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Commutative neutrosophic quadruple ideals of neutrosophic quadruple BCK-algebras

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Abstract

Commutative neutrosophic quadruple ideals and BCK-algebras are discussed, and related properties are investigated. Conditions for the neutrosophic quadruple BCK-algebra to be commutative are considered. Given subsets A and B of a neutrosophic quadruple BCK-algebra, conditions for the set $NQ(A, B)$ to be a commutative ideal of a neutrosophic quadruple BCK-algebra are provided.

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1 Introduction

The neutrosophic set which is developed by Smarandache ([17], [18] and [19]) is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [3], [8], [9], [10], [11], [13], [16] and [21]. Smarandache [20] considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [1, 2]. Jun et al. [12] used neutrosophic quadruple numbers based on a set, and constructed neutrosophic quadruple BCK/BCI-algebras. They investigated several properties, and considered ideal and positive

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implicative ideal in neutrosophic quadruple BCK-algebra, and closed ideal in neutrosophic quadruple BCI-algebra. Given subsets A and B of a neutrosophic quadruple BCK/BCI-algebra, they considered sets $NQ(A,B)$ which consists of neutrosophic quadruple BCK/BCI-numbers with a condition. They provided conditions for the set $NQ(A,B)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra, and the set $NQ(A,B)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra. They gave an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra, and then they considered conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra.

In this paper, we discuss a commutative neutrosophic quadruple ideal and BCK-algebra and investigate several properties. We consider conditions for the neutrosophic quadruple BCK-algebra to be commutative. Given subsets A and B of a neutrosophic quadruple BCK-algebra, we give conditions for the set $NQ(A,B)$ to be a commutative ideal of a neutrosophic quadruple BCK-algebra.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [6] and [7]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x,y,z \in X)\ (((x*y)*(x*z))*(z*y)=0)$,

(II) $(\forall x,y \in X)\ ((x*(x*y))*y=0)$,

(III) $(\forall x \in X)\ (x*x=0)$,

(IV) $(\forall x,y \in X)\ (x*y=0,\, y*x=0 \ \Rightarrow\ x=y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0*x=0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x*0=x)\,, \tag{1}$$
$$(\forall x,y,z \in X)\,(x\le y \ \Rightarrow\ x*z\le y*z,\ z*y\le z*x)\,, \tag{2}$$
$$(\forall x,y,z \in X)\,((x*y)*z=(x*z)*y)\,, \tag{3}$$
$$(\forall x,y,z \in X)\,((x*z)*(y*z)\le x*y) \tag{4}$$

where $x\le y$ if and only if $x*y=0$.

A BCK-algebra X is said to be *commutative* if the following assertion is valid.

$$(\forall x,y \in X)\,(x*(x*y)=y*(y*x))\,. \tag{5}$$

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0\in I, \tag{6}$$
$$(\forall x \in X)\,(\forall y \in I)\,(x*y\in I \ \Rightarrow\ x\in I)\,. \tag{7}$$

A subset I of a BCK-algebra X is called a *commutative ideal* of X if it satisfies (6) and

$$(\forall x, y \in X)(\forall z \in I)\left((x * y) * z \in I \;\Rightarrow\; x * (y * (y * x)) \in I\right). \tag{8}$$

Observe that every commutative ideal is an ideal, but the converse is not true (see [14]).

We refer the reader to the books [5, 14] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

3 Commutative neutrosophic quadruple BCK-algebras

In this section, we define commutative neutrosophic quadruple BCK-algebra under Theorem 3.3 and consider some properties of commutative neutrosophic quadruple BCK-algebra. Also, we investigate relation between commutative neutrosophic quadruple BCK-algebra and lattices.

