Two dimensional event set and its application in algebraic structures

Y.B. Jun

Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea
skywine@gmail.com

Abstract

Two dimensional event set is introduced, and it is applied to algebraic structures. Two dimensional BCK/BCI-eventful algebra, paired B-algebra and paired BCK/BCI-algebra are defined, and several properties are investigated. Conditions for two dimensional eventful algebra to be a B-algebra and a BCK/BCI-algebra are provided. The process of inducing a paired B-algebra using a group is discussed. Using two dimensional BCI-eventful algebra, a commutative group is established.

1 Introduction and Preliminaries

The notion of neutrosophic set is developed by Smarandache ([9], [10]), and is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Smarandache [11] considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part \( a \) and an unknown part \( (bT, cI, dF) \) where \( T, I, F \) have their usual neutrosophic logic meanings and \( a, b, c, d \) are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [1] and [2]. Neutrosophic quadruple BCK/BCI-algebra is studied in [5] and [7]. Using neutrosophic quadruple structures, Jun et al. [4] introduced the notion of events by considering two facts, and applied it to BCK/BCI-algebras. There are many things in our daily lives that we have to choose between two facts. For example, should I read a book or not, go to the movies or not, etc. To consider these two factors, we introduce two-dimensional event sets and try to apply them to algebraic structures. We introduce the notions of two dimensional BCK/BCI-eventful algebra, paired B-algebra and paired BCK/BCI-algebra, and investigate several properties. We provided conditions for two dimensional eventful algebra to be a...
B-algebra and a BCK/BCI-algebra. We discuss the process of inducing a paired B-algebra using a group, and establish a commutative group using two dimensional BCI-eventful algebra.

We describe the basic contents that will be needed in this paper. Let \((X, *, 0)\) be an algebra, i.e., let \(X\) be a set with a special element “0” and a binary operation “*”, and consider the following conditions.

We say that \(X := (X, *, 0)\) is

- a B-algebra (see [3]) if it satisfies (1), (2) and (4),
- a BCI-algebra (see [3]) if it satisfies (1), (5), (6) and (7),
- a BCK-algebra (see [3]) if it is a BCI-algebra satisfying (3).

A BCI-algebra \(X := (X, *, 0)\) is said to be \(p\)-semisimple (see [3]) if it satisfies:

\[
(\forall u \in X)(0 * (0 * u) = u).
\] (8)

2 Two dimensional event sets

Definition 2.1. Let \(\ell : X \to Q\) be a mapping from a set \(X\) to a set \(Q\). For any \(a, x \in X\), the ordered pair \((x, \ell_a)\) is called a two dimensional event on \(X\) where \(\ell_a\) is the image of \(a\) under \(\ell\).

The set of all two dimensional events on \(X\) is denoted by \((X, \ell_X)\), that is,

\[
(X, \ell_X) = \{(x, \ell_a) \mid x, a \in X\}.
\] (9)

and it is called a two dimensional \(X\)-event set. By a two dimensional \(X\)-eventful algebra we mean a two dimensional \(X\)-event set with a binary operation \& and \(\circ\), and it is denoted by \(\langle (X, \ell_X), \& \rangle\).

Let \(\langle (\mathbb{R}, \ell_{\mathbb{R}}), \oplus \rangle\), \(\langle (\mathbb{R}, \ell_{\mathbb{R}}), \ominus \rangle\) and \(\langle (\mathbb{R}, \ell_{\mathbb{R}}), \odot \rangle\) be two dimensional \(\mathbb{R}\)-eventful algebras in which “\(\oplus\)”, “\(\ominus\)” and “\(\odot\)” are defined as follows:

\[
\begin{align*}
(x, \ell_a) + (y, \ell_b) &= (x + y, \ell_{a+b}), \\
(x, \ell_a) - (y, \ell_b) &= (x - y, \ell_{a-b}), \\
(x, \ell_a) \cdot (y, \ell_b) &= (x \cdot y, \ell_{a\cdot b}),
\end{align*}
\]

respectively, for any two dimensional events \((x, \ell_a)\) and \((y, \ell_b)\) on \(\mathbb{R}\). For any \(t \in \mathbb{R}\) and a two dimensional event \((x, \ell_a)\) on \(\mathbb{R}\), we define

\[
t(x, \ell_a) = (tx, \ell_{ta}).
\] (10)

In particular, if \(t = -1\), then \(-1(x, \ell_a) = (-x, \ell_{-a})\) and \(-1(x, \ell_a)\) is simply denoted by \((-x, \ell_a)\).

Proposition 2.2. Let \(\langle (\mathbb{R}, \ell_{\mathbb{R}}), \oplus \rangle\), \(\langle (\mathbb{R}, \ell_{\mathbb{R}}), \ominus \rangle\) and \(\langle (\mathbb{R}, \ell_{\mathbb{R}}), \odot \rangle\) be two dimensional \(\mathbb{R}\)-eventful algebras. Then

\[
\begin{align*}
(1) \quad (x, \ell_a) \oplus (y, \ell_b) \oplus (z, \ell_c) &= (x, \ell_a) \oplus ((y, \ell_b) \oplus (z, \ell_c)),
\end{align*}
\]
Straightforward.

**Proof.** For any \((x, \ell_a) \in (X, \ell_X)\), we have the following theorem.

**Theorem 2.3.** Two dimensional \(\mathbb{R}\)-eventful algebras \(\langle (\mathbb{R}, \ell_\mathbb{R}), \oplus \rangle\) and \(\langle (\mathbb{R}, \ell_\mathbb{R}), \odot \rangle\) are commutative groups with identities \((0, \ell_0)\) and \((1, \ell_1)\), respectively.

### 3 Two dimensional eventful algebras

Let \((X, *, 0)\) be an algebra and \(\langle (X, \ell_X), \oplus \rangle\) be a two dimensional \(X\)-eventful algebra in which “\(\oplus\)” is defined by

\[
(x, \ell_a) \oplus (y, \ell_b) = (x * y, \ell_{a*b})
\]

respectively, for all \(x, y, a, b \in X\). In a two dimensional \(X\)-eventful algebra \(\langle (X, \ell_X), \oplus \rangle\), the order “\(\ll\)” is defined as follows:

\[
(x, \ell_a) \ll (y, \ell_b) \iff x \leq y \text{ and } a \leq b
\]

for all \(x, y, a, b \in X\) where \(x \leq y\) means \(x * y = 0\) and \(a \leq b\) means \(a * b = 0\).

**Theorem 3.1.** If \((X, *, 0)\) is a \(B\)-algebra, then the two dimensional \(X\)-eventful algebra \(\langle (X, \ell_X), \oplus \rangle\) is a \(B\)-algebra with the special element \((0, \ell_0)\).

