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Some properties of n-hyperideals in commutative hyperrings

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Abstract

In this paper, we reformulate several results in commutative algebra in terms of commutative hyperrings. We introduce n-hyperideals in commutative hyperrings and give its some basic properties. Based on new definitions and theorems, we obtain some results in the hyperring theory. Also, the paper is stated a characterization for fundamental n-hyperideals.

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1 Introduction

The theory of hyperstructures was introduced by Marty [10] at the 8^{th} congress of Scandinavian Mathematicians in 1934. Some review of the hyperstructure theory can be found in [3, 4, 5, 6, 15]. Mittas [11] introduced the notion of canonical hypergroups. Hyperrings and hyperfields were introduced by Krasner [9] in connection with his work on valued fields. Davvaz and Leoreanu studied hyperrings in more details in [6]. Several kinds of hyperrings are introduced and analyzed. Ameri and Norouzi [1] studied homomorphisms of hyperring and extension (contraction) of hyperideals in commutative hyperrings. In 2015, Jun [8] studied algebraic and geometric aspects of hyperrings. He introduced the notion of an integral hyperring scheme $(X, \mathcal{O}_X)$ and proved that $\Gamma(X, \mathcal{O}_X) \simeq R$ for any integral affine hyperring scheme $X = Spec(R)$. In [12], some results concerning ordered hyperstructures are proved. Some results on a derivation in hyperrings can be found in [2]. Recently, Tekir et al. [13] introduced the concept of n-ideals on commutative rings.

Let R be a commutative Krasner hyperring with nonzero identity. In this paper, we generalize some concepts of the ring theory such as n-ideals and r-ideals on hyperrings. Also, we investigate some properties of n-hyperideals analogous with prime hyperideals in commutative hyperrings.

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2 Preliminaries

Let H be a non-empty set and $\mathcal{P}^*(H)$ denotes the family of all non-empty subsets of H. A mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hypergroupoid $(H, \circ)$ is said to be a *semihypergroup* if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

A non-empty subset K of a semihypergroup $(H, \circ)$ is called a *subsemihypergroup* of H if $K \circ K \subseteq K$. A semihypergroup $(H, \circ)$ satisfying $x \circ H = H \circ x = H$ for any $x \in H$ is called a hypergroup. A non-empty subset K of H is a subhypergroup of H if $a \circ K = K \circ a = K$, for all $a \in K$.

Now, we introduce the notions of canonical hypergroups and Krasner hyperrings and we apply them in the next section.

Definition 2.1. [11] *A non-empty set R along with the hyperoperation $+$ is called a canonical hypergroup if the following axioms hold:*

(1) $x + (y + z) = (x + y) + z$, for any $x, y, z \in R$;

(2) $x + y = y + x$, for any $x, y \in R$;

(3) there exists $0 \in R$ such that $x + 0 = \{x\}$, for any $x \in R$;

(4) for any $x \in R$, there exists a unique element $x' \in R$, such that $0 \in x + x'$ (we shall write $-x$ for x' and we call it the opposite of x);

(5) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, that is $(R, +)$ is reversible.

Definition 2.2. [9] *A Krasner hyperring is an algebraic hypersructure $(R, +, \cdot)$ which satisfies the following axioms:*

(1) $(R, +)$ is a canonical hypergroup;

(2) $(R, \cdot)$ is a semigroup having 0 as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$, for all $x \in R$;

(3) $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$ and $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, for all $x, y, z \in R$.

A Krasner hyperring R is called with identity if there exists an element, say $1 \in R$, such that $1 \cdot x = x \cdot 1 = x$. An element x of a Krasner hyperring R is called a *unit* if there exists $y \in R$ such that $x \cdot y = y \cdot x = 1$. A Krasner hyperring R is called *commutative* (with unit element) if $(R, \cdot)$ is a commutative semigroup (with unit element). A Krasner hyperring R is called a *Krasner hyperfield*, if $(R \setminus \{0\}, \cdot)$ is a group. A Krasner hyperring R is called a *hyperdomain*, if R is a commutative hyperring with unit element and $a \cdot b = 0$ implies that $a = 0$ or $b = 0$, for all $a, b \in R$. A *subhyperring* of a Krasner hyperring $(R, +, \cdot)$ is a non-empty subset A of R which forms a Krasner hyperring containing 0 under the hyperoperation $+$ and the operation $\cdot$ on R, that is, A is a canonical subhypergroup of $(R,+)$ and $A \cdot A \subseteq A$. Then a non-empty subset A of R is a subhyperring of $(R,+,\cdot)$ if and only if, for all $x, y \in A$, $x + y \subseteq A$, $-x \in A$ and $x \cdot y \in A$. A non-empty subset I of $(R,+,\cdot)$ is called a *left* (resp. *right*) *hyperideal* of $(R,+,\cdot)$ if $(I,+)$ is a canonical subhypergroup of $(R,+)$ and for any $a \in I$ and $r \in R$, $r \cdot a \in I$ (resp. $a \cdot r \in I$). A *hyperideal* I of $(R,+,\cdot)$ is one which is a left as well as a right hyperideal of R, that is, $x + y \subseteq I$ and $-x \in I$, for all $x, y \in I$ and $x \cdot y, y \cdot x \in I$, for all $x \in I$ and $y \in R$. Throughout this paper, unless otherwise stated, R is always a commutative Krasner hyperring with nonzero identity.

