Multipolar fuzzy hyper BCK-ideals of hyper BCK-algebras

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Abstract

In this paper, we apply $m$-polar fuzzy set to hyper BCK-algebra. We introduce the notions of $k$-polar fuzzy hyper BCK-ideal, $k$-polar fuzzy weak hyper BCK-ideal, $k$-polar fuzzy $s$-weak hyper BCK-ideal, $k$-polar fuzzy strong hyper BCK-ideal and $k$-polar fuzzy reflexive hyper BCK-ideal, and investigate related properties and their relations. We discuss $k$-polar fuzzy (weak, $s$-weak, strong, reflexive) hyper BCK-ideal in relation to $k$-polar level set.

1 Introduction

The hyper algebraic structure was introduced by F. Marty [14] in 1934. Bolurian et al. [5] was introduced hyper BCK-algebra as an extension of BCK-algebra. Since then, many scholars have been studying hyper BCK-algebra and its infrastructure and so on. In addition, research using fuzzy and soft set is actively being carried out (see [4], [7], [8], [9], [11]). In 2014, Chen et al. [6] introduced an $m$-polar fuzzy set which is an extension of bipolar fuzzy set. The $m$-polar fuzzy set applied to decision making problem (see [4]) and BCK/BCI-algebra (see [2] [8] [15]).

The notion of $m$-polar fuzzy set is applied to hyper BCK-algebra. The concepts of $k$-polar fuzzy (weak, $s$-weak, strong, reflexive) hyper BCK-ideal are introduced, and the relations and properties are investigated in relation to $k$-polar level set.

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2 Preliminaries

Let \( H \) be a nonempty set endowed with a hyperoperation “\( \circ \)”. For two subsets \( A \) and \( B \) of \( H \), denote by \( A \circ B \) the set \( \bigcup_{a \in A, b \in B} a \circ b \). We shall use \( x \circ y \) instead of \( x \circ \{y\} \), \( \{x\} \circ y \), or \( \{x\} \circ \{y\} \).

By a hyper BCK-algebra (see \cite{13}) we mean a nonempty set \( H \) endowed with a hyperoperation “\( \circ \)” and a constant \( 0 \) satisfying the following axioms:

\begin{enumerate}
  \item[(HK1)] \((x \circ z) \circ (y \circ z) \ll x \circ y , \)
  \item[(HK2)] \((x \circ y) \circ z = (x \circ z) \circ y , \)
  \item[(HK3)] \(x \circ H \ll \{x\} , \)
  \item[(HK4)] \(x \ll y \text{ and } y \ll x \text{ imply } x = y , \)
\end{enumerate}

for all \( x, y, z \in H \), where \( x \ll y \) is defined by \( 0 \in x \circ y \) and for every \( A, B \subseteq H \), \( A \ll B \) is defined by \( \forall a \in A, \exists b \in B \text{ such that } a \ll b \). In such case, we call “\( \ll \)” the hyperorder in \( H \).

Note that the condition (HK3) is equivalent to the condition:

\[(\forall x, y \in H) (x \circ y \ll \{x\}).\] (1)

A subset \( A \) of a hyper BCK-algebra \( H \) is called

\begin{itemize}
  \item a hyper BCK-ideal of \( H \) (see \cite{13}) if
    \[(\forall x \in H) (0 \in A, (\forall x, y \in H) (x \circ y \ll A, y \in A \Rightarrow x \in A) ).\] (2)
  \item a weak hyper BCK-ideal of \( H \) (see \cite{13}) if it satisfies (2) and
    \[(\forall x, y \in H) (x \circ y \subseteq A, y \in A \Rightarrow x \in A).\] (3)
  \item a strong hyper BCK-ideal of \( H \) (see \cite{12}) if it satisfies (2) and
    \[(\forall x, y \in H) ((x \circ y) \cap A \neq \emptyset, y \in A \Rightarrow x \in A).\] (4)
  \item a reflexive hyper BCK-ideal of \( H \) (see \cite{12}) if it is a hyper BCK-ideal of \( H \) which satisfies:
    \[(\forall x \in H) (x \circ x \subseteq A).\] (5)
\end{itemize}

