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Fuzzy congruence relations on pseudo BE-algebras

A. Rezaei[1], A. Borumand Saeid[2] and Q. Zhan[3]

[1] *Department of Mathematics, Payame Noor University, P.o.box. 19395-3697, Tehran, Iran*
[2] *Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran*
[3] *Department of Mathematics, Beihang University (BUAA), BUA-Haidian School of Applied Technology, china*

rezaei@pnu.ac.ir, arsham@uk.ac.ir, by1709120@buaa.edu.cn

Abstract

In this paper, we introduce the concept of *fuzzy congruence relations* on a *pseudo BE-algebra* and some of properties are investigated. We show that the set of all fuzzy congruence relations is a *modular lattice* and the quotient structure induced by fuzzy congruence relations is studied.

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1 Introduction

The notion of a BE-algebra was introduced by H.S. Kim et al. [7]. A. Borumand Saeid et al. introduced some types of filters in BE-algebras [1]. Since developing algebraic models for non-commutative multiple-valued logics is a central topic in the study of fuzzy systems. R.A. Borzooei et al. generalized the notion of BE-algebras and introduced the notion of pseudo BE-algebras, pseudo subalgebras, pseudo filters and investigated some related properties [3]. A. Rezaei et al. introduced the notion of distributive pseudo BE-algebras and normal pseudo filters and proved some basic properties. They showed that in distributive pseudo BE-algebras normal pseudo filters and pseudo filters coincide [4]. L.C. Ciungu introduced the notion of commutative pseudo BE-algebras and proved that the class of commutative pseudo BE-algebras is term equivalent to the class of commutative pseudo BCK-algebras [5].

L.A. Zadeh introduced the notion of fuzzy sets and fuzzy relations [18]. Then many authors have studied about it. K.J. Lee defined the notion of ideals in pseudo BCI-algebras [9]. Fuzzy

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ideals of pseudo BCK-algebras were investigated in [6]. Also, A. Walendziak et al. consider fuzzy ideals theory in pseudo BCH-algebras and provided conditions for a fuzzy set to be a fuzzy ideal [17]. Since then V. Murali have studied fuzzy congruence relations on algebras [10, 11]. Further, M. Kondo has defined a fuzzy congruence relation on a group and showed that there is a lattice isomorphism between the set of fuzzy normal subgroups of a group and the set of fuzzy congruences on this group [8]. R.A. Borzooei et al. introduced the concept of a fuzzy filter of a BL-algebra, with respect to a t-norm and proved that there is a correspondence bijection between the set of all T-fuzzy filters of a BL-algebra and the set of all T-fuzzy congruences relations in that BL-algebra [2]. Recently, A. Rezaei et al. discussed on (fuzzy) congruence relations in (pseudo) CI/BE-algebras and studied some of their properties [13, 14, 15].

In this paper, since congruence relations are one of the important concept in algebraic structure, motivated by it, we apply the notion of fuzzy congruence relations on pseudo BE-algebras and discuss on the quotient algebras via this congruence relations. We show that quotient of a pseudo BE-algebra via a fuzzy congruence relation is a pseudo BE-algebra. Moreover, we show that in a distributive pseudo BE-algebra X for every fuzzy medial filter $\overline{\mu}$ there is a fuzzy congruence relation $\overline{\theta}$ such that $\overline{\theta}_1 = \overline{\mu}$.

2 Preliminaries

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.

Definition 2.1. [3] An algebra $(X; \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a *pseudo BE-algebra* if for all $x, y, z \in X$, it satisfies the following axioms:
(psBE$_1$) $x \to x = x \rightsquigarrow x = 1$,
(psBE$_2$) $x \to 1 = x \rightsquigarrow 1 = 1$,
(psBE$_3$) $1 \to x = 1 \rightsquigarrow x = x$,
(psBE$_4$) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
(psBE$_5$) $x \to y = 1$ if and only if $x \rightsquigarrow y = 1$.

In a pseudo BE-algebra $(X; \to, \rightsquigarrow, 1)$, one can introduce a binary relation $\leq$ on X by

$$x \leq y \Longleftrightarrow x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1,$$

for all $x, y \in X$.

Remark 2.2. If a pseudo BE-algebra X satisfies $x \to y = x \rightsquigarrow y$, for all $x, y \in X$, then X is called a BE-algebra.

Proposition 2.3. [3] In a pseudo BE-algebra X, the following statements hold:

(i) $x \to (y \rightsquigarrow x) = 1$, $x \rightsquigarrow (y \to x) = 1$,

(ii) $x \rightsquigarrow (y \rightsquigarrow x) = 1$, $x \to (y \to x) = 1$,

(iii) $x \rightsquigarrow [(x \rightsquigarrow y) \to y] = 1$, $x \to [(x \to y) \rightsquigarrow y] = 1$,

(iv) $x \to [(x \rightsquigarrow y) \to y] = 1$, $x \rightsquigarrow [(x \to y) \rightsquigarrow y] = 1$,

(v) if $x \leq y \to z$, then $y \leq x \rightsquigarrow z$,

(vi) if $x \leq y \rightsquigarrow z$, then $y \leq x \to z$,

(vii) $1 \leq x$ implies $x = 1$,

(viii) if $x \leq y$, then $x \leq z \to y$ and $x \leq z \rightsquigarrow y$, for all $x, y, z \in X$.

Definition 2.4. [4] A pseudo BE-algebra X is said to be *distributive,* if it satisfies *only one* of the following conditions:

(D_1) $x \to (y \rightsquigarrow z) = (x \to y) \rightsquigarrow (x \to z)$,

(D_2) $x \rightsquigarrow (y \to z) = (x \rightsquigarrow y) \to (x \rightsquigarrow z)$, for all $x, y, z \in X$.

Note that if $(X; \to, \rightsquigarrow, 1)$ is a pseudo BE-algebra, then $(X; \rightsquigarrow, \to, 1)$ is a pseudo BE-algebra, too. By [4, Theorem 2], if X satisfies (D_1) and (D_2), then $\to = \rightsquigarrow$ .So, in this paper, every distributive pseudo BE-algebra satisfies (D_1).
Also, note that if $x \to (z \rightsquigarrow y) = x \rightsquigarrow (z \to y)$, for all $x, y, z \in X$, then $\to = \rightsquigarrow$, since if $z := 1$ and using ($psBE_3$), we get

$$x \to y = x \to (1 \rightsquigarrow y) = x \rightsquigarrow (1 \to y) = x \rightsquigarrow y.$$

Definition 2.5. [16] A fuzzy set $\overline{\mu}$ of X is called a *fuzzy filter*, if for all $x, y \in X$, it satisfies the following conditions:

(FF_1) $\overline{\mu}(1) \geq \overline{\mu}(x)$,

(FF_2) $\overline{\mu}(y) \geq \min[\overline{\mu}(x), \overline{\mu}(x \to y)]$.

