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An overview of hyper logical algebras

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Abstract

Hyper logical algebras were first studied in 2000 by Borzooei et al. They applied the concept of hyperstructures to one of the logical algebraic structures known as the BCK-algebra, and introduced two generalizations of them called the hyper BCK-algebra and hyper K-algebra. Then many researchers in this field continued their research and used hyperstructures on other logical algebras and introduced the concepts of hyper residuated lattices, hyper BL-algebras, hyper MV-algebras, hyper EQ-algebras, hyper BE-algebras, hyper equality algebras, hyper hoops and etc. Moreover, they defined some new notions such as different kinds of hyper ideals, hyper filters and hyper congruence relations on these structures and studied some properties, the relation among them and the quotient structure.
Now, in this paper, we review the definitions of all those hyper logical algebras and investigate relations among them.

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1 Introduction

The hyper algebraic structure theory was introduced in 1934 [26], by Marty at the 8th congress of Scandinavian Mathematicians. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [25], Jun et al, introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algbera, and investigated some related propoerties. They also introduced the notion of a hyper BCK-ideal and a weak hyper BCK-ideal, and gave relations between hyper BCK-ideals and weak hyper BCK-ideals. In [8], Borzooei et al, defined the notions of hyper I-algebras and hyper K-algebras and the union of two hyper K-algebras(hyper

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BCK-algebras). Then they stated and proved some related theorems. In particular, by some examples they shew that these definiotions are different from the notion of hyper BCK-algberas, however any hyper BCK-algebra is a hyper K-algebra. Then by defining the concept of hyper K-algbera product(hyper BCK-product) of two hyper K-algebras(hyper BCK-algebras), they gave a theorem which shows that the relation between the hyper K-ideal of the given hyper K-algebras and the hyper K-ideals of their product. In [5], Borzooei et al, studied hyper BCC-algebras which are a common generalization of BCC-algebras and hyper BCK-algebras. In particular, they investigated different types of hyper BCC-ideals and described the relationship among them. Next, they calculated all non-isomorphic 22 hyper BCC-algebras of order 3 of which only three are not hyper BCK-algebras. In [30], Xin, introduced the concept of a hyper BCI-algebra which is a generalization of a BCK-algebra, and investigated some related properties. Moreover, he introduced a hyper BCI-ideal, weak hyper BCI-ideal, strong hyper BCI-ideal and reflexive hyper BCI-ideal in hyper BCI-algebras, and gave some relations among these hyper BCI-ideals. Finally, he discussed the relations between hyper BCI-algebras and hyper groups, and between hyper BCI-algebras and hyper H_v-groups. In [17], Ghorbani et al, defined the concept of hyper MV-algebra, weak hyper MV- filter, hyper MV-filter, hyper MV-subalgebra and homomorphism of hyper MV-algebras and they obtained some results. In particular, they shew that the class of hyper MV-algebras is a subclass of the class of bounded hyper K-algebras. In [36], Zahiri et al, defined the concept of regular compatible congruence on hyper residuated lattices. Then they attempted to construct quotient hyper residuted lattices. Finally, they stated and proved some theorems with appropriate results such as the isomorphism theorems. In [9], Borzooei et al, introduced the notion of weak hyper residuated lattices which is a generalization of residuated lattices and proved some related results. Moreover, they introduced deductive systems, (positive) implicative and fantastic deductive systems and investigated the relations among them. In [6], Borzooei et al, applied the hyper structures theory to EQ-algebras and introduced the notion of hyper EQ-algebra which is a generalization of EQ-algebra. In the following, they defined the notions of good and separated hyper EQ-algebras and stated and proved some properties of (good, separated) hyper EQ-algebras. Moreover, by defining the concept of (pre)filter, they constructed the quotient hyper EQ-algebra. Finally, they investigated the relation between hyper EQ-algebras and hyper BCK-algebras and (weak) hyper residuated lattices.

In [27] Radfar et al, introduced the notion of hyper BE-algebra and investigated some properties of it. Also, some types of hyper filters in hyper BE-algebras are studied and the relationship between them are stated. They tried to show that these notions are independent by some examples. Furthermore, they shew that under special condition hyper BE-algebras are equivalent to dual hyper K-algebras. In [10] Borzooei et al, applied the hyper structure theory to hoop-algebras and introduced the notion of (quasi) hyper hoop-algebra which is a generalization of hoop-algebra and investigated some related properties. They also introduced the notion of (weak)filters on hyper hoop-algebras, and gave several properties of them. Finally, they characterized the (weak) filter generated by a non-empty subset of a hyper hoop-algebra. In [13], Cheng introduced a new structure, called hyper equality algebras which are a generalization of equality algebras, and investigated some related properties. Then they defined (weak, strong) hyper filters and (weak, strong) hyper deductive systems, and gave relations between them. Moreover, he discussed relations between hyper equality algebras and other hyper structures, such as hyper EQ-algebras, hyper BCK-algebras and weak hyper residuated lattices. Finally, he also obtained quotient hyper equality algebras via regular hyper congruence relations.

In [18], Hashemi et al, investigated the relations among hyper equality algebras and other hyper algebraic structures such as hyper K(BE, MV)-algebras and hyper hoops. Specially, they proved that any linearly ordered hyper MV-algebra is a strongly commutative symmetric hyper equality algebra and under some conditions, any strongly commutative involutive hyper equality algebra is a hyper MV-algebra. In [32], Xin defined the concept of hyper BL-algebras which is a generalization of BL-algebras. He gave some non-trivial examples and properties of hyper BL-algebras. Moreover, he introduced weak filters and weak deductive systems of hyper BL-algebras and studied the relationships between them. Then he stated and proved some theorems about weak filters and weak deductive systems. In particular, he defined the concept of regular compatible congruence on hyper BL-algebras and constructed the quotient structure in hyper BL-algebras. Finally, he discussed the conditions in which a quotient hyper BL-algebra is an MV-algebra.

Now, in this paper, we review the definitions of all these hyper logical algebras and investigate the relations among them.

2 Preliminaries

Let H be a non-empty set and $\circ$ a function from $H \times H$ to $\mathcal{P}(H) - \{\emptyset\}$, where $\mathcal{P}(H)$ denotes the power set of H. For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup\limits_{a\in A, b\in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

Marty [26] defined a hypergroup as a hyperstructure $(H, \cdot)$ such that for all $x, y, z \in H$, the following axioms hold:
(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(ii) $x \cdot H = H \cdot x = H$.
A subset K of H is called a subhypergroup if $(K, \cdot)$ is a hypergroup.
Vougiouklis in [29] introduced an H_v-group which is a hyperstructure $(H, \cdot)$ such that for all $x, y, z \in H$, the following axioms hold:
(i) $((x \cdot y) \cdot z) \cap (x \cdot (y \cdot z)) \neq \emptyset$,
(ii) $x \cdot H = H \cdot x = H$.
If $(H, \cdot)$ satifies only the first axiom, it is called an H_v-semigroup.

Definition 2.1. *Let L be a non-empty set endowed with hyperoperations $\wedge$ and $\vee$. Then $(L, \wedge, \vee)$ is called a hyperlattice if for any $x, y, z \in L$, the following conditions hold:*
$(HL1)$ $x \in x \wedge x$, $x \in x \vee x$,
$(HL2)$ $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,
$(HL3)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
$(HL4)$ $x \in x \wedge (x \vee y)$, $x \in x \vee (x \wedge y)$.

