Commutative neutrosophic quadruple ideals of neutrosophic quadruple $BCK$-algebras

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Abstract

Commutative neutrosophic quadruple ideals and $BCK$-algebras are discussed, and related properties are investigated. Conditions for the neutrosophic quadruple $BCK$-algebra to be commutative are considered. Given subsets $A$ and $B$ of a neutrosophic quadruple $BCK$-algebra, conditions for the set $NQ(A, B)$ to be a commutative ideal of a neutrosophic quadruple $BCK$-algebra are provided.

Article Information

Corresponding Author: M. Mohseni Takallo;
Received: December 2019;
Accepted: January 2020.
Paper type: Original.

Keywords: $BCK$-algebra, ideal, neutrosophic quadruple ideal, commutative neutrosophic quadruple $BCK$-algebra.

1 Introduction

The neutrosophic set which is developed by Smarandache ([17], [18] and [19]) is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Neutrosophic algebraic structures in $BCK/BCI$-algebras are discussed in the papers [3], [8], [9], [10], [11], [13], [16] and [21]. Smarandache [20] considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part ($a$) and an unknown part ($bT, cI, dF$) where $T, I, F$ have their usual neutrosophic logic meanings and $a, b, c, d$ are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [1] [2]. Jun et al. [12] used neutrosophic quadruple numbers based on a set, and constructed neutrosophic quadruple $BCK/BCI$-algebras. They investigated several properties, and considered ideal and positive...
implicative ideal in neutrosophic quadruple $BCK$-algebra, and closed ideal in neutrosophic quadruple $BCI$-algebra. Given subsets $A$ and $B$ of a neutrosophic quadruple $BCK/BCI$-algebra, they considered sets $NQ(A, B)$ which consists of neutrosophic quadruple $BCK/BCI$-numbers with a condition. They provided conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple $BCK$-algebra, and the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple $BCI$-algebra. They gave an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple $BCK$-algebra, and then they considered conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple $BCK$-algebra.

In this paper, we discuss a commutative neutrosophic quadruple ideal and $BCK$-algebra and investigate several properties. We consider conditions for the neutrosophic quadruple $BCK$-algebra to be commutative. Given subsets $A$ and $B$ of a neutrosophic quadruple $BCK$-algebra, we give conditions for the set $NQ(A, B)$ to be a commutative ideal of a neutrosophic quadruple $BCK$-algebra.

2 Preliminaries

A $BCK/BCI$-algebra is an important class of logical algebras introduced by K. Iséki (see [6] and [7]) and was extensively investigated by several researchers.

By a $BCI$-algebra, we mean a set $X$ with a special element 0 and a binary operation $*$ that satisfies the following conditions:

\[(I) \; (\forall x, y, z \in X) \; (((x * y) * (x * z)) * (z * y) = 0),\]
\[(II) \; (\forall x, y \in X) \; ((x * (x * y)) * y = 0),\]
\[(III) \; (\forall x \in X) \; (x * x = 0),\]
\[(IV) \; (\forall x, y \in X) \; (x * y = 0, y * x = 0 \Rightarrow x = y).\]

If a $BCI$-algebra $X$ satisfies the following identity:

\[(V) \; (\forall x \in X) \; (0 * x = 0),\]

then $X$ is called a $BCK$-algebra. Any $BCK/BCI$-algebra $X$ satisfies the following conditions:

\[(\forall x \in X) \; (x * 0 = x),\]  \hspace{1cm} (1)
\[(\forall x, y, z \in X) \; (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x),\]  \hspace{1cm} (2)
\[(\forall x, y, z \in X) \; ((x * y) * z = (x * z) * y),\]  \hspace{1cm} (3)
\[(\forall x, y, z \in X) \; ((x * z) * (y * z) \leq x * y)\]  \hspace{1cm} (4)

where $x \leq y$ if and only if $x * y = 0$.

A $BCK$-algebra $X$ is said to be commutative if the following assertion is valid.

\[(\forall x, y \in X) \; (x * (x * y) = y * (y * x)).\]  \hspace{1cm} (5)

A subset $I$ of a $BCK/BCI$-algebra $X$ is called an ideal of $X$ if it satisfies:

\[0 \in I,\]
\[(\forall x \in X) \; (\forall y \in I) \; (x * y \in I \Rightarrow x \in I).\]  \hspace{1cm} (7)
A subset $I$ of a $BCK$-algebra $X$ is called a commutative ideal of $X$ if it satisfies (6) and
\[(\forall x, y \in X)(\forall z \in I)((x \ast y) \ast z \in I \Rightarrow x \ast (y \ast (y \ast x)) \in I).\] (8)
Observe that every commutative ideal is an ideal, but the converse is not true (see [14]).