Every hyper BCK-algebra \( H \) satisfies the following assertions.

\begin{enumerate}
  \item[(7)] \((\forall x \in H) (x \circ 0 \ll \{x\}, 0 \circ x = \{0\}, 0 \circ 0 = \{0\}),\)
  \item[(8)] \((\forall x \in H) (0 \ll x, x \ll x, x \in x \circ 0) ,\)
  \item[(9)] \((\forall x, y \in H) (x \circ 0 \ll \{y\} \Rightarrow x \ll y),\)
  \item[(10)] \((\forall x, y, z \in H) (y \ll z \Rightarrow x \circ z \ll x \circ y),\)
  \item[(11)] \((\forall x, y, z \in H) (x \circ y = \{0\} \Rightarrow x \circ z \ll y \circ z, (x \circ z) \circ (y \circ z) = \{0\}),\)
\end{enumerate}
For any subsets $A$, $B$ and $C$ of a hyper BCK-algebra $H$, the following assertions are valid.

\begin{align}
A \subseteq B & \Rightarrow A \ll B, \quad \text{(12)} \\
A \ll \{0\} & \Rightarrow A = \{0\}, \quad \text{(13)} \\
A \ll A, \ A \circ B \ll A, \ (A \circ B) \circ C = (A \circ C) \circ B, & \quad \text{(14)} \\
A \circ \{0\} = \{0\} & \Rightarrow A = \{0\}. \quad \text{(15)}
\end{align}

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

\[
\bigvee \{a_i \mid i \in \Lambda\} := \begin{cases} 
\max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite}, \\
\sup\{a_i \mid i \in \Lambda\} & \text{otherwise}.
\end{cases}
\]

\[
\bigwedge \{a_i \mid i \in \Lambda\} := \begin{cases} 
\min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite}, \\
\inf\{a_i \mid i \in \Lambda\} & \text{otherwise}.
\end{cases}
\]

If $\Lambda = \{1, 2\}$, then we will also use $a_1 \lor a_2$ and $a_1 \land a_2$ instead of $\bigvee \{a_i \mid i \in \Lambda\}$ and $\bigwedge \{a_i \mid i \in \Lambda\}$, respectively.

By a $k$-polar fuzzy set on a universe $H$ (see [9]), we mean a function $\hat{\varphi} : H \rightarrow [0, 1]^k$. The membership value of every element $x \in H$ is denoted by

\[
\hat{\varphi}(x) = (\pi_1 \circ \hat{\varphi}(x), \pi_2 \circ \hat{\varphi}(x), \ldots, \pi_k \circ \hat{\varphi}(x)),
\]

where $\pi_i : [0, 1]^k \rightarrow [0, 1]$ is the $i$-th projection for all $i = 1, 2, \ldots, k$.

Given a $k$-polar fuzzy set on a universe $H$, we consider the set

\[
U(\hat{\varphi}, \hat{t}) := \{x \in H \mid \hat{\varphi}(x) \geq \hat{t}\},
\]

where $\hat{t} = (t_1, t_2, \ldots, t_k) \in [0, 1]^k$, that is,

\[
U(\hat{\varphi}, \hat{t}) := \{x \in H \mid (\pi_i \circ \hat{\varphi})(x) \geq t_i \text{ for all } i = 1, 2, \ldots, k\}
\]

which is called a $k$-polar level set of $\hat{\varphi}$. It is clear that $U(\hat{\varphi}, \hat{t}) = \bigcap_{i=1}^k U(\hat{\varphi}, \hat{t})^i$ where

\[
U(\hat{\varphi}, \hat{t})^i = \{x \in H \mid (\pi_i \circ \hat{\varphi})(x) \geq t_i\}.
\]