3 Relation among different kinds of hyper logical algebras

In this section, we introduce different kinds of hyper logical algebras and investigate the relation between them.

In [25], Jun et al, introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra, and investigated some related propoerties.

Definition 3.1. [25] *By a hyper BCK-algebra (HBCK) it is meant a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 which for any $x, y, z \in H$ satisfiying in the following conditions:*
$(HBCK1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$,

$(HBCK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,
$(HBCK3)$ $x \circ H \ll \{x\}$,
$(HBCK4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$,
where $x \ll y$ *is defined by* $0 \in x \circ y$ *and for any* $A, B \subseteq H$, $A \ll B$ *is defined by for all* $a \in A$, *there exists* $b \in B$ *such that* $a \ll b$. *In this case,* $\ll$ *is called hyperorder in* H.

Example 3.2. (i) *Let* $(H, *, 0)$ *be a BCK-algebra and define a hyperoperation* $\circ$ *on* H *by* $x \circ y = \{x * y\}$ *for all* $x, y \in H$. *Then* $(H, \circ)$ *is a hyper BCK-algebra.*
(ii) *Define a hyperoperation* $\circ$ *on* $H = [0, \infty)$ *by*

$$x \circ y = \begin{cases} [0, x] & \text{if } x \leq y, \\ (0, y] & \text{if } x > y \neq 0, \\ \{x\} & \text{if } y = 0, \end{cases}$$

for all $x, y \in H$. *Then* $(H, \circ)$ *is a hyper BCK-algebra.*
(iii) *Let* $H = \{0, 1, 2\}$. *Consider the following table:*

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0,1}
2	{2}	{1,2}	{0,1,2}

Then $(H, \circ)$ *is a hyper BCK-algebra.*

Proposition 3.3. *Let* $(H, \circ, 0)$ *be a hyper BCK-algebra. Then for any* $x, y, z \in H$ *and* $A, B, C \subseteq H$, *the following statements hold:*
(i) $x \ll x$,
(ii) $0 \circ x = \{0\}$,
(iii) $x \circ 0 \ll \{y\}$,
(iv) *if* $x \circ y = \{0\}$, *then* $(x \circ z) \circ (y \circ z) = \{0\}$ *and* $x \circ z \ll y \circ z$,
(v) *if* $y \ll z$, *then* $x \circ z \ll x \circ y$,
(vi) *if* $(x \circ y) \circ z \ll A$, *then* $a \circ z \ll A$, *for any* $a \in x \circ y$,
(vii) $x \circ y \ll \{x\}$,
$(viii)$ $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ *and* $0 \circ A \ll \{0\}$.

In [8], Borzooei et al, defined the notions of hyper I-algebras and hyper K-algebras and the union of two hyper K-algebras(hyper BCK-algebras). Then they stated and proved some related theorems. In particular, by some examples they shew that these definitions are different from the notion of hyper BCK-algebras. However any hyper BCK-algebra is a hyper K-algebra.

Definition 3.4. [8] *A hyper structure* $(H, \circ)$ *is called a hyper I-algebra (HI) if it contains a constant* 0 *and for any* $x, y, z \in H$ *satisfies in the following axioms:*
$(HI1)$ $(x \circ z) \circ (y \circ z) \leq x \circ y$,
$(HI2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,
$(HI3)$ $x \leq x$,
$(HI4)$ $x \leq y$ *and* $y \leq x$ *imply* $x = y$,
where $x \leq y$ *is defined by* $0 \in x \circ y$ *and for any subsets* A *and* B *of* H, $A \leq B$ *is defined there exist* $a \in A$ *and* $b \in B$ *such that* $a \leq b$.

Definition 3.5. [8] *A hyper K-algebra (HK) is a non-empty set* H *endowed with a hyperoperation* "$\circ$" *and a constant* 0 *such that, for all* $x, y, z \in H$, *it satisfying in the following conditions:*

$(HK1)$ $(x \circ z) \circ (y \circ z) \leq x \circ y$.
$(HK2)$ $(x \circ z) \circ y = (x \circ y) \circ z$.
$(HK3)$ $x \leq x$.
$(HK4)$ *If* $x \leq y$ *and* $y \leq x$, *then* $x = y$.
$(HK5)$ $x \geq 0$,
where, for any $x, y \in H$, $x \leq y$ *if and only if* $0 \in x \circ y$, *and for any* $A, B \subseteq H$, $A \leq B$ *means that there exist* $a \in A$ *and* $b \in B$ *such that* $a \leq b$.

Theorem 3.6. *Every hyper K-algebra is a hyper I-algebra.*

Example 3.7. (i) *Let* $H = \{0, x, y\}$. *Consider the following table:*

$\circ$	0	x	y
0	$\{0, x, y\}$	$\{x, y\}$	$\{x, y\}$
x	$\{0, x, y\}$	$\{0, x, y\}$	$\{x, y\}$
y	$\{0, x, y\}$	$\{x, y\}$	$\{0, x, y\}$

Then $(H, \circ, 0)$ *is a hyper I-algebra.*
(ii) *Let* $H = \{0, x, y\}$. *Consider the following table:*

$\circ$	0	x	y
0	$\{0\}$	$\{0\}$	$\{0\}$
x	$\{x\}$	$\{0, x\}$	$\{0, x\}$
y	$\{y\}$	$\{x, y\}$	$\{0, x, y\}$

Then $(H, \circ, 0)$ *is a hyper K-algebra.*
(iii) *Let* $H = \{0, x, y\}$. *Consider the following table:*

$\circ$	0	x	y
0	$\{0\}$	$\{0, x, y\}$	$\{0, x, y\}$
x	$\{x\}$	$\{0, x, y\}$	$\{0, x, y\}$
y	$\{y\}$	$\{x, y\}$	$\{0, x, y\}$

Then $(H, \circ, 0)$ *is a hyper K-algebra which is not a hyper BCK-algebra.*

Proposition 3.8. *Let* $(H, \circ, 0)$ *be a hyper I-algebra. Then for all* $x, y, z \in H$ *and for all non-empty subsets* A, B *and* C *of* H *the following statements hold:*
(i) $x \circ (x \circ y) \leq y$,
(ii) $x \circ y \leq z$ *if and only if* $x \circ z \leq y$,
(iii) $(x \circ z) \circ (x \circ y) \leq y \circ z$,
(iv) $(A \circ B) \circ C = (A \circ C) \circ B$,
(v) $(A \circ C) \circ (B \circ C) \leq A \circ B$,
(vi) $(A \circ C) \circ (A \circ B) \leq B \circ C$.

Proposition 3.9. *Let* $(H, \circ, 0)$ *be a hyper K-algebra. Then for all* $x, y, z \in H$ *and for all non-empty subsets* A, B *and* C *of* H *the following statements hold:*
(i) $x \circ y \leq x$,
(ii) $0 \in x \circ (x \circ 0)$,
(iii) $x \leq x \circ 0$,
(iv) $A \circ B \leq A$,
(v) *if* $0 \in B$, *then* $A \leq A \circ B$.

Theorem 3.10. *Every hyper BCK-algebra is a hyper K-algebra.*

Corollary 3.11. *Every hyper BCK-algebra is a hyper I-algebra.*

In [5], Borzooei et al, studied hyper BCC-algebras which are a common generalization of BCC-algebras and hyper BCK-algebras.