**Proof.** For any \((x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)\), we have

\[
(x, \ell_a) \oplus (x, \ell_a) = (x * x, \ell_{a*a}) = (0, \ell_0), \quad (x, \ell_a) \oplus (0, \ell_0) = (x * 0, \ell_{a*0}) = (x, \ell_a),
\]

and

\[
((x, \ell_a) \oplus (y, \ell_b)) \oplus (z, \ell_c) = (x * y, \ell_{a*b}) \oplus (z, \ell_c)
\]

\[
= ((x * y) * z, \ell_{(a*b)*c})
\]

\[
= ((x * (z * 0), \ell_{a*0}))
\]

\[
= ((x, \ell_a) \oplus (z, \ell_c) \oplus (0 * y, \ell_{0*b}))
\]

\[
= ((x, \ell_a) \oplus ((z, \ell_c) \oplus (0, \ell_0) \oplus (y, \ell_b)))
\]

by 1, 2 and 4, respectively. Therefore \(\langle (X, \ell_X), \oplus \rangle\) is a \(B\)-algebra with the special element \((0, \ell_0)\).}

We say that \(\langle (X, \ell_X), \oplus, (0, \ell_0) \rangle\) is a **paired \(B\)-algebra**.
Example 3.2. Let $X = \{0, a, b\}$ be a set with the binary operation “$\ast$” as in Table 1.

For $Q = \{0, \frac{1}{2}, 1\}$, define a mapping $\ell$ as follows:

$$\ell : X \to Q, \ x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x = a, \\ 1 & \text{if } x = b. \end{cases}$$

Then

$$(X, \ell X) = \{(0, \ell_0), (0, \ell_a), (0, \ell_b), (a, \ell_0), (a, \ell_a), (a, \ell_b), (b, \ell_0), (b, \ell_a), (b, \ell_b)\}$$

$$= \{(0, 0), (0, \frac{1}{2}), (0, 1), (a, 0), (a, \frac{1}{2}), (a, 1), (b, 0), (b, \frac{1}{2}), (b, 1)\}$$

and the operation $\oplus$ is given by Table 2. It is routine to verify that $(X, \ell X, \oplus, (0, \ell_0))$ is a paired $B$-algebra.

Table 1: Cayley table for the binary operation “$\ast$”

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: Cayley table for the binary operation “$\ast$”

<table>
<thead>
<tr>
<th>$\oplus$</th>
<th>$(0, \ell_0)$</th>
<th>$(0, \ell_a)$</th>
<th>$(0, \ell_b)$</th>
<th>$(a, \ell_0)$</th>
<th>$(a, \ell_a)$</th>
<th>$(a, \ell_b)$</th>
<th>$(b, \ell_0)$</th>
<th>$(b, \ell_a)$</th>
<th>$(b, \ell_b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, \ell_0)$</td>
<td>$(0, 0)$</td>
<td>$(0, 1)$</td>
<td>$(0, \frac{1}{2})$</td>
<td>$(b, 0)$</td>
<td>$(b, 1)$</td>
<td>$(b, \frac{1}{2})$</td>
<td>$(a, 0)$</td>
<td>$(a, 1)$</td>
<td>$(a, \frac{1}{2})$</td>
</tr>
<tr>
<td>$(0, \ell_a)$</td>
<td>$(0, \frac{1}{2})$</td>
<td>$(0, 0)$</td>
<td>$(0, \frac{1}{2})$</td>
<td>$(b, 0)$</td>
<td>$(b, \frac{1}{2})$</td>
<td>$(b, 1)$</td>
<td>$(a, 0)$</td>
<td>$(a, \frac{1}{2})$</td>
<td>$(a, 1)$</td>
</tr>
<tr>
<td>$(0, \ell_b)$</td>
<td>$(0, 0)$</td>
<td>$(0, 1)$</td>
<td>$(0, \frac{1}{2})$</td>
<td>$(b, 0)$</td>
<td>$(b, \frac{1}{2})$</td>
<td>$(b, 1)$</td>
<td>$(a, 0)$</td>
<td>$(a, \frac{1}{2})$</td>
<td>$(a, 1)$</td>
</tr>
<tr>
<td>$(a, \ell_0)$</td>
<td>$(a, 0)$</td>
<td>$(a, 1)$</td>
<td>$(a, \frac{1}{2})$</td>
<td>$(0, 0)$</td>
<td>$(0, \frac{1}{2})$</td>
<td>$(0, 1)$</td>
<td>$(b, 0)$</td>
<td>$(b, \frac{1}{2})$</td>
<td>$(b, 1)$</td>
</tr>
<tr>
<td>$(a, \ell_a)$</td>
<td>$(a, \frac{1}{2})$</td>
<td>$(a, 0)$</td>
<td>$(a, \frac{1}{2})$</td>
<td>$(0, 0)$</td>
<td>$(0, \frac{1}{2})$</td>
<td>$(0, 1)$</td>
<td>$(b, 0)$</td>
<td>$(b, \frac{1}{2})$</td>
<td>$(b, 1)$</td>
</tr>
<tr>
<td>$(a, \ell_b)$</td>
<td>$(a, 0)$</td>
<td>$(a, 1)$</td>
<td>$(a, \frac{1}{2})$</td>
<td>$(0, 0)$</td>
<td>$(0, \frac{1}{2})$</td>
<td>$(0, 1)$</td>
<td>$(b, 0)$</td>
<td>$(b, \frac{1}{2})$</td>
<td>$(b, 1)$</td>
</tr>
<tr>
<td>$(b, \ell_0)$</td>
<td>$(b, 0)$</td>
<td>$(b, 1)$</td>
<td>$(b, \frac{1}{2})$</td>
<td>$(a, 0)$</td>
<td>$(a, \frac{1}{2})$</td>
<td>$(a, 1)$</td>
<td>$(0, 0)$</td>
<td>$(0, \frac{1}{2})$</td>
<td>$(0, 1)$</td>
</tr>
<tr>
<td>$(b, \ell_a)$</td>
<td>$(b, \frac{1}{2})$</td>
<td>$(b, 0)$</td>
<td>$(b, \frac{1}{2})$</td>
<td>$(a, 0)$</td>
<td>$(a, \frac{1}{2})$</td>
<td>$(a, 1)$</td>
<td>$(0, 0)$</td>
<td>$(0, \frac{1}{2})$</td>
<td>$(0, 1)$</td>
</tr>
<tr>
<td>$(b, \ell_b)$</td>
<td>$(b, 1)$</td>
<td>$(b, 0)$</td>
<td>$(b, \frac{1}{2})$</td>
<td>$(a, 0)$</td>
<td>$(a, \frac{1}{2})$</td>
<td>$(a, 1)$</td>
<td>$(0, 0)$</td>
<td>$(0, \frac{1}{2})$</td>
<td>$(0, 1)$</td>
</tr>
</tbody>
</table>

