Multipolar fuzzy hyper BCK-ideals of hyper BCK-algebras

Y.B. Jun

1Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

skywine@gmail.com

Abstract

In this paper, we apply $m$-polar fuzzy set to hyper BCK-algebra. We introduce the notions of $k$-polar fuzzy hyper BCK-ideal, $k$-polar fuzzy weak hyper BCK-ideal, $k$-polar fuzzy s-weak hyper BCK-ideal, $k$-polar fuzzy strong hyper BCK-ideal and $k$-polar fuzzy reflexive hyper BCK-ideal, and investigate related properties and their relations. We discuss $k$-polar fuzzy (weak, s-weak, strong, reflexive) hyper BCK-ideal in relation to $k$-polar level set.

1 Introduction

The hyper algebraic structure was introduced by F. Marty [14] in 1934. Bolurian et al. [5] was introduced hyper BCK-algebra as an extension of BCK-algebra. Since then, many scholars have been studying hyper BCK-algebra and its infrastructure and so on. In addition, research using fuzzy and soft set is actively being carried out (see [4], [7], [8], [9], [11]). In 2014, Chen et al. [6] introduced an $m$-polar fuzzy set which is an extension of bipolar fuzzy set. The $m$-polar fuzzy set applied to decision making problem (see [1]) and BCK/BCI-algebra (see [2], [3], [15]).

The notion of $m$-polar fuzzy set is applied to hyper BCK-algebra. The concepts of $k$-polar fuzzy (weak, s-weak, strong, reflexive) hyper BCK-ideal are introduced, and the relations and properties are investigated in relation to $k$-polar level set.

2 Preliminaries

Let $\mathcal{H}$ be a nonempty set endowed with a hyperoperation “$\circ$”. For two subsets $A$ and $B$ of $\mathcal{H}$, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.
By a hyper BCK-algebra (see \[13\]) we mean a nonempty set \( \mathcal{H} \) endowed with a hyperoperation “\( \circ \)" and a constant 0 satisfying the following axioms:

(HK1) \( (x \circ z) \circ (y \circ z) \ll x \circ y, \)

(HK2) \( (x \circ y) \circ z = (x \circ z) \circ y, \)

(HK3) \( x \circ \mathcal{H} \ll \{x\}, \)

(HK4) \( x \ll y \) and \( y \ll x \) imply \( x = y, \)

for all \( x, y, z \in \mathcal{H}, \) where \( x \ll y \) is defined by \( 0 \in x \circ y \) and for every \( A, B \subseteq \mathcal{H}, \) \( A \ll B \) is defined by \( \forall a \in A, \exists b \in B \) such that \( a \ll b. \) In such case, we call “\( \ll \)" the hyperorder in \( \mathcal{H}. \)

Note that the condition (HK3) is equivalent to the condition:

\[ (\forall x, y \in \mathcal{H})(x \circ y \ll \{x\}). \]

(1)

A subset \( A \) of a hyper BCK-algebra \( \mathcal{H} \) is called

- a hyper BCK-ideal of \( \mathcal{H} \) (see \[13\]) if
  \[ 0 \in A, \]
  \[ (\forall x, y \in \mathcal{H})(x \circ y \ll A, y \in A \Rightarrow x \in A). \]

(2)

- a weak hyper BCK-ideal of \( \mathcal{H} \) (see \[13\]) if it satisfies (2) and
  \[ (\forall x, y \in \mathcal{H})(x \circ y \subseteq A, y \in A \Rightarrow x \in A). \]

(3)

- a strong hyper BCK-ideal of \( \mathcal{H} \) (see \[12\]) if it satisfies (2) and
  \[ (\forall x, y \in \mathcal{H})(x \circ y \cap A \neq \emptyset, y \in A \Rightarrow x \in A). \]

(4)

- a reflexive hyper BCK-ideal of \( \mathcal{H} \) (see \[12\]) if it is a hyper BCK-ideal of \( \mathcal{H} \) which satisfies:
  \[ (\forall x \in \mathcal{H})(x \circ x \subseteq A). \]

(5)

Every hyper BCK-algebra \( \mathcal{H} \) satisfies the following assertions.

\[ (\forall x \in \mathcal{H})(x \circ 0 \ll \{x\}, 0 \circ x = \{0\}, 0 \circ 0 = \{0\}), \]

(7)

\[ (\forall x \in \mathcal{H})(0 \ll x, x \ll x, x \in x \circ 0), \]

(8)

\[ (\forall x, y \in \mathcal{H})(x \circ 0 \ll \{y\} \Rightarrow x \ll y), \]

(9)

\[ (\forall x, y, z \in \mathcal{H})(y \ll z \Rightarrow x \circ z \ll x \circ y), \]

(10)

\[ (\forall x, y, z \in \mathcal{H})(x \circ y = \{0\} \Rightarrow x \circ z \ll y \circ z, (x \circ z) \circ (y \circ z) = \{0\}), \]

(11)

For any subsets \( A, B \) and \( C \) of a hyper BCK-algebra \( \mathcal{H}, \) the following assertions are valid.

\[ A \subseteq B \Rightarrow A \ll B, \]