### 3 $k$-polar fuzzy hyper BCK-ideals

**Definition 3.1.** A $k$-polar fuzzy set $\hat{\varphi}$ on a hyper BCK-algebra $H$ is called a $k$-polar fuzzy hyper BCK-ideal of $H$ if it satisfies

\begin{align}
(\forall x, y \in H) \ (x \ll y & \Rightarrow \hat{\varphi}(x) \geq \hat{\varphi}(y)), \quad \text{(18)} \\
(\forall x, y \in H) \ (\hat{\varphi}(x) \geq \left(\bigwedge \{\hat{\varphi}(a) \mid a \in x \circ y\}\right) & \wedge \hat{\varphi}(y)), \quad \text{(19)}
\end{align}

that is, $(\pi_i \circ \hat{\varphi})(x) \geq (\pi_i \circ \hat{\varphi})(y)$ for all $x, y \in H$ with $x \ll y$ and

\[
(\pi_i \circ \hat{\varphi})(x) \geq \left(\bigwedge \{(\pi_i \circ \hat{\varphi})(a) \mid a \in x \circ y\}\right) \wedge (\pi_i \circ \hat{\varphi})(y)
\]

for all $x, y \in H$ and $i = 1, 2, \ldots, k$.

**Example 3.2.** Let $H = \{0, a, b\}$ be a set with the hyperoperation “$\circ$” in the following Cayley table
<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>$a$</td>
<td>${a}$</td>
<td>${0,a}$</td>
<td>${0,a}$</td>
</tr>
<tr>
<td>$b$</td>
<td>${b}$</td>
<td>${a,b}$</td>
<td>${0,a,b}$</td>
</tr>
</tbody>
</table>

Then $\mathcal{H}$ is a hyper BCK-algebra (see [13]). Let $\hat{\phi}$ be a 4-polar fuzzy set on $\mathcal{H}$ defined by

$$
\hat{\phi} : \mathcal{H} \rightarrow [0,1]^4, \ x \mapsto \begin{cases} 
\left( \frac{1}{n},0.9, \frac{1}{m-3},0.7 \right) & \text{if } x = 0, \\
\left( \frac{1}{2n},0.5, \frac{1}{2m-3},0.7 \right) & \text{if } x = a, \\
\left( \frac{1}{3n},0.2, \frac{1}{3m-3},0.4 \right) & \text{if } x = b 
\end{cases}
$$

where $m, n \in \mathbb{N}$ and $m \neq 3$. It is toutine to verify that $\hat{\phi}$ is a 4-polar fuzzy hyper BCK-ideal of $\mathcal{H}$.

**Proposition 3.3.** If $\hat{\phi}$ is a $k$-polar fuzzy hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$, then

(1) $(\forall x \in \mathcal{H})(\hat{\phi}(0) \geq \hat{\phi}(x))$, that is, $(\pi_i \circ \hat{\phi})(0) \geq (\pi_i \circ \hat{\phi})(x)$ for all $x \in \mathcal{H}$ and $i = 1,2,\cdots,k$.

(2) If $\hat{\phi}$ satisfies the condition

$$(\forall T \subseteq \mathcal{H}) \left( \exists x_0 \in T \text{ s.t. } \hat{\phi}(x_0) = \bigwedge_{x \in T} \hat{\phi}(x) \right),$$

then

$$(\forall x,y \in \mathcal{H}) \left( \exists a \in x \circ y \text{ s.t. } \hat{\phi}(x) \geq \hat{\phi}(a) \land \hat{\phi}(y) \right),$$

that is, for every $x,y \in \mathcal{H}$ there exists $a \in x \circ y$ such that

$$(\pi_i \circ \hat{\phi})(x) \geq (\pi_i \circ \hat{\phi})(a) \land (\pi_i \circ \hat{\phi})(y)$$

for $i = 1,2,\cdots,k$.

**Proof.** (1) Since $0 \ll x$ for all $x \in \mathcal{H}$, it follows from [13] that $\hat{\phi}(0) \geq \hat{\phi}(x)$ for all $x \in \mathcal{H}$.