We refer the reader to the books [5, 14] for further information regarding $BCK/BCI$-algebras, and to the site “http://fs.gallup.unm.edu/neutrosophy.htm” for further information regarding neutrosophic set theory.

3 Commutative neutrosophic quadruple $BCK$-algebras

In this section, we define commutative neutrosophic quadruple $BCK$-algebra under Theorem 3.3 and consider some properties of commutative neutrosophic quadruple $BCK$-algebra. Also, we investigate relation between commutative neutrosophic quadruple $BCK$-algebra and lattices.

**Definition 3.1 ([12]).** Let $X$ be a set. A neutrosophic quadruple $X$-number is an ordered quadruple $(a, x^T, y^I, z^F)$ where $a, x, y, z \in X$ and $T, I, F$ have their usual neutrosophic logic meanings.

The set of all neutrosophic quadruple $X$-numbers is denoted by $NQ(X)$, that is,

\[NQ(X) := \{(a, x^T, y^I, z^F) \mid a, x, y, z \in X\},\]

and it is called the neutrosophic quadruple set based on $X$. If $X$ is a $BCK/BCI$-algebra, a neutrosophic quadruple $X$-number is called a neutrosophic quadruple $BCK/BCI$-number and we say that $NQ(X)$ is the neutrosophic quadruple $BCK/BCI$-set.

Let $X$ be a $BCK/BCI$-algebra. We define a binary operation $\circ$ on $NQ(X)$ by

\[(a, x^T, y^I, z^F) \circ (b, u^T, v^I, w^F) = (a \ast b, (x \ast u)^T, (y \ast v)^I, (z \ast w)^F)\]

for all $(a, x^T, y^I, z^F), (b, u^T, v^I, w^F) \in NQ(X)$. Given $a_1, a_2, a_3, a_4 \in X$, the neutrosophic quadruple $BCK/BCI$-number $(a_1, a_2^T, a_3^I, a_4^F)$ is denoted by $\tilde{a}$, that is, \[\tilde{a} = (a_1, a_2^T, a_3^I, a_4^F),\]

and the zero neutrosophic quadruple $BCK/BCI$-number $(0, 0^T, 0^I, 0^F)$ is denoted by $\tilde{0}$, that is, \[\tilde{0} = (0, 0^T, 0^I, 0^F).\]

We define an order relation “$\ll$” and the equality “$=$” on $NQ(X)$ as follows:

\[\tilde{x} \ll \tilde{y} \iff x_i \leq y_i \text{ for } i = 1, 2, 3, 4,\]
\[\tilde{x} = \tilde{y} \iff x_i = y_i \text{ for } i = 1, 2, 3, 4,\]

for all $\tilde{x}, \tilde{y} \in NQ(X)$. It is easy to verify that “$\ll$” is a partial order on $NQ(X)$.

**Lemma 3.2 ([12]).** If $X$ is a $BCK/BCI$-algebra, then $(NQ(X); \circ, \tilde{0})$ is a $BCK/BCI$-algebra, which is called a neutrosophic quadruple $BCK/BCI$-algebra.

**Theorem 3.3.** The neutrosophic quadruple $BCK$-set $NQ(X)$ based on a commutative $BCK$-algebra $X$ is a commutative $BCK$-algebra, which is called a commutative neutrosophic quadruple $BCK$-algebra.
Proof. Let $X$ be a commutative $BCK$-algebra. Then $X$ is a $BCK$-algebra, and so $(NQ(X); \odot, \tilde{0})$ is a $BCK$-algebra by Lemma 3.2. Let $\tilde{x}, \tilde{y} \in NQ(X)$. Then

$$x_i * (x_i * y_i) = y_i * (y_i * x_i)$$

for all $i = 1, 2, 3, 4$ since $x_i, y_i \in X$ and $X$ is a commutative $BCK$-algebra. Hence $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) = \tilde{y} \odot (\tilde{y} \odot \tilde{x})$, and therefore $NQ(X)$ based on a commutative $BCK$-algebra $X$ is a commutative $BCK$-algebra.

Theorem 3.3 is illustrated by the following example.

