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From rings to minimal H_v-fields

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Abstract

The class of H_v-structures is the largest class of hyperstructures defined on the same set. For this reason, they have applications in mathematics and in other sciences, which range from biology, hadronic physics, leptons, linguistics, sociology, to mention but a few. They satisfy the weak axioms where the non-empty intersection replaces equality. The fundamental relations connect, by quotients, the H_v-structures with the classical ones. In order to specify the appropriate hyperstructure as a model for an application which fulfill a number of properties, the researcher can start from the basic ones. Thus, the researcher must know the minimal hyperstructures. H_v-numbers are elements of H_v-field, and they are used in representation theory. In this presentation we focus on minimal H_v-fields derived from rings.

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1 Introduction

The class of the H_v-structures, introduced by Vougiouklis in 1990 [13], is the largest class of hyperstructures. In the classical hyperstructures, in any axiom where the equality is used, if we replace the equality by the non-empty intersection, then we obtain a corresponding H_v-structures. The new axioms are called weak and this replacement leads to a partial order on the H_v-structures defined on the same set. A related generalization of H_v-structures are the h/v-structures. The number of H_v-structures defined on a set is extremely greater than the number of the classical hyperstructures defined on the same set. This fact leads the H_v-structures to admit more applications, because they can satisfy more weak properties. In applications in physics new H_v-fields are needed, especially defined on finite small sets where, moreover, the results have small number of elements. In this direction, we study small H_v-fields which are obtain from classical rings and have more properties as the existence of one only unit element and each element has only one inverse element.

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2 Preliminaries

In classical algebra, the quotient of a group by an invariant subgroup, is a group. In 1934 F. Marty introduced, for the first time, the hyperstructures defining the hypergroup which, in fact, is the quotient of a group by any subgroup. The largest class of hyperstructures where introduced by T. Vougiouklis in 1990 [13], [16] by defining the H_v-group. The motivation to introduce the H_v-structures is that the quotient of a group by any partition (equivalence) is an H_v-group.

The object of this paper is the class of H_v-structures, satisfy the weak axioms where the non-empty intersection replaces equality.

Definition 2.1. *Algebraic hyperstructure* $(H,\cdot)$ *is a set* H *equipped with a hyperoperation (abbreviated: hope)* $\cdot : H \times H \to P(H) - \{\varnothing\}$. *We abbreviate by WASS the weak associativity:* $(xy)z \cap x(yz) \neq \varnothing$, $\forall x,y,z \in H$ *and by COW the weak commutativity:* $xy \cap yx \neq \varnothing, \forall x,y \in H$. *The algebraic hyperstructure* $(H,\cdot)$ *is an* H_v*-semigroup if it is WASS, it is called* H_v*-group if it is reproductive* H_v*-semigroup, i.e.,* $xH = Hx = H$, $\forall x \in H$.

In an H_v-semigroup the *powers* are: $h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h$, where $(\circ)$ is the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An $(H,\cdot)$ is called *cyclic of period s*, if there exists an element h, called *generator*, and the minimum s such that $H = h^1 \cup h^2 ... \cup h^s$. Analogously the cyclicity for the infinite period is defined. If there are h and s, the minimum one such that $H = h^s$, then $(H,\cdot)$ is a *single-power cyclic of period s*.

Definition 2.2. *The hyperstructure* $(R,+,\cdot)$ *is called* H_v*-ring if* $(+)$ *and* $(\cdot)$ *are WASS, the reproduction axiom is valid for* $(+)$ *and* $(\cdot)$ *is weak distributive to* $(+)$*:*

$$x(y+z) \cap (xy+xz) \neq \varnothing,\ (x+y)z \cap (xz+yz) \neq \varnothing,\ \forall x,y,z \in R.$$

Let $(R,+,\cdot)$ *be an* H_v*-ring, a COW* H_v*-group* $(M,+)$ *is called* H_v*-module over* R*, if there is an external hope*

$$\cdot : R \times M \to P(M) : (a,x) \to ax$$

such that for all $a,b \in R$ *and for all* $x,y \in M$ *we have*

$$a(x+y) \cap (ax+ay) \neq \varnothing,\ (a+b)x \cap (ax+bx) \neq \varnothing \text{ and } (ab)x \cap a(bx) \neq \varnothing.$$

Definition 2.3. *Let* $(H,\cdot)$ *and* $(H,*)$ *be two* H_v*-semigroups, the hope* $(\cdot)$ *is smaller than* $(*)$*, and* $(*)$ *is greater than* $(\cdot)$*, iff there exists an automorphism*

$$f \in Aut(H,*) \text{ such that } xy \subset f(x*y),\ \forall x,y \in H.$$

We write $\cdot \leq *$ *and say that* $(H,*)$ *contains* $(H,\cdot)$*. If* $(H,\cdot)$ *is a classical structure, then it is basic structure and* $(H,*)$ *is* $H_b -$ *structure.*

The Little Theorem. Greater hopes than the ones which are WASS or COW, are WASS or COW, respectively. Thus, a partial order and posets on H_v*-structures is defined.*

Definition 2.4. *Minimal is called an* H_v*-group if contains no other* H_v*-group defined on the same set. We extend this definition to any* H_v*-structures with any more properties.*

The partial order on H_v-structures and the Little Theorem states that the number of the H_v-structures defined on a same finite set is extremely bigger comparing to the classical hyperstructures where the equality is valid. That means that the H_v-structures admit more axioms, thus, more applications [6], [7], [8], [16], [27].