(12)

\[ A \ll \{0\} \Rightarrow A = \{0\}, \]

(13)

\[ A \ll A, A \circ B \ll A, (A \circ B) \circ C = (A \circ C) \circ B, \]

(14)

\[ A \circ \{0\} = \{0\} \Rightarrow A = \{0\}. \]

(15)
For any family \( \{a_i \mid i \in \Lambda\} \) of real numbers, we define

\[
\bigvee \{a_i \mid i \in \Lambda\} := \begin{cases} 
\max \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\
\sup \{a_i \mid i \in \Lambda\} & \text{otherwise.}
\end{cases}
\]

\[
\bigwedge \{a_i \mid i \in \Lambda\} := \begin{cases} 
\min \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\
\inf \{a_i \mid i \in \Lambda\} & \text{otherwise.}
\end{cases}
\]

If \( \Lambda = \{1, 2\} \), then we will also use \( a_1 \vee a_2 \) and \( a_1 \wedge a_2 \) instead of \( \bigvee \{a_i \mid i \in \Lambda\} \) and \( \bigwedge \{a_i \mid i \in \Lambda\} \), respectively.

By a \( k \)-polar fuzzy set on a universe \( \mathcal{H} \) (see [6]), we mean a function \( \hat{\phi} : \mathcal{H} \to [0,1]^k \). The membership value of every element \( x \in \mathcal{H} \) is denoted by

\[
\hat{\phi}(x) = (\pi_1 \circ \hat{\phi}(x), \pi_2 \circ \hat{\phi}(x), \ldots, \pi_k \circ \hat{\phi}(x)),
\]

where \( \pi_i : [0,1]^k \to [0,1] \) is the \( i \)-th projection for all \( i = 1, 2, \ldots, k \).

Given a \( k \)-polar fuzzy set on a universe \( \mathcal{H} \), we consider the set

\[
U(\hat{\phi}, \hat{t}) := \{ x \in \mathcal{H} \mid \hat{\phi}(x) \geq \hat{t} \},
\]

where \( \hat{t} = (t_1, t_2, \ldots, t_k) \in [0,1]^k \), that is,

\[
U(\hat{\phi}, \hat{t}) := \{ x \in \mathcal{H} \mid (\pi_i \circ \hat{\phi})(x) \geq t_i \text{ for all } i = 1, 2, \ldots, k \}
\]

which is called a \( k \)-polar level set of \( \hat{\phi} \). It is clear that \( U(\hat{\phi}, \hat{t}) = \bigcap_{i=1}^k U(\hat{\phi}, \hat{t}_i) \) where

\[
U(\hat{\phi}, \hat{t}_i) = \{ x \in \mathcal{H} \mid (\pi_i \circ \hat{\phi})(x) \geq t_i \}.
\]

### 3 \( k \)-polar fuzzy hyper BCK-ideals

**Definition 3.1.** A \( k \)-polar fuzzy set \( \hat{\phi} \) on a hyper BCK-algebra \( \mathcal{H} \) is called a \( k \)-polar fuzzy hyper BCK-ideal of \( \mathcal{H} \) if it satisfies

\[
(\forall x, y \in \mathcal{H})(x \ll y \Rightarrow \hat{\phi}(x) \geq \hat{\phi}(y)), \tag{18}
\]

\[
(\forall x, y \in \mathcal{H})(\hat{\phi}(x) \geq \left( \bigwedge \{\hat{\phi}(a) \mid a \in x \circ y\} \right) \wedge \hat{\phi}(y)), \tag{19}
\]

that is, \( (\pi_i \circ \hat{\phi})(x) \geq (\pi_i \circ \hat{\phi})(y) \) for all \( x, y \in \mathcal{H} \) with \( x \ll y \) and

\[
(\pi_i \circ \hat{\phi})(x) \geq \left( \bigwedge \{(\pi_i \circ \hat{\phi})(a) \mid a \in x \circ y\} \right) \wedge (\pi_i \circ \hat{\phi})(y) \tag{20}
\]

for all \( x, y \in \mathcal{H} \) and \( i = 1, 2, \ldots, k \).

**Example 3.2.** Let \( \mathcal{H} = \{0, a, b\} \) be a set with the hyperoperation “*” in the following Cayley table

<table>
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<tr>
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<td>{0}</td>
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<tr>
<td>a</td>
<td>{a}</td>
<td>{0, a}</td>
<td>{0, a}</td>
</tr>
<tr>
<td>b</td>
<td>{b}</td>
<td>{a, b}</td>
<td>{0, a, b}</td>
</tr>
</tbody>
</table>
Then $H$ is a hyper BCK-algebra (see [13]). Let $\hat{\phi}$ be a 4-polar fuzzy set on $H$ defined by

\[ \hat{\phi} : H \to [0,1]^4, \quad x \mapsto \begin{cases} \left( \frac{1}{n}, 0.9, \frac{1}{m-3}, 0.7 \right) & \text{if } x = 0, \\ \left( \frac{1}{2n}, 0.5, \frac{1}{2m-3}, 0.7 \right) & \text{if } x = a, \\ \left( \frac{1}{3n}, 0.2, \frac{1}{3m-3}, 0.4 \right) & \text{if } x = b \end{cases} \]

where $m, n \in \mathbb{N}$ and $m \neq 3$. It is toutine to verify that $\hat{\phi}$ is a 4-polar fuzzy hyper BCK-ideal of $H$.