**Example 3.4.** Let $X = \{0, 1\}$ be a set with the binary operation $*$ which is given in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
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</tbody>
</table>

Then $(X, *, 0)$ is a commutative $BCK$-algebra (see [14]), and the neutrosophic quadruple $BCK$-set $NQ(X)$ is given as follows:

$$NQ(X) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$$

where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, 1F)$, $\tilde{2} = (0, 0T, 1I, 0F)$, $\tilde{3} = (0, 0T, 1I, 1F)$,
$\tilde{4} = (0, 1T, 0I, 0F)$, $\tilde{5} = (0, 1T, 0I, 1F)$, $\tilde{6} = (0, 1T, 1I, 0F)$, $\tilde{7} = (0, 1T, 1I, 1F)$,
$\tilde{8} = (1, 0T, 0I, 0F)$, $\tilde{9} = (1, 0T, 0I, 1F)$, $\tilde{10} = (1, 0T, 1I, 0F)$, $\tilde{11} = (1, 0T, 1I, 1F)$,
$\tilde{12} = (1, 1T, 0I, 0F)$, $\tilde{13} = (1, 1T, 0I, 1F)$, $\tilde{14} = (1, 1T, 1I, 0F)$, $\tilde{15} = (1, 1T, 1I, 1F)$.

Then $(NQ(X), \odot, \tilde{0})$ is a commutative $BCK$-algebra in which the operation $\odot$ is given by Table 2.
Proposition 3.5. The neutrosophic quadruple BCK-set $NQ(X)$ based on a commutative BCK-algebra $X$ satisfies the following assertions.

\[(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)) (\tilde{x} \ll \tilde{z}, \tilde{y} \ll \tilde{z} \circ \tilde{x} \Rightarrow \tilde{x} \ll \tilde{y}). \quad (9)\]

\[(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)) (\tilde{x} \ll \tilde{z}, \tilde{y} \ll \tilde{z}, \tilde{y} \ll \tilde{z} \circ \tilde{x} \Rightarrow \tilde{x} \ll \tilde{y}). \quad (10)\]

\[(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)) (\tilde{x} \ll \tilde{y} \Rightarrow \tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{x}). \quad (11)\]

\[(\forall \tilde{x}, \tilde{y} \in NQ(X)) (\tilde{x} \circ (\tilde{x} \circ \tilde{y}) = \tilde{y} \circ (\tilde{y} \circ (\tilde{x} \circ \tilde{y}))). \quad (12)\]

Proof. Assume that $\tilde{x} \ll \tilde{z}$ and $\tilde{z} \circ \tilde{y} \ll \tilde{z} \circ \tilde{x}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. Then $\tilde{x} \circ \tilde{z} = \tilde{0}$ and $(\tilde{z} \circ \tilde{y}) \circ (\tilde{z} \circ \tilde{x}) = \tilde{0}$. Since $NQ(X)$ is commutative, we have

$$\tilde{x} \circ \tilde{y} = (\tilde{x} \circ \tilde{0}) \circ \tilde{y} = (\tilde{x} \circ (\tilde{x} \circ \tilde{z})) \circ \tilde{y} = (\tilde{z} \circ (\tilde{z} \circ \tilde{x})) \circ \tilde{y} = (\tilde{z} \circ \tilde{y}) \circ (\tilde{z} \circ \tilde{x}) = \tilde{0},$$

that is, $\tilde{x} \ll \tilde{y}$. Condition (10) is clear by the condition (9). Suppose that $\tilde{x} \ll \tilde{y}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. Note that $\tilde{y} \circ (\tilde{y} \circ \tilde{x}) \ll \tilde{y}$ and $\tilde{y} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})) \ll \tilde{y} \circ \tilde{x}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. It follows from the condition (10) that $\tilde{x} \ll \tilde{y} \circ (\tilde{y} \circ \tilde{x})$. Obviously, $\tilde{y} \circ (\tilde{y} \circ \tilde{x}) \ll \tilde{x}$, and so $\tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{x}$. Condition (12) follows directly from the condition (11). \hfill \Box

Theorem 3.6. The neutrosophic quadruple BCK-set $NQ(X)$ based on a commutative BCK-algebra $X$ is a lower semilattice with respect to the order “$\ll$”.