Definition 3.1 ([12])**.** *Let X be a set. A neutrosophic quadruple X-number is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in X$ and T, I, F have their usual neutrosophic logic meanings.*

The set of all neutrosophic quadruple X-numbers is denoted by $NQ(X)$, that is,

$$NQ(X) := \{(a, xT, yI, zF) \mid a, x, y, z \in X\},$$

and it is called the *neutrosophic quadruple set* based on X. If X is a BCK/BCI-algebra, a neutrosophic quadruple X-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $NQ(X)$ is the *neutrosophic quadruple BCK/BCI-set.*

Let X be a BCK/BCI-algebra. We define a binary operation $\odot$ on $NQ(X)$ by

$$(a, xT, yI, zF) \odot (b, uT, vI, wF) = (a * b, (x * u)T, (y * v)I, (z * w)F)$$

for all $(a, xT, yI, zF), (b, uT, vI, wF) \in NQ(X)$. Given $a_1, a_2, a_3, a_4 \in X$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

We define an order relation "$\ll$" and the equality "$=$" on $NQ(X)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4,$$

for all $\tilde{x}, \tilde{y} \in NQ(X)$. It is easy to verify that "$\ll$" is a partial order on $NQ(X)$.

Lemma 3.2 ([12])**.** *If X is a BCK/BCI-algebra, then $(NQ(X); \odot, \tilde{0})$ is a BCK/BCI-algebra, which is called a neutrosophic quadruple BCK/BCI-algebra.*

Theorem 3.3. *The neutrosophic quadruple BCK-set $NQ(X)$ based on a commutative BCK-algebra X is a commutative BCK-algebra, which is called a commutative neutrosophic quadruple BCK-algebra.*

Proof. Let X be a commutative BCK-algebra. Then X is a BCK-algebra, and so $(NQ(X); \odot, \tilde{0})$ is a BCK-algebra by Lemma 3.2. Let $\tilde{x}$, $\tilde{y} \in NQ(X)$. Then

$$x_i * (x_i * y_i) = y_i * (y_i * x_i)$$

for all $i = 1, 2, 3, 4$ since $x_i, y_i \in X$ and X is a commutative BCK-algebra. Hence $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) = \tilde{y} \odot (\tilde{y} \odot \tilde{x})$, and therefore $NQ(X)$ based on a commutative BCK-algebra X is a commutative BCK-algebra. □

Theorem 3.3 is illustrated by the following example.

Example 3.4. *Let $X = \{0, 1\}$ be a set with the binary operation $*$ which is given in Table 1.*

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1
0	0	0
1	1	0

Then $(X, *, 0)$ is a commutative BCK-algebra (see [14]), and the neutrosophic quadruple BCK-set $NQ(X)$ is given as follows:

$$NQ(X) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

where
$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, 1F)$, $\tilde{2} = (0, 0T, 1I, 0F)$, $\tilde{3} = (0, 0T, 1I, 1F)$,
$\tilde{4} = (0, 1T, 0I, 0F)$, $\tilde{5} = (0, 1T, 0I, 1F)$, $\tilde{6} = (0, 1T, 1I, 0F)$, $\tilde{7} = (0, 1T, 1I, 1F)$,
$\tilde{8} = (1, 0T, 0I, 0F)$, $\tilde{9} = (1, 0T, 0I, 1F)$, $\tilde{10} = (1, 0T, 1I, 0F)$, $\tilde{11} = (1, 0T, 1I, 1F)$,
$\tilde{12} = (1, 1T, 0I, 0F)$, $\tilde{13} = (1, 1T, 0I, 1F)$, $\tilde{14} = (1, 1T, 1I, 0F)$, $\tilde{15} = (1, 1T, 1I, 1F)$.
Then $(NQ(X), \odot, \tilde{0})$ is a commutative BCK-algebra in which the operation $\odot$ is given by Table 2.

Table 2: Cayley table for the binary operation "$\odot$"