Proposition 3.3. If $(X, \ell X, \oplus, (0, \ell_0))$ is a paired $B$-algebra, then

(i) $(x, \ell_a) \oplus (y, \ell_b) = (x, \ell_a) \oplus ((0, \ell_0) \oplus ((0, \ell_0) \oplus (y, \ell_b)))$,

(ii) $(x, \ell_a) \oplus (y, \ell_b) \oplus ((0, \ell_0) \oplus (y, \ell_b)) = (x, \ell_a)$,

(iii) $(x, \ell_a) \oplus (z, \ell_c) = (y, \ell_b) \oplus (z, \ell_c)$ implies $(x, \ell_a) = (y, \ell_b)$,

(iv) $(x, \ell_a) \oplus ((y, \ell_b) \oplus (z, \ell_c)) = ((x, \ell_a) \oplus ((0, \ell_0) \oplus (z, \ell_c))) \oplus (y, \ell_b)$,

(v) $(x, \ell_a) \oplus (y, \ell_b) = (0, \ell_0)$ implies $(x, \ell_a) = (y, \ell_b)$,

(vi) $(0, \ell_0) \oplus (x, \ell_a) = (0, \ell_0) \oplus (y, \ell_b)$ implies $(x, \ell_a) = (y, \ell_b)$,

(vii) $(0, \ell_0) \oplus ((0, \ell_0) \oplus (x, \ell_a)) = (x, \ell_a)$

for all $x, y, z, a, b, c \in X$. 
Proof. Let \(x, y, z, a, b, c \in X\). Then
\[
(x, \ell_a) \oplus (y, \ell_b) = (x \ast y, \ell_{a \ast b}) = ((x \ast y) \ast 0, \ell_{(a \ast b) \ast 0})
\]
\[
= (x \ast (0 \ast (0 \ast y)), \ell_{a \ast (0 \ast (0 \ast b)))})
\]
\[
= (x, \ell_a) \oplus (0 \ast (0 \ast y), \ell_{a \ast (0 \ast b)})
\]
\[
= (x, \ell_a) \oplus ((0, \ell_a) \oplus (0, \ell_0) \oplus (y, \ell_b))
\]
which proves (i).
(ii) We have
\[
((x, \ell_a) \oplus (y, \ell_b)) \oplus ((0, \ell_0) \oplus (y, \ell_b)) = (x \ast y, \ell_{a \ast b}) \oplus (0 \ast y, \ell_{0 \ast b})
\]
\[
= ((x \ast y) \ast (0 \ast y), \ell_{(a \ast b) \ast (0 \ast b)})
\]
\[
= (x \ast (0 \ast y) \ast (0 \ast y), \ell_{a \ast (0 \ast (0 \ast b)))})
\]
\[
= (x \ast 0, \ell_{a \ast 0}) = (x, \ell_a).
\]
(iii) Assume that \((x, \ell_a) \oplus (z, \ell_c) = (y, \ell_b) \oplus (z, \ell_c)\). It follows from (ii) that
\[
(x, \ell_a) = ((x, \ell_a) \oplus (z, \ell_c)) \oplus ((0, \ell_0) \oplus (z, \ell_c))
\]
\[
= ((y, \ell_b) \oplus (z, \ell_c)) \oplus ((0, \ell_0) \oplus (z, \ell_c))
\]
\[
= (y, \ell_b).
\]
(iv) Using (i), we have
\[
((x, \ell_a) \oplus (0, \ell_0) \oplus (z, \ell_c)) \oplus (y, \ell_b)
\]
\[
= (x, \ell_a) \oplus ((y, \ell_b) \oplus ((0, \ell_0) \oplus (0, \ell_0) \oplus (z, \ell_c)))
\]
\[
= (x, \ell_a) \oplus ((y, \ell_b) \oplus (z, \ell_c)).
\]
(v) Suppose that \((x, \ell_a) \oplus (y, \ell_b) = (0, \ell_0)\). Then \((x, \ell_a) \oplus (y, \ell_b) = (y, \ell_b) \oplus (y, \ell_b)\), and so \((x, \ell_a) = (y, \ell_b)\) by (iii).
(vi) Assume that \((0, \ell_0) \oplus (x, \ell_a) = (0, \ell_0) \oplus (y, \ell_b)\). Then
\[
(0, \ell_0) = (x, \ell_a) \oplus (x, \ell_a) = (x, \ell_a) \oplus ((0, \ell_0) \oplus (0, \ell_0) \oplus (x, \ell_a))
\]
\[
= (x, \ell_a) \oplus ((0, \ell_0) \oplus (0, \ell_0) \oplus (y, \ell_b))
\]
\[
= (x, \ell_a) \oplus (y, \ell_b)
\]
and so \((x, \ell_a) = (y, \ell_b)\) by (v).
(vii) We have
\[
(0, \ell_0) \oplus (x, \ell_a) = ((0, \ell_0) \oplus (x, \ell_a)) \oplus (0, \ell_0)
\]
\[
= ((0 \ast x) \ast 0, \ell_{(0 \ast x) \ast 0})
\]
\[
= (0 \ast (0 \ast (0 \ast x)), \ell_{0 \ast (0 \ast (0 \ast x)))})
\]
\[
= (0, \ell_0) \oplus ((0, \ell_0) \oplus (0, \ell_0) \oplus (x, \ell_a))
\]
and so \((0, \ell_0) \oplus ((0, \ell_0) \oplus (x, \ell_a)) = (x, \ell_a)\) by (vi).
\]

We provide conditions for two dimensional \(X\)-eventful algebra to be a \(B\)-algebra.

**Theorem 3.4.** For an algebra \((X, \ast, 0)\), the two dimensional \(X\)-eventful algebra \(((X, \ell_X), \oplus)\) is a \(B\)-algebra with the special element \((0, \ell_0)\) if and only if it satisfies Proposition 3.3(vii) and
\[
(x, \ell_a) \oplus (x, \ell_a) = (0, \ell_0), \quad \text{(11)}
\]
\[
((x, \ell_a) \oplus (z, \ell_c)) \oplus ((y, \ell_b) \oplus (z, \ell_c)) = (x, \ell_a) \oplus (y, \ell_b), \quad \text{(12)}
\]
for all \((x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)\). \(\square\)
Proof. Assume that the two dimensional $X$-eventful algebra $\langle (X, \ell_X), \oplus \rangle$ is a B-algebra with the special element $(0, \ell_0)$. The condition Proposition 3.3(vii) is by Proposition 3.3. It is clear that (11) is true by the definition of B-algebra. Also, we have

\[
((x, \ell_a) \oplus (z, \ell_c)) \oplus ((y, \ell_b) \oplus (z, \ell_c)) = (x, \ell_a) \oplus ((0, \ell_0) \oplus (z, \ell_c))
\]

for all $(x, \ell_a), (y, \ell_b) \in (X, \ell_X)$. Conversely, suppose that $\langle (X, \ell_X), \oplus \rangle$ satisfies three conditions (11), (12) and Proposition 3.3(vii). Then

\[
(x, \ell_a) = (0, \ell_0) \oplus ((0, \ell_0) \oplus (x, \ell_a)) = ((0, \ell_0) \oplus (x, \ell_a)) \oplus (x, \ell_a)
\]

and

\[
((x, \ell_a) \oplus (y, \ell_b)) \oplus ((0, \ell_0) \oplus (y, \ell_b)) = (x, \ell_a) \oplus (0, \ell_0) = (x, \ell_a)
\]