Definition 2.5. *Let $(H,\cdot)$ be a hypergroupoid. We remove $h \in H$, if we take the restriction of $(\cdot)$ in $H-\{h\}$. $\underline{h} \in H$ absorbs $h \in H$ if we replace h by $\underline{h}$. $\underline{h} \in H$ merges with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ a class with representative $\underline{h}$.*

For more definitions and applications on H_v-structures one can see in books and papers as [1], [2], [4], [5], [7], [17], [18], [24].

M. Koskas in 1970, introduced in hypergroups the relation β^*, which connects hypergroups with groups and it is defined in H_v-groups as well. Vougiouklis [1], [12], [13], [16], [17], [19], [24], [26] introduced the γ^* and ϵ^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations, *fundamental*.

Definition 2.6. *The fundamental relations β^*, γ^* and ϵ^*, are defined, in H_v-groups, H_v-rings and H_v-vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.*

Let $(G,\cdot)$ be a group and R any partition in G, then $(G/R,\cdot)$ is H_v-group, therefore the quotient $(G/R,\cdot)/\beta^*$ is a group, the *fundamental* one.

Theorem 2.7. *In H_v-group $(H,\cdot)$ denote by U the set of all finite products. Define the relation β in H by: $x\beta y$ iff $\{x,y\} \subset u$ where $u \in U$. Then β^* is the transitive closure of β.*

Analogous theorems are for H_v-rings, H_v-vector spaces and so on [16].

Theorem 2.8. *Let $(R,+,\cdot)$ be an H_v-ring. Denote U all finite polynomials of elements of R. Define the relation γ in R by:*

$$x\gamma y \text{ iff } \{x,y\} \subset u \text{ where } u \in U.$$

Then the relation γ^* is the transitive closure of the relation γ.

Proof. Let $\underline{\gamma}$ be the transitive closure of γ, and denote by $\underline{\gamma}(a)$ the class of a. First, we prove that the quotient set $R/\underline{\gamma}$ is a ring.

In $R/\underline{\gamma}$ the sum $(\oplus)$ and the product $(\otimes)$ are defined in the usual manner:

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \{\underline{\gamma}(c) : c \in \underline{\gamma}(a) + \underline{\gamma}(b)\},$$

$$\underline{\gamma}^*(a) \otimes \underline{\gamma}(b) = \{\underline{\gamma}(d) : d \in \underline{\gamma}^*(a) \cdot \underline{\gamma}(b)\},\ \ \forall a,b \in R.$$

Take $a' \in \underline{\gamma}(a)$ and $b' \in \underline{\gamma}(b)$. Then we have

$$a'\underline{\gamma}a \text{ iff } \exists x_1,...,x_{m+1} \text{ with } x_1 = a', x_{m+1} = a \text{ and } u_1,...,u_m \in U,$$

$$\text{such that } \{x_i, x_{i+1}\} \subset u_i,\ i = 1,...,m, \text{ and}$$

$$b'\underline{\gamma}b \text{ iff } \exists y_1,...,y_{n+1} \text{ with } y_1 = b', y_{n+1} = b \text{ and } v_1,...,v_n \in U,$$

$$\text{such that } \{y_j, y_{j+1}\} \subset v_j,\ i = 1,...,n.$$

From the above we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1,\ \ i = 1,...,m-1,$$

$$x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j,\ \ j = 1,...,n.$$

The sums
$$u_i + v_1 = t_i,\ i = 1, ...m-1 \text{ and } u_m + v_j = t_{im+j-1},\ j = 1, ..., n,$$
are also polynomials, thus, $t_k \in U,\ \forall k \in \{1, ..., m+n-1\}$.

Now, pick up $z_1, ..., z_{m+n}$ such that $z_i \in x_i + y_1,\ i = 1, ..., n$ and $z_{m+j} \in x_{m+1} + y_{j+1},\ j = 1, ..., n$, thus, using the above relations we obtain $\{z_k, z_{k+1}\} \subset t_k,\ k = 1, ..., m+n-1$.

Thus, every $z_1 \in x_1 + y_1 = a' + b'$ is $\underline{\gamma}$ equivalent to every $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. So $\underline{\gamma}(a) \oplus \underline{\gamma}(b)$ is a singleton so we can write
$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \underline{\gamma}(c),\ \forall c \in \underline{\gamma}(a) + \underline{\gamma}(b).$$
By the similar way, we prove that
$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \underline{\gamma}(d),\ \forall d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b).$$
The WASS and the weak distributivity on R guarantee that associativity and distributivity are valid for R/γ^*. Therefore, R/γ^* is a ring.