$\odot$	$\tilde{0}$	$\tilde{1}$	$\tilde{2}$	$\tilde{3}$	$\tilde{4}$	$\tilde{5}$	$\tilde{6}$	$\tilde{7}$	$\tilde{8}$	$\tilde{9}$	$\tilde{10}$	$\tilde{11}$	$\tilde{12}$	$\tilde{13}$	$\tilde{14}$	$\tilde{15}$
$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{1}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{2}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$
$\tilde{3}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$
$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{5}$	$\tilde{5}$	$\tilde{4}$	$\tilde{5}$	$\tilde{4}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{5}$	$\tilde{4}$	$\tilde{5}$	$\tilde{4}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{6}$	$\tilde{6}$	$\tilde{6}$	$\tilde{4}$	$\tilde{4}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{6}$	$\tilde{6}$	$\tilde{4}$	$\tilde{4}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$
$\tilde{7}$	$\tilde{7}$	$\tilde{6}$	$\tilde{5}$	$\tilde{4}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{7}$	$\tilde{6}$	$\tilde{5}$	$\tilde{4}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$
$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{9}$	$\tilde{9}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{9}$	$\tilde{8}$	$\tilde{9}$	$\tilde{8}$	$\tilde{9}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{10}$	$\tilde{10}$	$\tilde{10}$	$\tilde{8}$	$\tilde{8}$	$\tilde{10}$	$\tilde{10}$	$\tilde{8}$	$\tilde{8}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$
$\tilde{11}$	$\tilde{11}$	$\tilde{10}$	$\tilde{9}$	$\tilde{8}$	$\tilde{11}$	$\tilde{10}$	$\tilde{9}$	$\tilde{8}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$
$\tilde{12}$	$\tilde{12}$	$\tilde{12}$	$\tilde{12}$	$\tilde{12}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{13}$	$\tilde{13}$	$\tilde{12}$	$\tilde{13}$	$\tilde{12}$	$\tilde{9}$	$\tilde{8}$	$\tilde{9}$	$\tilde{8}$	$\tilde{5}$	$\tilde{4}$	$\tilde{5}$	$\tilde{4}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{14}$	$\tilde{14}$	$\tilde{14}$	$\tilde{12}$	$\tilde{12}$	$\tilde{10}$	$\tilde{10}$	$\tilde{8}$	$\tilde{8}$	$\tilde{6}$	$\tilde{6}$	$\tilde{4}$	$\tilde{4}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$
$\tilde{15}$	$\tilde{15}$	$\tilde{14}$	$\tilde{13}$	$\tilde{12}$	$\tilde{11}$	$\tilde{10}$	$\tilde{9}$	$\tilde{8}$	$\tilde{7}$	$\tilde{6}$	$\tilde{5}$	$\tilde{4}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$

Proposition 3.5. *The neutrosophic quadruple BCK-set $NQ(X)$ based on a commutative BCK-algebra X satisfies the following assertions.*

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X))(\tilde{x} \ll \tilde{z}, \tilde{z} \odot \tilde{y} \ll \tilde{z} \odot \tilde{x} \Rightarrow \tilde{x} \ll \tilde{y}). \tag{9}$$

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X))(\tilde{x} \ll \tilde{z}, \tilde{y} \ll \tilde{z}, \tilde{z} \odot \tilde{y} \ll \tilde{z} \odot \tilde{x} \Rightarrow \tilde{x} \ll \tilde{y}). \tag{10}$$

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X))(\tilde{x} \ll \tilde{y} \ \Rightarrow \ \tilde{y} \odot (\tilde{y} \odot \tilde{x}) = \tilde{x}). \tag{11}$$

$$(\forall \tilde{x}, \tilde{y} \in NQ(X))(\tilde{x} \odot (\tilde{x} \odot \tilde{y}) = \tilde{y} \odot (\tilde{y} \odot (\tilde{x} \odot (\tilde{x} \odot \tilde{y})))). \tag{12}$$

Proof. Assume that $\tilde{x} \ll \tilde{z}$ and $\tilde{z} \odot \tilde{y} \ll \tilde{z} \odot \tilde{x}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. Then $\tilde{x} \odot \tilde{z} = \tilde{0}$ and $(\tilde{z} \odot \tilde{y}) \odot (\tilde{z} \odot \tilde{x}) = \tilde{0}$. Since $NQ(X)$ is commutative, we have

$$\tilde{x} \odot \tilde{y} = (\tilde{x} \odot \tilde{0}) \odot \tilde{y} = (\tilde{x} \odot (\tilde{x} \odot \tilde{z})) \odot \tilde{y} = (\tilde{z} \odot (\tilde{z} \odot \tilde{x})) \odot \tilde{y} = (\tilde{z} \odot \tilde{y}) \odot (\tilde{z} \odot \tilde{x}) = \tilde{0},$$