Proof. For any $\tilde{x}, \tilde{y} \in NQ(X)$, let $\tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{x} \land \tilde{y}$. Then $\tilde{x} \land \tilde{y} \ll \tilde{x}$ and $\tilde{x} \land \tilde{y} \ll \tilde{y}$. Let $\tilde{a} \in NQ(X)$ such that $\tilde{a} \ll \tilde{x}$ and $\tilde{a} \ll \tilde{y}$. Then

$$\tilde{a} = \tilde{a} \circ \tilde{0} = \tilde{a} \circ (\tilde{a} \circ \tilde{a}) = \tilde{x} \circ (\tilde{x} \circ \tilde{a}).$$
Similarly, we have $\tilde{a} = \tilde{y} \circ (\tilde{y} \circ \tilde{a})$. Thus
$$
\tilde{a} = \tilde{x} \circ (\tilde{x} \circ \tilde{a}) = \tilde{x} \circ (\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{a}))) \ll \tilde{x} \circ (\tilde{x} \circ \tilde{y}) = \tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{x} \land \tilde{y}.
$$
Hence $\tilde{x} \land \tilde{y}$ is the greatest lower bound, and therefore $(NQ(X), \ll)$ is a lower semilattice.

Given a neutrosophic quadruple $BCK$-algebra $NQ(X)$, we consider the following set.

$$
\Omega(\tilde{a}) := \{ \tilde{x} \in NQ(X) \mid \tilde{x} \ll \tilde{a} \}.
$$

**Proposition 3.7.** Every neutrosophic quadruple $BCK$-set $NQ(X)$ based on a commutative $BCK$-algebra $X$ satisfies the following identity.

$$
(\forall \tilde{a}, \tilde{b} \in NQ(X))(\Omega(\tilde{a}) \cap \Omega(\tilde{b}) = \Omega(\tilde{a} \land \tilde{b}))
$$

where $\tilde{a} \land \tilde{b} = \tilde{b} \circ (\tilde{b} \circ \tilde{a})$.

**Proof.** Let $\tilde{x} \in \Omega(\tilde{a}) \cap \Omega(\tilde{b})$. Then $\tilde{x} \ll \tilde{a}$ and $\tilde{x} \ll \tilde{b}$, and so $\tilde{x} \ll \tilde{a} \land \tilde{b}$. Thus $\tilde{x} \in \Omega(\tilde{a} \land \tilde{b})$, which shows that $\Omega(\tilde{a}) \cap \Omega(\tilde{b}) \subseteq \Omega(\tilde{a} \land \tilde{b})$. If $\tilde{x} \in \Omega(\tilde{a} \land \tilde{b})$, then $\tilde{x} \ll \tilde{a} \land \tilde{b}$. Hence $\tilde{x} \ll \tilde{a}$ and $\tilde{x} \ll \tilde{b}$, and thus $\tilde{x} \in \Omega(\tilde{a}) \cap \Omega(\tilde{b})$. This completes the proof.

We consider conditions for a neutrosophic quadruple $BCK$-algebra $NQ(X)$ to be commutative.

**Lemma 3.8.** If a neutrosophic quadruple $BCK$-algebra $NQ(X)$ satisfies the condition (11), then it is commutative.

**Proof.** Assume that $NQ(X)$ is a neutrosophic quadruple $BCK$-algebra which satisfies the condition (11). Note that $\tilde{y} \circ (\tilde{y} \circ \tilde{x}) \ll \tilde{x}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. It follows from the condition (11) that

$$
\tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{x} \circ (\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x}))).
$$

Hence

$$
(\tilde{y} \circ (\tilde{y} \circ \tilde{x})) \circ (\tilde{x} \circ (\tilde{x} \circ \tilde{y}))
= (\tilde{x} \circ (\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})))) \circ (\tilde{x} \circ (\tilde{x} \circ \tilde{y}))
= (\tilde{x} \circ (\tilde{x} \circ (\tilde{x} \circ \tilde{y}))) \circ (\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})))
= (\tilde{x} \circ \tilde{y}) \circ (\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})))
\ll (\tilde{y} \circ (\tilde{y} \circ \tilde{x})) \circ \tilde{y} = \tilde{0}
$$

for all $\tilde{x}, \tilde{y} \in NQ(X)$. Similarly, we get that $(\tilde{x} \circ (\tilde{x} \circ \tilde{y})) \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})) = \tilde{0}$ by changing the role of $\tilde{x}$ and $\tilde{y}$. Therefore $\tilde{x} \circ (\tilde{x} \circ \tilde{y}) = \tilde{y} \circ (\tilde{y} \circ \tilde{x})$ and so $NQ(X)$ is commutative.