**Proposition 3.3.** If $\hat{\phi}$ is a $k$-polar fuzzy hyper BCK-ideal of a hyper BCK-algebra $H$, then

1. $(\forall x \in H)(\hat{\phi}(0) \geq \hat{\phi}(x))$, that is, $(\pi_i \circ \hat{\phi})(0) \geq (\pi_i \circ \hat{\phi})(x)$ for all $x \in H$ and $i = 1, 2, \cdots, k$,

2. If $\hat{\phi}$ satisfies the condition

\[ (\forall T \subseteq H) \left( \exists x_0 \in T \text{ s.t. } \hat{\phi}(x_0) = \bigwedge_{x \in T} \hat{\phi}(x) \right), \tag{21} \]

then

\[ (\forall x, y \in H) \left( \exists a \in x \circ y \text{ s.t. } \hat{\phi}(x) \geq \hat{\phi}(a) \wedge \hat{\phi}(y) \right), \tag{22} \]

that is, for every $x, y \in H$ there exists $a \in x \circ y$ such that

\[ (\pi_i \circ \hat{\phi})(x) \geq (\pi_i \circ \hat{\phi})(a) \wedge (\pi_i \circ \hat{\phi})(y) \]

for $i = 1, 2, \cdots, k$.

**Proof.** (1) Since $0 \ll x$ for all $x \in H$, it follows from [18] that $\hat{\phi}(0) \geq \hat{\phi}(x)$ for all $x \in H$.

(2) Assume that $\hat{\phi}$ satisfies the condition (21). For any $x, y \in H$, there exists $a_0 \in x \circ y$ such that $(\pi_i \circ \hat{\phi})(a_0) = \bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\phi})(a)$. It follows from (20) that

\[ (\pi_i \circ \hat{\phi})(x) \geq \left( \bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\phi})(a) \right) \wedge (\pi_i \circ \hat{\phi})(y) = (\pi_i \circ \hat{\phi})(a_0) \wedge (\pi_i \circ \hat{\phi})(y) \]

for $i = 1, 2, \cdots, k$. which proves (2). \hfill \Box

**Theorem 3.4.** Let $\hat{\phi}$ be a $k$-polar fuzzy set in a hyper BCK-algebra $H$. If $\hat{\phi}$ is a $k$-polar fuzzy hyper BCK-ideal of $H$, then the $k$-polar level set $U(\hat{\phi}, \hat{t})$ is a hyper BCK-ideal of $H$ for all $\hat{t} \in [0,1]^k$.

**Proof.** Assume that $\hat{\phi}$ is a $k$-polar fuzzy hyper BCK-ideal of $H$ and let $\hat{t} \in [0,1]^k$. It is clear that $0 \in U(\hat{\phi}, \hat{t})$. Let $x, y \in H$ be such that $x \circ y \ll U(\hat{\phi}, \hat{t})$ and $y \in U(\hat{\phi}, \hat{t})$. Then $x \circ y \ll U(\hat{\phi}, \hat{t})^i$ and $y \in U(\hat{\phi}, \hat{t})^i$ for all $i = 1, 2, \cdots, k$. It follows that

\[ (\forall a \in x \circ y) \left( \exists a_0 \in U(\hat{\phi}, \hat{t})^i \text{ s.t. } a \ll a_0 \text{ and so } (\pi_i \circ \hat{\phi})(a) \geq (\pi_i \circ \hat{\phi})(a_0) \right), \]

which implies that $(\pi_i \circ \hat{\phi})(a) \geq t_i$ for all $a \in x \circ y$. Hence $\bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\phi})(a) \geq t_i$, and so

\[ (\pi_i \circ \hat{\phi})(x) \geq \left( \bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\phi})(a) \right) \wedge (\pi_i \circ \hat{\phi})(y) \geq t_i \]

for all $i = 1, 2, \cdots, k$. Thus $x \in \bigcap_{i=1}^k U(\hat{\phi}, \hat{t})^i = U(\hat{\phi}, \hat{t})$, and therefore $U(\hat{\phi}, \hat{t})$ is a hyper BCK-ideal of $H$ for all $\hat{t} \in [0,1]^k$. \hfill \Box
In order to consider the converse of Theorem 3.4, we need the following lemma.

**Lemma 3.5** ([10]). Let $A$ be a subset of a hyper BCK-algebra $H$. If $K$ is a hyper BCK-ideal of $H$ such that $A \ll K$, then $A$ is contained in $K$.

**Theorem 3.6.** Let $\hat{\varphi}$ be a $k$-polar fuzzy set in a hyper BCK-algebra $H$ in which the $k$-polar level set $U(\hat{\varphi}, \hat{t})$ is a hyper BCK-ideal of $H$ for all $\hat{t} \in [0, 1]^k$. Then $\hat{\varphi}$ is a $k$-polar fuzzy hyper BCK-ideal of $H$.