(13)

Combining (12) with (13) induces

\[
(x, \ell_a) \oplus ((z, \ell_c) \oplus ((0, \ell_0) \oplus (y, \ell_b))) = ((x, \ell_a) \oplus (y, \ell_b)) \oplus ((0, \ell_0) \oplus (z, \ell_c))
\]

for all $(x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)$. Therefore $\langle (X, \ell_X), \oplus \rangle$ is a B-algebra with the special element $(0, \ell_0)$.

\[\square\]

Theorem 3.5. For an algebra $(X, \ast, 0)$, the two dimensional $X$-eventful algebra $\langle (X, \ell_X), \oplus \rangle$ is a B-algebra with the special element $(0, \ell_0)$ if and only if it satisfies (11) and

\[
(x, \ell_a) \oplus (((0, \ell_0) \oplus (y, \ell_b)) \oplus (z, \ell_c)) = (y, \ell_b)
\]

(14)

for all $(x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)$.

Proof. Assume that the two dimensional $X$-eventful algebra $\langle (X, \ell_X), \oplus \rangle$ is a B-algebra with the special element $(0, \ell_0)$. Then (11) is valid in Theorem 3.4. Using (12), we get

\[
(((0, \ell_0) \oplus (y, \ell_b)) \oplus (z, \ell_c)) \oplus ((0, \ell_0) \oplus (x, \ell_a)) = (y, \ell_b)
\]

(15)

which proves (14).

Conversely, suppose that $\langle (X, \ell_X), \oplus \rangle$ satisfies (11) and (14). If we substitute $(y, \ell_b)$ for $(x, \ell_a)$ in (14) and use (11), then

\[
(x, \ell_a) = (x, \ell_a) \oplus (((0, \ell_0) \oplus (y, \ell_b)) \oplus (z, \ell_c)) = (x, \ell_a)
\]

(16)
If we put \((x, \ell_a) = (0, \ell_0) = (z, \ell_c)\) and \((y, \ell_b) = (x, \ell_a)\) in (14), then
\[
(x, \ell_a) = (0, \ell_0) \oplus (((0, \ell_0) \oplus (x, \ell_a)) \oplus (0, \ell_0)) \oplus (((0, \ell_0) \oplus (0, \ell_0)) \oplus (0, \ell_0))
\]
\[
= (0, \ell_0) \oplus ((0, \ell_0) \oplus (x, \ell_a))
\] (17)
by (16). Assume that \((0, \ell_0) \oplus (x, \ell_a) = (0, \ell_0) \oplus (y, \ell_b)\). Then
\[
(x, \ell_a) = (0, \ell_0) \oplus (0, \ell_0) \oplus (x, \ell_a)) = (0, \ell_0) \oplus ((0, \ell_0) \oplus (y, \ell_b)) = (y, \ell_b)
\]
by (17) which proves
\[
(0, \ell_0) \oplus (x, \ell_a) = (0, \ell_0) \oplus (y, \ell_b) \Rightarrow (x, \ell_a) = (y, \ell_b).
\] (18)

Putting \((x, \ell_a) = (0, \ell_0), (y, \ell_b) = (0, \ell_0) \oplus (y', \ell_{b'})\) and \((z, \ell_c) = (z', \ell_{c'})\) in (14) induces
\[
(0, \ell_0) \oplus (y', \ell_{b'}) = (0, \ell_0) \oplus (((0, \ell_0) \oplus (0, \ell_0) \oplus (y', \ell_{b'})) \oplus (z', \ell_{c'})) \oplus ((0, \ell_0) \oplus (z', \ell_{c'}))
\]
\[
= (0, \ell_0) \oplus (((y', \ell_{b'}) \oplus (z', \ell_{c'})) \oplus (0, \ell_0) \oplus (z', \ell_{c'})).
\] (19)

It follows from (18) that
\[
(y', \ell_{b'}) = (((y', \ell_{b'}) \oplus (z', \ell_{c'})) \oplus (0, \ell_0) \oplus (z', \ell_{c'})).
\]

If we substitute \((x, \ell_a), (y, \ell_b)\) and \((z, \ell_c)\) for \((x', \ell_{a'}), (0, \ell_0) \oplus ((z', \ell_{c'}) \oplus (x', \ell_{a'}))\) and \((0, \ell_0)\), respectively, in (14), then
\[
(0, \ell_0) \oplus ((z', \ell_{c'}) \oplus (x', \ell_{a'}))
\]
\[
= (x', \ell_{a'}) \oplus (((0, \ell_0) \oplus (0, \ell_0) \oplus ((z', \ell_{c'}) \oplus (x', \ell_{a'}))) \oplus (0, \ell_0))
\]
\[
\oplus ((0, \ell_0) \oplus (x', \ell_{a'}) \oplus (0, \ell_0))
\] (20)
\[
= (x', \ell_{a'}) \oplus (((z', \ell_{c'}) \oplus (x', \ell_{a'})) \oplus (0, \ell_0) \oplus (x', \ell_{a'}))
\]
\[
= ((x', \ell_{a'}) \oplus (z', \ell_{c'}))
\]

In (14), taking \((x, \ell_a) = (y', \ell_{b'}) \oplus (w, \ell_d), (y, \ell_b) = (y', \ell_{b'}) \oplus (z', \ell_{c'})\) and \((z, \ell_c) = (0, \ell_0) \oplus (y', \ell_{b'})\) imply that
\[
(y', \ell_{b'}) \oplus (z', \ell_{c'}) = (((y', \ell_{b'}) \oplus (w, \ell_d)) \oplus [((0, \ell_0) \oplus (y', \ell_{b'}) \oplus (z', \ell_{c'}))
\]
\[
\oplus ((0, \ell_0) \oplus (y', \ell_{b'})) \oplus (((0, \ell_0) \oplus (y', \ell_{b'}) \oplus (w, \ell_d)))
\]
\[
\oplus ((0, \ell_0) \oplus (y', \ell_{b'}))]
\] (21)