Lemma 2.3. [6] *A non-empty subset A of a Krasner hyperring R is a left (resp. right) hyperideal if and only if*

(1) $a, b \in A$ implies $a - b \subseteq A$.

(2) $a \in A$ and $r \in R$ imply $r \cdot a \in A$ (resp. $a \cdot r \in A$).

Definition 2.4. *A homomorphism from a Krasner hyperring $(R,+,\cdot)$ into a Krasner hyperring $(S,\oplus,\odot)$ is a mapping $\varphi : R \to S$ such that we have:*

(1) $\varphi(a + b) \subseteq \varphi(a) \oplus \varphi(b)$;

(2) $\varphi(a \cdot b) = \varphi(a) \odot \varphi(b)$.

Also, φ is called a *good homomorphism* if in the previous condition (1), the equality is valid.

3 n-Hyperideals of commutative hyperrings

Recall that a proper hyperideal $\mathfrak{p}$ of a commutative hyperring $(R,+,\cdot)$ is called *prime* if $a \cdot b \in \mathfrak{p}$ implies that either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Let R be a commutative hyperring with identity. By $Spec(R)$ we mean the set of all the prime hyperideals of R. For hyperideal I of R we define $V(I)$ as follows:

$$V(I) := \{\mathfrak{p} \in Spec(R) \mid I \subseteq \mathfrak{p}\}.$$

For $a \in R$, we set $V(a) := \{\mathfrak{p} \in Spec(R) \mid a \in \mathfrak{p}\}$. Then, $V(I) = \bigcap_{a \in I} V(a)$.

Lemma 3.1. [8] *Let I be a hyperideal of a hyperring R. Then*

$$\sqrt{I} := \{r \in R \mid \exists n \in \mathbb{N} \text{ such that } r^n \in I\}.$$

is a hyperideal.

Lemma 3.2. [8] *Let I be a hyperideal of a hyperring R. Then*

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

Definition 3.3. *A hyperideal I of a Krasner hyperring $(R,+,\cdot)$, such that $I \neq R$, is called an n-hyperideal if for a, b of R, $a \cdot b \in I$ and $a \notin \sqrt{0}$ implies that $b \in I$.*

Example 3.4. *Let $R = \{0, a, b\}$ be a set with the hyperaddition $+$ and the multiplication $\cdot$ defined as follows:*

$+$	0	a	b
0	0	a	b
a	a	R	a
b	b	a	$\{0,b\}$

$\cdot$	a	b	c
0	0	0	0
a	0	a	b
b	0	b	0

Then, $(R,+,\cdot)$ is a Krasner hyperring. It is easy to see that $\{0\}$ and $\{0, b\}$ are n-hyperideals of R.

Lemma 3.5. *Let $(R, +, \cdot)$ be a hyperring. Then,*

(1) If $\{I_k \mid k \in \Omega\}$ is a family of n-hyperideals of R such that $I_i \subseteq I_j$ or $I_j \subseteq I_i$ for all $i, j \in \Omega$, then $\bigcup\limits_{k\in\Omega} I_k$ is an n-hyperideal of R.

(2) If $\{I_k \mid k \in \Omega\}$ is a family of n-hyperideals of R, then $\bigcap\limits_{k\in\Omega} I_k$ is an n-hyperideal of R.