**Proof.** Suppose that the $k$-polar level set $U(\hat{\varphi}, \hat{t})$ is a hyper BCK-ideal of $H$ for all $\hat{t} \in [0, 1]^k$. Let $x, y \in H$ be such that $x \ll y$ and $\hat{\varphi}(y) = \hat{t}$. Then $y \in U(\hat{\varphi}, \hat{t})$ and so $\{x\} \ll U(\hat{\varphi}, \hat{t})$. It follows from Lemma 3.5 that $\{x\} \subseteq U(\hat{\varphi}, \hat{t})$, i.e., $x \in U(\hat{\varphi}, \hat{t})$. Hence $\hat{\varphi}(x) \geq \hat{t} = \hat{\varphi}(y)$. For any $x, y \in H$, let $\hat{t} := \left( \bigwedge_{a \in x \circ y} \hat{\varphi}(a) \right) \land \hat{\varphi}(y)$. Then $y \in U(\hat{\varphi}, \hat{t})$ and

$$\hat{\varphi}(c) \geq \bigwedge_{a \in x \circ y} \hat{\varphi}(a) \geq \left( \bigwedge_{a \in x \circ y} \hat{\varphi}(a) \right) \land \hat{\varphi}(y) = \hat{t}$$

for all $c \in x \circ y$, i.e., $c \in U(\hat{\varphi}, \hat{t})$. Thus $x \circ y \subseteq U(\hat{\varphi}, \hat{t})$ and so $x \circ y \ll U(\hat{\varphi}, \hat{t})$ by [12]. Since $y \in U(\hat{\varphi}, \hat{t})$ and $U(\hat{\varphi}, \hat{t})$ is a hyper BCK-ideal of $H$, we have $x \in U(\hat{\varphi}, \hat{t})$ which implies that $\hat{\varphi}(x) \geq \hat{t} = \left( \bigwedge_{a \in x \circ y} \hat{\varphi}(a) \right) \land \hat{\varphi}(y)$. Therefore $\hat{\varphi}$ is a $k$-polar fuzzy hyper BCK-ideal of $H$. $\square$

**Definition 3.7.** A $k$-polar fuzzy set $\hat{\varphi}$ on a hyper BCK-algebra $H$ is called a

- $k$-polar fuzzy weak hyper BCK-ideal of $H$ if it satisfies Proposition 3.3(1) and (19).
- $k$-polar fuzzy s-weak hyper BCK-ideal of $H$ if it satisfies Proposition 3.3(1) and (22).
- $k$-polar fuzzy strong hyper BCK-ideal of $H$ if it satisfies

$$\forall x, y \in H \left( \bigwedge_{a \in x \circ y} \hat{\varphi}(a) \geq \hat{\varphi}(x) \geq \left( \bigvee_{b \in x \circ y} \hat{\varphi}(b) \right) \land \hat{\varphi}(y) \right),$$

that is, $\bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(x) \geq \left( \bigvee_{b \in x \circ y} (\pi_i \circ \hat{\varphi})(b) \right) \land (\pi_i \circ \hat{\varphi})(y)$ for all $x, y \in H$ and $i = 1, 2, \ldots, k$.

**Example 3.8.** Let $H = \{0, a, b\}$ be a set with the hyperoperation “$\circ$” in the following Cayley table

<table>
<thead>
<tr>
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<th>a</th>
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<td>{0}</td>
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<tr>
<td>a</td>
<td>{a}</td>
<td>{0}</td>
<td>{a}</td>
</tr>
<tr>
<td>b</td>
<td>{b}</td>
<td>{b}</td>
<td>{0, b}</td>
</tr>
</tbody>
</table>
Theorem 3.9. In a hyper BCK-algebra, every $k$-polar fuzzy s-weak hyper BCK-ideal is a $k$-polar fuzzy weak hyper BCK-ideal.

Proof. Let $\hat{\varphi}$ be a $k$-polar fuzzy s-weak hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ and let $x, y \in \mathcal{H}$. Then there exists $a \in x \circ y$ such that $\hat{\varphi}(x) \geq \hat{\varphi}(a) \land \hat{\varphi}(y)$ by (22). Since $\hat{\varphi}(a) \geq \bigwedge_{b \in x \circ y} \hat{\varphi}(b)$, it follows that

$$\hat{\varphi}(x) \geq \left(\bigwedge_{b \in x \circ y} \hat{\varphi}(b) \mid b \in x \circ y\right) \land \hat{\varphi}(y).$$

Therefore $\hat{\varphi}$ is a $k$-polar fuzzy weak hyper BCK-ideal of $\mathcal{H}$.

Theorem 3.10. Let $\hat{\varphi}$ be a $k$-polar fuzzy weak hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ which satisfies the condition (21). Then $\hat{\varphi}$ is a $k$-polar fuzzy s-weak hyper BCK-ideal of $\mathcal{H}$.