Let σ be an equivalence relation in R such that R/σ is a ring and $\sigma(a)$ the class of a. Then $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons $\forall a, b \in R$, i.e.
$$\sigma(a) \oplus \sigma(b) = \sigma(c), \forall c \in \sigma(a) + \sigma(b) \text{ and } \sigma(a) \otimes \sigma(b) = \sigma(d), \forall d \in \sigma(a) \cdot \sigma(b).$$
Therefore, we write, for every $a, b \in R$, $A \subset \sigma(a)$ and $B \subset \sigma(b)$, and so,
$$\sigma(a) \oplus \sigma(b) = \sigma(a+b) = \sigma(A+B),\ \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B).$$
By induction, we extend these relations on finite sums and products. Thus, for all $u \in U$, we have the relation $\sigma(x) = \sigma(u)$, for all $x \in u$. Consequently, if $x \in \gamma(a)$, then $x \in \sigma(a), \forall x \in R$. But σ is transitively closed, so we obtain: If $x \in \underline{\gamma}(x)$, then $x \in \sigma(a)$. That $\underline{\gamma}$ is the smallest equivalence relation in R such that $R/\underline{\gamma}$ is a ring, i.e. $\underline{\gamma} = \gamma^*$. □

An element is called single if its fundamental class is singleton [16].

Several classes of general hyperstructures can be defined by using the fundamental structures. From 1990 there is the following [13, 16]:

Definition 2.9. *An H_v-ring $(R, +, \cdot)$ is called H_v-field if R/γ^* is a field. An H_v-module over an H_v-field F, it is called H_v-vector space.*

The analogous to Theorem 2.8, on H_v-vector spaces, can be proved as well:

Theorem 2.10. *Let $(V, +)$ be an H_v-vector space over the H_v-field F. Denote U the set of all expressions of finite hopes either on F and V or the external hope applied on finite sets of elements of F and V. Define the relation ϵ in V as follows: $x\epsilon y$ iff $\{x, y\} \subset u$ where $u \in U$. Then ϵ^* is the transitive closure of the relation ϵ.*

Definition 2.11. *Let $(L, +)$ be an H_v-vector space over the H_v-field F, $\phi : F \to F/\gamma^*$ canonical; $\omega_F = \{x \in F : \phi(x) = 0\}$, the core, 0 is the zero of F/γ. Let ω_L be the core of $\phi' : L \to L/\epsilon^*$ and denote by 0 the zero of L/ϵ^*, as well. Take the bracket (commutator) hope:*
$$[,] : L \times L \to P(L) : (x, y) \to [x, y]$$
then L is an H_v-Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e., $\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$,

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \varnothing$$
$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \varnothing,$$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \quad \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \quad \forall x, y, z \in L$

Definition 2.12. *The H_v-semigroup $(H, \cdot)$ is an h/v-group if H/β^* is a group [20].*

The h/v-group is a generalization of an H_v-group where a *reproductivity of classes* is valid: if $\sigma(x), \forall x \in H$ are equivalence classes, then $x\sigma(y) = \sigma(xy) = \sigma(x)y, \forall x, y \in H$. Similarly *$h/v$-rings*, *$h/v$-fields* and *$h/v$-vector spaces* etc, are defined.

3 Hopes, representations and applications

An extreme class of H_v-structures and related constructions, introduced in [14], [16], [18], [20], [23], [24], [25], are the following:

Definition 3.1. *An H_v-structure is called very thin if there exists a pair $(a, b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products are singletons.*

From the very thin hopes the Attach Construction is obtained: Let $(H, \cdot)$ be an H_v-semigroup and $v \notin H$. We extend the hope $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by:

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

The $(\underline{H}, \cdot)$ is an H_v-group, where $(\underline{H}, \cdot)/\beta^* \cong Z_2$ and v is a single. Denote $[x]$ the fundamental class of $\forall x \in H$, then the Unit class is the class $[e]$ if

$$([e] \cdot [x]) \cap [x] \neq \varnothing \text{ and } ([x] \cdot [e]) \cap [x] \neq \varnothing, \forall x \in H,$$

and $\forall x \in H$, we call inverse class of $[x]$, the class $[x]^{-1}$, if

$$([x] \cdot [x]^{-1}) \cap [e] \neq \varnothing \text{ and } ([x]^{-1} \cdot [x]) \cap [e] \neq \varnothing.$$