Definition 2.6. [12] A fuzzy set $\overline{\mu}$ of X is called a *fuzzy medial filter* if for all $x, y, z \in X$, it satisfies (FF_1) together with the following conditions:

(FMF_1) $\overline{\mu}(x \to y) \geq \min[\overline{\mu}(x \to z), \overline{\mu}(z \to y)]$,

(FMF_2) $\overline{\mu}(x \rightsquigarrow y) \geq \min[\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)]$.

Note. From now on, X denote a pseudo BE-algebra, unless otherwise is stated.

3 Fuzzy congruence relations in pseudo BE-algebras

In this section, we discussed the basic properties of fuzzy congruence relations on pseudo BE-algebras. Let X be a pseudo BE-algebra. A fuzzy relation $\overline{\theta}$ on X is a map $\overline{\theta} : X \times X \to [0, 1]$.

Definition 3.1. A fuzzy relation $\overline{\theta}$ is called a *fuzzy congruence relation* on X if it satisfies the following conditions: for all $x, y, z, u \in X$,

(FC_1) $\overline{\theta}(x, x) = \overline{\theta}(1, 1)$,

(FC_2) $\overline{\theta}(x, y) = \overline{\theta}(y, x)$,

(FC_3) $\overline{\theta}(x, z) \geq \sup_{y \in X} \min[\overline{\theta}(x, y), \overline{\theta}(y, z)]$,

(FC_4) $\overline{\theta}(x \to u, y \to u) \geq \overline{\theta}(x, y)$ and $\overline{\theta}(x \rightsquigarrow u, y \rightsquigarrow u) \geq \overline{\theta}(x, y)$ (right compatible),

(FC_5) $\overline{\theta}(u \to x, u \to y) \geq \overline{\theta}(x, y)$ and $\overline{\theta}(u \rightsquigarrow x, u \rightsquigarrow y) \geq \overline{\theta}(x, y)$ (left compatible).

Let FCon(X) denote the set of all fuzzy congruence relations on X.

Example 3.2. Let $X = \{1, a, b, c\}$ and the binary operations $\to$ and $\rightsquigarrow$ defined as follows:

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

$\rightsquigarrow$	1	a	b	c
1	1	a	b	c
a	1	1	c	1
b	1	1	1	1
c	1	a	b	1

Then $(X; \to, \rightsquigarrow, 1)$ is a pseudo BE-algebra. Define $\overline{\theta} : X \times X \to [0, 1]$ as follows:

$\overline{\theta}$	1	a	b	c
1	0.87	0.32	0.32	0.32
a	0.32	0.87	0.32	0.32
b	0.32	0.32	0.87	0.32
c	0.32	0.32	0.32	0.87

Then $\overline{\theta}$ is a fuzzy congruence relation on X.

Lemma 3.3. *If $\overline{\theta}$ satisfies* (FC_2)*,* (FC_3) *and* (FC_4)*, then* (FC_1) *is equivalent to*

$$\overline{\theta}(1, 1) \geq \overline{\theta}(x, y), \text{ for all } x, y \in X.$$

Proof. Assume that $\overline{\theta}(1, 1) = \overline{\theta}(x, x)$. Since $\overline{\theta}$ satisfies (FC_2) and (FC_3), we have

$$\overline{\theta}(1, 1) = \overline{\theta}(x, x) \geq \sup_{y \in X} \min[\overline{\theta}(x, y), \overline{\theta}(y, x)] = \sup_{y \in X} \overline{\theta}(x, y) \geq \overline{\theta}(x, y).$$

Conversely, using (FC_4) we get $\overline{\theta}(x, x) = \overline{\theta}(1 \to x, 1 \to x) \geq \overline{\theta}(1, 1)$. On the other hand, since $\overline{\theta}(x, x) \leq \overline{\theta}(1, 1)$, we get $\overline{\theta}(x, x) = \overline{\theta}(1, 1)$. □

Proposition 3.4. *If $\overline{\theta} \in \mathrm{FCon}(X)$, then for all $x, y \in X$*

$$\overline{\theta}(x, y) \leq \overline{\theta}(x \to y, 1) = \overline{\theta}(x \rightsquigarrow y, 1).$$

Proof. Let $x, y \in X$. Using (FC_4) and $(psBE_1)$, we have

$$\overline{\theta}(x, y) \leq \overline{\theta}(x \to y, y \to y) = \overline{\theta}(x \to y, 1).$$

Applying (FC_4), (FC_5), $(psBE_4)$, $(psBE_1)$ and $(psBE_2)$ we get

$$\begin{aligned} \overline{\theta}(x \to y, 1) &\leq \overline{\theta}((x \to y) \to y, 1 \to y) \\ &\leq \overline{\theta}(x \rightsquigarrow ((x \to y) \to y), x \rightsquigarrow (1 \to y)) \\ &= \overline{\theta}((x \to y) \rightsquigarrow (x \to y), 1 \to (x \rightsquigarrow y)) \\ &= \overline{\theta}(1, x \rightsquigarrow y) \end{aligned}$$

Thus, $\overline{\theta}(x \to y, 1) \leq \overline{\theta}(1, x \rightsquigarrow y) = \overline{\theta}(x \rightsquigarrow y, 1)$. By a similar argument $\overline{\theta}(x \rightsquigarrow y, 1) \leq \overline{\theta}(1, x \to y) = \overline{\theta}(x \to y, 1)$. Therefore, $\overline{\theta}(x \to y, 1) = \overline{\theta}(x \rightsquigarrow y, 1)$. □

Note that, in the Proposition 3.4, if $\overline{\theta}(x, y) = \overline{\theta}(x \to y, 1)$, for all $x, y \in X$, then $\overline{\theta}(x, y) = \overline{\theta}(1, 1)$. Using $(psBE_3)$, we have $\overline{\theta}(x, 1) = \overline{\theta}(x \to 1, 1) = \overline{\theta}(1, 1)$, and so, for all $x \in X$, we have $\overline{\theta}(x, 1) = \overline{\theta}(1, 1)$. Since $x \to y \in X$, we obtain $\overline{\theta}(x \to y, 1) = \overline{\theta}(1, 1)$. Thus, $\overline{\theta}(x, y) = \overline{\theta}(1, 1)$.