that is, $\tilde{x} \ll \tilde{y}$. Condition (10) is clear by the condition (9). Suppose that $\tilde{x} \ll \tilde{y}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. Note that $\tilde{y} \odot (\tilde{y} \odot \tilde{x}) \ll \tilde{y}$ and $\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \ll \tilde{y} \odot \tilde{x}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. It follows from the condition (10) that $\tilde{x} \ll \tilde{y} \odot (\tilde{y} \odot \tilde{x})$. Obviously, $\tilde{y} \odot (\tilde{y} \odot \tilde{x}) \ll \tilde{x}$, and so $\tilde{y} \odot (\tilde{y} \odot \tilde{x}) = \tilde{x}$. Condition (12) follows directly from the condition (11). □

Theorem 3.6. *The neutrosophic quadruple BCK-set $NQ(X)$ based on a commutative BCK-algebra X is a lower semilattice with respect to the order "$\ll$".*

Proof. For any $\tilde{x}, \tilde{y} \in NQ(X)$, let $\tilde{y} \odot (\tilde{y} \odot \tilde{x}) = \tilde{x} \wedge \tilde{y}$. Then $\tilde{x} \wedge \tilde{y} \ll \tilde{x}$ and $\tilde{x} \wedge \tilde{y} \ll \tilde{y}$. Let $\tilde{a} \in NQ(X)$ such that $\tilde{a} \ll \tilde{x}$ and $\tilde{a} \ll \tilde{y}$. Then

$$\tilde{a} = \tilde{a} \odot \tilde{0} = \tilde{a} \odot (\tilde{a} \odot \tilde{x}) = \tilde{x} \odot (\tilde{x} \odot \tilde{a}).$$

Similarly, we have $\tilde{a} = \tilde{y} \odot (\tilde{y} \odot \tilde{a})$. Thus

$$\tilde{a} = \tilde{x} \odot (\tilde{x} \odot \tilde{a}) = \tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{a}))) \ll \tilde{x} \odot (\tilde{x} \odot \tilde{y}) = \tilde{y} \odot (\tilde{y} \odot \tilde{x}) = \tilde{x} \wedge \tilde{y}.$$

Hence $\tilde{x} \wedge \tilde{y}$ is the greatest lower bound, and therefore $(NQ(X), \ll)$ is a lower semilattice. □

Given a neutrosophic quadruple BCK-algebra $NQ(X)$, we consider the following set.

$$\Omega(\tilde{a}) := \{\tilde{x} \in NQ(X) \mid \tilde{x} \ll \tilde{a}\}. \tag{13}$$

Proposition 3.7. *Every neutrosophic quadruple BCK-set $NQ(X)$ based on a commutative BCK-algebra X satisfies the following identity.*

$$(\forall \tilde{a}, \tilde{b} \in NQ(X))(\Omega(\tilde{a}) \cap \Omega(\tilde{b}) = \Omega(\tilde{a} \wedge \tilde{b})) \tag{14}$$

where $\tilde{a} \wedge \tilde{b} = \tilde{b} \odot (\tilde{b} \odot \tilde{a})$.

Proof. Let $\tilde{x} \in \Omega(\tilde{a}) \cap \Omega(\tilde{b})$. Then $\tilde{x} \ll \tilde{a}$ and $\tilde{x} \ll \tilde{b}$, and so $\tilde{x} \ll \tilde{a} \wedge \tilde{b}$. Thus $\tilde{x} \in \Omega(\tilde{a} \wedge \tilde{b})$, which shows that $\Omega(\tilde{a}) \cap \Omega(\tilde{b}) \subseteq \Omega(\tilde{a} \wedge \tilde{b})$. If $\tilde{x} \in \Omega(\tilde{a} \wedge \tilde{b})$, then $\tilde{x} \ll \tilde{a} \wedge \tilde{b}$. Hence $\tilde{x} \ll \tilde{a}$ and $\tilde{x} \ll \tilde{b}$, and thus $\tilde{x} \in \Omega(\tilde{a}) \cap \Omega(\tilde{b})$. This completes the proof. □

We consider conditions for a neutrosophic quadruple BCK-algebra $NQ(X)$ to be commutative.