(2) Assume that $\hat{\phi}$ satisfies the condition (21). For any $x,y \in \mathcal{H}$, there exists $a_0 \in x \circ y$ such that $(\pi_i \circ \hat{\phi})(a_0) = \bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\phi})(a)$. It follows from (20) that

$$(\pi_i \circ \hat{\phi})(x) \geq \left( \bigwedge \{(\pi_i \circ \hat{\phi})(a) \mid a \in x \circ y \} \right) \land (\pi_i \circ \hat{\phi})(y) = (\pi_i \circ \hat{\phi})(a_0) \land (\pi_i \circ \hat{\phi})(y)$$

for $i = 1,2,\cdots,k$ which proves (2). \hfill \Box

**Theorem 3.4.** Let $\hat{\phi}$ be a $k$-polar fuzzy set in a hyper BCK-algebra $\mathcal{H}$. If $\hat{\phi}$ is a $k$-polar fuzzy hyper BCK-ideal of $\mathcal{H}$, then the $k$-polar level set $U(\hat{\phi}, \hat{t})$ is a hyper BCK-ideal of $\mathcal{H}$ for all $\hat{t} \in [0,1]^k$.

**Proof.** Assume that $\hat{\phi}$ is a $k$-polar fuzzy hyper BCK-ideal of $\mathcal{H}$ and let $\hat{t} \in [0,1]^k$. It is clear that $0 \in U(\hat{\phi}, \hat{t})$. Let $x, y \in \mathcal{H}$ be such that $x \circ y \ll U(\hat{\phi}, \hat{t})$ and $y \in U(\hat{\phi}, \hat{t})$. Then $x \circ y \ll U(\hat{\phi}, \hat{t})^i$ and $y \in U(\hat{\phi}, \hat{t})^i$ for all $i = 1,2,\cdots,k$. It follows that

$$(\forall a \in x \circ y) \left( \exists a_0 \in U(\hat{\phi}, \hat{t})^i \text{ s.t. } a \ll a_0 \text{ and so } (\pi_i \circ \hat{\phi})(a) \geq (\pi_i \circ \hat{\phi})(a_0) \right),$$
which implies that \((\pi_i \circ \hat{\varphi})(a) \geq t_i\) for all \(a \in x \circ y\). Hence \(\bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \geq t_i\), and so
\[
(\pi_i \circ \hat{\varphi})(x) \geq \left(\bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \mid a \in x \circ y\right) \land (\pi_i \circ \hat{\varphi})(y) \geq t_i
\]
for all \(i = 1, 2, \cdots, k\). Thus \(x \in \bigcap_{i=1}^k U(\hat{\varphi}, \hat{t})^i = U(\hat{\varphi}, \hat{t})\), and therefore \(U(\hat{\varphi}, \hat{t})\) is a hyper BCK-ideal of \(H\) for all \(\hat{t} \in [0, 1]^k\). 

In order to consider the converse of Theorem 3.4, we need the following lemma.

**Lemma 3.5** ([10]). Let \(A\) be a subset of a hyper BCK-algebra \(H\). If \(K\) is a hyper BCK-ideal of \(H\) such that \(A \lessdot K\), then \(A\) is contained in \(K\).

**Theorem 3.6.** Let \(\hat{\varphi}\) be a k-polar fuzzy set in a hyper BCK-algebra \(H\) in which the k-polar level set \(U(\hat{\varphi}, \hat{t})\) is a hyper BCK-ideal of \(H\) for all \(\hat{t} \in [0, 1]^k\). Then \(\hat{\varphi}\) is a k-polar fuzzy hyper BCK-ideal of \(H\).