Proposition 3.5. *Let* $\overline{\theta}, \overline{\eta} \in \mathrm{FCon}(X)$*. Then* $\overline{\theta} \cap \overline{\eta} \in \mathrm{FCon}(X)$*, where*

$$(\overline{\theta} \cap \overline{\eta})(x, y) = \min[\overline{\theta}(x, y), \overline{\eta}(x, y)].$$

Remark 3.6. *However, the following example shows that union of two fuzzy congruence relation* $\overline{\theta}$ *and* $\overline{\eta}$ *is not necessarily a fuzzy congruence relation.*

Example 3.7. Let $X = \{1, a, b\}$. Define the binary operation $\to$ on X as follows:

$\to$	1	a	b
1	1	a	b
a	1	1	b
b	1	1	1

Then $(X; \to, 0)$ is a BE-algebra. If put $\rightsquigarrow := \to$, then $(X; \to, \rightsquigarrow, 0)$ is a pseudo BE-algebra. Define the fuzzy relations $\overline{\theta}$ and $\overline{\eta}$ as follows:

$\overline{\theta}$	1	a	b
1	0.52	0.42	0.42
a	0.42	0.52	0.42
b	0.42	0.42	0.52

and

$\overline{\eta}$	1	a	b
1	0.52	0.3	0.3
a	0.3	0.52	0.3
b	0.3	0.3	0.52

Then $(\overline{\theta} \cup \overline{\eta})(x, y) = \max[\overline{\theta}(x, y), \overline{\eta}(x, y)]$ is not a fuzzy congruence relation on X.

Definition 3.8. Let $\overline{\theta}, \overline{\eta} \in \mathrm{FCon}(X)$. Define the composition $\overline{\theta} \circ \overline{\eta}$ by:

$$(\overline{\theta} \circ \overline{\eta})(x, y) = \sup_{z \in X} \min[\overline{\theta}(x, z), \overline{\eta}(z, y)].$$

Example 3.9. Consider the pseudo BE-algebra given in Example 3.2. Define fuzzy congruence relations $\overline{\theta}$ and $\overline{\eta}$ as follows:

$\overline{\theta}$	1	a	b	c
1	0.75	0.44	0.44	0.44
a	0.44	0.75	0.44	0.44
b	0.44	0.44	0.75	0.44
c	0.44	0.44	0.44	0.75

and

$\overline{\eta}$	1	a	b	c
1	0.65	0.24	0.24	0.24
a	0.24	0.65	0.24	0.24
b	0.24	0.24	0.65	0.24
c	0.24	0.24	0.24	0.65

Then $\overline{\theta} \circ \overline{\eta}$ is a fuzzy congruence relation on X by the following table.

$\overline{\theta} \circ \overline{\eta}$	1	a	b	c
1	0.65	0.44	0.44	0.44
a	0.44	0.65	0.44	0.44
b	0.44	0.44	0.65	0.44
c	0.44	0.44	0.44	0.65

By induction, we have:

Theorem 3.10. *Let* $\overline{\theta} \in \mathrm{FCon}(X)$*. Then* $\bigcup_{n=1}^{\infty} \overline{\theta}^n$ *is so, where,* $\overline{\theta}^n = \overline{\theta} \circ \overline{\theta} \circ \ldots \circ \overline{\theta}$*.*

Theorem 3.11. $(\mathrm{FCon}(X), \subseteq)$ *is a complete lattice, where* $\subseteq$ *is defined by:*

$$\overline{\theta} \subseteq \overline{\eta} \text{ if and only if } \overline{\theta}(x, y) \leq \overline{\eta}(x, y), \text{for all } x, y \in X.$$

Proof. Clearly $\subseteq$ is a partial order relation. It is easy to check that the relation $\overline{\sigma}$ defined by $\overline{\sigma}(x,y)=1$, for all $x,y\in X$ is in FCon(X) and the relation $\overline{\lambda}$ defined by $\overline{\lambda}(x,x)=\overline{\lambda}(1,1)$, for all $x\in X$ and $\overline{\lambda}(x,y)=0$ for $x\neq y$ is in FCon(X). Also, $\overline{\sigma}$ is the greatest element and $\overline{\lambda}$ is the least element of FCon(X) w.r.t. $\subseteq$. Let $\{\overline{\theta}_i\}_{i\in I}$ be a non-empty collection of fuzzy congruence relations in FCon(X). Let $\overline{\theta}(x,y)=\inf_{i\in X}\overline{\theta}_i(x,y)$, for all $x,y\in X$. It is easy to see that (FC_1), (FC_2), (FC_3), (FC_4) and (FC_5). Also, we have

$$\begin{aligned}
\overline{\theta}\circ\overline{\theta}(x,y) &= \sup_{z\in X}\min\{\overline{\theta}(x,z),\overline{\theta}(z,y)\}\\
&= \sup_{z\in X}\min\{\inf_{i\in I}\overline{\theta}_i(x,z),\inf_{i\in I}\overline{\theta}_i(z,y)\}\\
&= \sup_{z\in X}\inf_{i\in I}\{\min[\overline{\theta}_i(x,z),\overline{\theta}_i(z,y)]\}\\
&\leq \inf_{i\in I}\sup_{z\in X}\{\min[\overline{\theta}_i(x,z),\overline{\theta}_i(z,y)]\}\\
&= \inf(\overline{\theta}_i\circ\overline{\theta}_i)(x,y)\\
&\leq \inf_{i\in I}\overline{\theta}_i(x,y)\\
&= \overline{\theta}(x,y).
\end{aligned}$$

That is, $\overline{\theta}\in \mathrm{FCon}(X)$. Since $\overline{\theta}$ is the greatest lower bound of $\{\overline{\theta}_i\}_{i\in I}$, hence $(\mathrm{FCon}(X),\subseteq)$ is a complete lattice. □

Theorem 3.12. $(\mathrm{FCon}(X),\subseteq)$ *is a modular lattice.*

Proof. Assume that $\overline{\theta},\overline{\eta},\overline{\zeta}\in\mathrm{FCon}(X)$ and $\overline{\theta}\subseteq\overline{\zeta}$. It is sufficient to prove that $(\overline{\theta}\circ\overline{\eta})\cap\overline{\zeta}\subseteq\overline{\theta}\circ(\overline{\eta}\cap\overline{\zeta})$. For every $(x,y)\in X\times X$ and $z\in X$, since $\min\{\overline{\theta}(x,z),\overline{\zeta}(x,z)\}=\overline{\theta}(x,z)$, applying ($FC_3$) for $\overline{\zeta}$, we get

$$\begin{aligned}
[(\overline{\theta}\circ\overline{\eta})\cap\overline{\zeta}](x,y) &= \min[(\overline{\theta}\circ\overline{\eta})(x,y),\overline{\zeta}(x,y)]\\
&= \min[\sup_{z\in X}\min\{\overline{\theta}(x,z),\overline{\theta}(z,y)\},\overline{\zeta}(x,y)]\\
&= \sup_{z\in X}\{\min[\overline{\theta}(x,z),\overline{\theta}(z,y)],\overline{\zeta}(x,y)\}\\
&= \sup_{z\in X}\{\min[\overline{\theta}(x,z),\overline{\theta}(z,y)],\overline{\zeta}(x,z),\overline{\zeta}(x,y)\}\\
&\leq \sup_{z\in X}\{\min[\overline{\theta}(x,z),\overline{\theta}(z,y)],\overline{\zeta}(z,y)\}\\
&= \sup_{z\in X}\{\overline{\theta}(x,z),\min[\overline{\theta}(z,y),\overline{\zeta}(z,y)]\}\\
&= \sup_{z\in X}\{\overline{\theta}(x,z),[\overline{\theta}\cap\overline{\zeta}](z,y)\}\\
&= [\overline{\theta}\circ(\overline{\theta}\cap\overline{\zeta})](x,y).
\end{aligned}$$