Definition 3.2. *Enlarged hopes is one if new elements appear in results. Let $(H, \cdot)$ be an H_v-semigroup, $v \notin H$. We extend $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by: $x \cdot v = v \cdot x = v, \forall x \in H$, and $v \cdot v = H$. The $(\underline{H}, \cdot)$ is an h/v-group, called attach, where $(\underline{H}, \cdot)/\beta^* \cong Z_2$ and v is single. If $(H, \cdot)$ is COW then $(\underline{H}, \cdot)$ is COW. Let $(H, \cdot)$ be an H_v-semigroup, $v \notin H$ and $(\underline{H}, \cdot)$ its attached h/v-group. Take $0 \notin H$ and define in $\underline{H}_\circ = H \cup \{v, 0\}$ a hypersum $\forall x, y \in H$,*

$$(+) : 0 + 0 = x + v = v + x = 0, \; 0 + v = v + 0 = x + y = v, \; 0 + x = x + 0 = v + v = H,$$

and a hyperproduct is the same in $\underline{H}$ and $0 \cdot 0 = v \cdot x = x \cdot 0 = 0, \forall x \in \underline{H}$. Then $(\underline{H}_o, +, \cdot)$ is an h/v-field with $(\underline{H}_o, +, \cdot)/\gamma^* \cong Z_3$. The operations $(+)$ is associative, $(\cdot)$ is WASS and weak distributive to $(+)$. 0 is zero absorbing in $(+)$. Hence, $(\underline{H}_o, +, \cdot)$ is the attached h/v-field of $(H, \cdot)$.

Definition 3.3. [7, 18, 21, 23, 25] *Let $(G, \cdot)$ be a groupoid and $f : G \to G$ be a map. We define a hope (∂) called theta-hope, we write ∂-hope, on G as follows,*

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \quad \forall x, y \in G.$$

If $(\cdot)$ is commutative, then ∂ is commutative. If $(\cdot)$ is COW, then ∂ is COW. Property: If $(G, \cdot)$ is a groupoid and $f : G \to P(G) - \{\varnothing\}$ be a multivalued map, then we define the ∂-hope on G as follows, $x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G$. Motivation for ∂-hope is the derivative, where only the product of functions is used.

Basic property: if $(G, \cdot)$ is a semigroup, then $\forall f$, the ∂-hope is WASS.

Example 3.4. *(a) In integers $(Z, +, \cdot)$ fix $n \neq 0$, a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x, \forall x \in Z - \{0\}$. Then $(Z, \partial_+, \partial.)$, where ∂_+ and $\partial.$ are the ∂-hopes refereed to the addition and the multiplication, respectively, is an H_v-near-ring, with*

$$(Z, \partial_+, \partial.)/\gamma^* \cong Z_n.$$

(b) In $(Z, +, \cdot)$ with $n \neq 0$, take f such that $f(n) = 0$ and $f(x) = x, \forall x \in Z - \{n\}$. Then $(Z, \partial_+, \partial.)$ is an H_v-ring, moreover, $(Z, \partial_+, \partial.)/\gamma^* \cong Z_n$.

Special case of the above is for $n = p$, prime, then $(Z, \partial_+, \partial.)$ is an H_v-field.

Combining uniting elements with the enlarging theory we obtain analogous results.

Theorem 3.5. *In the ring $(Z_n, +, \cdot)$, with $n = ms$ we enlarge the multiplication only in the product of the elements $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$ and the rest results remain the same. Then*

$$(Z_n, +, \otimes)/\gamma^* \cong (Z_m, +, \cdot).$$

Remark that we can enlarge other products as well, for example $2 \cdot m$ by setting $2 \otimes m = \{2, m + 2\}$, then the result remains the same. In this case 0 and 1 are scalars.

Corollary 3.6. *In the ring $(Z_n, +, \cdot)$, with $n = ps$, where p is prime, we enlarge only the product $0 \cdot p$ by $0 \oplus p = \{0, p\}$ and the rest results remain the same. Then $(Z_n, +, \oplus)$ is a very thin H_v-field.*

Hopes defined on classical structures are the following [7], [10], [11], [16]:

Definition 3.7. *Let $(G, \cdot)$ be a groupoid, then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-hopes: $\forall x, y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P})$,$(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P-hyperstructures. If $(G, \cdot)$ is a semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup.

H_v-structures used in Representation (abbr. rep) Theory of H_v-groups can be achieved by generalized permutations [15] or by H_v-matrices [7], [10], [16], [19], [23].

H_v-matrix is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices (a_{ij}) and (b_{ij}), of type $m \times n$ and $n \times r$, respectively, is defined in the usual manner, and it is a set of $m \times r$ H_v-matrices. The sum of products of elements of the H_v-ring is the *n-ary circle hope* on the hyper-sum.

The problem of the H_v-matrix (orh/v-group) reps is the following:

Definition 3.8. *Let $(H,\cdot)$ be an H_v-group. Find an H_v-ring $(R,+,\cdot)$, a set $M_R{=}\{(a_{ij})|a_{ij}{\in}R\}$, and a map $T: H \to M_R : h \mapsto T(h)$ such that*

$$T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing,\ \forall h_1, h_2 \in H.$$

T is an H_v-matrix rep. If $T(h_1h_2) \subset T(h_1)T(h_2), \forall h_1, h_2 \in H$, then T is an inclusion rep. If $T(h_1h_2) = T(h_1)T(h_2), \forall h_1, h_2 \in H$, then T is a good rep. If T is a good rep and one to one, then it is a faithful rep.

The rep problem is simplified in cases such as if the h/v-rings have scalars 0 and 1.