Definition 3.12. [5] *By a hyper BCC-algebra (HBCC) it is meant a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 which for any $x, y, z \in H$ satisfiying in the following conditions:*
$(HBCC1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$,
$(HBCC2)$ $0 \circ x = \{0\}$,
$(HBCC3)$ $x \circ 0 \ll \{x\}$,
$(HBCC4)$ $x \ll y$ *and* $y \ll x$ *imply* $x = y$,
where $x \ll y$ *is defined by* $0 \in x \circ y$ *and for any* $A, B \subseteq H$, $A \ll B$ *is defined by for all* $a \in A$, *there exists* $b \in B$ *such that* $a \ll b$. *In this case,* $\ll$ *is called hyperorder in* H.

Example 3.13. (i) *Let* $H = \{0, 1, 2, 3, ...\}$ *and hyperoperation* $\circ$ *on* H *is defined as follows:*

$$x \circ y = \begin{cases} \{0, x\} & \text{if } x \leq y, \\ \{x\} & \text{if } x > y, \end{cases}$$

for all $x, y \in H$. *Then* $(H, \circ)$ *is a hyper BCC-algebra.*
(ii) *Let* $H = \{0, 1, 2, 3\}$ *be a set. Define a hyper operations* $\circ$ *on* H *as follows:*

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0}	{0}	{0}
2	{2}	{2}	{0,1}	{2}
3	{3}	{1,3}	{0,1,3}	{0,1,3}

Then $(H, \circ)$ *is a hyper BCC-algebras.*

Theorem 3.14. (i) *Any hyper BCK-algebra is a hyper BCC-algebra.*
(ii) *Let* H *be a hyper BCC-algebra. Then* H *is a hyper BCK-algebra if and only if for any* $x, y, z \in H$,

$$(x \circ y) \circ z = (x \circ z) \circ y. \tag{3.1}$$

(iii) *Let* H *be a hyper BCC-algebra such that for all* $x, y \in H$,

$$x \circ (x \circ y) \ll \{y\}. \tag{3.2}$$

Then $(H, \circ)$ *is a hyper BCK-algebra.*

In [30], Xin et al, introduced the concept of a hyper BCI-algebra which is a generalization of a BCI-algebra, and investigated some related properties.

Definition 3.15. [30] *By a hyper BCI-algebra (HBCI) we mean a hypergroupoid* $(H, \circ)$ *that contains a constant* 0 *and for any* $x, y, z \in H$ *it satisfies the following axioms:*
$(HBCI1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$,
$(HBCI2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,

$(HBCI3)$ *$x \ll x$,*
$(HBCI4)$ *$x \ll y$ and $y \ll x$ imply $x = y$,*
$(HBCI5)$ *$0 \circ (0 \circ x) \ll x$,*
where for any $A, B \subseteq H$, we define $A \ll B$ if and only if for all $x \in A$, there exists $y \in B$ such that $x \ll y$ which means that $0 \in x \circ y$ and $A \circ B = \bigcup\limits_{x \in A, y \in B} x \circ y$.

Example 3.16. *Let $H = \{0, 1, 2\}$ be a set. Consider the following table:*

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$

Then $(H, \circ)$ is a hyper BCI-algebra.

Proposition 3.17. *Let $(H, \circ, 0)$ be a hyper BCI-algebra. Then for any $x, y, z \in H$ and A and B are nonempty subset of H, the following conditions hold:*
(i) $0 \in x \circ (x \circ 0)$,
(ii) $0 \circ (x \circ y) \ll y \circ x$,
(iii) $x \circ 0 \ll \{y\}$,
(iv) if $A \subseteq B$, then $A \ll B$,
(v) if $A \ll \{0\}$, then $A = \{0\}$.

Theorem 3.18. *Every hyper BCK-algebra is a hyper BCI-algebra.*

In the following diagram we show the relation among these hyper logical algebras:

$$\begin{tikzcd}
\text{HBCK} \arrow[r] \arrow[d] \arrow[dr] & \text{HK} \arrow[dr] & \\
\text{HBCC} \arrow[u, bend left, "(3.1)\text{ or }(3.2)"] & \text{HBCI} & \text{HI}
\end{tikzcd}$$

Now, we study the relation between hyper BCI-algebras and hypergroups, hyper BCI-algebras and H_v-groups.

At first we define $x \cdot y$ by $x \cdot y = y \circ (0 \circ x)$, for all $x, y \in H$ in a hyper BCI-algebras.

Theorem 3.19. *Let $(H, \circ)$ be a hyper BCI-algebra and for any $x, y, z \in H$, it satisfies in the following conditions:*
(H_1) *$x \in y \circ (y \circ x)$,*
(H_2) *$(x \cdot y) \cap (y \cdot x) \neq \emptyset$,*
(H_3) *$(x \cdot y) \cdot z = (x \cdot z) \cdot y$.*
Then $(H, \cdot)$ is an H_ν-group.

Theorem 3.20. *Let $(H, \circ)$ be a hyper BCI-algebra and for any $x, y \in H$, it satisfies in the following conditions:*
(H_1) *$x \in y \circ (y \circ x)$,*
(H_4) *$x \circ (0 \circ y) = y \circ (0 \circ x)$.*
Then $(H, \cdot)$ is a hyper group.

In the following diagram we show the relation between hypergroup and hyper BCI-algebra.

$$\begin{tikzcd}
\text{Commutative Hgroup} \arrow[d] & \text{HI} \arrow[l, "0\circ(0\circ x)=x \text{ and } H^{+}=\{0\}"'] & H_\nu\text{-group} \\
\text{Hgroup} & \text{HBCI} \arrow[l, "(H_1)+(H_4)"'] \arrow[ur, "(H_1)+(H_2)+(H_3)"]
\end{tikzcd}$$

In [17], Ghorbani et al, defined the concept of hyper MV-algebra, and they obtained some results. In particular, they shew that the class of hyper MV-algebras is a subclass of the class of bounded hyper K-algebras.

Definition 3.21. [17] *A hyper MV-algebra (HMV) is a non-empty set M endowed with a binary hyper operation $\oplus$, a unary operation $*$ and a constant 0 which for all $x, y, z \in M$, it satisfies in the following conditions:*
$(HMV1)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
$(HMV2)$ $x \oplus y = y \oplus x$,
$(HMV3)$ $(x^*)^* = x$,
$(HMV4)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
$(HMV5)$ $0^* \in x \oplus 0^*$,
$(HMV6)$ $0^* \in x \oplus x^*$,
$(HMV7)$ *if* $x \ll y$ *and* $y \ll x$, *then* $x = y$,
where $x \ll y$ *is denoted by* $0^* \in x^* \oplus y$.
For every $A, B \subseteq M$, *we define* $A \ll B$ *if and only if there exist* $x \in A$ *and* $y \in B$ *such that* $x \ll y$ *and* $A \oplus B = \bigcup\limits_{x\in A, y\in B} x \oplus y$. *Also, we define* $0^* = 1$ *and* $A^* = \{x^* \mid x \in A\}$.