□

Proposition 3.13. *Let* $\overline{\theta}$, $\overline{\eta}$, $\overline{\zeta}\in\mathrm{FCon}(X)$. *Then* $\overline{\theta}\circ(\overline{\eta}\cap\overline{\zeta})\subseteq(\overline{\theta}\circ\overline{\eta})\cap(\overline{\theta}\circ\overline{\zeta})$.

Proof. Assume that $\overline{\theta}$, $\overline{\eta}$, $\overline{\zeta} \in \mathrm{FCon}(X)$. Let $(x, y) \in X \times X$. Then

$$\begin{aligned}
[\overline{\theta} \circ (\overline{\eta} \cap \overline{\zeta})](x, y) &= \sup_{z\in X}\{\min[\overline{\theta}(x, z), (\overline{\eta} \cap \overline{\zeta})(z, y)\} \\
&= \sup_{z\in X}\{\min[\overline{\theta}(x, z), \min\{\overline{\eta}(z, y), \overline{\zeta}(z, y)\}]\} \\
&\le \min\{\sup_{z\in X}\{\min[\overline{\theta}(x, z), \overline{\eta}(z, y)]\}, \sup_{z\in X}\{\min[\overline{\theta}(x, z), \overline{\zeta}(z, y)]\}\} \\
&= \min\{(\overline{\theta} \circ \overline{\eta})(x, y), (\overline{\theta} \circ \overline{\zeta})(x, y)\} \\
&= [(\overline{\theta} \circ \overline{\eta}) \cap (\overline{\theta} \circ \overline{\zeta})](x, y).
\end{aligned}$$

□

Definition 3.14. Let $\overline{\theta} \in \mathrm{FCon}(X)$ and $\alpha \in [0, 1]$. Then the *level congruence relation* $\overline{\theta}^{\alpha}$ of $\overline{\theta}$ and *strong level congruence* $\overline{\theta}^{\alpha}_{>}$ of X are defined as the following:

$$\overline{\theta}^{\alpha} := \{(x, y) \in X \times X : \overline{\theta}(x, y) \ge \alpha\} \text{ and } \overline{\theta}^{\alpha}_{>} := \{(x, y) \in X \times X : \overline{\theta}(x, y) > \alpha\}.$$

Example 3.15. Consider the pseudo BE-algebra given in Example 3.2. Then

(i) if $\alpha \in (0, 0.4]$, then $\overline{\theta}^{\alpha} = X \times X$,

(ii) if $\alpha \in (0.4, 0.7]$, then $\overline{\theta}^{\alpha} = \Delta$, where $\Delta = \{(x, x) : x \in X\}$,

(iii) if $\alpha \in (0.7, 1]$, then $\overline{\theta}^{\alpha} = \varnothing$,

(iv) if $\alpha \in (0, 0.4)$, then $\overline{\theta}^{\alpha}_{>} = \{(1, a), (1, b), (1, c), (a, b), (b, a), (a, c), (c, a), (c, b), (b, c)\}$,

(v) if $\alpha \in [0.4, 0.7)$, then $\overline{\theta}^{\alpha}_{>} = \Delta$,

(vi) if $\alpha \in [0.7, 1]$, then $\overline{\theta}^{\alpha}_{>} = \varnothing$.

Proposition 3.16. *Let* $\overline{\theta} \in \mathrm{FCon}(X)$ *and* $\alpha \in [0, 1]$. *Then*

(i) if $\overline{\theta}^{\alpha} \neq \varnothing$, *then* $\overline{\theta}(1, 1) \ge \alpha$,

(ii) if $\overline{\theta}^{\alpha} := \{(x, y) : \overline{\theta}(x, y) = \overline{\theta}(y, x) \ge \alpha\}$, *then* $\overline{\theta}^{\alpha} \neq \varnothing$ *and* $\overline{\theta}^{\alpha}$ *is a congruence relation on* X.

Proof. We only prove (i). Since $\overline{\theta}^{\alpha} \neq \varnothing$, there exists $(x, y) \in \overline{\theta}^{\alpha}$. Applying Lemma 3.3, we get $\overline{\theta}(1, 1) \ge \overline{\theta}(x, y) \ge \alpha$. □

Lemma 3.17. *Let* $\overline{\theta} \in \mathrm{FCon}(X)$ *and* $\alpha \in (0, 1)$. *Then*

$$\overline{\theta}^{\alpha} = \bigcap_{0\le t<\alpha} \overline{\theta}^{t}_{>} \quad \text{and} \quad \overline{\theta}^{\alpha}_{>} = \bigcup_{\alpha<t\le 1} \overline{\theta}^{t}.$$

Proposition 3.18. *Let* $\overline{\theta}$ *be a fuzzy relation on* X *and* $\alpha \in (0, 1)$. *Then*

(i) $\overline{\theta}$ *is a fuzzy left (right) compatible relation if and only if* $\overline{\theta}^{\alpha}$ $(\overline{\theta}^{\alpha}_{>})$ *is a left (right) compatible relation on* X,

(ii) $\overline{\theta}$ *is a fuzzy congruence relation if and only if* $\overline{\theta}^{\alpha}$ $(\overline{\theta}^{\alpha}_{>})$ *is a congruence relation on* X.