**Proof.** Suppose that the k-polar level set \(U(\hat{\varphi}, \hat{t})\) is a hyper BCK-ideal of \(H\) for all \(\hat{t} \in [0, 1]^k\). Let \(x, y \in H\) be such that \(x \lessdot y\) and \(\hat{\varphi}(y) = \hat{t}\). Then \(y \in U(\hat{\varphi}, \hat{t})\) and so \(\{x\} \lessdot U(\hat{\varphi}, \hat{t})\). It follows from Lemma 3.5 that \(\{x\} \subseteq U(\hat{\varphi}, \hat{t})\), i.e., \(x \in U(\hat{\varphi}, \hat{t})\). Hence \(\hat{\varphi}(x) \geq \hat{t} = \hat{\varphi}(y)\). For any \(x, y \in H\), let \(\hat{t} := \left(\bigwedge_{a \in x \circ y} \hat{\varphi}(a)\right) \land \hat{\varphi}(y)\). Then \(y \in U(\hat{\varphi}, \hat{t})\) and
\[
\hat{\varphi}(c) \geq \bigwedge_{a \in x \circ y} \hat{\varphi}(a) \geq \left(\bigwedge_{a \in x \circ y} \hat{\varphi}(a)\right) \land \hat{\varphi}(y) = \hat{t}
\]
for all \(c \in x \circ y\), i.e., \(c \in U(\hat{\varphi}, \hat{t})\). Thus \(x \circ y \subseteq U(\hat{\varphi}, \hat{t})\) and so \(x \circ y \lessdot U(\hat{\varphi}, \hat{t})\) by (12). Since \(y \in U(\hat{\varphi}, \hat{t})\) and \(U(\hat{\varphi}, \hat{t})\) is a hyper BCK-ideal of \(H\), we have \(x \in U(\hat{\varphi}, \hat{t})\) which implies that \(\hat{\varphi}(x) \geq \hat{t} = \left(\bigwedge_{a \in x \circ y} \hat{\varphi}(a)\right) \land \hat{\varphi}(y)\). Therefore \(\hat{\varphi}\) is a k-polar fuzzy hyper BCK-ideal of \(H\). 

**Definition 3.7.** A k-polar fuzzy set \(\hat{\varphi}\) on a hyper BCK-algebra \(H\) is called a

- k-polar fuzzy weak hyper BCK-ideal of \(H\) if it satisfies Proposition 3.3(1) and (19).
- k-polar fuzzy s-weak hyper BCK-ideal of \(H\) if it satisfies Proposition 3.3(1) and (22).
- k-polar fuzzy strong hyper BCK-ideal of \(H\) if it satisfies
\[
(\forall x, y \in H) \left(\bigwedge_{a \in x \circ y} \hat{\varphi}(a) \geq \hat{\varphi}(x) \geq \left(\bigvee_{b \in x \circ y} \hat{\varphi}(b)\right) \land \hat{\varphi}(y)\right),
\]
that is, \(\bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(x) \geq \left(\bigvee_{b \in x \circ y} (\pi_i \circ \hat{\varphi})(b)\right) \land (\pi_i \circ \hat{\varphi})(y)\) for all \(x, y \in H\) and \(i = 1, 2, \cdots, k\).

**Example 3.8.** Let \(H = \{0, a, b\}\) be a set with the hyperoperation “\(\circ\)” in the following Cayley table

\[
\begin{array}{ccc}
0 & a & b \\
a & a & b \\
b & b & 0 \\
\end{array}
\]
For any

\begin{center}
\begin{array}{c|ccc}
\circ & 0 & a & b \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
a & \{a\} & \{0\} & \{a\} \\
b & \{b\} & \{b\} & \{0, b\} \\
\end{array}
\end{center}

Then \( H \) is a hyper BCK-algebra (see [13]). Let \( \hat{\varphi} \) be a 4-polar fuzzy set on \( H \) defined by

\[
\hat{\varphi} : H \to [0, 1]^4, \ x \mapsto \begin{cases} 
(2\pi, \mu(x), \frac{1}{m+3}, 0.8) & \text{if } x = 0, \\
(\pi, \mu(2x), \frac{1}{m+5}, 0.7) & \text{if } x = a, \\
(\frac{1}{2}\pi, \mu(3x), \frac{1}{m+7}, 0.4) & \text{if } x = b
\end{cases}
\]

where \( m, n \in \mathbb{N} \) and \( \mu : [0, 1] \to [0, 1], \ x \mapsto \frac{1}{\pi} \). It is toutine to verify that \( \hat{\varphi} \) is a 4-polar fuzzy strong hyper BCK-ideal of \( H \).