Example 3.22. (i) *Let* $M = \{0, a, b, c, 1\}$ *be a set. Consider the following tables:*

$\oplus$	0	a	b	c	1
0	{0}	{0,a}	{0,a,b}	{0,c}	{0,a,b,c,1}
a	{0,a}	{0,a}	{0,a,b,c,1}	{0,a,c}	{0,a,b,c,1}
b	{0,a,b}	{0,a,b,c,1}	{0,a,b,1}	{0,a,b,c}	{0,a,b,c,1}
c	{0,c}	{0,a,c}	{0,a,b,c}	{0,a,b,c,1}	{0,a,b,c,1}
1	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}	{0,a,b,c,1}

$*$	0	a	b	c	1
	1	b	a	c	1

Then $(M, \oplus, *, 0)$ *is a hyper MV-algebra.*
(ii) *Let* $M = [0, 1]$. *Define a unary operation* $*$ *on* M *by* $x^* = 1 - x$ *and a hyper operation* $\oplus$ *on* M *by* $x \oplus y = [0, \min(1, x + y)]$. *Then* $(M, \oplus, *, 0)$ *is a hyper MV-algebra.*

Proposition 3.23. *Let* $(M, \oplus, *, 0)$ *be a hyper MV-algebra. Then for all* $x, y, z \in M$ *and for all non-empty subset* A, B *and* C *of* M *the following statements hold:*
(i) $0 \ll x$,
(ii) $x \ll x$ *and* $A \ll A$,
(iii) *if* $x \ll y$, *then* $y^* \ll x^*$ *and if* $A \ll B$, *then* $B^* \ll A^*$,
(iv) $x \ll 1$,
(v) $x \ll x \oplus y$,
(vi) $(A^*)^* = A$,

(vii) $x \in x \oplus 0$,
$(viii)$ *if* $y \in x \oplus 0$, *then* $y \ll x$,
(ix) *if* $x \oplus 0 = y \oplus 0$, *then* $x = y$.

Theorem 3.24. *Let* $(H, \circ, 0)$ *be a bounded hyper K-algebra with a unite element* e *such that for any* $x, y \in H$ *satisfying the following conditions:*
(i) $e \circ (e \circ x) = \{x\}$,
(ii) $e \circ ((e \circ (x \circ y)) \circ y) = e \circ ((e \circ (y \circ x)) \circ x)$.
Then $e \circ x$ *has only one element. We define a unary operation "* $*$ *" on* H *such that if* $e \circ x = \{a\}$, *then* $x^* = a$ *and a hyperoperation* $\oplus$ *on* H *by* $x \oplus y := (x^* \circ y)^*$. *Then* $(H, \oplus, *, 0)$ *is a hyper MV-algebra.*

Theorem 3.25. *If* $(H, \oplus, *, 0)$ *is a hyper MV-algebra such that* $x \oplus 0 = \{x\}$, *then* $(H, \circ, 0)$ *is a bounded hyper K-algebra and for any* $x, y \in H$ *satisfying in the following conditions:*
(i) $e \circ (e \circ x) = \{x\}$,
(ii) $e \circ ((e \circ (x \circ y)) \circ y) = e \circ ((e \circ (y \circ x)) \circ x)$.
Moreover we have $x^* = 1 \circ x$ *and* $x \oplus y = (x^* \circ y)^*$.

Theorem 3.26. *If* M *is a finite hyper MV-algebra such that* $x^* = x$, *for any* $x \in M - \{0, 1\}$, *then* M *is a hyperlattice.*

In [36], Zahiri et al, defined the concept of regular compatible congruence on hyper residuated lattices. Then they attempted to construct quotient hyper residuted lattices.

Definition 3.27. [36] *By a hyper residuated lattice (HRL) we mean a non-empty set* L *endowed with four binary hyperoperations* $\wedge, \vee, \odot$ *and* $\rightarrow$ *and two constant* 0 *and* 1 *satisfiying the following conditions:*
$(HRL1)$ $(L, \leq, \wedge, \vee, 0, 1)$ *is a bounded superlattice.*
$(HRL2)$ $(L, \odot, 1)$ *is a commutative semihypergroup with* 1 *as the identity.*
$(HRL3)$ $x \odot z \ll y$ *if and only if* $z \ll x \rightarrow y$,
where $A \ll B$ *means that there exist* $a \in A$ *and* $b \in B$ *such that* $a \ll b$ *for any non-empty subsets* A *and* B *of* L.
A hyper residuated lattice L *is called nontrivial if* $0 \neq 1$.

Example 3.28. *Let* $(L = \{0, a, b, c, 1\}, \leq)$ *be a partially ordered set such that* $0 \leq a \leq b \leq c \leq 1$. *Consider the following tables:*

$\vee$	0	a	b	c	1
0	{0,a,c,1}	{a,c,1}	{b,c,1}	{c,1}	{1}
a	{a,c,1}	{a,c,1}	{b,c,1}	{c,1}	{1}
b	{b,c,1}	{b,c,1}	{b,c,1}	{c,1}	{1}
c	{c,1}	{c,1}	{c,1}	{c,1}	{1}
1	{1}	{1}	{1}	{1}	{1}

$\wedge$	0	a	b	c	1
0	{0}	{0}	{0}	{0}	{0}
a	{0}	{0,a}	{0,a}	{0,a}	{0,a}
b	{0}	{0,a}	{0,b}	{0,a,b}	{0,a,b}
c	{0}	{0,a}	{0,a,b}	{0,a,c}	{0,a,c}
1	{0}	{0,a}	{0,a,b}	{0,a,c}	{0,a,c,1}

$\rightarrow$	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{a,c,1}	{a,1}	{b,1}	{c,1}	{1}
b	{b,c,1}	{b,1}	{b,1}	{c,1}	{1}
c	{c,1}	{c}	{c}	{c,1}	{1}
1	{c,1}	{c,1}	{c,1}	{c,1}	{1}

Suppose $\odot = \wedge$. *Then* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *is a hyper residuated lattice.*

Theorem 3.29. *Let $\mathcal{H} = (H, \circ, 0)$ be a commutative and bounded hyper BCK-algebra with unit e and $e \circ (e \circ x) = x$, for all $x \in H$. Then $\mathcal{H}$ is a hyper residuated lattice.*

In [9], Borzooei et al, introduced the notion of weak hyper residuated lattices which is a generalization of residuated lattices and proved some related results.

Definition 3.30. [9] *By a weak hyper residuated lattice (WHRL), we mean a non-empty set L endowed with two binary operations $\vee$, $\wedge$ and two binary hyperoperations $\otimes$, $\to$ and two constants 0 and 1 satisfying the following conditions:*
$(WHRL1)$ *$(L, \leq, \vee, \wedge, 0, 1)$ is a bounded lattice,*
$(WHRL2)$ *$(L, \otimes, 1)$ is a commutative semihypergroup with 1 as the identity,*
$(WHRL3)$ *$x \otimes z \ll y$ if and only if $z \ll x \to y$,*
where $A \ll B$ means that for any $x \in A$, there exists $y \in B$ such that $x \leq y$.

Example 3.31. *Suppose $L = [0, 1]$ with the natural ordering is a bounded lattice. For any $x, y \in L$ define the hyperoperations $\odot$ and $\to$ on L as follows:*

$$x \odot y = x \times y \quad \textit{and} \quad x \to y = \begin{cases} \{1\} & \text{if } x \leq y, \\ [y, 1] & \text{if } x > y, \end{cases}$$

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a weak hyper residuated lattice.

Proposition 3.32. *Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a weak hyper residuated lattice. Then for non-empty subsets A, B and C of L and $x, y, z \in L$ we have,*
(i) if $x \leq y$, then $1 \in x \to y$,
(ii) $1 \in (x \to x) \cap (x \to 1) \cap (0 \to x)$,
(iii) $x \to (y \to z) \leq (x \odot y) \to z \leq y \to (x \to z)$,
(iv) $x \leq y \to (x \odot y)$ and $x \leq (y \to x) \to x$,
(v) $x \to y \leq (y \to z) \to (x \to z)$,
(vi) $A \ll B \to C$ if and only if $A \odot B \ll C$ if and only if $B \ll A \to C$.