Proposition 3.19. *Let* $\overline{\theta}, \overline{\eta} \in \mathrm{FCon}(X)$ *and* $\alpha \in [0, 1)$. *Then* *(i)* $\overline{\theta} = \overline{\eta}$ *if and only if* $\overline{\theta}^{\alpha}_{>} = \overline{\eta}^{\alpha}_{>}$,

(ii) $(\overline{\theta} \circ \overline{\eta})^{\alpha}_{>} = \overline{\theta}^{\alpha}_{>} \circ \overline{\eta}^{\alpha}_{>}$,

(iii) $\overline{\theta} \circ \overline{\eta} = \overline{\eta} \circ \overline{\theta}$ *if and only if* $\overline{\theta}^{\alpha}_{>} \circ \overline{\eta}^{\alpha}_{>} = \overline{\eta}^{\alpha}_{>} \circ \overline{\theta}^{\alpha}_{>}$, *for all* $\alpha \in [0, 1)$, *where,* $\overline{\theta}^{\alpha}_{>} \neq \emptyset$ *and* $\overline{\eta}^{\alpha}_{>} \neq \emptyset$.

Proof. We only prove (i). Assume that $(x, y) \in \overline{\theta}^{\alpha}_{>}$. Then $\overline{\eta}_{>}(x, y) = \overline{\theta}_{>}(x, y) > \alpha$ and so $(x, y) \in \overline{\eta}^{\alpha}_{>}$. Hence $\overline{\theta}^{\alpha}_{>} \subseteq \overline{\eta}^{\alpha}_{>}$. Similarly, $\overline{\eta}^{\alpha}_{>} \subseteq \overline{\theta}^{\alpha}_{>}$.

Conversely, let $\overline{\theta}^{\alpha}_{>} = \overline{\eta}^{\alpha}_{>}$, but there exists $(x, y) \in X \times X$ such that $\overline{\theta}(x, y) \neq \overline{\eta}(x, y)$. Let $\overline{\theta}(x, y) = t_1$ and $\overline{\eta}(x, y) = t_2$. Then $t_1 > t_2$ or $t_2 > t_1$. If $t_1 > t_2$, then $\overline{\theta}(x, y) = t_1 > t_2$, and so $(x, y) \in \overline{\theta}^{t_1}_{>} = \overline{\eta}^{t_1}_{>}$. Hence $\overline{\eta}(x, y) > t_1$, and so $t_2 > t_1$, which is a contradiction. If $t_2 > t_1$, by a similar argument we have a contradiction. □

Theorem 3.20. *If* $\overline{\theta}$ *and* $\overline{\eta}$ *are fuzzy left (right) compatible (congruence) relation on* X*. Then* $\overline{\theta} \times \overline{\eta}$ *is a left (right) compatible (congruence) relation on* $X \times X$*.*

In this section, we investigate fuzzy congruence relations induced by fuzzy medial filters in a pseudo BE-algebra.

Theorem 3.21. *Let* f *be an endomorphism of* X*. If* $\theta \in \mathrm{FCon}(X)$*, then* $\overline{\theta}$ *is defined by* $\overline{\theta}(x, y) := \theta(f(x), f(y))$ *is so.*

Proof. It is obvious that $\overline{\theta}$ well-defined. Let $x, y, z, u \in X$.

(FC$_1$) $\overline{\theta}(x, x) = \theta(f(x), f(x)) = \theta(1, 1) = \overline{\theta}(1, 1)$.

(FC$_2$) $\overline{\theta}(x, y) = \theta(f(x), f(y)) = \theta(f(y), f(x)) = \overline{\theta}(y, x)$.

(FC$_3$)
$$\begin{aligned}\overline{\theta}(x, y) = \theta(f(x), f(y)) &\geq \min[\theta(f(x), f(z)), \theta(f(z), f(y))] \\ &= \min[\overline{\theta}(x, z), \overline{\theta}(z, y)].\end{aligned}$$

(FC$_4$)
$$\begin{aligned}\overline{\theta}(x \to u, y \to u) &= \theta(f(x \to u), f(y \to u)) \\ &= \theta(f(x) \to f(u), f(y) \to f(u)) \\ &\geq \theta(f(x), f(y)) = \overline{\theta}(x, y).\end{aligned}$$

Similarly, $\overline{\theta}(v \to x, v \to y) \geq \overline{\theta}(x, y)$.

(FC$_5$)
$$\begin{aligned}\overline{\theta}(x \rightsquigarrow u, y \rightsquigarrow u) &= \theta(f(x \rightsquigarrow u), f(y \rightsquigarrow u)) \\ &= \theta(f(x) \rightsquigarrow f(u), f(y) \rightsquigarrow f(u)) \\ &\geq \theta(f(x), f(y)) = \overline{\theta}(x, y).\end{aligned}$$

Similarly, $\overline{\theta}(v \rightsquigarrow x, v \rightsquigarrow y) \geq \overline{\theta}(x, y)$. □

Remark 3.22. *The fuzzy subset* $\overline{\theta}_x : X \to [0, 1]$*, which is defined by* $\overline{\theta}_x(y) = \overline{\theta}(x, y)$*, is called the fuzzy congruence class containing* x*.*

By a routine calculation we can see that:

Proposition 3.23. *Let* $\overline{\theta} \in \mathrm{FCon}(X)$*. Then for all* $x, y, z, u \in X$

(i) $\overline{\theta}_x(x) = \overline{\theta}_1(1) = \overline{\theta}_1(x)$,

(ii) $\overline{\theta}_x(y) = \overline{\theta}_y(x) = \overline{\theta}_{x \to y}(1) = \overline{\theta}_{x \rightsquigarrow y}(1)$,

(iii) $\overline{\theta}_x(y) \geq \overline{\theta}_x(y \to z)$,

(iv) $\overline{\theta}_x(z) \geq \min[\overline{\theta}_x(y), \overline{\theta}_y(z)]$,

(v) $\overline{\theta}_x(z) \geq \min[\overline{\theta}_x(y), \overline{\theta}_x(y \to z)]$,

(vi) $\overline{\theta}_{x\to u}(y \to u) \geq \overline{\theta}_x(y)$ and $\overline{\theta}_{x\rightsquigarrow u}(y \rightsquigarrow u) \geq \overline{\theta}_x(y)$,

(vii) $\overline{\theta}_{u\to x}(u \to y) \geq \overline{\theta}_x(y)$ and $\overline{\theta}_{u\rightsquigarrow x}(u \rightsquigarrow y) \geq \overline{\theta}_x(y)$,

(viii) if $x \leq y$, then $\overline{\theta}_x(y) = \overline{\theta}_y(x) = \overline{\theta}_1(1)$,

(ix) $\overline{\theta}_x = \overline{\theta}_y$ if and only if $\overline{\theta}_{x\to z}(1) = \overline{\theta}_{y\to z}(1)$,

(x) $\overline{\theta}_x = \overline{\theta}_y$ if and only if $\overline{\theta}_{x\rightsquigarrow z}(1) = \overline{\theta}_{y\rightsquigarrow z}(1)$.

Proposition 3.24. *Let $\overline{\theta} \in \mathrm{FCon}(X)$ and $x \in X$. Then $\overline{\theta}_x$ is a fuzzy filter of X.*

The following example shows that the converse of Proposition 3.24, is not valid in general.