Proof. (1): Since $0 \in \bigcup\limits_{k\in\Omega} I_k$, it follows that $\bigcup\limits_{k\in\Omega} I_k \neq \emptyset$. Let $x, y \in \bigcup\limits_{k\in\Omega} I_k$. Then $x, y \in I_k$ for some $k \in \Omega$. Since I_k is a hyperideal of R, we obtain $x - y \subseteq I_k$ for some $k \in \Omega$. Thus $x - y \subseteq \bigcup\limits_{k\in\Omega} I_k$. Also, $(\bigcup\limits_{k\in\Omega} I_k) \cdot R = \bigcup\limits_{k\in\Omega} I_k \cdot R \subseteq \bigcup\limits_{k\in\Omega} I_k$ and $R \cdot (\bigcup\limits_{k\in\Omega} I_k) = \bigcup\limits_{k\in\Omega} R \cdot I_k \subseteq \bigcup\limits_{k\in\Omega} I_k$. So, for each $x \in \bigcup\limits_{k\in\Omega} I_k$ and $s \in R$, $x \cdot s \in \bigcup\limits_{k\in\Omega} I_k$. Similarly, $s \cdot x \in \bigcup\limits_{k\in\Omega} I_k$. Now, let $a \cdot b \in \bigcup\limits_{k\in\Omega} I_k$ and $a \notin \sqrt{0}$ for $a, b \in R$. Then, $a \cdot b \in I_i$ for some $i \in \Omega$. Since I_i is an n-hyperideal of R, it follows that $b \in I_i \subseteq \bigcup\limits_{k\in\Omega} I_k$. Therefore, $\bigcup\limits_{k\in\Omega} I_k$ is an n-hyperideal of R.

(2): The proof is straightforward. □

The set $ann(x) = \{a \in R \mid a \cdot x = 0\}$ is called the *annihilator of x in R*. A proper hyperideal I of a hyperring $(R, +, \cdot)$ is said to be an *r-hyperideal* of R if $x \cdot y \in I$ and $ann(x) = 0$ imply that $y \in I$ for any $x, y \in R$. Every n-hyperideal of a hyperring R is an r-hyperideal of R. The converse is not true, in general, that is, an r-hyperideal may not be an n-hyperideal of R. The following example denotes such a situation.

Example 3.6. *Let $R = \{0, a, b, c\}$ be a set with the hyperaddition $+$ and the multiplication $\cdot$ defined as follows:*

$+$	0	a	b	c
0	0	a	b	c
a	a	$\{0, b\}$	$\{a, c\}$	b
b	b	$\{a, c\}$	$\{0, b\}$	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

Then, $(R, +, \cdot)$ is a Krasner hyperring [2]*. Clearly, $\{0\}$, $\{0, b\}$ and $\{0, c\}$ are proper hyperideals of R. It is easy to see that $\{0, b\}$ is an r-hyperideal of R, but it is not an n-hyperideal of R. Indeed:*

$$b \cdot c = 0 \in \{0, b\} \text{ and } b \notin \sqrt{0_R} \text{ but } c \notin \{0, b\}.$$

Theorem 3.7. *Let $\mathfrak{p}$ be a prime hyperideal of a hyperring $(R, +, \cdot)$. Then $\mathfrak{p}$ is an n-hyperideal of R if and only if $\mathfrak{p} = \sqrt{0}$.*

Proof. By Lemma 3.2, $\sqrt{0} = \bigcap\limits_{\mathfrak{p}\in Spec(R)} \mathfrak{p} \subseteq \mathfrak{p}$. Let $\mathfrak{p} \nsubseteq \sqrt{0}$. Then there exists $a \in \mathfrak{p}$ such that $a \notin \sqrt{0}$. Since $\mathfrak{p}$ is an n-hyperideal of R and $a \cdot 1 = a \in \mathfrak{p}$, we get $1 \in \mathfrak{p}$. Thus, $I = R$, a contradiction. Hence, $\mathfrak{p} \subseteq \sqrt{0}$ which implies that $\mathfrak{p} = \sqrt{0}$.

Conversely, let $a \cdot b \in \mathfrak{p}$ and $a \notin \sqrt{0} = \mathfrak{p}$ for $a, b \in R$. Since $\mathfrak{p}$ is a prime hyperideal of R, we have $b \in \mathfrak{p}$. Therefore, $\mathfrak{p}$ is a prime hyperideal of R. □

Example 3.8. *In Example 3.6, $\{0, b\}$ is a prime hyperideal of R, but it is not an n-hyperideal of R.*

For a (multiplicative) submonoid S of a hyperring R, let us consider the following relation in $R \times S$:

$$(r, s) \sim (r', s') \Leftrightarrow \exists x \in S \text{ s.t. } xrs' = xr's.$$