**Theorem 3.9.** If a neutrosophic quadruple $BCK$-algebra $NQ(X)$ satisfies the condition (12), then it is commutative.

**Proof.** Assume that $NQ(X)$ is a neutrosophic quadruple $BCK$-algebra which satisfies the condition (12). Let $\tilde{x}, \tilde{y} \in NQ(X)$ such that $\tilde{x} \ll \tilde{y}$. Then

$$
\tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{y} \circ (\tilde{y} \circ (\tilde{x} \circ (\tilde{x} \circ \tilde{y}))) = \tilde{x} \circ (\tilde{x} \circ \tilde{y}) = \tilde{x} \circ \tilde{0} = \tilde{x},
$$
and so $NQ(X)$ is commutative by Lemma 3.8.
Lemma 3.10. A neutrosophic quadruple BCK-algebra $NQ(X)$ is commutative if and only if the following assertion is valid.

\[(\forall \tilde{x}, \tilde{y} \in NQ(X)) (\tilde{y} \circ (\tilde{y} \circ \tilde{x}) \preccurlyeq (\tilde{x} \circ (\tilde{x} \circ \tilde{y}))).\] \hspace{1cm} (15)

**Proof.** It is straightforward. \hfill \qed

**Theorem 3.11.** If a neutrosophic quadruple BCK-algebra $NQ(X)$ satisfies the condition (14), then it is commutative.

**Proof.** Let $NQ(X)$ be a neutrosophic quadruple BCK-algebra which satisfies the condition (14). Let $\tilde{x} \land \tilde{y} := \tilde{y} \circ (\tilde{y} \circ \tilde{x})$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. Then

\[
\Omega(\tilde{x} \land \tilde{y}) = \Omega(\tilde{x}) \cap \Omega(\tilde{y}) = \Omega(\tilde{y}) \cap \Omega(\tilde{x}) = \Omega(\tilde{y} \land \tilde{x})
\]

for all $\tilde{x}, \tilde{y} \in NQ(X)$, and thus $\tilde{x} \land \tilde{y} \in \Omega(\tilde{y} \land \tilde{x})$. Hence $\tilde{x} \land \tilde{y} \preccurlyeq \tilde{y} \land \tilde{x}$, that is, $\tilde{y} \circ (\tilde{y} \circ \tilde{x}) \preccurlyeq \tilde{x} \circ (\tilde{x} \circ \tilde{y})$. It follows from Lemma 3.10 that $NQ(X)$ is a commutative neutrosophic quadruple BCK-algebra. \hfill \qed

**Theorem 3.12.** Given a nonempty set $X$, if a neutrosophic quadruple set $NQ(X)$ satisfies the following assertions

\[(\forall \tilde{x} \in NQ(X)) (\tilde{x} \circ \tilde{0} = \tilde{x}, \tilde{x} \circ \tilde{x} = \tilde{0}),\] \hspace{1cm} (16)

\[(\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)) (\tilde{x} \circ (\tilde{y} \circ \tilde{z}) = (\tilde{x} \circ \tilde{y}) \circ \tilde{z}),\] \hspace{1cm} (17)

\[(\tilde{x}, \tilde{y} \in NQ(X)) (\tilde{x} \land \tilde{y} = \tilde{y} \land \tilde{x}),\] \hspace{1cm} (18)

where $\tilde{x} \land \tilde{y} = \tilde{y} \circ (\tilde{y} \circ \tilde{x})$, then it is a commutative neutrosophic quadruple BCK-algebra.

**Proof.** Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. Using conditions (16) and (17) imply that

\[ (\tilde{x} \circ (\tilde{x} \circ \tilde{y})) \circ \tilde{y} = (\tilde{x} \circ \tilde{y}) \circ (\tilde{x} \circ \tilde{y}) = \tilde{0}.\]

Assume that $\tilde{x} \circ \tilde{y} = \tilde{0}$ and $\tilde{y} \circ \tilde{x} = \tilde{0}$. Then

\[ \tilde{x} = \tilde{x} \circ \tilde{0} = \tilde{x} \circ (\tilde{x} \circ \tilde{y}) = \tilde{y} \land \tilde{x} = \tilde{x} \land \tilde{y} = \tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{y} \circ \tilde{0} = \tilde{y}.\]