Example 3.25. (i) ([16]) Let $X = \{a, b, c, d, 1\}$. Define the operations $\to$ and $\rightsquigarrow$ on X as follows:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	c	c	1

$\rightsquigarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	1
b	1	d	1	1	d
c	1	d	1	1	d
d	1	1	b	c	1

Then $(X; \to, \rightsquigarrow, 1)$ is a pseudo-BE algebra. Define $\overline{\theta} : X \times X \longrightarrow [0, 1]$ as follows:

$\overline{\theta}$	1	a	b	c	d
1	0.7	0.5	0.6	0.6	0.5
a	0.5	0.7	0.2	0.3	0.1
b	0.6	0.2	0.7	0.1	0.2
c	0.6	0.3	0.1	0.7	0.4
d	0.5	0.1	0.2	0.4	0.7

Then $\overline{\theta}$ is not a fuzzy congruence relation. Since

$$\overline{\theta}(a, d) = 0.1 \ngeq \min[\overline{\theta}(a, c), \overline{\theta}(c, d)] = \min\{0.3, 0.4\} = 0.3.$$

Routine calculations show that $\overline{\theta}_1(1) = 0.7$, $\overline{\theta}_1(a) = \overline{\theta}_1(d) = 0.5$ and $\overline{\theta}_1(b) = \overline{\theta}_1(c) = 0.6$. It is easily seen that $\overline{\theta}_1 : X \longrightarrow [0, 1]$ is a fuzzy filter of X.

(ii) ([4]) Let $X = \{a, b, c, d, 1\}$. Define the operations $\to$ and $\rightsquigarrow$ on X as follows:

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	b	c
c	1	a	1	1	b
d	1	a	1	1	1

$\rightsquigarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	b	c
c	1	a	1	1	b
d	1	a	1	1	1

Then $(X; \to, \rightsquigarrow, 1)$ is a pseudo BE-algebra, but it is not distributive. Since

$$c \to (c \rightsquigarrow d) = c \to b = 1 \neq b = 1 \rightsquigarrow b = (c \to c) \rightsquigarrow (c \to d).$$

Define a fuzzy relation $\overline{\theta} : X \times X \longrightarrow [0, 1]$ as follows:

$\overline{\theta}$	1	a	b	c	d
1	0.4	0.2	0.2	0.2	0.2
a	0.3	0.1	0.2	0.3	0.1
b	0.1	0.2	0.2	0.4	0.1
c	0.2	0.3	0.1	0.3	0.2
d	0.4	0.1	0.1	0.4	0.2

Then $\overline{\theta}$ is not a fuzzy congruence relation. Since

$$\overline{\theta}(a,d) = 0.1 \not\geq \min[\overline{\theta}(a,c), \overline{\theta}(c,d)] = \min\{0.3, 0.2\} = 0.2.$$

Routine calculation shows that $\overline{\theta}_1(1) = 0.4$, $\overline{\theta}_1(a) = \overline{\theta}_1(b) = \overline{\theta}_1(c) = \overline{\theta}_1(d) = 0.2$. It is easy to see that $\overline{\theta}_1 : X \longrightarrow [0,1]$ is a fuzzy filter of X.

In the following theorem we show that if $\overline{\mu}$ is a *fuzzy medial filter* and X is *distributive*, then the converse of Proposition 3.24, holds.

Theorem 3.26. *Let $\overline{\mu}$ be a fuzzy medial filter in distributive pseudo BE-algebra X. Then there is a fuzzy congruence relation $\overline{\theta}$ in X such that $\overline{\theta}_1 = \overline{\mu}$.*

Proof. Assume that $\overline{\mu}$ is a fuzzy medial filter. Define a fuzzy relation in X by:

$$\overline{\theta}(x,y) = \min[\overline{\mu}(x \to y), \overline{\mu}(y \to x)], \text{ for all } x, y \in X.$$

Then, for all $x, y \in X$, we have

(FC$_1$) $$\begin{aligned}\overline{\theta}(x,x) = \min[\overline{\mu}(x \to x), \overline{\mu}(x \to x)] &= \min[\overline{\mu}(1), \overline{\mu}(1)] \\ &= \min[\overline{\mu}(1 \to 1), \overline{\mu}(1 \to 1)] \\ &= \overline{\theta}(1,1).\end{aligned}$$

(FC$_2$) $$\begin{aligned}\overline{\theta}(x,y) = \min[\overline{\mu}(x \to y), \overline{\mu}(y \to x)] &= \min[\overline{\mu}(y \to x), \overline{\mu}(x \to y)] \\ &= \overline{\theta}(y,x).\end{aligned}$$

(FC$_3$) Since $\overline{\mu}$ is a fuzzy medial filter, we have

$$\begin{aligned}\overline{\theta}(x,z) &= \min[\overline{\mu}(x \to z), \overline{\mu}(z \to x)] \\ &\geq \min[\min(\overline{\mu}(x \to y), \overline{\mu}(y \to z)), \min(\overline{\mu}(z \to y), \overline{\mu}(y \to x))] \\ &= \min[\min(\overline{\mu}(x \to y), \overline{\mu}(y \to x)), \min(\overline{\mu}(z \to y), \overline{\mu}(y \to z))] \\ &= \min[\overline{\theta}(x,y), \overline{\theta}(y,z)].\end{aligned}$$

(FC$_4$) For the right compatible condition, let $u \in X$. Since

$$\begin{aligned}(y \to x) \to ((x \to u) \rightsquigarrow (y \to u)) &= (x \to u) \rightsquigarrow ((y \to x) \rightsquigarrow (y \to u)) \\ &= (x \to u) \rightsquigarrow (y \to (x \rightsquigarrow u)) = 1.\end{aligned}$$

Thus, $(y \to x) \to ((x \to u) \to (y \to u)) = 1$. Therefore, $\overline{\mu}((x \to u) \to (y \to u)) \geq \overline{\mu}(y \to x)$. Similarly, we get $\overline{\mu}((y \to u) \to (x \to u)) \geq \overline{\mu}(x \to y)$. Then

$$\begin{aligned}\overline{\theta}(x \to u, y \to u) &= \min[\overline{\mu}((x \to u) \to (y \to u)), \overline{\mu}((y \to u) \to (x \to u))] \\ &\geq \min[\overline{\mu}(y \to x), \overline{\mu}(x \to y)] \\ &= \overline{\theta}(x,y).\end{aligned}$$

By a similar argument, $\overline{\theta}(x \rightsquigarrow u, y \rightsquigarrow u) \geq \overline{\theta}(x,y)$.