Lemma 3.8. *If a neutrosophic quadruple BCK-algebra $NQ(X)$ satisfies the condition* (11)*, then it is commutative.*

Proof. Assume that $NQ(X)$ is a neutrosophic quadruple BCK-algebra which satisfies the condition (11). Note that $\tilde{y} \odot (\tilde{y} \odot \tilde{x}) \ll \tilde{x}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. It follows from the condition (11) that

$$\tilde{y} \odot (\tilde{y} \odot \tilde{x}) = \tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))).$$

Hence

$$\begin{aligned}
&(\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \\
&= (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) \odot (\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \\
&= (\tilde{x} \odot (\tilde{x} \odot (\tilde{x} \odot \tilde{y}))) \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\
&= (\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\
&\ll (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot \tilde{y} = \tilde{0}
\end{aligned}$$

for all $\tilde{x}, \tilde{y} \in NQ(X)$. Similarly, we get that $(\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) = \tilde{0}$ by changing the role of $\tilde{x}$ and $\tilde{y}$. Therefore $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) = \tilde{y} \odot (\tilde{y} \odot \tilde{x})$ and so $NQ(X)$ is commutative. □

Theorem 3.9. *If a neutrosophic quadruple BCK-algebra $NQ(X)$ satisfies the condition* (12)*, then it is commutative.*

Proof. Assume that $NQ(X)$ is a neutrosophic quadruple BCK-algebra which satisfies the condition (12). Let $\tilde{x}, \tilde{y} \in NQ(X)$ such that $\tilde{x} \ll \tilde{y}$. Then

$$\tilde{y} \odot (\tilde{y} \odot \tilde{x}) = \tilde{y} \odot (\tilde{y} \odot (\tilde{x} \odot (\tilde{x} \odot \tilde{y}))) = \tilde{x} \odot (\tilde{x} \odot \tilde{y}) = \tilde{x} \odot \tilde{0} = \tilde{x},$$

and so $NQ(X)$ is commutative by Lemma 3.8. □

Lemma 3.10. *A neutrosophic quadruple BCK-algebra $NQ(X)$ is commutative if and only if the following assertion is valid.*

$$(\forall \tilde{x}, \tilde{y} \in NQ(X)) \, (\tilde{y} \odot (\tilde{y} \odot \tilde{x}) \ll (\tilde{x} \odot (\tilde{x} \odot \tilde{y}))) \,. \tag{15}$$

Proof. It is straightforward. □

Theorem 3.11. *If a neutrosophic quadruple BCK-algebra $NQ(X)$ satisfies the condition* (14), *then it is commutative.*

Proof. Let $NQ(X)$ be a neutrosophic quadruple BCK-algebra which satisfies the condition (14). Let $\tilde{x} \wedge \tilde{y} := \tilde{y} \odot (\tilde{y} \odot \tilde{x})$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. Then

$$\Omega(\tilde{x} \wedge \tilde{y}) = \Omega(\tilde{x}) \cap \Omega(\tilde{y}) = \Omega(\tilde{y}) \cap \Omega(\tilde{x}) = \Omega(\tilde{y} \wedge \tilde{x})$$

for all $\tilde{x}, \tilde{y} \in NQ(X)$, and thus $\tilde{x} \wedge \tilde{y} \in \Omega(\tilde{y} \wedge \tilde{x})$. Hence $\tilde{x} \wedge \tilde{y} \ll \tilde{y} \wedge \tilde{x}$, that is, $\tilde{y} \odot (\tilde{y} \odot \tilde{x}) \ll \tilde{x} \odot (\tilde{x} \odot \tilde{y})$. It follows from Lemma 3.10 that $NQ(X)$ is a commutative neutrosophic quadruple BCK-algebra. □

Theorem 3.12. *Given a nonempty set X, if a neutrosophic quadruple set $NQ(X)$ satisfies the following assertions*

$$(\forall \tilde{x} \in NQ(X)) \left(\tilde{x} \odot \tilde{0} = \tilde{x}, \ \tilde{x} \odot \tilde{x} = \tilde{0}\right), \tag{16}$$

$$(\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)) \, ((\tilde{x} \odot \tilde{y}) \odot \tilde{z} = (\tilde{x} \odot \tilde{z}) \odot \tilde{y}) \,, \tag{17}$$

$$(\tilde{x}, \tilde{y} \in NQ(X)) \, (\tilde{x} \wedge \tilde{y} = \tilde{y} \wedge \tilde{x}) \tag{18}$$

where $\tilde{x} \wedge \tilde{y} = \tilde{y} \odot (\tilde{y} \odot \tilde{x})$, then it is a commutative neutrosophic quadruple BCK-algebra.