Using (17) and (18), we have

\[ (\tilde{x} \circ \tilde{y}) \circ (\tilde{z} \circ \tilde{z}) = (\tilde{x} \circ (\tilde{x} \circ \tilde{z})) \circ \tilde{y} = (\tilde{z} \land \tilde{x}) \circ \tilde{y} = (\tilde{z} \circ \tilde{z}) \circ \tilde{y} = (\tilde{z} \circ \tilde{y}) \circ (\tilde{z} \circ \tilde{x}).\] \hspace{1cm} (19)

If we take $\tilde{y} = \tilde{x}$ and $\tilde{z} = \tilde{0}$ in (19), then

\[ \tilde{0} \circ \tilde{x} = (\tilde{x} \circ \tilde{x}) \circ (\tilde{x} \circ \tilde{0}) = (\tilde{0} \circ \tilde{x}) \circ (\tilde{0} \circ \tilde{x}) = \tilde{0}.\]

It follows from (19) and (16) that

\[ ((\tilde{x} \circ \tilde{y}) \circ (\tilde{x} \circ \tilde{z})) \circ (\tilde{z} \circ \tilde{y}) = ((\tilde{z} \circ \tilde{y}) \circ (\tilde{z} \circ \tilde{x})) \circ (\tilde{z} \circ \tilde{y} \circ \tilde{0}) = (\tilde{0} \circ (\tilde{z} \circ \tilde{x})) \circ (\tilde{0} \circ (\tilde{z} \circ \tilde{y})) = \tilde{0} \circ \tilde{0} = \tilde{0}.\]

Therefore $(NQ(X), \circ, \tilde{0})$ is a commutative neutrosophic quadruple BCK-algebra. \hfill \qed
Given subsets \( A \) and \( B \) of a BCK-algebra \( X \), consider the set
\[
NQ(A, B) := \{(a, xT, yI, zF) \in NQ(X) \mid a, x \in A; y, z \in B\}.
\]

**Theorem 3.13.** If \( A \) and \( B \) are commutative ideals of a BCK-algebra \( X \), then the set \( NQ(A, B) \) is a commutative ideal of \( NQ(X) \), which is called a commutative neutrosophic quadruple ideal.

**Proof.** Assume that \( A \) and \( B \) are commutative ideals of a BCK-algebra \( X \). Obviously, \( \tilde{0} \in NQ(A, B) \). Let \( \tilde{x} = (x_1, x_2T, x_3I, x_4F), \tilde{y} = (y_1, y_2T, y_3I, y_4F) \) and \( \tilde{z} = (z_1, z_2T, z_3I, z_4F) \) be elements of \( NQ(X) \) such that \( \tilde{z} \in NQ(A, B) \) and \((\tilde{x} \circ \tilde{y}) \circ \tilde{z} \in NQ(A, B)\). Then
\[
(\tilde{x} \circ \tilde{y}) \circ \tilde{z} = ((x_1 \ast y_1) * z_1, ((x_2 \ast y_2) * z_2)T, (x_3 \ast (y_3 \ast z_3)I, (x_4 \ast (y_4 \ast z_4)F) \in NQ(A, B),
\]
and so \((x_1 \ast y_1) * z_1 \in A, (x_2 \ast y_2) * z_2 \in A, (x_3 \ast y_3) * z_3 \in B \) and \((x_4 \ast y_4) * z_4 \in B \). Since \( \tilde{z} \in NQ(A, B) \), we have \( z_1, z_2 \in A \) and \( z_3, z_4 \in B \). Since \( A \) and \( B \) are commutative ideals of \( X \), it follows that \( x_1 \ast (y_1 \ast (y_1 \ast x_1)) \in A, x_2 \ast (y_2 \ast (y_2 \ast x_2)) \in A, x_3 \ast (y_3 \ast (y_3 \ast x_3)) \in B \) and \( x_4 \ast (y_4 \ast (y_4 \ast x_4)) \in B \). Hence
\[
\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})) = (x_1 \ast (y_1 \ast (y_1 \ast x_1)), (x_2 \ast (y_2 \ast (y_2 \ast x_2)))T, (x_3 \ast (y_3 \ast (y_3 \ast x_3)))I, (x_4 \ast (y_4 \ast (y_4 \ast x_4)))F \in NQ(A, B),
\]
and therefore \( NQ(A, B) \) is a commutative ideal of \( NQ(X) \).

**Lemma 3.14 (T2).** If \( A \) and \( B \) are ideals of a BCK-algebra \( X \), then the set \( NQ(A, B) \) is an ideal of \( NQ(X) \), which is called a neutrosophic quadruple ideal.