(FC$_5$) For the left compatible condition, let $u \in X$. Then

$$\begin{aligned}\overline{\theta}(u \to x, u \to y) &= \min[\overline{\mu}((u \to x) \to (u \to y)), \overline{\mu}((u \to y) \to (u \to x))] \\ &\geq \min[\overline{\mu}(x \to y), \overline{\mu}(y \to x)] \\ &= \overline{\theta}(x, y).\end{aligned}$$

Similarly, $\overline{\theta}(u \rightsquigarrow x, u \rightsquigarrow y) \geq \overline{\theta}(x, y)$. Also, for all $x \in X$,

$$\overline{\theta}_1(x) = \overline{\theta}(1, x) = \min[\overline{\mu}(1 \to x), \overline{\mu}(x \to 1)] = \min[\overline{\mu}(x), \overline{\mu}(1)] = \overline{\mu}(x).$$

Therefore, $\overline{\theta}_1 = \overline{\mu}$. □

Remark 3.27. *Let $\overline{\theta} \in \mathrm{FCon}(X)$. For every element $x \in X$, define:*

$$\overline{\theta}_x = \{y \in X : \overline{\theta}_x(y) = \overline{\theta}_1(1)\}$$

of X and $X/\overline{\theta} = \{\overline{\theta}_x : x \in X\}$. It is obviously that $\overline{\theta}_x \neq \emptyset$, for all $x \in X$ (since $x \in \overline{\theta}_x$) and $X = \bigcup_{x \in X} \overline{\theta}_x$. Also, define the binary operations $\to$ and $\rightsquigarrow$ on $X/\overline{\theta}$ as follows:

$$\overline{\theta}_x \to \overline{\theta}_y = \overline{\theta}_{x \to y} \text{ and } \overline{\theta}_x \rightsquigarrow \overline{\theta}_y = \overline{\theta}_{x \rightsquigarrow y}.$$

These operations are well-defined. Because, if $\overline{\theta}_x = \overline{\theta}_{x'}$ and $\overline{\theta}_y = \overline{\theta}_{y'}$, then we have $\overline{\theta}(x, x') = \overline{\theta}(y, y') = \overline{\theta}(1, 1)$. Since

$$\overline{\theta}(1,1) = \overline{\theta}(x, x') \leq \overline{\theta}(x \to y, x' \to y) \text{ and } \overline{\theta}(1,1) = \overline{\theta}(y, y') \leq \overline{\theta}(x' \to y, x' \to y'),$$

we have

$$\overline{\theta}(1,1) \leq \min[\overline{\theta}(x \to y, x' \to y), \overline{\theta}(x' \to y, x' \to y')] \leq \overline{\theta}(x \to y, x' \to y') \leq \overline{\theta}(1,1).$$

This means that $\overline{\theta}(x \to y, x' \to y') = \overline{\theta}(1,1)$ and $\overline{\theta}_{x \to y} = \overline{\theta}_{x' \to y'}$. By a similar argument, $\overline{\theta}_{x \rightsquigarrow y} = \overline{\theta}_{x' \to y'}$. So, the binary operations $\to$ and $\rightsquigarrow$ are well-defined.

Theorem 3.28. *If $\overline{\theta} \in \mathrm{FCon}(X)$, then $(X/\overline{\theta}; \to, \rightsquigarrow, \overline{\theta}_1)$ is a pseudo BE-algebra.*

Example 3.29. Consider the fuzzy congruence relation $\overline{\theta}$ given in Example 3.2, and $1_{\overline{\theta}} = \{1\}$, $a_{\overline{\theta}} = \{a\}$, $b_{\overline{\theta}} = \{b\}$ and $c_{\overline{\theta}} = \{c\}$. Then $X/\overline{\theta} = \{\{1\}, \{a\}, \{b\}, \{c\}\}$ with the following tables:

$\to$	$1_{\overline{\theta}}$	$a_{\overline{\theta}}$	$b_{\overline{\theta}}$	$c_{\overline{\theta}}$
$1_{\overline{\theta}}$	$1_{\overline{\theta}}$	$a_{\overline{\theta}}$	$b_{\overline{\theta}}$	$c_{\overline{\theta}}$
$a_{\overline{\theta}}$	$1_{\overline{\theta}}$	$1_{\overline{\theta}}$	$a_{\overline{\theta}}$	$1_{\overline{\theta}}$
$b_{\overline{\theta}}$	$1_{\overline{\theta}}$	$1_{\overline{\theta}}$	$1_{\overline{\theta}}$	$1_{\overline{\theta}}$
$c_{\overline{\theta}}$	$1_{\overline{\theta}}$	$a_{\overline{\theta}}$	$a_{\overline{\theta}}$	$1_{\overline{\theta}}$

$\rightsquigarrow$	$1_{\overline{\theta}}$	$a_{\overline{\theta}}$	$b_{\overline{\theta}}$	$c_{\overline{\theta}}$
$1_{\overline{\theta}}$	$1_{\overline{\theta}}$	$a_{\overline{\theta}}$	$b_{\overline{\theta}}$	$c_{\overline{\theta}}$
$a_{\overline{\theta}}$	$1_{\overline{\theta}}$	$1_{\overline{\theta}}$	$c_{\overline{\theta}}$	$1_{\overline{\theta}}$
$b_{\overline{\theta}}$	$1_{\overline{\theta}}$	$1_{\overline{\theta}}$	$1_{\overline{\theta}}$	$1_{\overline{\theta}}$
$c_{\overline{\theta}}$	$1_{\overline{\theta}}$	$a_{\overline{\theta}}$	$b_{\overline{\theta}}$	$1_{\overline{\theta}}$

Then $(X/\overline{\theta}; \to, \rightsquigarrow, 1_{\overline{\theta}})$ is a pseudo BE-algebra.

Let $\overline{\theta} \in \mathrm{FCon}(X)$. The natural map of X onto $X/\overline{\theta}$ is $\pi : X \longrightarrow X/\overline{\theta}$ given by $\pi(x) = \overline{\theta}_x$.

Remark 3.30. *Assume that $\overline{\theta}$, $\overline{\eta} \in \mathrm{FCon}(X)$. Let $\pi_1 : X \longrightarrow X/\overline{\theta}$ and $\pi_2 : X \longrightarrow X/\overline{\eta}$ be the natural homomorphisms. Combining these, we obtain a homomorphism $\pi_1 \times \pi_2 : X \longrightarrow X/\overline{\theta} \times X/\overline{\eta}$. Then $X/(\overline{\theta} \cap \overline{\eta}) \cong X/\overline{\theta} \times X/\overline{\eta}$.*

Definition 3.31. Let θ be an equivalence relation and $\overline{\theta}$ be a fuzzy relation on X. Then $\overline{\theta}$ is called *θ-invariant* if $\overline{\theta}_x = \overline{\theta}_a$ and $\overline{\theta}_y = \overline{\theta}_b$ imply $\theta(x, y) = \theta(a, b)$.