Theorem 3.9. *A necessary condition in order to have an inclusion rep T of an h/v-group $(H,\cdot)$ by $n \times n$ h/v-matrices over the h/v-ring $(R,+,\cdot)$ is the following: $\forall \beta^*(x)$, $x \in H$ there must exist elements $a_{ij} \in H, i,j \in \{1,...,n\}$ such that*

$$T(\beta^*(a)) \subset \{A = (a'_{ij})|a'_{ij} \in \gamma^*(a_{ij}), i,j \in \{1,...,n\}\},$$

The inclusion rep $T: H \to M_R : a \mapsto T(a) = (a_{ij})$ induces an homomorphic T^* of H/β^* on R/γ^* by $T^*(\beta^*(a)) = [\gamma^*(a_{ij})], \forall \beta^*(a) \in H/\beta^*$, where $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of $T^*(\beta^*(a))$.

An important hope on non-square matrices is defined [9], [28]:

Definition 3.10. *Let $A = (a_{ij}) \in M_{m\times n}$ and $s,t \in N$ such that $1 \le s \le m$, $1 \le t \le n$. Define a mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st} = (\underline{a}_{ij})$ with entries the sets*

$$\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} | 1 \le i \le s, 1 \le j \le t \text{ and } \kappa, \lambda \in N, i+\kappa s \le m, j+\lambda t \le n\}.$$

The map $\underline{st} : M_{m\times n}M_{s\times t} : A \to A\underline{st}(\underline{a}_{ij})$, is called helix-projection of type $\underline{st}$. $A\underline{st}$ is a set of $s \times t$-matrices $X = (x_{ij})$ such that $x_{ij} \in a_{ij}, \forall i,j$. Obviously $A\underline{mn} = A$.

Let $A = (a_{ij}) \in M_{m\times n}$ and $s,t \in N$, where $1 \le s \le m$, $1 \le t \le n$. We apply the helix-projection first on the columns and then on the rows and the result is the same: $(A\underline{sn})\underline{st} = (A\underline{mt})\underline{st} = A\underline{st}$.

Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{u\times v}$ be matrices. Denote $s = \min(m,u)$ and $t = \min(n,u)$, then we define the helix-sum by

$$\oplus : M_{m\times n} \times M_{u\times v} \to P(M_{s\times t}) : (A,B) \to A \oplus B = A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s\times t},$$

where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij})|a_{ij} \in \underline{a}_{ij}$ and $b_{ij} \in \underline{b}_{ij}\}$. Denote s=min(n,u), then we define the helix-product by

$$\otimes : M_{m\times n} \times M_{u\times v} \to P(M_{s\times t}) : (A,B) \to A \otimes B = A\underline{ms} \cdot B\underline{sv} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{m\times v},$$

where $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{ij}b_{ij})|a_{ij} \in \underline{a}_{ij}$ and $b_{ij} \in \underline{b}_{ij}\}$.

Last decades H_v-structures have applications in mathematics and in other sciences. Applications range from biology and hadronic physics or leptons to mention but a few. The hyperstructure theory is related to fuzzy one; consequently, can be widely applicable in industry and production, too [2], [5], [6], [7], [8], [22], [26].

An application, which combines ∂-hyperstructures and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis (V&V bar) [27]. They suggest the following:

Definition 3.11. *In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

0 ________________ *1*

The participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point she/he feels expresses her/his answer to the specific question.

The use of V&V bar instead of a Likert scale has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm.

The Lie-Santilli theory on isotopies was born to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n-dimensional trivial unit matrix into an appropriate new matrix. The original theory is reconstructed to admit the new matrix as left and right unit. The isofields needed in this theory correspond into the hyperstructures called e-hyperfields, introduced by Santilli & Vougiouklis in 1996 [6], [8].

Definition 3.12. *The* $(H,\cdot)$ *is called an e-hyperstructure if contains a unique scalar unit* e *and for all* x *there exists an inverse* x^{-1}, *i.e.* $e \in (x\cdot x^{-1}) \cap (x^{-1}\cdot x)$. $(F,+,\cdot)$, *where* $(+)$ *is operation and* $(\cdot)$ *is hope, is an e-hyperfield: The* $(F,+)$ *is an Abelian group with unit* 0, $(\cdot)$ *is WASS,* $(\cdot)$ *is weak distributive to* $(+)$, 0 *is absorbing:* $0\cdot x = x\cdot 0 = 0, \forall x \in F$, *there exists a scalar unit* 1, *i.e.* $1\cdot x = x\cdot 1 = x, \forall x \in F$, *and* $\forall x \in F$ *there is a unique inverse* x^{-1}.