The following theorem describes the relation between \( k \)-polar fuzzy weak hyper BCK-ideal and \( k \)-polar fuzzy \( s \)-weak hyper BCK-ideal.

**Theorem 3.9.** In a hyper BCK-algebra, every \( k \)-polar fuzzy \( s \)-weak hyper BCK-ideal is a \( k \)-polar fuzzy weak hyper BCK-ideal.

**Proof.** Let \( \hat{\varphi} \) be a \( k \)-polar fuzzy \( s \)-weak hyper BCK-ideal of a hyper BCK-algebra \( H \) and let \( x, y \in H \). Then there exists \( a \in x \circ y \) such that \( \hat{\varphi}(x) \geq \hat{\varphi}(a) \wedge \hat{\varphi}(y) \) by [22]. Since \( \hat{\varphi}(a) \geq \bigwedge_{b \in x \circ y} \hat{\varphi}(b) \), it follows that

\[
\hat{\varphi}(x) \geq \left( \bigwedge_{b \in x \circ y} \hat{\varphi}(b) \right) \wedge \hat{\varphi}(y).
\]

Therefore \( \hat{\varphi} \) is a \( k \)-polar fuzzy weak hyper BCK-ideal of \( H \).

**Theorem 3.10.** Let \( \hat{\varphi} \) be a \( k \)-polar fuzzy weak hyper BCK-ideal of a hyper BCK-algebra \( H \) which satisfies the condition [21]. Then \( \hat{\varphi} \) is a \( k \)-polar fuzzy \( s \)-weak hyper BCK-ideal of \( H \).

**Proof.** For any \( x, y \in H \), there exists \( a_0 \in x \circ y \) such that \( \hat{\varphi}(a_0) = \bigwedge_{a \in x \circ y} \hat{\varphi}(a) \), that is, \( (\pi_i \circ \hat{\varphi})(a_0) = \bigwedge_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \) by [21]. It follows that

\[
(\pi_i \circ \hat{\varphi})(x) \geq \left( \bigwedge \{(\pi_i \circ \hat{\varphi})(a) \mid a \in x \circ y\} \right) \wedge (\pi_i \circ \hat{\varphi})(y) = (\pi_i \circ \hat{\varphi})(a_0) \wedge (\pi_i \circ \hat{\varphi})(y).
\]

Therefore \( \hat{\varphi} \) is a \( k \)-polar fuzzy \( s \)-weak hyper BCK-ideal of \( H \).

**Proposition 3.11.** Every \( k \)-polar fuzzy strong hyper BCK-ideal \( \hat{\varphi} \) of a hyper BCK-algebra \( H \) satisfies the following assertions.

1. \((\forall x \in H)(\hat{\varphi}(0) \geq \hat{\varphi}(x))\), that is, \((\pi_i \circ \hat{\varphi})(0) \geq (\pi_i \circ \hat{\varphi})(x)\) for all \( x \in H \) and \( i = 1, 2, \cdots, k \).
2. \((\forall x, y \in H)(x \ll y \Rightarrow \hat{\varphi}(x) \geq \hat{\varphi}(y))\), that is, \((\pi_i \circ \hat{\varphi})(x) \geq (\pi_i \circ \hat{\varphi})(y)\) for all \( x, y \in H \) with \( x \ll y \) and \( i = 1, 2, \cdots, k \).
3. \((\forall a, x, y \in H)(a \in x \circ y \Rightarrow \hat{\varphi}(x) \geq \hat{\varphi}(a) \wedge \hat{\varphi}(y))\).
Proof. (1) Since $0 \in x \circ x$ for all $x \in \mathcal{H}$, we get

$$\hat{\phi}(0) \geq \bigwedge_{a \in x \circ x} \hat{\phi}(a) \geq \hat{\phi}(x)$$

for all $x \in \mathcal{H}$.