Definition 3.32. Let θ be a congruence relation and $\overline{\theta}$ be a θ-invariant fuzzy relation on X. Define a fuzzy relation $\overline{\overline{\theta}}$ on X/θ as follows:

$$\overline{\overline{\theta}}(\overline{\theta}_x, \overline{\theta}_y) = \overline{\theta}_x(y).$$

Proposition 3.33. *Let $\overline{\theta}, \overline{\eta} \in \mathrm{FCon}(X)$, $\overline{\theta}$ be a θ-invariant, $\overline{\eta}$ be a η-invariant and $\overline{\theta} \subseteq \overline{\eta}$. Then $\overline{\overline{\theta}} \subseteq \overline{\overline{\eta}}$.*

Proof. Assume that $x, y \in X$. Then

$$\overline{\overline{\theta}}(\overline{\theta}_x, \overline{\theta}_y) = \overline{\theta}_x(y) = \overline{\theta}(x, y) \leq \overline{\eta}(x, y) = \overline{\eta}_x(y) = \overline{\overline{\eta}}(\overline{\eta}_x, \overline{\eta}_y).$$

□

Theorem 3.34. *If $\overline{\theta}$ is a θ-invariant fuzzy left (right) compatible (congruence) relation on X, then $\overline{\overline{\theta}}$ is so on X/θ.*

Proof. Since $\overline{\theta}$ is a θ-invariant fuzzy left relation on X, we get $\overline{\overline{\theta}}$ is well defined. Let $x, y, z, u \in X$. Then

(FC$_1$) $\overline{\overline{\theta}}(\overline{\theta}_x, \overline{\theta}_x) = \overline{\theta}_x(x) = \overline{\theta}_1(1)$, on the other hand, $\overline{\overline{\theta}}(\overline{\theta}_1, \overline{\theta}_1) = \overline{\theta}_1(1)$. So, $\overline{\overline{\theta}}(\theta_x, \theta_x) = \overline{\theta}(\overline{\theta}_1, \overline{\theta}_1)$.

(FC$_2$) $\overline{\overline{\theta}}(\overline{\theta}_x, \overline{\theta}_y) = \overline{\theta}(x, y) = \overline{\theta}(y, x) = \overline{\overline{\theta}}(\overline{\theta}_y, \overline{\theta}_x)$.

(FC$_3$) $\overline{\overline{\theta}}(\overline{\theta}_x, \overline{\theta}_y) = \overline{\theta}(x, y) \geq \min[\overline{\theta}(x, z), \overline{\theta}(z, y)] = \min[\overline{\overline{\theta}}(\overline{\theta}_x, \overline{\theta}_z), \overline{\overline{\theta}}(\overline{\theta}_z, \overline{\theta}_y)]$.

(FC$_4$) $$\begin{aligned}\overline{\overline{\theta}}(\overline{\theta}_x \to \overline{\theta}_u, \overline{\theta}_y \to \overline{\theta}_u) = \overline{\overline{\theta}}(\overline{\theta}_{x\to u}, \overline{\theta}_{y\to u}) &= \overline{\theta}(x \to u, y \to u)\\ &\geq \overline{\theta}(x, y) = \overline{\overline{\theta}}(\overline{\theta}_x, \overline{\theta}_y).\end{aligned}$$

By a similar argument, $\overline{\overline{\theta}}(\overline{\theta}_u \to \overline{\theta}_x, \overline{\theta}_u \to \overline{\theta}_y) \geq \overline{\overline{\theta}}(\overline{\theta}_x, \overline{\theta}_y)$.

(FC$_5$) $$\begin{aligned}\overline{\overline{\theta}}(\overline{\theta}_x \rightsquigarrow \overline{\theta}_u, \overline{\theta}_y \rightsquigarrow \overline{\theta}_u) = \overline{\overline{\theta}}(\overline{\theta}_{x\rightsquigarrow u}, \overline{\theta}_{y\rightsquigarrow u}) &= \overline{\theta}(x \rightsquigarrow u, y \rightsquigarrow u)\\ &\geq \overline{\theta}(x, y)\\ &= \overline{\overline{\theta}}(\overline{\theta}_x, \overline{\theta}_y)\end{aligned}$$

Similarly, $\overline{\overline{\theta}}(\overline{\theta}_u \rightsquigarrow \overline{\theta}_x, \overline{\theta}_u \rightsquigarrow \overline{\theta}_y) \geq \overline{\overline{\theta}}(\overline{\theta}_x, \overline{\theta}_y)$. Also, assume that $\overline{\overline{\theta}}_{\overline{\theta}_x} = \overline{\overline{\theta}}_{\overline{\theta}_a}$ and $\overline{\overline{\theta}}_{\overline{\theta}_y} = \overline{\overline{\theta}}_{\overline{\theta}_b}$, for some $a, b \in X$. Then $\overline{\overline{\theta}}_{\overline{\theta}_x}(\overline{\theta}_c) = \overline{\overline{\theta}}_{\overline{\theta}_a}(\overline{\theta}_c)$ and $\overline{\overline{\theta}}_{\overline{\theta}_y}(\overline{\theta}_c) = \overline{\overline{\theta}}_{\overline{\theta}_b}(\overline{\theta}_c)$, for all $c \in X$. Hence $\overline{\theta}_x(c) = \overline{\theta}(x, c) = \overline{\theta}(a, c) = \overline{\theta}_a(c)$ and $\overline{\theta}_y(c) = \overline{\theta}(y, c) = \overline{\theta}(b, c) = \overline{\theta}_b(c)$, and so $\overline{\theta}_x = \overline{\theta}_a$ and $\overline{\theta}_y = \overline{\theta}_b$. Since $\overline{\theta}$ is θ-invariant fuzzy relation, we have $\theta(x, y) = \theta(a, b)$. Therefore, $\overline{\overline{\theta}}$ is a θ-invariant fuzzy relation. □

4 Conclusions

A fuzzy congruence relation is a generalization of a congruence relation on an algebraic structure. In this paper, we introduced the notion of the fuzzy congruence relation on a pseudo BE-algebra and investigated some of their properties. Moreover, we have showed that $(\mathrm{FCon}(X), \subseteq)$ is a modular lattice. Also, fuzzy congruence relation derived from a fuzzy medial filter is investigated.

As future work, the relation between fuzzy congruence relations and *fuzzy homomorphisms* will be study. Also, the *fuzzy homomorphism theorems* an extension of homomorphism theorems can be investigated.

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