**Theorem 3.15.** Let \( A \) and \( B \) be ideals of a BCK-algebra \( X \) such that
\[
(\forall x, y \in X) (x \ast y \in A \text{ (resp., } B) \Rightarrow x \ast (y \ast (y \ast x)) \in A \text{ (resp., } B)). \tag{20}
\]
Then \( NQ(A, B) \) is a commutative ideal of \( NQ(X) \).

**Proof.** If \( A \) and \( B \) are ideals of a BCK-algebra \( X \), then \( NQ(A, B) \) is an ideal of \( NQ(X) \) by Lemma 3.14. Let \( \tilde{x} = (x_1, x_2T, x_3I, x_4F), \tilde{y} = (y_1, y_2T, y_3I, y_4F) \) and \( \tilde{z} = (z_1, z_2T, z_3I, z_4F) \) be elements of \( NQ(X) \) such that \( (\tilde{x} \circ \tilde{y}) \circ \tilde{z} \in NQ(A, B) \) and \( \tilde{z} \in NQ(A, B) \). Then
\[
(\tilde{x} \circ \tilde{y}) \circ \tilde{z} = ((x_1 \ast y_1) * z_1, ((x_2 \ast y_2) * z_2)T, (x_3 \ast (y_3 \ast z_3)I, (x_4 \ast (y_4 \ast z_4)F) \in NQ(A, B),
\]
and \( \tilde{z} = (z_1, z_2T, z_3I, z_4F) \in NQ(A, B) \), so \((x_1 \ast y_1) * z_1 \in A, (x_2 \ast y_2) * z_2 \in A, (x_3 \ast y_3) * z_3 \in B, (x_4 \ast y_4) * z_4 \in B \). Since \( A \) and \( B \) are ideals of \( X \), we get that \( x_1 \ast y_1 \in A, x_2 \ast y_2 \in A, x_3 \ast y_3 \in B \) and \( x_4 \ast y_4 \in B \). It follows from (20) that \( x_1 \ast (y_1 \ast (y_1 \ast x_1)) \in A, x_2 \ast (y_2 \ast (y_2 \ast x_2)) \in A, x_3 \ast (y_3 \ast (y_3 \ast x_3)) \in B \) and \( x_4 \ast (y_4 \ast (y_4 \ast x_4)) \in B \). Hence
\[
\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})) = (x_1 \ast (y_1 \ast (y_1 \ast x_1)), (x_2 \ast (y_2 \ast (y_2 \ast x_2)))T, (x_3 \ast (y_3 \ast (y_3 \ast x_3)))I, (x_4 \ast (y_4 \ast (y_4 \ast x_4)))F \in NQ(A, B),
\]
and therefore \( NQ(A, B) \) is a commutative ideal of \( NQ(X) \).

**Corollary 3.16.** For any ideals \( A \) and \( B \) of a BCK-algebra \( X \), if the set \( NQ(A, B) \) satisfies
\[
(\forall \tilde{x}, \tilde{y} \in NQ(A, B)) (\tilde{x} \circ \tilde{y} \in NQ(A, B) \Rightarrow \tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})) \in NQ(A, B)),
\]
then \( NQ(A, B) \) is a commutative ideal of \( NQ(X) \).
**Theorem 3.17.** Let $I$, $J$, $A$ and $B$ be ideals of a $BCK$-algebra $X$ such that $I \subseteq A$ and $J \subseteq B$. If $I$ and $J$ are commutative ideals of $X$, then the set $NQ(A, B)$ is a commutative ideal of $NQ(X)$.

**Proof.** If $I$ and $J$ are commutative ideals of $X$, then $NQ(I, J)$ is a commutative ideal of $NQ(X)$ by Theorem 3.13. Note that $NQ(A, B)$ is an ideal of $NQ(X)$ by Lemma 3.14 and $NQ(I, J) \subseteq NQ(A, B)$. Assume that $x \ast y \in A$ (resp., $B$) for all $x, y \in X$ and let $a := x \ast y$. Then

$$(x \ast a) \ast y = (x \ast y) \ast a = 0 \in I \text{ (resp., } J),$$

and so $((x \ast a) \ast y) \ast 0 = (x \ast a) \ast y \in I \text{ (resp., } J)$. Since $I$ and $J$ are commutative ideals of $X$ with $I \subseteq A$ and $J \subseteq B$, it follows that

$$(x \ast (y \ast (x \ast a))) \ast a = (x \ast a) \ast (y \ast (x \ast a))) \in I \subseteq A \text{ (resp., } J \subseteq B),$$

thus, $x \ast (y \ast (x \ast a))) \in A$ (resp., $B$). On the other hand,

$$
\begin{align*}
(x \ast (y \ast (y \ast x))) \ast (x \ast (y \ast (x \ast a))) & \leq (y \ast (y \ast (x \ast a))) \ast (y \ast (y \ast x)) \\
& \leq (y \ast x) \ast (y \ast (x \ast a)) \leq (x \ast a) \ast x = 0 \ast a = 0.
\end{align*}
$$