Proof. Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. Using conditions (16) and (17) imply that

$$(\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot \tilde{y} = (\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot \tilde{y}) = \tilde{0}.$$

Assume that $\tilde{x} \odot \tilde{y} = \tilde{0}$ and $\tilde{y} \odot \tilde{x} = \tilde{0}$. Then

$$\tilde{x} = \tilde{x} \odot \tilde{0} = \tilde{x} \odot (\tilde{x} \odot \tilde{y}) = \tilde{y} \wedge \tilde{x} = \tilde{x} \wedge \tilde{y} = \tilde{y} \odot (\tilde{y} \odot \tilde{x}) = \tilde{y} \odot \tilde{0} = \tilde{y}.$$

Using (17) and (18), we have

$$\begin{aligned}(\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot \tilde{z}) &= (\tilde{x} \odot (\tilde{x} \odot \tilde{z})) \odot \tilde{y} = (\tilde{z} \wedge \tilde{x}) \odot \tilde{y} = (\tilde{x} \wedge \tilde{z}) \odot \tilde{y} \\ &= (\tilde{z} \odot (\tilde{z} \odot \tilde{x})) \odot \tilde{y} = (\tilde{z} \odot \tilde{y}) \odot (\tilde{z} \odot \tilde{x}).\end{aligned} \tag{19}$$

If we take $\tilde{y} = \tilde{x}$ and $\tilde{z} = \tilde{0}$ in (19), then

$$\tilde{0} \odot \tilde{x} = (\tilde{x} \odot \tilde{x}) \odot (\tilde{x} \odot \tilde{0}) = (\tilde{0} \odot \tilde{x}) \odot (\tilde{0} \odot \tilde{x}) = \tilde{0}.$$

It follows from (19) and (16) that

$$\begin{aligned}((\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot \tilde{z})) \odot (\tilde{z} \odot \tilde{y}) &= ((\tilde{z} \odot \tilde{y}) \odot (\tilde{z} \odot \tilde{x})) \odot ((\tilde{z} \odot \tilde{y}) \odot \tilde{0}) \\ &= (\tilde{0} \odot (\tilde{z} \odot \tilde{x})) \odot (\tilde{0} \odot (\tilde{z} \odot \tilde{y})) \\ &= \tilde{0} \odot \tilde{0} = \tilde{0}.\end{aligned}$$

Therefore $(NQ(X), \odot, \tilde{0})$ is a commutative neutrosophic quadruple BCK-algebra. □

Given subsets A and B of a BCK-algebra X, consider the set

$$NQ(A,B) := \{(a, xT, yI, zF) \in NQ(X) \mid a, x \in A; y, z \in B\}.$$

Theorem 3.13. *If A and B are commutative ideals of a BCK-algebra X, then the set $NQ(A,B)$ is a commutative ideal of $NQ(X)$, which is called a commutative neutrosophic quadruple ideal.*

Proof. Assume that A and B are commutative ideals of a BCK-algebra X. Obviously, $\tilde{0} \in NQ(A,B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $\tilde{z} \in NQ(A,B)$ and $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A,B)$. Then

$$\begin{aligned}(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = ((x_1 * y_1) * z_1, &((x_2 * y_2) * z_2)T,\\ &((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A,B),\end{aligned}$$

and so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$ and $(x_4 * y_4) * z_4 \in B$. Since $\tilde{z} \in NQ(A,B)$, we have $z_1, z_2 \in A$ and $z_3, z_4 \in B$. Since A and B are commutative ideals of X, it follows that $x_1 * (y_1 * (y_1 * x_1)) \in A$, $x_2 * (y_2 * (y_2 * x_2)) \in A$, $x_3 * (y_3 * (y_3 * x_3)) \in B$ and $x_4 * (y_4 * (y_4 * x_4)) \in B$. Hence