Theorem 3.33. *Every weak hyper residuated lattice is a hyper residuated lattice.*

In [6], Borzooei et al, applied the hyper structures theory to EQ-algebras and introduced the notion of hyper EQ-algebra which is a generalization of EQ-algebra. In the following, they defined the notions of good and separated hyper EQ-algebras and stated and proved some properties of (good, separated) hyper EQ-algebras. Moreover, by defining the concept of (pre)filter, they constructed the quotient hyper EQ-algebra. Finally, they investigated the relation between hyper EQ-algebras and hyper BCK-algebras and (weak) hyper residuated lattices.

Definition 3.34. [6] *A hyper EQ-algebra (HEQ) $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a non-empty set H with a binary operations $\wedge$ and two binary hyper operations $\otimes$, $\sim$ and top element "1" satisfying the following conditions, for all $x, y, z, t \in H$:*
$(HEQ1)$ *$\langle H, \wedge, 1\rangle$ is a commutative idempotent monoid with top element 1,*
$(HEQ2)$ *$\langle H, \otimes, 1\rangle$ is a commutative semihypergroup with 1 as an identity and $\otimes$ is isotone,*
$(HEQ3)$ $((x \wedge y) \sim z) \otimes (t \sim x) \ll z \sim (t \wedge y)$,
$(HEQ4)$ $(x \sim y) \otimes (z \sim t) \ll (x \sim z) \sim (y \sim t)$,
$(HEQ5)$ $(x \wedge y \wedge z) \sim x \ll (x \wedge y) \sim x$,
$(HEQ6)$ $(x \wedge y) \sim x \ll (x \wedge y \wedge z) \sim (x \wedge z)$,
$(HEQ7)$ $x \otimes y \ll x \sim y$,
where $A \ll B$, means that, for all $a \in A$ there exists $b \in B$ such that $a \leq b$.

The hyper EQ-algebra H is called
(i) separated (S) if $1 \in x \sim y$, then $x = y$, for all $x, y \in H$, in other words $1 \in x \sim y$ if and only if $x = y$.
(ii) good(G) if $x \sim 1 = x = 1 \sim x$, for all $x \in H$.

Example 3.35. *(i) Let $H = [0, 1]$. Then for any $x, y \in H$ define the operations $\wedge$, $\otimes$ and the hyper operation $\sim$ on H as follows:*

$$x \otimes y = \begin{cases} \{0\} & \text{if } x + y \leq 1, \\ \{x \wedge y\} & \text{if } x > y, \end{cases} \quad , \qquad x \sim y = \{1, x \wedge y\}, \qquad \textit{and} \quad x \wedge y = \min\{x, y\}.$$

Then $H = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra.
(ii) Let $H = \{0, a, b, 1\}$ be a chain such that $0 \leq a \leq b \leq 1$. Then for any $x, y \in H$ define the operations $\wedge$, $\otimes$ and the hyper operation $\sim$ on H as follows:

$\otimes$	0	a	b	1
0	{0}	{0}	{0}	{0}
a	{0}	{0,a}	{0,a}	{0,a}
b	{0}	{0,a}	{0,b}	{0,b}
1	{0}	{0,a}	{0,b}	{0,1}

$\sim$	0	a	b	1
0	{1}	{a,b,1}	{a,1}	{0,1}
a	{a,b,1}	{a,1}	{a,b,1}	{a,1}
b	{a,1}	{a,b,1}	{b,1}	{b,1}
1	{0,1}	{a,1}	{b,1}	{1}

$x \wedge y = \min\{x, y\}$.

Then $H = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra.

Proposition 3.36. *Let $(H, \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra such that $x \to y = (x \wedge y) \sim x$ and $\overline{x} = x \sim 1$. Then for any $x, y, z \in H$ the following statements hold:*
(i) $1 \in x \sim x$, $1 \ll x \to x$ and $1 \in A \sim A$,
(ii) $z \otimes (x \wedge y) \ll (z \otimes x) \wedge (z \otimes y)$,
(iii) $x \sim y \ll x \to y$,
(iv) $(x \sim y) \otimes (y \sim z) \ll x \sim z$ and $(x \to y) \otimes (y \to z) \ll x \to z$,
(v) $(z \to (x \wedge y)) \otimes (x \sim t) \ll z \to (t \wedge y)$,
(vi) $(x \sim y) \otimes (z \sim t) \ll (x \wedge z) \sim (y \wedge t)$,
(vii) $x \sim y \ll ((x \wedge t) \sim z) \sim ((y \wedge t) \sim z$.

Proposition 3.37. *Every good hyper EQ-algebra is separated.*

Theorem 3.38. *Let $\mathcal{H}$ be a separated hyper EQ-algebra such that for all $x, y, z \in H$,*

$$z \to (y \to x) = y \to (z \to x). \tag{3.3}$$

Then $\mathcal{H}$ is a hyper BCK-algebra.

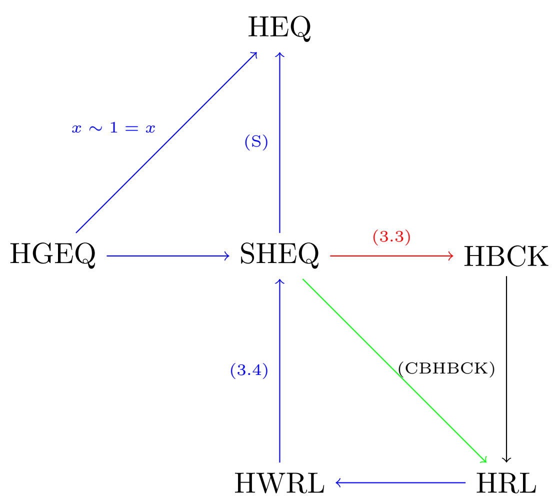
Theorem 3.39. *Let $H = (H, \circ, 0)$ be a commutative and bounded hyper BCK-algebra or (CBH-BCK) for short, with unit e and $e \circ (e \circ x) = x$, for all $x \in H$. Then H is a hyper residuated lattice.*

Theorem 3.40. *Let $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ be a weak hyper residuated lattice such that*

$$\text{if } A \ll B, \text{ then } A \leq B. \tag{3.4}$$

Then $L = (L, \wedge, \odot, \leftrightarrow, 1)$ is a separated hyper EQ-algebra, where for any $x, y \in L$, $x \leftrightarrow y = (x \to y) \wedge (y \to x)$.

In the following diagram we show the relation among hyper EQ-algbera with other hyper logical algebras, where (SHEQ) means separated hyper EQ-algebra, (HGEQ) means hyper good EQ-algebra, (CBHBCK) is a commutative bounded hyper BCK-algebra.



In [27], Radfar et al, introduced the notion of hyper BE-algebra and investigated some properties of it. Also, they shew that under a special condition hyper BE-algebras are equivalent to the dual hyper K-algebras.

Definition 3.41. [27] *A hyper BE-algebra (HBE) is a hypergroupoid $(H, \circ, 1)$ such that, for all $x, y, z \in H$, it satisfies in the following conditions:*
$(HBE1)$ *$x \leq 1$ and $x \leq x$.*
$(HBE2)$ *$x \circ (y \circ z) = y \circ (x \circ z)$.*
$(HBE3)$ *$x \in 1 \circ x$.*
$(HBE4)$ *$1 \leq x$ implies $x = 1$,*
where, $x \leq y$ if and only if $1 \in x \circ y$ and for any $A, B \subseteq H$, $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$.