Using (19) and (20), we get
\[
(((0, \ell_0) \oplus (y', \ell_{b'}) \oplus (z', \ell_{c'})) \oplus (0, \ell_0) \oplus (y', \ell_{b'}))
\]
\[
= ((z', \ell_{c'}) \oplus (y', \ell_{b'})) \oplus ((0, \ell_0) \oplus (y', \ell_{b'})) = (z', \ell_{c'}).
\] (22)

Similarly, we have
\[
((0, \ell_0) \oplus (y', \ell_{b'}) \oplus (w, \ell_d)) \oplus ((0, \ell_0) \oplus (y', \ell_{b'})) = (w, \ell_d).
\] (23)

Combining (21), (22) and (23) induces
\[
(y', \ell_{b'}) \oplus (z', \ell_{c'}) = ((y', \ell_{b'}) \oplus (w, \ell_d)) \oplus ((z', \ell_{c'}) \oplus (w, \ell_d)).
\]

Therefore \(\langle X, \ell_X, \otimes \rangle\) is a B-algebra with the special element \((0, \ell_0)\) by Theorem 3.4.

The following theorem shows the process of inducing a paired B-algebra using a group.
Theorem 3.6. If \( (X, \circ, 0) \) is a group, then the two dimensional \( X \)-eventful algebra \( \langle (X, \ell_X), \oplus, (0, \ell_0) \rangle \) is a paired \( B \)-algebra where

\[
\oplus : (X, \ell_X) \times (X, \ell_X) \to (X, \ell_X), \quad ((x, \ell_a), (y, \ell_b)) \mapsto (x \circ y^{-1}, \ell_{aob-1}) = (x \ast y, \ell_{a+b}).
\]

**Proof.** Let \((x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X)\). Then \((x, \ell_a) \oplus (x, \ell_a) = (x \circ x^{-1}, \ell_{aa-1}) = (0, \ell_0)\) and \((x, \ell_a) \oplus (0, \ell_0) = (x \circ 0^{-1}, \ell_{a0-1}) = (x, \ell_a)\). Also

\[
((x, \ell_a) \oplus (y, \ell_b)) \oplus (z, \ell_c) = (x \circ y^{-1}, \ell_{aob-1}) \oplus (z, \ell_c)
= ((x \circ y^{-1}) \circ z^{-1}, \ell_{(aob-1)c-1})
= (x \circ (z \circ y)^{-1}, \ell_{aoc(b-1)-1})
\]

and

\[
(x, \ell_a) \oplus ((z, \ell_c) \oplus ((0, \ell_0) \oplus (y, \ell_b))) = (x, \ell_a) \oplus ((z, \ell_c) \oplus (0 \circ y^{-1}, \ell_{0ob-1}))
= (x, \ell_a) \oplus ((z, \ell_c) \oplus (y^{-1} \circ b^{-1})) = (x, \ell_a) \oplus ((y^{-1} \circ b^{-1}), \ell_{coc(b^{-1})-1})
= (x, \ell_a) \oplus (z \circ y, \ell_{cob}) = (x \circ (z \circ y)^{-1}, \ell_{aoc(b-1)-1}).
\]

Hence \((x, \ell_a) \oplus (y, \ell_b)) \oplus (z, \ell_c) = (x, \ell_a) \oplus ((z, \ell_c) \oplus ((0, \ell_0) \oplus (y, \ell_b)))\). Therefore \(\langle (X, \ell_X), \oplus, (0, \ell_0) \rangle\) is a paired \( B \)-algebra.

Let \( X := (X, \ast, 0) \) be an algebra. In a two dimensional \( X \)-eventful algebra \( \langle (X, \ell_X), \oplus, (0, \ell_0) \rangle \), we consider the following assertions.

\[
\begin{align*}
((x, \ell_a) \oplus (y, \ell_b)) \oplus ((x, \ell_a) \oplus (z, \ell_c)) & = (0, \ell_0), \\
(x, \ell_a) \oplus ((x, \ell_a) \oplus (y, \ell_b)) \oplus (y, \ell_b) & = (0, \ell_0), \\
(0, \ell_0) \oplus (x, \ell_a) & = (0, \ell_0), \\
(x, \ell_a) \oplus (y, \ell_b) & = (0, \ell_0), \\
(y, \ell_b) \oplus (x, \ell_a) & = (0, \ell_0) \Rightarrow (x, \ell_a) = (y, \ell_b)
\end{align*}
\]

for all \( x, y, z, a, b, c \in X \).

**Definition 3.7.** Given an algebra \( X := (X, \ast, 0) \), let \( \langle (X, \ell_X), \oplus \rangle \) be a two dimensional \( X \)-eventful algebra with a special element \((0, \ell_0)\). If it satisfies (11), (24) and (25), we say that \( \langle (X, \ell_X), \oplus, (0, \ell_0) \rangle \) is a two dimensional \( BCI \)-eventful algebra. If a two dimensional \( BCI \)-eventful algebra \( \langle (X, \ell_X), \oplus, (0, \ell_0) \rangle \) satisfies the condition (26), it is called a two dimensional \( BCK \)-eventful algebra.

**Example 3.8.** (1) Consider an algebra \( X := (X, \ast, 0) \) where \( X = \{0, a\} \) and the binary operation “\( \ast \)” is given by Table 3. Given a set \( Q = \{\alpha, \beta\} \), define a mapping \( \ell \) as follows:

| Table 3: Cayley table for the binary operation “\( \ast \)” |
|-----------------|-----------------|
| \( \ast \)      | 0               | \( a \)            |
| 0               | 0               | 0                 |
| \( a \)         | \( a \)         | \( a \)            |

\[
\ell : X \to Q, \quad x \mapsto \begin{cases} 
\alpha & \text{if } x = 0, \\
\beta & \text{if } x = a.
\end{cases}
\]

Then \( (X, \ell_X) = \{(0, \ell_0), (0, \ell_a), (a, \ell_0), (a, \ell_a)\} \) and the operation \( \oplus \) is given by Table 4. It is routine to verify that \( \langle (X, \ell_X), \oplus, (0, \ell_0) \rangle \) is a two dimensional \( BCK \)-eventful algebra.

(2) The paired \( B \)-algebra \( \langle (X, \ell_X), \oplus, (0, \ell_0) \rangle \) in Example 3.2 is two dimensional \( BCI \)-eventful algebra.
Given a nonempty set $\mathcal{Q}$.

**Example 3.11.** Consider a BCK-algebra $(\mathcal{Q}, \ell, 0)$ as seen in the following examples.

If $\mathcal{Q}$ is a BCK/BCI-eventful algebra by Lemma 3.10. It is routine to verify that $\langle \mathcal{Q}, \ell \rangle$ satisfies the condition (27).