The Main e-Construction. Given a group $(G,\cdot)$, e *unit, defines hopes* $(\otimes)$ *by:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\},$$

$(G,\otimes)$ *is an* H_b*-group which contains the* $(G,\cdot)$. $(G,\otimes)$ *is an e-hypergroup. Moreover if for all* x, y *such that* $xy = e$, *and so* $x \otimes y = xy$, *then* $(G,\otimes)$ *becomes a strong e-hypergroup.*

Example 3.13. *In quaternions* $Q = \{1,-1,i,-i,j,-j,k,-k\}$ *with* $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$ *denote* $\underline{i} = \{i,-i\}, \underline{j} = \{j,-j\}, \underline{k} = \{k,-k\}$, *define hopes* $(*)$ *by enlarging few products. For example,* $(-1) * k = \underline{k}, k * i = \underline{j}$ *and* $i * j = \underline{k}$. *Then* $(Q,*)$ *is a strong e-hypergroup.*

A generalization of P-hopes used in Santilli's isotheory, is the following: Let $(G,\cdot)$ *be an Abelian group,* P *a subset of* G *with* $\#P > 1$. *Define the hope* $\times_p$ *as follows:*

$$x \times_p y = \begin{cases} x\cdot P\cdot y = \{x\cdot h\cdot y | h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x\cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e*-hope. The hyperstructure* $(G,\times_p)$ *is an Abelian* H_v*-group.*

4 Very thin minimal h/v-fields

The uniting elements method, introduced by Corsini & Vougiouklis in 1989, is the following [3]: Let G be a structure and a not valid property d, described by a set of equations. Take the partition in G for which put in the same class, all pairs of elements that causes the non-validity of d. The quotient by this partition G/d is an H_v-structure. Then, quotient out G/d by β^*, is a stricter structure $(G/d)/\beta^*$ for which d is valid.

Theorem 4.1. *Let $(R, +, \cdot)$ be a ring, $F = \{f_1, ..., f_m, f_{m+1}, ..., f_{m+n}\}$ be a system of equations on R consisting of subsystems $F_m = \{f_1, ..., f_m\}$ and $F_n = f_{m+1}, ..., f_{m+n}\}$. Let σ and σ_m be the equivalence relations defined by the uniting elements using the F and F_m respectively, and σ_n be the equivalence defined using on F_n on the ring $R_m = (R_m/\sigma_n)/\gamma^*$. Then*

$$(R/\sigma)/\gamma^* \cong (R_m/\sigma_n)/\gamma^*$$

Basic general results on the topic can be found in [3], [16]. In this paper, we need some properties, so we focus on them.

Commutativity. Let $(G, \cdot)$ be a groupoid. We unite any two elements a and b of G if there exist a pair $(x, y) \in G^2$ such that $xy = a, yx = b$, and we take the transitive closure. Then, the quotient set is an h/v-commutative groupoid so, divided by β^*, a commutative groupoid is obtained.

Theorem 4.2. *Let $(S, \cdot)$ be a commutative semigroup with at least one element $u \in S$ such that the set uS is finite. Consider the transitive closure R^* of the relation R defined as follows:*

$$s_1 R s_2 \text{ iff there exists } x \in S \text{ such that } xs_1 = xs_2.$$

Then $< S/R^*, \circ > /\beta^*$ is a finite commutative group, where $(\circ)$ is the induced operation on classes of S/R^*.

Proof. Lets suppose that a and b are equivalent elements of S, i.e. aR^*b. That means that there are elements $x_1, \ldots, x_{n+1}, \mu_1, \ldots \mu_n$ of S such that

$$x_1 a = x_1 \mu_1, x_2 \mu_1 = x_2 \mu_2, \ldots x_n \mu_{n-1} = x_n \mu_n, x_{n+1} \mu_n = x_{n+1} b.$$

From these relations we obtain that

$$x_{n=1} \ldots x_2 x_1 a = x_{n+1} \ldots x_2 x_1 \mu_1 = x_{n+1} \ldots x_2 x_1 \mu_2 = x_{n+1} \ldots x_2 x_1 b,$$

so setting $x = x_{n+1} \ldots x_2 x_1$, we have $xa = xb$. Therefore, it is proved that $R^* \cong R$.

In the following, let's denote by a the class of the element a and $\underline{S} = S/R = S/R^*$. Consider the mapping $f : \underline{S} \to \{\underline{z} | z \in xS\} : \underline{s} \to \underline{xs}$, for which we have

$$\underline{s} = \underline{s}' \Rightarrow xR's \Rightarrow \exists p : ps = ps' \Rightarrow xps = xps' \Rightarrow p(xs) = p(xs)' \Rightarrow xs = xs'.$$

Vice versa let $\underline{ys} = \underline{ys}'$. Then there exists q such that

$$qys = qys' \Rightarrow (qy)s = qys' \Rightarrow \underline{s} = \underline{s}'.$$

Therefore, f is a bijection. From the above we have

$$|us| \geq |\{\underline{z} : z \in xS\}| = |\underline{S}|.$$

Consider the subset $\underline{xS} = \{\underline{xs} : x \in S\}$ of $\underline{S}$. Then we have $|\underline{xS}| = \underline{S}$ and since $|\underline{S}|$ is finite, it follows that $\underline{xS} = \underline{S}$. So, for every $\underline{y} \in \underline{S}$, there exists an element $z \in S$ such that $\underline{y} = \underline{xz}$ but $xz \in \underline{x} \cdot \underline{z}$, so $\underline{xz} \in x \circ z$ or $\underline{y} \in \underline{x} \circ \underline{z}$.

