3 Large classes of hopes

A class of  $H_v$ -structures, introduced in [8], is the following:

**Definition 3.1.** *An  $H_v$ -structure is called very thin if all hopes are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one. Thus, in a very thin  $H_v$ -structure in  $H$  there exists a hope  $(\cdot)$  and a pair  $(a, b) \in H^2$  for which  $ab = A$ , with  $\text{card}A > 1$ , and all the other products, are singletons.*

From the very thin hopes the *Attach Construction* is obtained [11]:

Let  $(H, \cdot)$  be an  $H_v$ -semigroup,  $v \notin H$ . We extend the hope  $(\cdot)$  into  $\underline{H} = H \cup \{v\}$  by:

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

The  $(\underline{H}, \cdot)$  is an  $H_v$ -group, where  $(\underline{H}, \cdot)/\beta^* \cong Z_2$  and  $v$  is a single.

The *enlarged* hopes are the ones where a new element appears in one result. The useful enlargements or reductions are those  $h/v$ -structures with the same fundamental structure.

**Definition 3.2.** [10] *Let  $(G, \cdot)$  be a groupoid and  $f : G \rightarrow G$  be a map. We define a hope  $(\partial)$ , called *theta-hope*, we write  $\partial$ -hope, on  $G$  as follows:*

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \forall x, y \in G.$$

*If  $(\cdot)$  is commutative, then  $\partial$  is commutative. If  $(\cdot)$  is COW, then  $\partial$  is COW.*

*If  $(G, \cdot)$  is a groupoid and  $f : G \rightarrow P(G) - \{\emptyset\}$  be a multivalued map, then we define the  $\partial$ -hope on  $G$  as follows:*

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \forall x, y \in G.$$

Motivation for  $\partial$ -hope is the *derivative* where only the product of functions, is used.

Basic property: if  $(G, \cdot)$  is a semigroup, then for all  $f$ , the  $\partial$ -hope is WASS.

**Example 3.3.** (a) *In integers  $(Z, +, \cdot)$  fix  $n \neq 0$ , a natural number. Consider the map  $f$  such that  $f(0) = n$  and  $f(x) = x, \forall x \in Z - \{0\}$ . Then  $(Z, \partial_+, \partial \cdot)$ , where  $\partial_+$  and  $\partial \cdot$  are the  $\partial$ -hopes refereed to the addition and the multiplication respectively, is an  $H_v$ -near-ring, with*

$$(Z, \partial_+, \partial \cdot)/\gamma^* \cong Z_n.$$

(b) *In  $(Z, +, \cdot)$  with  $n \neq 0$ , take  $f$  such that  $f(n) = 0$  and  $f(x) = x, \forall x \in Z - \{n\}$ . Then  $(Z, \partial_+, \partial \cdot)$  is an  $H_v$ -ring, moreover,  $(Z, \partial_+, \partial \cdot)/\gamma^* \cong Z_n$ .*

Special case of the above is for  $n = p$ , prime, then  $(Z, \partial_+, \partial \cdot)$  is an  $H_v$ -field.

Combining the *uniting elements* procedure [2], with the enlarging theory or the  $\partial$ -theory, we can obtain analogous results.

**Theorem 3.4.** *In the ring  $(Z_n, +, \cdot)$ , with  $n = ms$  enlarge the multiplication only in the product of the elements  $0 \cdot m$  by setting  $0 \otimes m = \{0, m\}$  and the rest results remain the same. Then*

$$(Z_n, +, \otimes)/\gamma^* \cong (Z_m, +, \cdot).$$

Remark that we can enlarge other products as well, for example  $2 \cdot m$  by setting  $2 \otimes m = \{2, m + 2\}$ , then the result remains the same. In this case 0 and 1 are scalars.

In the ring  $(Z_n, +, \cdot)$ , with  $n = ps$  where  $p$  is prime, we enlarge only the product  $0 \cdot p$  by  $0 \otimes p = \{0, p\}$  and the rest remain the same. Then  $(Z_n, +, \otimes)$  is a very thin  $H_v$ -field.

Well known hopes defined on classical structures is the following [6], [8]:

**Definition 3.5.** Let  $(G, \cdot)$  be a groupoid, then for every  $P \subset G, P \neq \emptyset$ , we define the following hopes called  $P$ -hopes:  $\forall x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py), \underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The  $(G, \underline{P}), (G, \underline{P}_r)$  and  $(G, \underline{P}_l)$  are called  $P$ -hyperstructures.

If  $(G, \cdot)$  is a semigroup, then

$$x\underline{P}y = (xP)y \cup x(Py) = xPy$$

and the  $(G, \underline{P})$  is a semihypergroup.