$$\begin{aligned}\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) = (x_1 * (y_1 * (y_1 * x_1)), &(x_2 * (y_2 * (y_2 * x_2)))T,\\ &(x_3 * (y_3 * (y_3 * x_3)))I, (x_4 * (y_4 * (y_4 * x_4)))F) \in NQ(A,B),\end{aligned}$$

and therefore $NQ(A,B)$ is a commutative ideal of $NQ(X)$. □

Lemma 3.14 ([12])**.** *If A and B are ideals of a BCK-algebra X, then the set $NQ(A,B)$ is an ideal of $NQ(X)$, which is called a neutrosophic quadruple ideal.*

Theorem 3.15. *Let A and B be ideals of a BCK-algebra X such that*

$$(\forall x, y \in X)\,(x * y \in A \text{ (resp., } B) \;\Rightarrow\; x * (y * (y * x)) \in A \text{ (resp., } B)). \tag{20}$$

Then $NQ(A,B)$ is a commutative ideal of $NQ(X)$.

Proof. If A and B are ideals of a BCK-algebra X, then $NQ(A,B)$ is an ideal of $NQ(X)$ by Lemma 3.14. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A,B)$ and $\tilde{z} \in NQ(A,B)$. Then

$$(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = ((x_1 * y_1) * z_1, ((x_2 * y_2) * z_2)T, ((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A,B),$$

and $\tilde{z} = (z_1, z_2T, z_3I, z_4F) \in NQ(A,B)$, so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$, $z_1 \in A$, $z_2 \in A$, $z_3 \in B$ and $z_4 \in B$. Since A and B are ideals of X, we get that $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$ and $x_4 * y_4 \in B$. It follows from (20) that $x_1 * (y_1 * (y_1 * x_1)) \in A$, $x_2 * (y_2 * (y_2 * x_2)) \in A$, $x_3 * (y_3 * (y_3 * x_3)) \in B$ and $x_4 * (y_4 * (y_4 * x_4)) \in B$. Hence

$$\begin{aligned}\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) = (x_1 * (y_1 * (y_1 * x_1)), &(x_2 * (y_2 * (y_2 * x_2)))T,\\ &(x_3 * (y_3 * (y_3 * x_3)))I, (x_4 * (y_4 * (y_4 * x_4)))F) \in NQ(A,B).\end{aligned}$$

Therefore $NQ(A,B)$ is a commutative ideal of $NQ(X)$. □

Corollary 3.16. *For any ideals A and B of a BCK-algebra X, if the set $NQ(A,B)$ satisfies*

$$(\forall\, \tilde{x}, \tilde{y} \in NQ(A,B))\,(\tilde{x} \odot \tilde{y} \in NQ(A,B) \;\Rightarrow\; \tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \in NQ(A,B)),$$

then $NQ(A,B)$ is a commutative ideal of $NQ(X)$.

Theorem 3.17. *Let I, J, A and B be ideals of a BCK-algebra X such that $I \subseteq A$ and $J \subseteq B$. If I and J are commutative ideals of X, then the set $NQ(A, B)$ is a commutative ideal of $NQ(X)$.*

Proof. If I and J are commutative ideals of X, then $NQ(I, J)$ is a commutative ideal of $NQ(X)$ by Theorem 3.13. Note that $NQ(A, B)$ is an ideal of $NQ(X)$ by Lemma 3.14 and $NQ(I, J) \subseteq NQ(A, B)$. Assume that $x * y \in A$ (resp., B) for all $x, y \in X$ and let $a := x * y$. Then

$$(x * a) * y = (x * y) * a = 0 \in I \text{ (resp., } J),$$

and so $((x * a) * y) * 0 = (x * a) * y \in I$ (resp., J). Since I and J are commutative ideals of X with $I \subseteq A$ and $J \subseteq B$, it follows that

$$(x * (y * (y * (x * a)))) * a = (x * a) * (y * (y * (x * a))) \in I \subseteq A \text{ (resp., } J \subseteq B),$$

thus, $x * (y * (y * (x * a))) \in A$ (resp., B). On the other hand,

$$\begin{aligned}&(x * (y * (y * x))) * (x * (y * (y * (x * a)))) \leq (y * (y * (x * a))) * (y * (y * x))\\&\leq (y * x) * (y * (x * a)) \leq (x * a) * x = 0 * a = 0.\end{aligned}$$