In general, two dimensional BCK/BCI-eventful algebra $\langle (\mathcal{Q}, \ell_x), \odot, (0, \ell_0) \rangle$ does not satisfy the condition (27) as seen in the following examples.

**Example 3.9.** Consider an algebra $\mathcal{X} = (\mathcal{X}, \star, 0)$ where $\mathcal{X} = \{0, a, b, c\}$ and the binary operation “$\star$” is given by Table 3.

Given a set $\mathcal{Q} = \{0.2, 0.5, 0.7\}$, define a mapping $\ell$ as follows:

$$\ell : \mathcal{X} \to \mathcal{Q}, x \mapsto \begin{cases} 0.2 & \text{if } x \in \{0, a\} \\ 0.7 & \text{if } x \in \{b, c\}. \end{cases}$$

Then the two dimensional $\mathcal{X}$-event set is given as follows:

$$(\mathcal{X}, \ell_x) = \{(0, 0.2), (0, 0.7), (a, 0.2), (a, 0.7), (b, 0.2), (b, 0.7), (c, 0.2), (c, 0.7)\}$$

and it is routine to check that $\langle (\mathcal{X}, \ell_x), \odot, (0, \ell_0) \rangle$ is a two dimensional BCI-eventful algebra. But it is not a two dimensional BCK-eventful algebra since $\odot (0, \ell_0) @ (b, \ell_b) = (c, \ell_c) \neq (0, \ell_0)$. Note that $\odot (0, \ell_0) @ (0, \ell_0) = (0, \ell_0) = (0, \ell_0)$, and $\odot (0, \ell_b) @ (0, \ell_0) = (0, \ell_b) = (0, \ell_0)$, but $\odot (0, \ell_0) @ (0, \ell_b)$. Hence $\langle (\mathcal{X}, \ell_x), \odot, (0, \ell_0) \rangle$ does not satisfy the condition (27).

**Lemma 3.10.** If $\mathcal{X} := (\mathcal{X}, \star, 0)$ is a BCK/BCI-algebra, then $\langle (\mathcal{X}, \ell_x), \odot, (0, \ell_0) \rangle$ is a two dimensional BCK/BCI-eventful algebra.

Proof. Straightforward.

By a paired BCK/BCI-algebra we mean a two dimensional BCK/BCI-eventful algebra $\langle (\mathcal{X}, \ell_x), \odot, (0, \ell_0) \rangle$ which satisfies the condition (27).

**Example 3.11.** Consider a BCK-algebra $\mathcal{X} = (\mathcal{X}, \star, 0)$ where $\mathcal{X} = \{0, a, b, c\}$ and the binary operation “$\star$” is given by Table 4.

Given a nonempty set $\mathcal{Q}$, define a mapping $\ell$ as follows:

$$\ell : \mathcal{X} \to \mathcal{Q}, x \mapsto u.$$ 

Then $(\mathcal{X}, \ell_x) = \{(0, u), (a, u), (b, u), (c, u)\}$, and it is clear that $\langle (\mathcal{X}, \ell_x), \odot, (0, u) \rangle$ is a two dimensional BCK-eventful algebra by Lemma 3.10. It is routine to verify that $\langle (\mathcal{X}, \ell_x), \odot, (0, u) \rangle$ satisfies the condition (27). Hence it is a paired BCK-algebra.

**Table 4:** Cayley table for the binary operation “$\star$”

<table>
<thead>
<tr>
<th>$\oplus$</th>
<th>$0$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, \ell_0)$</td>
<td>$(0, \alpha)$</td>
<td>$(0, \alpha)$</td>
<td>$(0, \alpha)$</td>
<td>$(0, \alpha)$</td>
</tr>
<tr>
<td>$(0, \ell_a)$</td>
<td>$(0, \beta)$</td>
<td>$(0, \alpha)$</td>
<td>$(0, \beta)$</td>
<td>$(0, \alpha)$</td>
</tr>
<tr>
<td>$(a, \ell_0)$</td>
<td>$(a, \alpha)$</td>
<td>$(a, \alpha)$</td>
<td>$(0, \alpha)$</td>
<td>$(0, \alpha)$</td>
</tr>
<tr>
<td>$(a, \ell_a)$</td>
<td>$(a, \beta)$</td>
<td>$(a, \alpha)$</td>
<td>$(0, \beta)$</td>
<td>$(0, \alpha)$</td>
</tr>
</tbody>
</table>

**Table 5:** Cayley table for the binary operation “$\star$”

<table>
<thead>
<tr>
<th>$\odot$</th>
<th>$0$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$c$</td>
<td>$b$</td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$0$</td>
<td>$c$</td>
<td>$b$</td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>$0$</td>
<td>$c$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$c$</td>
<td>$b$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Table 6: Cayley table for the binary operation “∗”

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

(2) Consider the algebra $X = (X, *, 0)$ in Example 3.9 and let $ℓ$ be a mapping from $X$ to a nonempty set $Q$ given by $ℓ(x) = v ∈ Q$ for all $x ∈ X$. Then $(X, ℓ_X) = \{ (0, v), (a, v), (b, v), (c, v) \}$, we recall that $X = (X, *, 0)$ is a BCI-algebra, and thus $\langle (X, ℓ_X), ⊞, (0, v) \rangle$ is a two dimensional BCI-eventful algebra by Lemma 3.10. It is routine to verify that $\langle (X, ℓ_X), ⊞, (0, u) \rangle$ satisfies the condition (27). Hence it is a paired BCI-algebra.

We consider a generalization of Example 3.11.

**Theorem 3.12.** Let $X = (X, *, 0)$ be a BCK/BCI-algebra. Given a nonempty set $Q$, let $ℓ : X → Q$ be a constant mapping, say $ℓ(x) = q$ for all $x ∈ X$. Then $\langle (X, ℓ_X), ⊞, (0, q) \rangle$ is a paired BCK/BCI-algebra.

**Proof.** Using Lemma 3.10, we know that $\langle (X, ℓ_X), ⊞, (0, q) \rangle$ is a two dimensional BCK/BCI-eventful algebra. Let $x, y, a, b ∈ X$ be such that $(x, ℓ_a) ∗ (y, ℓ_b) = (0, q)$ and $(y, ℓ_b) ∗ (x, ℓ_a) = (0, q)$. Then $(0, q) = (x, ℓ_a) ∗ (y, ℓ_b) = (x ∗ y, ℓ_{a+b}) = (x ∗ y, q)$ and $(0, q) = (y, ℓ_b) ∗ (x, ℓ_a) = (y ∗ x, ℓ_{b+a}) = (y ∗ x, q)$. It follows that $x ∗ y = 0$ and $y ∗ x = 0$. Hence $x = y$, and so $(x, ℓ_a) = (q, q) = (y, ℓ_b)$. This shows that $\langle (X, ℓ_X), ⊞, (0, u) \rangle$ satisfies the condition (27). Therefore it is a paired BCK/BCI-algebra.