Example 3.42. *Let $H = \{a, b, 1\}$. Define the hyper operation $\circ$ on H as follows:*

$\circ$	1	a	b
1	{1}	{a,b}	{b}
a	{1}	{1,a}	{1,b}
b	{1}	{1,a,b}	{1}

Then $(H, \circ)$ is a hyper BE-algebra.
(ii) *Define the hyperoperation "$\circ$" on $\mathbb{R}$ as follows:*

$$x \circ y = \begin{cases} \{y\} & \text{if } x = 1, \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

Then $(H, \circ)$ is a hyper BE-algebra.

Proposition 3.43. *Let H be a hyper BE-algebra. Then for any $x, y, z \in H$ the following statements hold:*
(i) *$x \leq y \circ x$,*
(ii) *if $x \leq y \circ z$, then $y \leq x \circ z$,*
(iii) *$x \leq (x \circ y) \circ y$,*
(iv) *if $z \in x \circ y$, then $x \leq z \circ y$,*
(v) *if $y \in 1 \circ x$, then $y \leq x$.*

Definition 3.44. *The algebraic structure $(H; \circ, 1)$ is called a dual hyper K-algebra if for all $x, y, z \in H$ it satisfies in the following conditions:*
$(DHK1)$ $x \preceq 1$ *and* $x \preceq x$.
$(DHK2)$ $(x \circ z) \circ y = (x \circ y) \circ z$.
$(DHK3)$ $x \circ y \preceq (y \circ z) \circ (x \circ z)$.
$(DHK4)$ $x \preceq y$ *and* $y \preceq x$ *imply that* $x = y$.

Where the relation $\preceq$ is defined by $x \preceq y$ if and only if $1 \in x \circ y$. For any $A, B \subseteq H$, $A \preceq B$ means that there exist $a \in A$ and $b \in B$ such that $a \preceq b$.

Theorem 3.45. *Let $(X; *, 1)$ be a (dual BCK-algebra) BE-algebra. Let define $x \circ y = \{x * y\}$, for all $x, y \in X$, then $(X; \circ, 1)$ is a (dual hyper K-algebra) hyper BE-algebra.*

Theorem 3.46. *Let $(H, \diamond, 0)$ be a hyper K-algebra. Then $(H, \circ, 1)$ is a dual hyper K-algebra, whenever $1 := 0$ and $x \circ y := y \diamond x$, for all $x, y \in H$.*

Proposition 3.47. *Every dual hyper K-algebra is a hyper BE-algebra.*

In [10] Borzooei et al, applied the hyper structure theory to hoop-algebras and introduced the notion of (quasi) hyper hoop-algebra which is a generalization of hoop-algebra and investigated some related properties.

Definition 3.48. [10] *A quasi-hyper hoop (QHH) is a non-empty set H endowed with two binary hyperoperations $\odot$, $\to$ and a constant 1 such that, for all $x, y, z \in H$, the following conditions hold:*
(HHA1) $(H, \odot, 1)$ *is a commutative semihypergroup with 1 as the unit.*
(HHA2) $1 \in x \to x$.
(HHA3) $(x \to y) \odot x = (y \to x) \odot y$.
(HHA4) $x \to (y \to z) = (x \odot y) \to z$.
A quasi-hyper hoop $(H, \odot, \to, 1)$ is called a hyper hoop (HH) if the following conditions hold:
(HHA5) $1 \in x \to 1$.
(HHA6) *If* $1 \in x \to y$ *and* $1 \in y \to x$, *then* $x = y$.
(HHA7) *If* $1 \in x \to y$ *and* $1 \in y \to z$, *then* $1 \in x \to z$,
where we have $x \leq y$ if and only if $1 \in x \to y$, and for any $A, B \subseteq H$, $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$. If $\mathcal{H} = (H, \odot, \to, 1)$ is a hyper hoop, then $\leq$ is a partial order relation on H. Moreover, for all $A, B \subseteq H$, we define $A \ll B$ if there exist $a \in A$ and $b \in B$ such that $a \leq b$ and $A \leq B$, if for all $a \in A$ there exists $b \in B$ such that $a \leq b$. A hyper hoop $\mathcal{H}$ is bounded if, for all $x \in H$, there is an element $0 \in H$ such that $0 \leq x$. For a bounded hyper hoop H and for any $x \in H$, we consider $x' = x \to 0$.

Example 3.49. *Let $H = \{a, b, c, 1\}$ be a set. Define the hyper operations $\odot$ and $\to$ on H as follows:*

$\odot$	a	b	c	1
a	$\{a\}$	$\{a,b,c\}$	$\{a,c\}$	$\{a\}$
b	$\{a,b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b\}$
c	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{c\}$
1	$\{a\}$	$\{b\}$	$\{c\}$	$\{1\}$

$\to$	a	b	c	1
a	$\{a,1\}$	$\{b\}$	$\{c,1\}$	$\{1\}$
b	$\{a\}$	$\{b,c,1\}$	$\{b,c\}$	$\{1\}$
c	$\{a\}$	$\{b\}$	$\{b,c,1\}$	$\{1\}$
1	$\{a\}$	$\{b\}$	$\{c\}$	$\{1\}$

Then $(H, \odot, \to, 1)$ is an unbounded hyper hoop algebra.

Proposition 3.50. *Let H be a quasi hyper hoop. Then for all $x, y, z \in H$ and $A, B, C \subseteq H$ the following statements hold:*

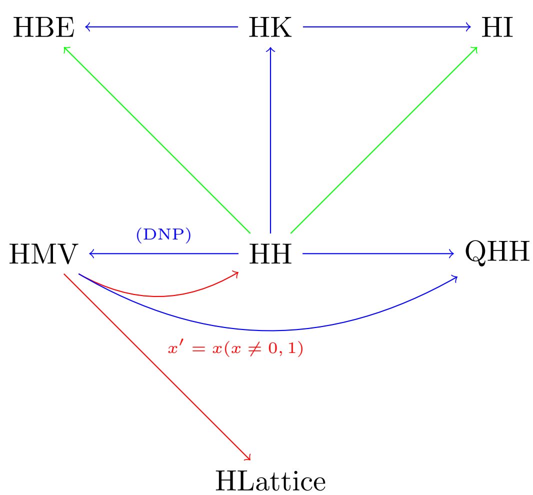
(i) $A \leq B$ *if and only if* $1 \in A \to B$,
(ii) $(A \odot B) \to C = A \to (B \to C)$,
(iii) $A \odot B \leq C$ *if and only if* $A \leq B \to C$,
(iv) $x \to (y \to z) = y \to (x \to z)$,
(v) $\{x\} \leq y \to z$ *if and only if* $\{y\} \leq x \to z$,
(vi) $x \odot (x \to y) \leq \{y\}$.

Theorem 3.51. *(i) Every hyper hoop is a hyper K-algebra.*
(ii) Every hyper hoop is a hyper I-algebra.
(iii) Every hyper hoop is a hyper BE-algebra.