(2) Let $x, y \in \mathcal{H}$ be such that $x \ll y$. Then $0 \in x \circ y$ and thus $\bigvee_{b \in x \circ y} (\pi_i \circ \hat{\phi})(b) \geq (\pi_i \circ \hat{\phi})(0)$ for $i = 1, 2, \cdots, k$. It follows from (1) that

$$(\pi_i \circ \hat{\phi})(x) \geq \left( \bigvee_{b \in x \circ y} (\pi_i \circ \hat{\phi})(b) \right) \land (\pi_i \circ \hat{\phi})(y) \geq (\pi_i \circ \hat{\phi})(0) \land (\pi_i \circ \hat{\phi})(y) = (\pi_i \circ \hat{\phi})(y),$$

for $i = 1, 2, \cdots, k$, that is, $\hat{\phi}(x) \geq \hat{\phi}(y)$ for all $x, y \in \mathcal{H}$ with $x \ll y$.

(3) Let $a, x, y \in \mathcal{H}$ be such that $a \in x \circ y$. Then

$$(\pi_i \circ \hat{\phi})(x) \geq \left( \bigvee_{b \in x \circ y} (\pi_i \circ \hat{\phi})(b) \right) \land (\pi_i \circ \hat{\phi})(y) \geq (\pi_i \circ \hat{\phi})(a) \land (\pi_i \circ \hat{\phi})(y),$$

for $i = 1, 2, \cdots, k$. Hence $\hat{\phi}(x) \geq \hat{\phi}(a) \land \hat{\phi}(y)$ for all $a, x, y \in \mathcal{H}$ with $a \in x \circ y$. \qed

Corollary 3.12. If $\hat{\phi}$ is a $k$-polar fuzzy strong hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$, then

$$(\forall x, y \in \mathcal{H}) \left( \hat{\phi}(x) \geq \hat{\phi}(y) \land \left( \bigwedge_{a \in x \circ y} \hat{\phi}(a) \right) \right).$$

Proof. It is straightforward by Proposition 3.11(3). \qed

Corollary 3.13. Every $k$-polar fuzzy strong hyper BCK-ideal is a $k$-polar fuzzy hyper BCK-ideal and a $k$-polar fuzzy $s$-weak hyper BCK-ideal (and hence a $k$-polar fuzzy weak hyper BCK-ideal).

In general, a $k$-polar fuzzy (weak) hyper BCK-ideal may not be a $k$-polar fuzzy strong hyper BCK-ideal. In fact, the 4-polar fuzzy hyper BCK-ideal $\hat{\phi}$ of $\mathcal{H}$ in Example 3.2 is not a 4-polar fuzzy strong hyper BCK-ideal of $\mathcal{H}$ since

$$(\pi_3 \circ \hat{\phi})(b) = \frac{1}{3m-3} < \frac{1}{2m-3} = (\pi_3 \circ \hat{\phi})(a) = (\pi_3 \circ \hat{\phi})(a) \land \bigvee_{x \in \text{boa}} (\pi_3 \circ \hat{\phi})(x).$$

It is clear that every $k$-polar fuzzy hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$ is a $k$-polar fuzzy weak hyper BCK-ideal of $\mathcal{H}$. But the converse is not true in general as seen in the following example.

Example 3.14. Let $\mathcal{H} = \{0, a, b\}$ be a hyper BCK-algebra as in Example 3.2. Let $\hat{\phi}$ be a 3-polar fuzzy set on $\mathcal{H}$ defined by

$$\hat{\phi} : \mathcal{H} \to [0, 1]^3, \ x \mapsto \begin{cases} (5n, \frac{1}{m-3}, 0.7) & \text{if } x = 0, \\ (n, \frac{1}{3m-3}, 0.1) & \text{if } x = a, \\ (3n, \frac{1}{2m-3}, 0.5) & \text{if } x = b \end{cases}$$

where $m, n \in \mathbb{N}$ and $m \neq 3$. Then $\hat{\phi}$ is a 3-polar fuzzy weak hyper BCK-ideal of $\mathcal{H}$. But it is not a 3-polar fuzzy hyper BCK-ideal of $\mathcal{H}$ since $a \ll b$ and $\hat{\phi}(a) \nleq \hat{\phi}(b)$. 
Then