Therefore, the reproduction axiom is satisfied for $< \underline{S}, \circ >$ and S/β^* is a finite commutative group. □

Algorithm 4.3. *It is clear, that the uniting elements method can be applied to obtain several properties in different order. Moreover, if we apply the method to obtain one property the result covers more properties. On the other hand, some of the properties are easy to apply, for example the commutativity, and maybe it is not necessary to apply the total relation, in all pairs of elements, but we can apply in one only pair and we can reach the property or the properties. Therefore, we suggest in applications to apply the uniting elements method according to the following algorithm: We select the simplest, for the uniting elements, property and apply the method for one pair of elements and then for the second pair and so on. In every step we check all properties which are valid.*

Now, we focus on *Very Thin minimal* H_v*-fields* obtained by a classical field.

Theorem 4.4. *In a field* $(F,+,\cdot)$*, we enlarge only in the product of the special elements* a *and* b*, by setting* $a\otimes b=\{ab,c\}$*, where* $c\neq ab$*, and the rest results remain the same. Then we obtain the degenerate, minimal very thin,* H_v*-field*

$$(F,+,\otimes)/\gamma^* \cong \{0\}.$$

Thus, there is no non-degenerate H_v*-field obtained by a field by enlarging any product.*

Proof. Take any $x\in F-\{0\}$, then from $a\otimes b=\{ab,c\}$ we obtain

$$(a\otimes b)-ab=\{0,c-ab\} \text{ and then } (x(c-ab)^{-1})\otimes((a\otimes b)-ab)=\{0,x\},$$

thus, $0\gamma x, x\in F-\{0\}$. Which means that every x is in the same fundamental class with the element 0. Thus, $(F+\otimes)/\gamma^*\cong\{0\}$. □

Theorem 4.5. *In a field* $(F,+,\cdot)$*, we enlarge only in the sum of the special elements* a *and* b*, by setting* $a\oplus b=\{ab,c\}$*, where* $c\neq a+b$*, and the rest results remain the same. Then we obtain the degenerate, minimal very thin,* H_v*-field* $(F,\oplus,\cdot)/\gamma^*\cong\{0\}$*. Thus, there is no non-degenerate* H_v*-field obtained by a field by enlarging any sum.*

Proof. Take any $x\in F-\{0\}$, then from $a\oplus b=\{a+b,c\}$ we obtain

$$(a\oplus b)-(a+b)=\{0,c-(a+b)\}$$

and then

$$[x(c-(a+b))^{-1}]\oplus[(a\oplus b)-(a+b)]=\{0,x\},$$

thus, $0\gamma x, x\in F-\{0\}$. Which means that every x is in the same fundamental class with the element 0. Thus,

$$(F\oplus, cdot)/\gamma^*\cong\{0\}.$$

□

The above two theorems state that there is no non-degenerate H_v-field obtained by a field by enlarging any sum or product.

The *small non-degenerate h/v-fields* on $(Z_n,+,\cdot)$ in iso-theory, satisfy the following:

1. Very thin minimal,
2. COW,

3. They have the elements 0 and 1, scalars,

4. If an element has inverse element, this is unique.

Therefore, we cannot enlarge the result if it is 1 and we cannot put 1 in enlargement.

Theorem 4.6. [25] *All multiplicative h/v-fields defined on $(Z_4,+,\cdot)$, with non-degenerate fundamental field, satisfying the above four conditions, are the following isomorphic cases: The only product which is set is $2\otimes 3=\{0,2\}$ or $3\otimes 2=\{0,2\}$. Fundamental classes: $[0]=\{0,2\},[1]=\{1,3\}$ and we have*

$$(Z_4,+,\otimes)/\gamma^*\cong(Z_2,+,\cdot).$$

Example 4.7. *Denote E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Take the 2×2 upper triangular h/v-matrices on the above h/v-field $(Z_4,+,\otimes)$ of the case that only $2\otimes 3=\{0,2\}$ is a hyperproduct:*

$$I=E_{11}+E_{22},a=E_{11}+E_{12}+E_{22},b=E_{11}+2E_{12}+E_{22},c=E_{11}+3E_{12}+E_{22},$$

$$d=E_{11}+3E_{22},e=E_{11}+E_{12}+3E_{22},f=E_{11}+2E_{12}+3E_{22},g=E_{11}+3E_{12}+3E_{22},$$

then, we obtain for $X=\{I,a,b,c,d,e,f,g\}$, that $(X,\otimes)$ is non-COW, H_v-group where the fundamental classes are $\underline{a}=\{a,c\},\underline{d}=\{d,f\},\underline{e}=\{e,g\}$ and the fundamental group is isomorphic to $(Z_2\times Z_2,+)$. There is only one unit and every element has unique double inverse. Only f has one more right inverse element d, since $f\otimes d=\{I,b\}$. $(X,\otimes)$ is not cyclic.