Hence $(x \ast (y \ast (y \ast x))) \ast (x \ast (y \ast (x \ast a))) = 0 \in A$ (resp., $B$), and thus $x \ast (y \ast (y \ast x)) \in A$ (resp., $B$). Therefore $A$ and $B$ are commutative ideals of $X$, and so $NQ(A, B)$ is a commutative ideal of $NQ(X)$ by Theorem 3.13. \qed

The following examples illustrate Theorem 3.13.

**Example 3.18.** Consider a $BCK$-algebra $X = \{0, 1, 2\}$ with the binary operation $\ast$ which is given in Table 3.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: Cayley table for the binary operation “$\ast$”

Then the neutrosophic quadruple $BCK$-algebra $NQ(X)$ has 81 elements. If we take commutative ideals $A = \{0, 1\}$ and $B = \{0, 2\}$ of $X$, then

$$NQ(A, B) = \{(0, 0T, 0I, 0F), (0, 0T, 0I, 2F), (0, 0T, 2I, 0F), (0, 0T, 2I, 2F), (0, 1T, 0I, 0F), (0, 1T, 0I, 2F), (0, 1T, 2I, 0F), (0, 1T, 2I, 2F), (1, 0T, 0I, 0F), (1, 0T, 0I, 2F), (1, 0T, 2I, 0F), (1, 0T, 2I, 2F), (1, 1T, 0I, 0F), (1, 1T, 0I, 2F), (1, 1T, 2I, 0F), (1, 1T, 2I, 2F)\}$$

which is a commutative ideal of $NQ(X)$.

**Example 3.19.** Consider a $BCK$-algebra $X = \{0, a, b, c\}$ with the binary operation $\ast$ which is given in Table 4.
Table 4: Cayley table for the binary operation “∗”

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X, ∗, 0)\) is a commutative BCK-algebra (see [14]), and the neutrosophic quadruple BCK-set \(NQ(X)\) based on \(X\) has 256 elements and it is a commutative BCK-algebra by Theorem 3.3.

If we take commutative ideals \(A = \{0, a, b\}\) and \(B = \{0, c\}\) of \(X\), then the set \(NQ(A, B)\) consists of 36 elements, which is a commutative ideal of \(NQ(X)\) by Theorem 3.13, and it is given as follows.

\[
NQ(A, B) = \{(0, 0T, 0I, 0F), (0, 0T, 0I, cF), (0, 0T, cI, 0F), (0, 0T, cI, cF), (0, bT, 0I, 0F), (0, bT, 0I, cF), (0, bT, cI, 0F), (0, bT, cI, cF), (a, 0T, 0I, 0F), (a, 0T, 0I, cF), (a, 0T, cI, 0F), (a, 0T, cI, cF), (a, aT, 0I, 0F), (a, aT, 0I, cF), (a, aT, cI, 0F), (a, aT, cI, cF), (a, bT, 0I, 0F), (a, bT, 0I, cF), (a, bT, cI, 0F), (a, bT, cI, cF), (b, 0T, 0I, 0F), (b, 0T, 0I, cF), (b, 0T, cI, 0F), (b, 0T, cI, cF), (b, aT, 0I, 0F), (b, aT, 0I, cF), (b, aT, cI, 0F), (b, aT, cI, cF), (b, bT, 0I, 0F), (b, bT, 0I, cF), (b, bT, cI, 0F), (b, bT, cI, cF)\}.

4 Conclusions

We have considered a commutative neutrosophic quadruple ideals and BCK-algebras are discussed, and investigated several related properties are investigated. Conditions for the neutrosophic quadruple BCK-algebra to be commutative are considered. Given subsets \(A\) and \(B\) of a neutrosophic quadruple BCK algebra, conditions for the set \(NQ(A, B)\) to be a commutative ideal of a neutrosophic quadruple BCK-algebra are provided.

References


Commutative neutrosophic quadruple ideals of neutrosophic quadruple BCK-algebras