Proof. For any $x, y \in \mathcal{H}$, there exists $a_{0} \in x \circ y$ such that $\hat{\varphi}(a_{0}) = \bigwedge_{a \in x \circ y} \hat{\varphi}(a)$, that is, $(\pi_{i} \circ \hat{\varphi})(a_{0}) = \bigwedge_{a \in x \circ y} (\pi_{i} \circ \hat{\varphi})(a)$ by (21). It follows that

$$(\pi_{i} \circ \hat{\varphi})(x) \geq \left(\bigwedge_{a \in x \circ y} (\pi_{i} \circ \hat{\varphi})(a) \mid a \in x \circ y\right) \land (\pi_{i} \circ \hat{\varphi})(y) = (\pi_{i} \circ \hat{\varphi})(a_{0}) \land (\pi_{i} \circ \hat{\varphi})(y).$$

Therefore $\hat{\varphi}$ is a $k$-polar fuzzy s-weak hyper BCK-ideal of $\mathcal{H}$.

Proposition 3.11. Every $k$-polar fuzzy strong hyper BCK-ideal $\hat{\varphi}$ of a hyper BCK-algebra $\mathcal{H}$ satisfies the following assertions.

1. $(\forall x \in \mathcal{H})(\hat{\varphi}(0) \geq \hat{\varphi}(x))$, that is, $(\pi_{i} \circ \hat{\varphi})(0) \geq (\pi_{i} \circ \hat{\varphi})(x)$ for all $x \in \mathcal{H}$ and $i = 1, 2, \ldots, k$,

2. $(\forall x, y \in \mathcal{H})(x \ll y \Rightarrow \hat{\varphi}(x) \geq \hat{\varphi}(y))$, that is, $(\pi_{i} \circ \hat{\varphi})(x) \geq (\pi_{i} \circ \hat{\varphi})(y)$ for all $x, y \in \mathcal{H}$ with $x \ll y$ and $i = 1, 2, \ldots, k$.

3. $(\forall a, x, y \in \mathcal{H})(a \in x \circ y \Rightarrow \hat{\varphi}(x) \geq \hat{\varphi}(a) \land \hat{\varphi}(y))$.

Proof. (1) Since $0 \in x \circ x$ for all $x \in \mathcal{H}$, we get

$$\hat{\varphi}(0) \geq \bigwedge_{a \in x \circ x} \hat{\varphi}(a) \geq \hat{\varphi}(x)$$

for all $x \in \mathcal{H}$.
(2) Let \(x, y \in H\) be such that \(x \ll y\). Then \(0 \in x \circ y\) and thus \(\bigvee_{b \in x \circ y} (\pi_i \circ \hat{\varphi})(b) \geq (\pi_i \circ \hat{\varphi})(0)\) for \(i = 1, 2, \ldots, k\). It follows from (1) that
\[
(\pi_i \circ \hat{\varphi})(x) \geq \left( \bigvee_{b \in x \circ y} (\pi_i \circ \hat{\varphi})(b) \right) \wedge (\pi_i \circ \hat{\varphi})(y) \geq (\pi_i \circ \hat{\varphi})(0) \wedge (\pi_i \circ \hat{\varphi})(y) = (\pi_i \circ \hat{\varphi})(y),
\]
for \(i = 1, 2, \ldots, k\), that is, \(\hat{\varphi}(x) \geq \hat{\varphi}(y)\) for all \(x, y \in H\) with \(x \ll y\).

(3) Let \(a, x, y \in H\) be such that \(a \in x \circ y\). Then
\[
(\pi_i \circ \hat{\varphi})(x) \geq \left( \bigvee_{b \in x \circ y} (\pi_i \circ \hat{\varphi})(b) \right) \wedge (\pi_i \circ \hat{\varphi})(y) \geq (\pi_i \circ \hat{\varphi})(a) \wedge (\pi_i \circ \hat{\varphi})(y),
\]
for \(i = 1, 2, \ldots, k\). Hence \(\hat{\varphi}(x) \geq \hat{\varphi}(a) \wedge \hat{\varphi}(y)\) for all \(a, x, y \in H\) with \(a \in x \circ y\).

**Corollary 3.12.** If \(\hat{\varphi}\) is a \(k\)-polar fuzzy strong hyper BCK-ideal of a hyper BCK-algebra \(H\), then
\[
(\forall x, y \in H) \left( \hat{\varphi}(x) \geq \hat{\varphi}(y) \wedge \left( \bigwedge_{a \in x \circ y} \hat{\varphi}(a) \right) \right).
\]

**Proof.** It is straightforward by Proposition 3.11(3). \(\square\)

**Corollary 3.13.** Every \(k\)-polar fuzzy strong hyper BCK-ideal is a \(k\)-polar fuzzy hyper BCK-ideal and a \(k\)-polar fuzzy \(s\)-weak hyper BCK-ideal (and hence a \(k\)-polar fuzzy weak hyper BCK-ideal).

In general, a \(k\)-polar fuzzy (weak) hyper BCK-ideal may not be a \(k\)-polar fuzzy strong hyper BCK-ideal. In fact, the 4-polar fuzzy hyper BCK-ideal \(\hat{\varphi}\) of \(H\) in Example 3.2 is not a 4-polar fuzzy strong hyper BCK-ideal of \(H\) since
\[
(\pi_3 \circ \hat{\varphi})(b) = \frac{1}{6m-3} < \frac{1}{5m-3} = (\pi_3 \circ \hat{\varphi})(a) = (\pi_3 \circ \hat{\varphi})(a) \wedge \bigvee_{x \in \text{boa}} (\pi_3 \circ \hat{\varphi})(x).
\]

It is clear that every \(k\)-polar fuzzy hyper BCK-ideal of a hyper BCK-algebra \(H\) is a \(k\)-polar fuzzy weak hyper BCK-ideal of \(H\). But the converse is not true in general as seen in the following example.