Theorem 3.12 shows that it can induce many other BCK/BCI-algebras from given a BCK/BCI-algebra. These induced BCK/BCI-algebras are isomorphic each other. Thus a BCK/BCI-algebra induce a unique paired BCK/BCI-algebra up to isomorphism.

**Theorem 3.13.** Let $X = (X, *, 0)$ be a BCK/BCI-algebra. Given a nonempty set $Q$, if a mapping $ℓ : X → Q$ is one-to-one, then $\langle (X, ℓ_X), ⊞, (0, ℓ_0) \rangle$ is a paired BCK/BCI-algebra.

**Proof.** Using Lemma 3.10, we know that $\langle (X, ℓ_X), ⊞, (0, ℓ_0) \rangle$ is a two dimensional BCK/BCI-eventful algebra. Let $x, y, a, b ∈ X$ be such that $(x, ℓ_a) ∗ (y, ℓ_b) = (0, ℓ_0)$ and $(y, ℓ_b) ∗ (x, ℓ_a) = (0, ℓ_0)$. Then $(0, ℓ_0) = (x, ℓ_a) ∗ (y, ℓ_b) = (x ∗ y, ℓ_{a+b})$ and $(0, ℓ_0) = (y, ℓ_b) ∗ (x, ℓ_a) = (y ∗ x, ℓ_{b+a})$. It follows that $x ∗ y = 0$, $y ∗ x = 0$, $ℓ_{a+b} = ℓ_0$ and $ℓ_{b+a} = ℓ_0$. Since $ℓ$ is one-to-one, we have $a ∗ b = 0$ and $b ∗ a = 0$. It follows that $x = y$ and $a = b$. Hence $(x, ℓ_a) = (y, ℓ_b)$. This shows that $\langle (X, ℓ_X), ⊞, (0, ℓ_0) \rangle$ satisfies the condition (27). Therefore it is a paired BCK/BCI-algebra.

**Theorem 3.14.** Let $X = (X, *, 0)$ be a BCK/BCI-algebra. Given a nonempty set $Q$, if a mapping $ℓ : X → Q$ satisfies $ℓ^{-1}(ℓ_0) = \{ 0 \}$, then $\langle (X, ℓ_X), ⊞, (0, ℓ_0) \rangle$ is a paired BCK/BCI-algebra.

**Proof.** Using Lemma 3.10, we know that $\langle (X, ℓ_X), ⊞, (0, ℓ_0) \rangle$ is a two dimensional BCK/BCI-eventful algebra. Let $x, y, a, b ∈ X$ be such that $(x, ℓ_a) ∗ (y, ℓ_b) = (0, ℓ_0)$ and $(y, ℓ_b) ∗ (x, ℓ_a) = (0, ℓ_0)$. Then $(0, ℓ_0) = (x, ℓ_a) ∗ (y, ℓ_b) = (x ∗ y, ℓ_{a+b})$ and $(0, ℓ_0) = (y, ℓ_b) ∗ (x, ℓ_a) = (y ∗ x, ℓ_{b+a})$. It follows that $x ∗ y = 0$, $y ∗ x = 0$, $ℓ_{a+b} = ℓ_0$ and $ℓ_{b+a} = ℓ_0$. Hence $a ∗ b ∈ ℓ^{-1}(ℓ_0) = \{ 0 \}$ and $b ∗ a ∈ ℓ^{-1}(ℓ_0) = \{ 0 \}$ which shows that $a ∗ b = 0$ and $b ∗ a = 0$. It follows that $x = y$ and $a = b$. Hence $(x, ℓ_a) = (y, ℓ_b)$. This shows that $\langle (X, ℓ_X), ⊞, (0, ℓ_0) \rangle$ satisfies the condition (27). Therefore it is a paired BCK/BCI-algebra.

**Lemma 3.15.** Given a BCI-algebra $X = (X, *, 0)$, the following are equivalent,

(i) $X$ is p-semisimple.

(ii) $x ∗ (0 ∗ y) = y ∗ (0 ∗ x)$ for all $x, y ∈ X$. 

Theorem 3.16. The two dimensional BCI-eventful algebra \( ((X, \ell_X), \oplus, (0, \ell_0)) \) induced by a p-semisimple BCI-algebra \( X = (X, *, 0) \) is a commutative group under the operation \( \odot \) which is given by

\[
\odot : (X, \ell_X) \times (X, \ell_X) \to (X, \ell_X), \quad ((x, \ell_a), (y, \ell_b)) \mapsto (x, \ell_a) \oplus ((0, \ell_0) \oplus (y, \ell_b)).
\]

**Proof.** By Lemma 3.10 we know that \( ((X, \ell_X), \oplus, (0, \ell_0)) \) is a two dimensional BCI-eventful algebra. Let \( (x, \ell_a), (y, \ell_b), (z, \ell_c) \in (X, \ell_X) \). Then

\[
(x, \ell_a) \odot (y, \ell_b) = (x, \ell_a) \oplus ((0, \ell_0) \oplus (y, \ell_b)) = (x, \ell_a) \oplus (0 \ast y, \ell_{0 \ast b}) = (x \ast (0 \ast y), \ell_{(b \ast (0 \ast a))}) = (y \ast (0 \ast x), \ell_{(b \ast (0 \ast a))}) = (y, \ell_b) \oplus (0 \ast x, \ell_{0 \ast a}) = (y, \ell_b) \odot (0 \ast x, \ell_{0 \ast a}) = (y, \ell_b) \oplus ((0, \ell_0) \oplus (x, \ell_a)) = (y, \ell_b) \odot (x, \ell_a)
\]

by Lemma 3.15 and

\[
((y, \ell_b) \odot (z, \ell_c)) \odot (x, \ell_a) = ((y, \ell_b) \oplus ((0, \ell_0) \oplus (z, \ell_c))) \oplus ((0, \ell_0) \oplus (x, \ell_a)) = ((y \ast (0 \ast z)) \ast (0 \ast x), \ell_{(b \ast (0 \ast c)) \ast (0 \ast a)}) = ((y \ast (0 \ast x)) \ast (0 \ast z), \ell_{(b \ast (0 \ast a)) \ast (0 \ast c)}) = ((y, \ell_b) \oplus ((0, \ell_0) \oplus (x, \ell_a))) \oplus ((0, \ell_0) \oplus (z, \ell_c)) = ((y, \ell_b) \odot (x, \ell_a)) \odot (z, \ell_c).
\]