Theorem 3.52. *(i) Let* $(M, \oplus, *, 0)$ *be a hyper MV-algebra such that for all* $x, y, z \in M$

$$0^* \in x^* \oplus y \quad \textit{and} \quad 0^* \in y^* \oplus z \ \textit{imply}\ 0^* \in x^* \oplus z.$$

Then $(M, \odot, \to, 1)$ *is a bounded hyper hoop, where for any* $x, y \in M$*, we define* $x \to y := x^* \oplus y$, $x \odot y := (x^* \oplus y^*)^*$ *and* $1 := 0^*$.
(ii) Let $(A, \to, \odot, 0, 1)$ *be a finite bounded hyper hoop in which* $x^{**} = \{x\}$*, for all* $x \in A$*, where* $x^* = x \to 0$*. Let for any* $x, y \in A$*, we define* $x \oplus y := (x^* \odot y^*)^*$*. Then* $(A, \oplus, *, 0)$ *is a hyper MV-algebra.*



In [13], Cheng et al, introduced a new structure, called hyper equality algebras which are a generalization of equality algebras, and they discussed relations between hyper equality algebras and other hyper structures, such as hyper EQ-algebras, hyper BCK-algebras and weak hyper residuated lattices. In [18], Hashemi et al, investigated the relations among hyper equality algebras and other hyper logical algebraic structures such as hyper K(BE, MV)-algebras and hyper hoops.

Definition 3.53. [13] *A hyper equality algebra (Heq)* $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ *is a non-empty set* H *endowed with a binary operation* $\wedge$*, a binary hyperoperation* $\sim$ *and a top element 1 such that, for all* $x, y, z \in H$*, the following axioms are fulfilled:*
(HE1) $\langle H, \wedge, 1\rangle$ *is a meet-semilattice with top element 1.*
(HE2) $x \sim y \ll y \sim x$.
(HE3) $1 \in x \sim x$.
(HE4) $x \in 1 \sim x$.
(HE5) $x \leq y \leq z$ *implies* $x \sim z \ll y \sim z$ *and* $x \sim z \ll x \sim y$.

(HE6) $x \sim y \ll (x \wedge z) \sim (y \wedge z)$.
(HE7) $x \sim y \ll (x \sim z) \sim (y \sim z)$,
where $x \leq y$ if and only if $x \wedge y = x$, and for any non-empty subsets $A, B \subseteq H$, $A \ll B$ is defined by, for all $x \in A$, there exists $y \in B$ such that $x \leq y$. A hyper equality algebra $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is called good *if, for all $x \in H$, $x = 1 \sim x$. A hyper equality algebra $\mathcal{H}$ is called* bounded, *if there is a bottom element 0 in $\mathcal{H}$. Define the following two derived operations, the "implication" and the "equivalence operation" of hyper equality algebra $\langle H, \sim, \wedge, 1\rangle$, for any $x, y \in H$, by*

$$x \to y = x \sim (x \wedge y) \quad \text{and} \quad x \leftrightarrow y = (x \to y) \wedge (y \to x).$$

If hyper equality algebra $\mathcal{H}$ is bounded, then the unary operation $*$ on $\mathcal{H}$ which, for all $x \in H$, is defined by $x^* = x \sim 0$ is called a negation. *It is clear that $x^* = x \to 0$.*

Example 3.54. *Let $H = \{0, a, b, 1\}$ be a poset such that $0 \leq a, b \leq 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows:*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\sim$	0	a	b	1
0	{1}	{1}	{b,1}	{0,a}
a	{1}	{1}	{a,1}	{a}
b	{b,1}	{a,1}	{1}	{b,1}
1	{0,a}	{a}	{b,1}	{1}

Then $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a hyper equality algebra.

Note. Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra. We say that H has (I) or (II) properties, if, for any $x, y \in H$, H satisfies the following conditions:

$$x \to (y \to z) = y \to (x \to z) \qquad , \qquad \text{(I)}$$

$$x \to y \ll (z \to x) \to (z \to y) \qquad , \qquad \text{(II)}$$

Proposition 3.55. *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra. Then, for all $x, y, z \in H$ and $A, B, C \subseteq H$, we have the following properties:*
(i) $x \leq y$ *and* $y \leq x$, *imply* $x = y$,
(ii) $1 \in x \to x$, $1 \in x \to 1$, $x \ll x \sim 1$, $x \in 1 \to x$ *and* $1 \in x \leftrightarrow x$,
(iii) $x \sim y \ll x \to y$ *and* $x \sim y \ll y \to x$,
(iv) *if* $x \leq y$, *then* $1 \in x \to y$,
(v) *if* $x \leq y \leq z$, *then* $z \sim x \ll z \sim y$ *and* $z \sim x \ll y \sim x$,
(vi) $x \ll y \to x$ *and* $A \ll B \to A$,
(vii) *if* $A \ll B$, *then* $C \to A \ll C \to B$ *and* $B \to C \ll A \to C$,
(viii) *if* $x \leq y$, *then* $x \ll y \sim x$,
(ix) $x \to y \ll (y \to z) \to (x \to z)$.

Definition 3.56. [13] *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra. Then H is called separated (S), if $1 \in x \sim y$ implies $x = y$, for all $x, y \in H$.*

Theorem 3.57. *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a separated hyper equality algebra with properties (I) and (II). Then $(H, \circ, 0)$ is a hyper K-algebra, where, for any $x, y \in H$, $x \circ y = y \sim (y \wedge x)$.*

Corollary 3.58. *Every separated and symmetric hyper equality algebra that satisfies in the condition (I), is a hyper K-algebra.*

Theorem 3.59. *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a separated hyper equality algebra that satisfies in the condition (I). Then $(H, \circ, 1)$ is a hyper BE-algebra, where, for any $x, y \in H$, $x \circ y := x \to y$.*

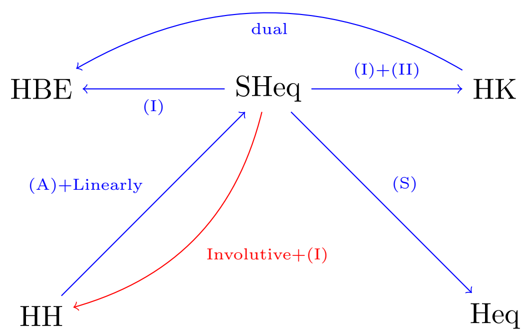
Theorem 3.60. *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be an involutive separated hyper equality algebra such that, for all $x, y \in H$, $x \to y = y^* \to x^*$ and satisfies in condition (I). Then $(H, \odot, \to, 1)$ is a hyper hoop, where for all $x, y \in H$, $x \odot y = (x \to y^*)^*$ and $(x \to y) \odot x = x \wedge y$.*

Note. Let H be a hyper hoop. We say that H satisfies in condition (A), if for any $x, y, z \in H$,

$$x \to ((y \to z) \odot y) = ((x \to y) \to (x \to z)) \odot (x \to y). \qquad \text{(A)}$$

Theorem 3.61. *Every linearly ordered hyper hoop $(H, \odot, \to, 1)$ that satisfying the condition (A), is a separated hyper equality algebra.*

In the following diagram we show the relation among hyper equality algebra with other hyper logical algebras.



In [32], Xin put forth the concept of hyper BL-algebras which is a generalization of BL-algebras. He gave some non-trivial examples and properties of hyper BL-algebras. He discussed the conditions in which a quotient hyper BL-algebra is an MV-algebra.

Definition 3.62. [32] *By a hyper BL-algebra (HBL) we mean a non-empty set L endowed with four binary hyperoperations $\wedge, \vee, \odot, \to$ and two constants 0 and 1 satisfying the following conditions:*
$(HBL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded superlattice,
$(HBL2)$ $(L, \odot, 1)$ is a commutative semihypergroup with 1 as an identity,
$(HBL3)$ $x \circ z \ll y$ if and only if $z \ll x \to y$,
$(HBL4)$ $x \wedge y \ll x \odot (x \to y)$,
$(HBL5)$ $1 \in (x \to y) \vee (y \to x)$,

where $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$, for all non-empty subsets A and B of L.