**Example 3.14.** Let \(\mathcal{H} = \{0, a, b\}\) be a hyper BCK-algebra as in Example 3.2. Let \(\hat{\varphi}\) be a 3-polar fuzzy set on \(\mathcal{H}\) defined by
\[
\hat{\varphi} : \mathcal{H} \to [0,1]^3, x \mapsto \begin{cases} 
(5n, \frac{1}{m-3}, 0.7) & \text{if } x = 0, \\
(n, \frac{1}{5m-3}, 0.1) & \text{if } x = a, \\
(3n, \frac{1}{2m-3}, 0.5) & \text{if } x = b
\end{cases}
\]
where \(m, n \in \mathbb{N}\) and \(m \neq 3\). Then \(\hat{\varphi}\) is a 3-polar fuzzy weak hyper BCK-ideal of \(\mathcal{H}\). But it is not a 3-polar fuzzy hyper BCK-ideal of \(\mathcal{H}\) since \(a \ll b\) and \(\hat{\varphi}(a) \neq \hat{\varphi}(b)\).

By the similar way to the proofs of Theorems 3.4 and 3.6 we have a characterization of a \(k\)-polar fuzzy weak hyper BCK-ideal.
Theorem 3.15. Given a $k$-polar fuzzy set $\hat{\phi}$ in a hyper BCK-algebra $H$, the following are equivalent.

1. $\hat{\phi}$ is a $k$-polar fuzzy weak hyper BCK-ideal of $H$.

2. The $k$-polar level set $U(\hat{\phi}, \hat{t})$ is a weak hyper BCK-ideal of $H$ for all $\hat{t} \in [0,1]^k$.

Theorem 3.16. Let $\hat{\phi}$ be a $k$-polar fuzzy set in a hyper BCK-algebra $H$. If $\hat{\phi}$ is a $k$-polar fuzzy strong hyper BCK-ideal of $H$, then the $k$-polar level set $U(\hat{\phi}, \hat{t})$ is a strong hyper BCK-ideal of $H$ for all $\hat{t} \in [0,1]^k$.

Proof. Assume that $\hat{\phi}$ is a $k$-polar fuzzy strong hyper BCK-ideal of $H$ and let $\hat{t} \in [0,1]^k$ be such that the $k$-polar level set $U(\hat{\phi}, \hat{t})$ is nonempty. Then there exists $a \in U(\hat{\phi}, \hat{t})$ and so $\hat{\phi}(a) \geq \hat{t}$, that is, $(\pi_i \circ \hat{\phi})(a) \geq t_i$ for all $i = 1, 2, \ldots, k$. It is clear that $0 \in U(\hat{\phi}, \hat{t})$ by Proposition 3.11(1). Let $x, y \in H$ be such that $y \in U(\hat{\phi}, \hat{t})$ and $(x \circ y) \cap U(\hat{\phi}, \hat{t}) \neq \emptyset$. Then there exists $a_0 \in (x \circ y) \cap U(\hat{\phi}, \hat{t})$ and so $\hat{\phi}(a_0) \geq \hat{t}$, i.e., $(\pi_i \circ \hat{\phi})(a_0) \geq t_i$ for $i = 1, 2, \ldots, k$. It follows that

$$(\pi_i \circ \hat{\phi})(x) \geq \left( \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\phi})(a) \right) \wedge (\pi_i \circ \hat{\phi})(y) \geq (\pi_i \circ \hat{\phi})(a_0) \wedge (\pi_i \circ \hat{\phi})(y) \geq t_i$$

for all $i = 1, 2, \ldots, k$. Hence $x \in \bigcap_{i=1}^k U(\hat{\phi}, \hat{t}) = U(\hat{\phi}, \hat{t})$. Therefore $U(\hat{\phi}, \hat{t})$ is a strong hyper BCK-ideal of $H$. 

Theorem 3.17. Let $\hat{\phi}$ be a $k$-polar fuzzy set on a hyper BCK-algebra $H$ which satisfies the condition

$$\left( \forall T \subseteq H \right) \exists x_0 \in T \text{ s.t. } \hat{\phi}(x_0) = \bigvee_{x \in T} \hat{\phi}(x)$$

(24)

If the $k$-polar level set $U(\hat{\phi}, \hat{t})$ is a strong hyper BCK-ideal of $H$ for all $\hat{t} \in [0,1]^k$, then $\hat{\phi}$ is a $k$-polar fuzzy strong hyper BCK-ideal of $H$.