Using (28) and (29), we get

\[
(x, \ell_a) \odot ((y, \ell_b) \odot (z, \ell_c)) = ((y, \ell_b) \odot (z, \ell_c)) \odot (x, \ell_a) = ((y, \ell_b) \odot (x, \ell_a)) \odot (z, \ell_c) = ((x, \ell_a) \odot (y, \ell_b)) \odot (z, \ell_c).
\]

Now,

\[
(0, \ell_0) \odot (x, \ell_a) = (0, \ell_0) \oplus ((0, \ell_0) \oplus (x, \ell_a)) = (0, \ell_0) \oplus (0 \ast x, \ell_{0 \ast a}) = (0 \ast (0 \ast x), \ell_{0 \ast (0 \ast a)}) = (x, \ell_a),
\]

which shows that \( (0, \ell_0) \) is the identity element of \( (X, \ell_X) \). Finally, we show that \( (0, \ell_0) \oplus (x, \ell_a) \) is the inverse of any element \( (x, \ell_a) \). In fact,

\[
(x, \ell_a) \odot ((0, \ell_0) \oplus (x, \ell_a)) = (x, \ell_a) \oplus ((0, \ell_0) \oplus ((0, \ell_0) \oplus (x, \ell_a))) = (x \ast (0 \ast x), \ell_{a \ast (0 \ast a)}) = (x \ast x, \ell_{a \ast a}) \oplus (0, \ell_0).
\]

Therefore \( ((X, \ell_X), \odot, (0, \ell_0)) \) is a commutative group.

**Corollary 3.17.** Let \( X = (X, *, 0) \) be a BCI-algebra which satisfies any one of the following assertions.

\[
\forall x \in X)(0 \ast x = 0 \Rightarrow x = 0), \quad (30)
\]

\[
\forall a \in X)(X = \{a \ast x \mid x \in X\}), \quad (31)
\]

\[
\forall a, x \in X)(a \ast (a \ast x) = x), \quad (32)
\]

\[
\forall a, x, y, z \in X)((x \ast y) \ast (z \ast a) = (x \ast (y \ast (0 \ast a))) \ast (z \ast (0 \ast a))), \quad (33)
\]

\[
\forall x, y \in X)(0 \ast (y \ast x) = x \ast y), \quad (34)
\]

\[
\forall x, y, z \in X)((x \ast y) \ast (x \ast z) = z \ast y). \quad (35)
\]

Then the two dimensional BCI-eventful algebra \( ((X, \ell_X), \oplus, (0, \ell_0)) \) is a commutative group under the operation \( \odot \).
Theorem 3.18. Let \( f : X \to Y \) be an onto homomorphism of BCK/BCI-algebras. If \( \langle (X, \ell_X), \odot, (0, \ell_0) \rangle \) satisfies the condition \((27)\), then \( \langle (Y, \ell_Y), \odot, (0, \zeta_0) \rangle \) is a paired BCK/BCI-algebra where \( \zeta \) is a mapping from \( Y \) to \( Q \).

Proof. By Lemma 3.10, we know that \( \langle (X, \ell_X), \odot, (0, \ell_0) \rangle \) is a two dimensional BCK/BCI-eventful algebra. Let \((x', \zeta_{y'}), (y', \zeta_y), (z', \zeta_z) \in (Y, \ell_Y)\). Then there exist \( x, y, z, a, b, c \in X \) such that \( f(x) = x' \), \( f(y) = y' \), \( f(z) = z' \), \( f(a) = a' \), \( f(b) = b' \) and \( f(c) = c' \). Hence

\[
(((x', \zeta_{y'})) \odot (y', \zeta_y)) \odot ((z', \zeta_z)) \odot (y', \zeta_y)) = ((f(x), \zeta_{f(y)})) \odot ((f(x), \zeta_{f(y)})) \odot ((f(z), \zeta_{f(c)})) \odot (f(y), \zeta_{f(b)})
\]

and \((x', \zeta_{y'}) \odot (x', \zeta_{y'}) = (f(x), \zeta_{f(y)}) \odot (f(x), \zeta_{f(y)}) = (f(x) \ast f(x), \zeta_{f(a)}) = (0, \zeta_0)\). Hence \( \langle (Y, \ell_Y), \odot, (0, \zeta_0) \rangle \) is a two dimensional BCI-eventful algebra. Since \((0, \zeta_0) \odot (x', \zeta_{y'}) = (f(0), \zeta_{f(0)}) \odot (f(x), \zeta_{f(a)}) = (f(0) \ast f(x), \zeta_{f(0)}) = (0, \zeta_0)\), we know that \( \langle (Y, \ell_Y), \odot, (0, \zeta_0) \rangle \) is a two dimensional BCI-eventful algebra. Assume that \((x', \zeta_{y'}) \odot (y', \zeta_y) = (0, \zeta_0)\) and \((y', \zeta_{y'}) \odot (x', \zeta_{y'}) = (0, \zeta_0)\). Then

\[
(0, \zeta_0) = (x', \zeta_{y'}) \odot (y', \zeta_y) = (f(x), \zeta_{f(y)}) \odot (f(y), \zeta_{f(b)}) = (f(x) \ast f(y), \zeta_{f(a)} \ast f(b))
\]

and

\[
(0, \zeta_0) = (y', \zeta_{y'}) \odot (x', \zeta_{y'}) = (f(y), \zeta_{f(b)}) \odot (f(x), \zeta_{f(a)}) = (f(y) \ast f(x), \zeta_{f(b)} \ast f(a)),
\]

which imply that \( f(x) \ast f(y) = 0 \), \( f(y) \ast f(x) = 0 \), \( f(a) \ast f(b) = 0 \) and \( f(b) \ast f(a) = 0 \). Hence \( x' = f(x) = f(y) = y' \) and \( a' = f(a) = f(b) = b' \). Therefore \((x', \zeta_{y'}) = (y', \zeta_{y'})\). Consequently, \( \langle (Y, \ell_Y), \odot, (0, \zeta_0) \rangle \) is a paired BCK/BCI-algebra.

4 Conclusions

We have introduced two-dimensional event sets and have applied it to algebraic structures. We have introduced the notions of two dimensional BCK/BCI-eventful algebra, paired B-algebra and paired BCK/BCI-algebra, and have investigated several properties. We have considered conditions for two dimensional eventful algebra to be a B-algebra and a BCK/BCI-algebra. We have discussed the process of inducing a paired B-algebra using a group, and have established a commutative group using two dimensional BCI-eventful algebra. We have presented examples to show that a two dimensional eventful BCK/BCI-algebra is not a BCK/BCI-algebra, and then we have considered conditions for a two dimensional eventful BCK/BCI-algebra to be a BCK/BCI-algebra. We have studied a paired BCK/BCI-algebra in relation to the BCK/BCI-homomorphism.

\[\square\]
References