Clearly, $\sim$ is an equivalence relation on $R \times S$. Let $[(r, s)]$ be the equivalence relation of $(r, s) \in R \times S$. $S^{-1}R$ is the set $(R \times S/\sim)$. Now, we define the following hyperoperation $\oplus$ and operation $\odot$ on $S^{-1}R$,

$$[(r, s)] \oplus [(r', s')] = \{[(y, s \cdot s')] \mid y \in r \cdot s' + r' \cdot s\}$$

and

$$[(r, s)] \odot [(r', s')] = \{[(r \cdot r', s \cdot s')]\}.$$

Clearly, $(S^{-1}R, \oplus, \odot)$ is a commutative hyperring [7]. The mapping $\varphi : R \to S^{-1}R$ given by $\varphi(r) = r/1$ is a homomorphism. If I is a hyperideal of R, then

$$\varphi(I) = S^{-1}I = \{\lambda \in S^{-1}R \mid \lambda = a/s, \exists a \in I, \exists s \in S\}$$

is a hyperideal of $S^{-1}R$. $S^{-1}I$ is called the extension of I in $S^{-1}R$.

Theorem 3.9. *If I is an n-hyperideal of a hyperring $(R, +, \cdot)$, then $S^{-1}I$ is an n-hyperideal of $S^{-1}R$.*

Proof. Let $r/s \odot r'/s' \in S^{-1}I$ and $r/s \notin \sqrt{0_{S^{-1}R}}$ for $r, r' \in R$ and $s, s' \in S$. Then there exists $u \in S$ such that $urr' \in I$. Next, we show that $r \notin \sqrt{0_R}$. If $r \in \sqrt{0_R}$, then there exists $n \in \mathbb{N}$ such that $r^n = 0_R$. This means that $(r/1)^n = r^n/1 = 0_R/1 = 0_{S^{-1}R} = 0_R/s$, and so $r/1 \in \sqrt{0_{S^{-1}R}}$. Since $r/s = 1/s \odot r/1$, we get $(r/s)^n = (1/s)^n \odot 0_{S^{-1}R} = 0_{S^{-1}R}$. Hence, $r/s \in \sqrt{0_{S^{-1}R}}$, which is a contradiction. This implies that $r \notin \sqrt{0_R}$. Now, since I is an n-hyperideal of R, we have $ur' \in I$ and so $r'/s' = ur'/us' \in S^{-1}I$. Therefore, $S^{-1}I$ is an n-hyperideal of $S^{-1}R$. □

Theorem 3.10. *Let I be an n-hyperideal of the hyperring $(R, +, \cdot)$ and $\varphi : R \to S$ a good epimorphism such that $Ker\varphi \subseteq I$. Then $\varphi(I)$ is an n-hyperideal of the hyperring $(S, \oplus, \odot)$.*

Proof. Clearly, $\varphi(I)$ is a hyperideal of S. Let $s_1 \odot s_2 \in \varphi(I)$ and $s_1 \notin \sqrt{0_S}$ for $s_1, s_2 \in S$. Then, there exist $r_1, r_2 \in R$ such that $s_1 = \varphi(r_1)$ and $s_2 = \varphi(r_2)$ (since φ is onto) which

$$s_1 \odot s_2 = \varphi(r_1) \odot \varphi(r_2) = \varphi(r_1 \cdot r_2) = \varphi(x) \in \varphi(I)$$

for some $x \in I$. So, we have

$$0 \in \varphi(r_1 \cdot r_2) \ominus \varphi(x) = \varphi(r_1 \cdot r_2 - x).$$

Hence, there exists $t \in r_1 \cdot r_2 - x$ such that $\varphi(t) = 0$. By hypothesis, we have

$$r_1 \cdot r_2 \in t + x \subseteq Ker\varphi + I \subseteq I + I \subseteq I.$$

So, $r_1 \cdot r_2 \in I$. Next, we show that $r_1 \notin \sqrt{0_R}$. If $r_1 \in \sqrt{0_R}$, then there exists $n \in \mathbb{N}$ such that ${r_1}^n = 0_R$. This means that $\varphi(r_1^n) = \varphi(0) = 0_S$, and so $(\varphi(r_1))^n = 0_S$. Hence, $s_1 = \varphi(r_1) \in \sqrt{0_S}$, which is a contradiction. This implies that $r_1 \notin \sqrt{0_R}$. Now, since I is an n-hyperideal of R, we get $r_2 \in I$ and so $s_2 = \varphi(r_2) \in \varphi(I)$. This completes the proof. □