Proof. Assume that the $k$-polar level set $U(\hat{\phi}, \hat{t})$ is a strong hyper BCK-ideal of $H$ for all $\hat{t} \in [0,1]^k$. Then $x \in U(\hat{\phi}, \hat{t})$ for some $x \in H$, and so $x \circ x \ll \{x\} \subseteq U(\hat{\phi}, \hat{t})$. This implies from Lemma 3.5 that $x \circ x \subseteq U(\hat{\phi}, \hat{t})$. Hence for every $a \in x \circ x$, we get $a \in U(\hat{\phi}, \hat{t})$ and so $(\pi_i \circ \hat{\phi})(a) \geq t_i$ for all $i = 1, 2, \ldots, k$. It follows that

$$\bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\phi})(a) \geq t_i = (\pi_i \circ \hat{\phi})(x)$$

for $i = 1, 2, \ldots, k$. For any $x, y \in H$, put $\hat{d} := \left( \bigvee_{a \in x \circ y} \hat{\phi}(a) \right) \wedge \hat{\phi}(y)$, that is, $d_i := \left( \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\phi})(a) \right) \wedge (\pi_i \circ \hat{\phi})(y)$ for $i = 1, 2, \ldots, k$. Then $U(\hat{\phi}, \hat{d})$ is a strong hyper BCK-ideal of $H$ by hypothesis. The condition (24) implies that there exists $a_0 \in x \circ y$ such that $\hat{\phi}(a_0) = \bigvee_{a \in x \circ y} \hat{\phi}(a)$, i.e.,

$$(\pi_i \circ \hat{\phi})(a_0) = \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\phi})(a)$$

for $i = 1, 2, \ldots, k$. Hence

$$(\pi_i \circ \hat{\phi})(a_0) = \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\phi})(a) \geq \left( \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\phi})(a) \right) \wedge (\pi_i \circ \hat{\phi})(y) = d_i$$
for $i = 1, 2, \cdots, k$, which implies that $a_0 \in \bigcap_{i=1}^{k} U(\hat{\varphi}, \hat{d})^i = U(\hat{\varphi}, \hat{d})$. Hence $(x \circ y) \cap U(\hat{\varphi}, \hat{d}) \neq \emptyset$, and thus $x \in U(\hat{\varphi}, \hat{d})$. It follows that

$$(\pi_i \circ \hat{\varphi})(x) \geq d_i = \left( \bigvee_{a \in \pi x y} (\pi_i \circ \hat{\varphi})(a) \right) \land (\pi_i \circ \hat{\varphi})(y)$$

for $i = 1, 2, \cdots, k$. Therefore $\hat{\varphi}$ is a $k$-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$. \hfill $\square$

**Definition 3.18.** A $k$-polar fuzzy set $\hat{\varphi}$ on a hyper BCK-algebra $\mathcal{H}$ is called a $k$-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ if it satisfies:

$$\begin{align*}
(\forall x,y \in \mathcal{H}) \left( \hat{\varphi}(y) \leq \bigwedge_{a \in \pi x y} \hat{\varphi}(a) \right), & \quad (25) \\
(\forall x,y \in \mathcal{H}) \left( \hat{\varphi}(x) \geq \left( \bigvee_{a \in \pi x y} \hat{\varphi}(a) \right) \land \hat{\varphi}(y) \right), & \quad (26)
\end{align*}$$

that is, $(\pi_i \circ \hat{\varphi})(y) \leq \bigwedge_{a \in \pi x y} (\pi_i \circ \hat{\varphi})(a)$ and $(\pi_i \circ \hat{\varphi})(x) \geq \left( \bigvee_{a \in \pi x y} (\pi_i \circ \hat{\varphi})(a) \right) \land (\pi_i \circ \hat{\varphi})(y)$ for all $x, y \in \mathcal{H}$ and $i = 1, 2, \cdots, k$.

The following theorem is straightforward.

**Theorem 3.19.** Every $k$-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ is a $k$-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$.

**Theorem 3.20.** If $\hat{\varphi}$ is a $k$-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$, then the $k$-polar level set $U(\hat{\varphi}, \hat{t})$ is a reflexive hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$.

**Proof.** Assume that $\hat{\varphi}$ is a $k$-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$. Then $\hat{\varphi}$ is a $k$-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$ by Theorem 3.19, and so $\hat{\varphi}$ is a $k$-polar fuzzy hyper BCK-ideal of $\mathcal{H}$. It follows from Theorem 3.19 that the $k$-polar level set $U(\hat{\varphi}, \hat{t})$ is a hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$. Let $\hat{t} \in [0, 1]^k$ be such that $U(\hat{\varphi}, \hat{t})$ is nonempty. Then $\hat{\varphi}(c) \geq \hat{t}$ for some $c \in \mathcal{H}$. For any $x \in \mathcal{H}$, let $b \in x \circ x$. The condition (25) implies that $\hat{\varphi}(b) \geq \bigwedge_{a \in \pi x y} \hat{\varphi}(a) \geq \hat{\varphi}(c) \geq \hat{t}$, that is, $b \in U(\hat{\varphi}, \hat{t})$. Thus $x \circ x \subseteq U(\hat{\varphi}, \hat{t})$ for all $x \in \mathcal{H}$, and therefore $U(\hat{\varphi}, \hat{t})$ is a reflexive hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$. \hfill $\square$

**Lemma 3.21**. Every reflexive hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ is a strong hyper BCK-ideal of $\mathcal{H}$.

In order to consider the converse of Theorem 3.20 we need an additional condition.