Hence $(x*(y*(y*x)))*(x*(y*(y*(x*a)))) = 0 \in A$ (resp., B), and thus $x * (y * (y * x)) \in A$ (resp., B). Therefore A and B are commutative ideals of X, and so $NQ(A, B)$ is a commutative ideal of $NQ(X)$ by Theorem 3.13. □

The following examples illustrate Theorem 3.13.

Example 3.18. *Consider a BCK-algebra $X = \{0, 1, 2\}$ with the binary operation $*$ which is given in Table 3,*

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then the neutrosophic quadruple BCK-algebra $NQ(X)$ has 81 elements. If we take commutative ideals $A = \{0, 1\}$ and $B = \{0, 2\}$ of X, then

$$\begin{aligned}NQ(A, B) =\{&(0, 0T, 0I, 0F), (0, 0T, 0I, 2F), (0, 0T, 2I, 0F), (0, 0T, 2I, 2F),\\&(0, 1T, 0I, 0F), (0, 1T, 0I, 2F), (0, 1T, 2I, 0F), (0, 1T, 2I, 2F),\\&(1, 0T, 0I, 0F), (1, 0T, 0I, 2F), (1, 0T, 2I, 0F), (1, 0T, 2I, 2F),\\&(1, 1T, 0I, 0F), (1, 1T, 0I, 2F), (1, 1T, 2I, 0F), (1, 1T, 2I, 2F)\}\end{aligned}$$

which is a commutative ideal of $NQ(X)$.

Example 3.19. *Consider a BCK-algebra $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 4.*

Table 4: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then $(X, *, 0)$ is a commutative BCK-algebra (see [14]), and the neutrosophic quadruple BCK-set $NQ(X)$ based on X has 256 elements and it is a commutative BCK-algebra by Theorem 3.3. If we take commutative ideals $A = \{0, a, b\}$ and $B = \{0, c\}$ of X, then the set $NQ(A, B)$ consists of 36 elements, which is a commutative ideal of $NQ(X)$ by Theorem 3.13, and it is given as follows.

$$\begin{aligned} NQ(A,B) = \{&(0,0T,0I,0F),(0,0T,0I,cF),(0,0T,cI,0F),(0,0T,cI,cF),\\ &(0,aT,0I,0F),(0,aT,0I,cF),(0,aT,cI,0F),(0,aT,cI,cF),\\ &(0,bT,0I,0F),(0,bT,0I,cF),(0,bT,cI,0F),(0,bT,cI,cF),\\ &(a,0T,0I,0F),(a,0T,0I,cF),(a,0T,cI,0F),(a,0T,cI,cF),\\ &(a,aT,0I,0F),(a,aT,0I,cF),(a,aT,cI,0F),(a,aT,cI,cF),\\ &(a,bT,0I,0F),(a,bT,0I,cF),(a,bT,cI,0F),(a,bT,cI,cF),\\ &(b,0T,0I,0F),(b,0T,0I,cF),(b,0T,cI,0F),(b,0T,cI,cF),\\ &(b,aT,0I,0F),(b,aT,0I,cF),(b,aT,cI,0F),(b,aT,cI,cF),\\ &(b,bT,0I,0F),(b,bT,0I,cF),(b,bT,cI,0F),(b,bT,cI,cF)\}. \end{aligned}$$

4 Conclusions

We have considered a commutative neutrosophic quadruple ideals and BCK-algebras are discussed, and investigated several related properties are investigated. Conditions for the neutrosophic quadruple BCK-algebra to be commutative are considered. Given subsets A and B of a neutrosophic quadruple BCK algebra, conditions for the set NQ(A, B) to be a commutative ideal of a neutrosophic quadruple BCK-algebra are provided.

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