Let $\varphi : R \to S$ be a homomorrphism of hyperrings and I a hyperideal of R. The hyperideal $\langle\varphi(I)\rangle$ of S generated by the set $\varphi(I)$ is called the extension of I, and is denoted by I^e. We have

$$\langle \varphi(I) \rangle = \{x \in S \mid x \in \sum_{i=1}^{n} s_i \cdot \varphi(a_i), s_i \in S, a_i \in I, n \in \mathbb{N}\}.$$

The mapping $\varphi : R \to S^{-1}R$ given by $\varphi(r) = r/1$ is a homomorphism. Consider $\lambda \in S^{-1}I$. Then $\lambda = i/s$, where $i \in I$ and $s \in S$. Hence, $i/1 \in \varphi(I)$. This implies that $i/1 \in I^e$. Since I^e is a hyperideal of $S^{-1}R$, we get $i/s = 1/s \odot i/1 \in I^e$. So, $\lambda = i/s \in I^e$. Thus, $S^{-1}I \subseteq I^e$. Now, suppose that $\lambda \in \varphi(I)$. Then there exists $a \in I$ such that $\lambda = a/1$. Hence, $\lambda \in S^{-1}I$ which implies that $\varphi(I) \subseteq S^{-1}I$. Thus, $I^e = \langle \varphi(I) \rangle \subseteq S^{-1}I$. Hence, $S^{-1}I = I^e$.

Theorem 3.11. *Let I be an n-hyperideal of the hyperring $(R, +, \cdot)$ and $\varphi : R \to S$ a good epimorphism such that $Ker\varphi \subseteq I$. Then I^e is an n-hyperideal of the hyperring $(S, \oplus, \odot)$.*

Proof. The proof is similar to the proof of Theorem 3.10. □

Theorem 3.12. *Let J be an n-hyperideal of the hyperring $(S, \oplus, \odot)$ and $\varphi : R \to S$ a good monomorphism. Then $\varphi^{-1}(J) = \{a \in R \mid \varphi(a) \in J\}$ is an n-hyperideal of the hyperring $(R, +, \cdot)$. $\varphi^{-1}(J)$ is called the contraction of J, and is denoted by J^c.*

Proof. Since $0 \in \varphi^{-1}(J)$, it follows that $\varphi^{-1}(J) \neq \emptyset$. Let $x \in R$. Since φ is a homomorphism and $0 \in x - x$, we have $0 = \varphi(0) \in \varphi(x - x) \subseteq \varphi(x) \oplus \varphi(-x)$. So $0 \in \varphi(x) \oplus \varphi(-x)$. Thus, $\varphi(-x)$ is the inverse of $\varphi(x)$ in the canonical hypergroup $(S, \oplus)$. Since $0 \in \varphi(x) \oplus \varphi(-x)$, it follows that $\varphi(-x) = -\varphi(x)$. Now, let $a_1, a_2 \in \varphi^{-1}(J)$. Then $\varphi(a_1), \varphi(a_2) \in J$. Since J is a hyperideal of T, we have $\varphi(a_1 - a_2) \subseteq \varphi(a_1) \ominus \varphi(a_2) \subseteq J$. Hence $a_1 - a_2 \subseteq \varphi^{-1}(J)$. Let $x \in R$ and $a \in \varphi^{-1}(J)$. Then $\varphi(a) \in J$. Since φ is a homomorphism, it follows that $\varphi(x \cdot a) = \varphi(x) \odot \varphi(a) \in J$. Thus $x \cdot a \in \varphi^{-1}(J)$. Hence, $\varphi^{-1}(J)$ is a hyperideal of R. Now, let $a \cdot b \in \varphi^{-1}(J)$ and $a \notin \sqrt{0_R}$. Then $\varphi(a) \odot \varphi(b) = \varphi(a \cdot b) \in J$. Next, we show that $\varphi(a) \notin \sqrt{0_S}$. If $\varphi(a) \in \sqrt{0_S}$, then there exists $n \in \mathbb{N}$ such that $(\varphi(a))^n = 0_S$. This means that $\varphi(a^n) = 0_S = \varphi(0_R)$, and so $a^n = 0_R$. Hence, $a \in \sqrt{0_R}$, which is a contradiction. This leads to $\varphi(a) \notin \sqrt{0_S}$. Now, since J is an n-hyperideal of S, we get $\varphi(b) \in J$ and so $b \in \varphi^{-1}(J)$. Therefore, $\varphi^{-1}(J)$ is an n-hyperideal of R. □