**Theorem 3.22.** Let $\hat{\varphi}$ be a $k$-polar fuzzy set on a hyper BCK-algebra $\mathcal{H}$ which satisfies the condition (24). If the $k$-polar level set $U(\hat{\varphi}, \hat{t})$ is a reflexive hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$, then $\hat{\varphi}$ is a $k$-polar fuzzy reflexive hyper BCK-ideal of $\mathcal{H}$.

**Proof.** Assume that the $k$-polar level set $U(\hat{\varphi}, \hat{t})$ is a reflexive hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0, 1]^k$. Then $U(\hat{\varphi}, \hat{t})$ is a strong hyper BCK-ideal of $\mathcal{H}$ by Lemma 3.21. Using Theorem 3.17 we know that $\hat{\varphi}$ is a $k$-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$ and so (26) is valid. Let $x, y \in \mathcal{H}$
and \((\pi_i \circ \hat{\varphi})(y) = t_i\) for \(i = 1, 2, \cdots, k\). Since \(U(\hat{\varphi}, \hat{t})\) is a reflexive hyper BCK-ideal of \(\mathcal{H}\), we get \(x \circ x \subseteq U(\hat{\varphi}, \hat{t})\) and thus \(c \in U(\hat{\varphi}, \hat{t})\) for all \(c \in x \circ x\). Hence \((\pi_i \circ \hat{\varphi})(c) \geq t_i\) which implies that

\[
\bigwedge_{c \in x \circ x} (\pi_i \circ \hat{\varphi})(c) \geq t_i = (\pi_i \circ \hat{\varphi})(y)
\]

for all \(i = 1, 2, \cdots, k\). Therefore \(\hat{\varphi}\) is a \(k\)-polar fuzzy reflexive hyper BCK-ideal of \(\mathcal{H}\).

**Theorem 3.23.** Let \(\hat{\varphi}\) be a \(k\)-polar fuzzy strong hyper BCK-ideal of \(\mathcal{H}\) which satisfies the condition (24). Then \(\hat{\varphi}\) is a \(k\)-polar fuzzy reflexive hyper BCK-ideal of \(\mathcal{H}\) if and only if \(\bigwedge_{a \in \text{polar}} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(0)\) for all \(x \in \mathcal{H}\) and \(i = 1, 2, \cdots, k\).

**Proof.** Assume that \(\hat{\varphi}\) is a \(k\)-polar fuzzy strong hyper BCK-ideal of \(\mathcal{H}\) which satisfies the condition (24). The necessity is clear. Assume that \(\bigwedge_{a \in \text{polar}} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(0)\) for all \(x \in \mathcal{H}\) and \(i = 1, 2, \cdots, k\). Since \(\hat{\varphi}\) is a \(k\)-polar fuzzy hyper BCK-ideal of \(\mathcal{H}\) by Corollary 3.13, we have \((\pi_i \circ \hat{\varphi})(0) \geq (\pi_i \circ \hat{\varphi})(y)\) for all \(y \in \mathcal{H}\) and \(i = 1, 2, \cdots, k\). It follows that \(\bigwedge_{a \in \text{polar}} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(y)\) for all \(x, y \in \mathcal{H}\) and \(i = 1, 2, \cdots, k\). For any \(x, y \in \mathcal{H}\) and \(i = 1, 2, \cdots, k\), let \(t_i := (\pi_i \circ \hat{\varphi})(y) \land \left( \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right)\). The condition (24) implies that there exists \(a_0 \in x \circ y\) such that \(\hat{\varphi}(a_0) = \bigvee_{a \in x \circ y} \hat{\varphi}(a)\) and so \(\hat{\varphi}(a_0) \geq \hat{t}, i.e., a_0 \in U(\hat{\varphi}, \hat{t})\). Hence \((x \circ y) \cap U(\hat{\varphi}, \hat{t}) \neq \emptyset\). Since \(U(\hat{\varphi}, \hat{t})\) is a strong hyper BCK-ideal of \(\mathcal{H}\) by Theorem 3.16, it follows that \(x \in U(\hat{\varphi}, \hat{t})\). Hence \((\pi_i \circ \hat{\varphi})(x) \geq t_i = (\pi_i \circ \hat{\varphi})(y) \land \left( \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right)\). Therefore \(\hat{\varphi}\) is a \(k\)-polar fuzzy reflexive hyper BCK-ideal of \(\mathcal{H}\).

4 Conclusion

We have applied the \(m\)-polar fuzzy set to hyper BCK-algebra. We have introduced the notions of \(k\)-polar fuzzy hyper BCK-ideal, \(k\)-polar fuzzy weak hyper BCK-ideal, \(k\)-polar fuzzy \(s\)-weak hyper BCK-ideal, \(k\)-polar fuzzy strong hyper BCK-ideal and \(k\)-polar fuzzy reflexive hyper BCK-ideal, and have investigated related properties and their relations. We have discussed \(k\)-polar fuzzy (weak, \(s\)-weak, strong, reflexive) hyper BCK-ideal in relation to \(k\)-polar level set. In the future work, we will use the idea and results in this paper to study hyper MV-algebra, hyper hoop, hyper equality algebra, hyper BCI-algebra etc.

References