Theorem 4.8. *All multiplicative h/v-fields on $(Z_6,+,\cdot)$, with non-degenerate fundamental field, satisfying the above four conditions, are the following isomorphic cases: We have the only one hyperproduct,*

(I) $2\otimes 3=\{0,3\},2\otimes 4=\{2,5\},3\otimes 4=\{0,3\},3\otimes 5=\{0,3\},4\otimes 5=\{2,5\}$. *The fundamental classes are* $[0]=\{0,3\},\ [1]=\{1,4\},\ [2]=\{2,5\}$ *and we have*

$$(Z_6,+,\otimes)/\gamma^*\cong(Z_3,+,\cdot).$$

(II) $2\otimes 3=\{0,2\}$ *or* $2\otimes 3=\{0,4\}$, $2\otimes 4=\{0,2\}$ *or* $\{2,4\},2\otimes 5=\{0,4\}$ *or* $2\otimes 5=\{2,4\}$, $3\otimes 4=\{0,2\}$ *or* $\{0,4\},3\otimes 5=\{3,5\},\ 4\otimes 5=\{0,2\}$ *or* $\{2,4\}$. *In all these cases the fundamental classes are* $[0]=\{0,2,4\},\ [1]=\{1,3,5\}$ *and we have*

$$(Z_6,+,\otimes)/\gamma^*\cong(Z_2,+,\cdot).$$

Example 4.9. *In the h/v-field $(Z_6,+,\otimes)$ where only the hyperproduct is $5\otimes 4=\{2,5\}$ take the h/v-matrices of type $\underline{i}=E_{11}+iE_{12}+4E_{22}$, where $i=0,1,...,5$, then the multiplicative table of the hyperproduct of those h/v-matrices is*

$\otimes$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{1}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{2}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$
$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{5}$	$\underline{2},\underline{5}$	$\underline{0},\underline{3}$	$\underline{1},\underline{4}$	$\underline{2},\underline{5}$	$\underline{0},\underline{3}$	$\underline{1},\underline{4}$

The fundamental classes are $[0] = \{\underline{0}, \underline{3}\}$, $[1] = \{\underline{1}, \underline{4}\}$, $[2] = \{\underline{2}, \underline{5}\}$ *and the fundamental group is isomorphic to* $(Z_3, +)$. *The* $(Z_6, \otimes)$ *is an h/v-group which is cyclic where* $\underline{2}$ *and* $\underline{4}$ *are generators of period* 4.

One can specify the analogous h/v-fields for other 'small cases' as the following.

Theorem 4.10. *All multiplicative h/v-fields defined on* $(Z_9, +, \cdot)$, *which have non-degenerate fundamental field, and satisfy the above* 4 *conditions, are the following isomorphic cases: We have the only one hyperproduct,*
$2 \otimes 3 = \{0, 6\}$ *or* $\{3, 6\}, 2 \otimes 4 = \{2, 8\}$ *or* $\{5, 8\}, 2 \otimes 6 = \{0, 3\}$ *or* $\{3, 6\}$,
$2 \otimes 7 = \{2, 5\}$ *or* $\{5, 8\}, 2 \otimes 8 = \{1, 7\}$ *or* $\{4, 7\}, 3 \otimes 4 = \{0, 3\}$ *or* $\{3, 6\}$,
$3 \otimes 5 = \{0, 6\}$ *or* $\{3, 6\}, 3 \otimes 6 = \{0, 3\}$ *or* $\{0, 6\}, 3 \otimes 7 = \{0, 3\}$ *or* $\{3, 6\}$,
$3 \otimes 8 = \{0, 6\}$ *or* $\{3, 6\}, 4 \otimes 5 = \{2, 5\}$ *or* $\{2, 8\}, 4 \otimes 6 = \{0, 6\}$ *or* $\{3, 6\}$,
$4 \otimes 8 = \{2, 5\}$ *or* $\{5, 8\}, 5 \otimes 6 = \{0, 3\}$ *or* $\{3, 6\}, 5 \otimes 7 = \{2, 8\}$ *or* $\{5, 8\}$,
$5 \otimes 8 = \{1, 4\}$ *or* $\{4, 7\}, 6 \otimes 7 = \{0, 6\}$ *or* $\{3, 6\}, 6 \otimes 8 = \{0, 3\}$ *or* $\{3, 6\}$,
$7 \otimes 8 = \{2, 5\}$ *or* $\{2, 8\}$.
In all the above cases the fundamental classes are

$$[0] = \{0, 3, 6\}, [1] = \{1, 4, 7\}, [2] = \{2, 5, 8\}, \text{ and we have}$$

$$(Z_9, +, \otimes)/\gamma^* \cong (Z_3, +, \cdot).$$

5 Conclusion

On the class of the H_v-structures, the largest class of hyperstructures, one can apply the uniting elements procedure to obtain hyperstructures where more properties, are valid. Moreover, one can add or take out elements in order to obtain hyperstructures with more new properties. In applications on physics, mainly in Lie-Santilli theory on isotopies, new H_v-fields and h/v-fields, are needed. We present some large classes of H_v-fields and h/v-fields by determining the minimal H_v-fields, obtained from small finite classical rings. We present some examples on the topic which lead to new classes of hyperstructures, as well.

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