Example 3.63. *Let $(L = \{0, a, b, c, 1\}, \leq)$ be a partially ordered set such that $0 \leq c \leq a \leq b \leq 1$. Then for all $x, y \in L$ define the binary hyperoperations $\wedge$, $\vee$ and $\odot$ on L as follows:*

$$x \vee y = \{z \mid x \leq z, y \leq z\} \text{ and } x \odot y = x \wedge y = \{z \mid z \leq x, z \leq y\}.$$

Now, let $\to$ be a hyperoperation on L defined by the following table:

$\to$	0	a	b	c	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{0,1\}$	$\{1\}$	$\{1\}$	$\{c,1\}$	$\{1\}$
b	$\{0,1\}$	$\{a,b,1\}$	$\{1\}$	$\{c,1\}$	$\{1\}$
c	$\{0,1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
1	$\{0,1\}$	$\{a,b,1\}$	$\{b,1\}$	$\{c,1\}$	$\{1\}$

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ *is a hyper BL-algebra.*

Proposition 3.64. *Let* $(L, \wedge, \vee, \odot, \to, 0, 1)$ *be a hyper BL-algebra. Then for any* $x, y, z \in L$ *the following conditions hold:*
(i) $x \to y \ll ((x \to y) \to y) \to y$,
(ii) $0 \in x \odot (x \to 0)$,
(iii) if $x \leq y$ *and* $x \leq z$, *then* $x \ll y \vee z$,
(iv) $x \wedge y \ll x \ll x \vee y$,
(v) $x \to (y \wedge x) \ll x \to y$.

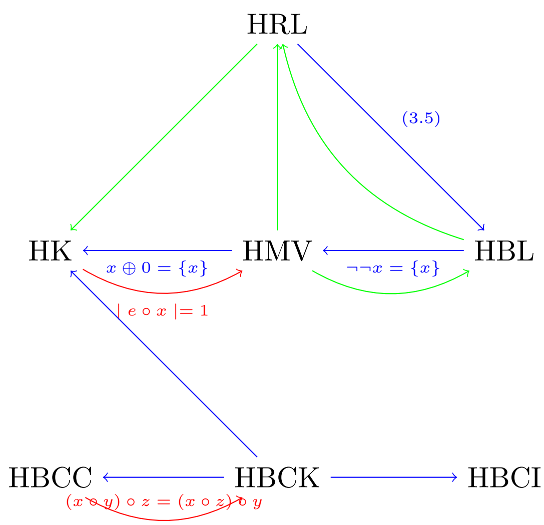
Theorem 3.65. *Any hyper BL-algebra is a hyper residuated lattice.*

Theorem 3.66. *Let* $(L, \wedge, \vee, \odot, \to, 0, 1)$ *be a hyper residuated lattice such that for any* $x, y \in L$ *we have*

$$\neg(x \odot \neg y) \odot \neg y = \neg(y \odot \neg x) \odot \neg x, \tag{3.5}$$

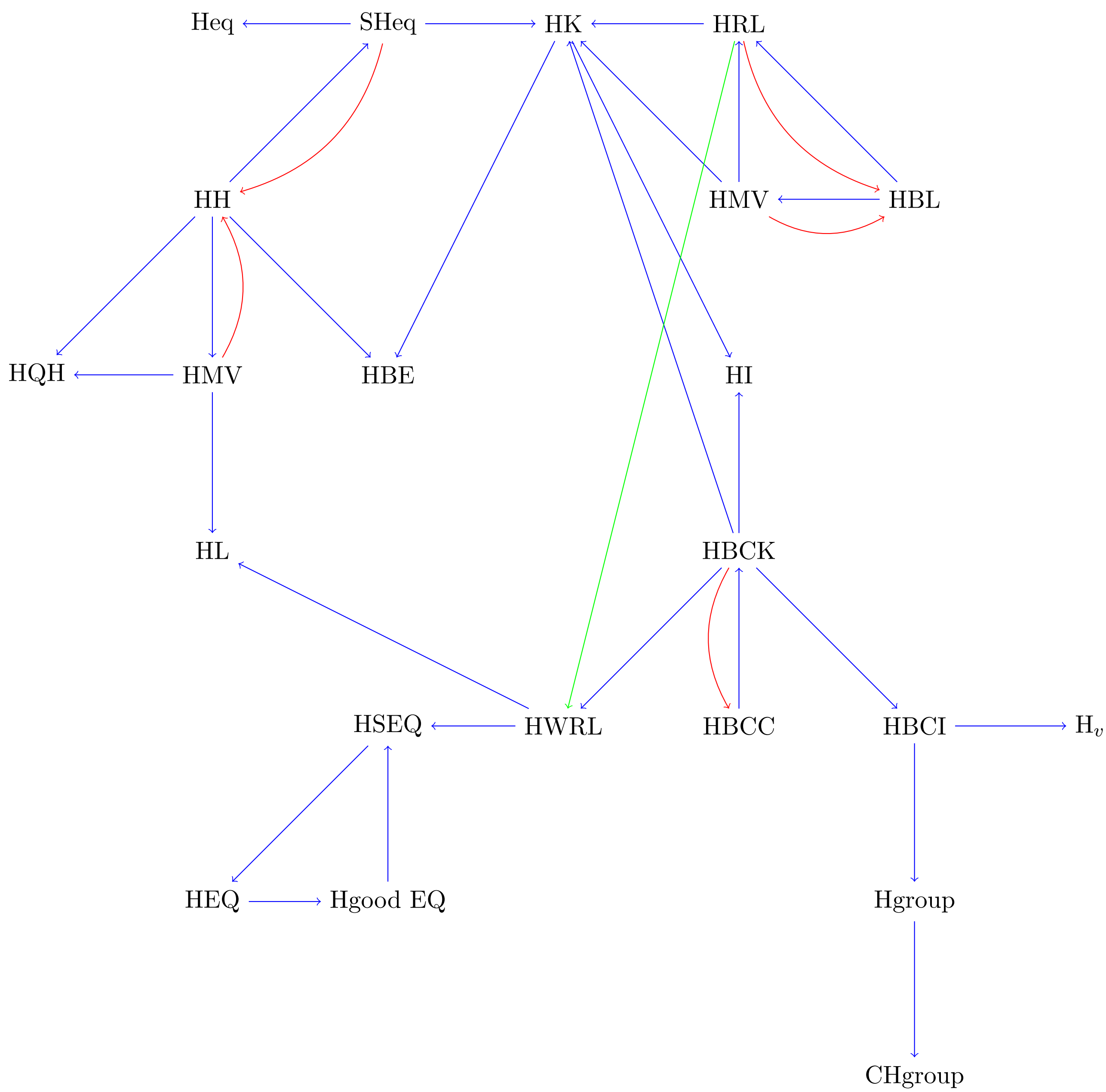
where $\neg x = x \to 0$. *Then* L *is a hyper BL-algebra.*

In the following diagram, we show the relation among hyper residuated lattice with other hyper logical algebras.



4 Conclusions

In this section we summarize the relations among some hyper logical algebras such as hyper BCK-algebra, hyper K-algebra, hyper residuated lattices, hyper BL-algebras, hyper MV-algebras, hyper EQ-algebras, hyper BE-algebras, hyper equality algebras, hyper hoops and etc. in the following diagram:



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