## 4 Cyclicity with respect to fundamental classes

A large class of hyperstructures is the following [9], [13]:

**Definition 4.1.** The  $H_v$ -semigroup  $(H, \cdot)$  is called a  $h/v$ -group if  $H/\beta^*$  is a group.

The  $h/v$ -group is a generalization of the  $H_v$ -group, where the following property is valid: Let  $(H, \cdot)$  be an  $h/v$ -group and devote  $[x]$  the fundamental class of the element  $x \in H$ , then we have

$$x[y] = [xy] = [x]y, \forall x, y \in H.$$

**Definition 4.2.** Let  $(H, \cdot)$  be an  $h/v$ -group and devote  $[x]$  the fundamental class of the element  $x \in H$ . We define the property of the reproductivity of classes, if the product of any element  $x$ , on the left, with all elements  $y$  of  $H$  gives representatives of all fundamental classes  $[y], \forall y \in H$ .

Similarly,  $h/v$ -rings,  $h/v$ -fields,  $h/v$ -vector spaces etc. are defined.

Combining the definitions of cyclicity and reproductivity of classes, we give a new definition: *cyclicity with respect to fundamental classes*.

**Definition 4.3.** An  $h/v$ -group is called *cyclic with respect to fundamental classes* if there exist an element  $h \in H$  and a number  $s$ , the minimum one, which is called *generator*, such that the union of the powers  $h^s$  contain elements of all fundamental classes.

**Definition 4.4.** An  $h/v$ -group is called *single-power cyclic with respect to fundamental classes* if there exists an element  $h \in H$ , which is called *generator*, and a number  $s$ , the minimum one, such that the power  $h^s$  contains elements of all fundamental classes.

According to this definition we remark the cyclicity with respect to fundamental classes is a generalization of the cyclicity. Therefore, an immediately corollary is that all *cyclic  $h/v$ -groups* are *cyclic with respect to fundamental classes*, as well. However, the period normally is smaller.

**Example 4.5.** Consider the  $h/v$ -group, based on  $h/v$ -matrices based  $(Z_6, \otimes)$ , which is given by the following table [13]:

$\otimes$	<u>0</u>	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>
<u>0</u>	<u>0</u>	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>
<u>1</u>	<u>4</u>	<u>5</u>	<u>0</u>	<u>1</u>	<u>2</u>	<u>3</u>
<u>2</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>0</u>	<u>1</u>
<u>3</u>	<u>0</u>	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u>5</u>
<u>4</u>	<u>4</u>	<u>5</u>	<u>0</u>	<u>1</u>	<u>2</u>	<u>3</u>
<u>5</u>	<u>2,5</u>	<u>0, 3</u>	<u>1,4</u>	<u>2, 5</u>	<u>0,3</u>	<u>1, 4</u>

This  $h/v$ -group is a cyclic where the element  $\underline{1}$  is a generator of period 5 and the element  $\underline{5}$  is a generator of period 4. The elements  $\underline{2}$ ,  $\underline{3}$  and  $\underline{4}$  are not generators.

We denote  $[0]$ ,  $[1]$  and  $[2]$  the classes of the elements 0, 1 and 2, which are the sets  $\{0, 3\}$ ,  $\{1, 4\}$  and  $\{2, 5\}$  respectively. Thus the fundamental classes are

$$[0] = \{0, 3\}, [1] = \{1, 4\}, [2] = \{2, 5\}.$$

We remark that the re-productivity from the left is satisfied, for example:

$$1 \otimes 0 = 4, 1 \otimes 1 = 5, 1 \otimes 2 = 0, 1 \otimes 3 = 1, 1 \otimes 4 = 2, 1 \otimes 5 = 3.$$

The re-productivity from the right is not satisfied, however representatives of all fundamental classes are given, for example:

$$0 \otimes 1 = 1, 1 \otimes 1 = 5, 2 \otimes 1 = 3, 3 \otimes 1 = 1, 4 \otimes 1 = 5, 5 \otimes 1 = \{0, 3\}.$$

The  $(Z_6, \otimes)$  is an  $h/v$ -group which is cyclic with respect to fundamental classes, where the elements  $\underline{1}$ ,  $\underline{2}$ ,  $\underline{4}$  and  $\underline{5}$  are generators of period 3 with respect to fundamental classes:

$$\underline{2}^1 = \underline{2}, \underline{2}^2 = \underline{4}, \underline{2}^3 = \underline{0}.$$

The element  $\underline{3}$  is not a generator.

## 5 Conclusion

We introduced the concept of cyclicity with respect to the fundamental classes. The cyclicity, generally in hyperstructures, is related to the fundamental classes, but the above one is based on them. Some examples are presented as well, in order to see the new field which is opened.

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