A relation σ^* is the *transitive closure* of a binary relation σ if (1) σ^* is transitive; (2) $\sigma \subseteq \sigma^*$ and (3) for any relation σ', if $\sigma \subseteq \sigma'$ and σ' is transitive, then $\sigma^* \subseteq \sigma'$, that is, σ^* is the smallest relation that satisfies (1) and (2). Let $(R, +, \cdot)$ be a hyperring. We define the relation γ as follows:

$$x\gamma y \Leftrightarrow \exists n \in \mathbb{N}, \exists k_i \in \mathbb{N}, \exists (x_{i1}, \cdots, x_{ik_i}) \in R^{k_i}, 1 \leq i \leq n,$$

such that

$$\{x, y\} \subseteq \sum_{i=1}^{n} \Big(\prod_{j=1}^{k_i} x_{ij} \Big).$$

Theorem 3.13. [14] *Let R be a hyperring and γ^* be the transitive closure of γ. Then, we have:*

(1) γ^* is a strongly regular relation both on $(R, +)$ and $(R, \cdot)$.

(2) The quotient R/γ^* is a ring.

(3) The relation γ^* is the smallest equivalence relation such that the quotient R/γ^* is a ring.

Clearly, $\varphi : R \to R/\gamma^*$ defined by $\varphi(x) = \gamma^*(x)$ for all $x \in R$, is a homomorphism. The kernel of φ, kerφ, is defined by ker$\varphi = \{x \in R \mid \gamma^*(x) = \gamma^*(0)\}$. We denote by $0_{R/\gamma^*}$ the zero element of R/γ^*. If R is a Krasner hyperring, then $\gamma^*(0) = 0_{R/\gamma^*}$ and $\gamma^*(-x) = -\gamma^*(x)$ for all $x \in R$.

Theorem 3.14. *Let $(R, +, \cdot)$ be a Krasner hyperring and γ^* a fundamental relation on R. If I is an n-hyperideal of R such that $Ker\varphi \subseteq I$, then $\gamma^*(I) = \{\gamma^*(a) \mid a \in I\}$ is an n-hyperideal of R/γ^*.*

Proof. Clearly, $\gamma^*(I)$ is a hyperideal of R/γ^*. Let $\gamma^*(a) \odot \gamma^*(b) \in \gamma^*(I)$ and $\gamma^*(a) \notin \sqrt{0_{R/\gamma^*}}$ for $\gamma^*(a), \gamma^*(b) \in R/\gamma^*$. Then, there exists $x \in I$ such that $\gamma^*(a \cdot b) = \gamma^*(a) \odot \gamma^*(b) = \gamma^*(x)$. So, we have

$$\gamma^*(0) = \gamma^*(a \cdot b) \ominus \gamma^*(x) = \varphi(a \cdot b) \ominus \varphi(x) = \varphi(a \cdot b - x) = \gamma^*(a \cdot b - x).$$

Hence, $a \cdot b - x \subseteq Ker\varphi \subseteq I$. Since $(R, +)$ is a canonical hypergroup, we have

$$a \cdot b \in a \cdot b + 0 \subseteq a \cdot b + x - x \subseteq I + x \subseteq I.$$

So, $a \cdot b \in I$. Next, we show that $a \notin \sqrt{0_R}$. By hypothesis, we have

$$\gamma^*(a) \notin \sqrt{0_{R/\gamma^*}} = \sqrt{\gamma^*(0)}.$$

If $a \in \sqrt{0_R}$, then there exists $n \in \mathbb{N}$ such that $a^n = 0$. This means that $\gamma^*(a^n) = \gamma^*(0)$, and so $(\gamma^*(a))^n = 0_{R/\gamma^*}$. Hence, $\gamma^*(a) \in \sqrt{0_{R/\gamma^*}}$, which is a contradiction. This leads to $a \notin \sqrt{0_R}$. Now, since I is an n-hyperideal of R, we get $b \in I$ and so $\gamma^*(b) \in \gamma^*(I)$. Therefore, $\gamma^*(I)$ is an n-hyperideal of R/γ^*. □

4 Conclusions

In this paper, we introduced and studied some properties of n-hyperideals of commutative hyperrings. Also, we proved that some results on extension (contraction) of n-hyperideals in commutative hyperrings. Moreover, we described the behavior of n-hyperideals under fundamental relations. We hope that this paper would offer foundation for further study of the theory on commutative hyperrings.

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