Proof. Assume that \( \hat{\varphi} \) is a \( k \)-polar fuzzy strong hyper BCK-ideal of \( \mathcal{H} \) and let \( \hat{t} \in [0,1]^k \) be such that the \( k \)-polar level set \( U(\hat{\varphi}, \hat{t}) \) is nonempty. Then there exists \( a \in U(\hat{\varphi}, \hat{t}) \) and so \( \hat{\varphi}(a) > \hat{t} \), that is, \( (\pi_i \circ \hat{\varphi})(a) \geq t_i \) for all \( i = 1, 2, \ldots, k \). It is clear that \( 0 \in U(\hat{\varphi}, \hat{t}) \) by Proposition 3.11(1). Let \( x, y \in \mathcal{H} \) be such that \( y \in U(\hat{\varphi}, \hat{t}) \) and \( (x \circ y) \cap U(\hat{\varphi}, \hat{t}) \neq \emptyset \). Then there exists \( a_0 \in (x \circ y) \cap U(\hat{\varphi}, \hat{t}) \) and so \( \hat{\varphi}(a_0) \geq \hat{t} \), i.e., \( (\pi_i \circ \hat{\varphi})(a_0) \geq t_i \) for \( i = 1, 2, \ldots, k \). It follows that

\[
(\pi_i \circ \hat{\varphi})(x) \geq \left( \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right) \land (\pi_i \circ \hat{\varphi})(y) \geq (\pi_i \circ \hat{\varphi})(a_0) \land (\pi_i \circ \hat{\varphi})(y) \geq t_i
\]

for all \( i = 1, 2, \ldots, k \). Hence \( x \in \bigcap_{i=1}^k U(\hat{\varphi}, \hat{t}) = U(\hat{\varphi}, \hat{t}) \). Therefore \( U(\hat{\varphi}, \hat{t}) \) is a strong hyper BCK-ideal of \( \mathcal{H} \).

**Theorem 3.17.** Let \( \hat{\varphi} \) be a \( k \)-polar fuzzy set on a hyper BCK-algebra \( \mathcal{H} \) which satisfies the condition

\[
(\forall T \subseteq \mathcal{H}) \left( \exists x_0 \in T \text{ s.t. } \hat{\varphi}(x_0) = \bigvee_{x \in T} \hat{\varphi}(x) \right). \tag{24}
\]

If the \( k \)-polar level set \( U(\hat{\varphi}, \hat{t}) \) is a strong hyper BCK-ideal of \( \mathcal{H} \) for all \( \hat{t} \in [0,1]^k \), then \( \hat{\varphi} \) is a \( k \)-polar fuzzy strong hyper BCK-ideal of \( \mathcal{H} \).

Proof. Assume that the \( k \)-polar level set \( U(\hat{\varphi}, \hat{t}) \) is a strong hyper BCK-ideal of \( \mathcal{H} \) for all \( \hat{t} \in [0,1]^k \). Then \( x \in U(\hat{\varphi}, \hat{t}) \) for some \( x \in \mathcal{H} \), and so \( x \circ x \ll \{x\} \subseteq U(\hat{\varphi}, \hat{t}) \). This implies from Lemma 3.5 that \( x \circ x \subseteq U(\hat{\varphi}, \hat{t}) \). Hence for every \( a \in x \circ x \), we get \( a \in U(\hat{\varphi}, \hat{t}) \) and so \( (\pi_i \circ \hat{\varphi})(a) \geq t_i \) for all \( i = 1, 2, \ldots, k \). It follows that

\[
\bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\varphi})(a) \geq t_i = (\pi_i \circ \hat{\varphi})(x)
\]

for \( i = 1, 2, \ldots, k \). For any \( x, y \in \mathcal{H} \), put \( \hat{d} := \left( \bigvee_{a \in x \circ y} \hat{\varphi}(a) \right) \land \hat{\varphi}(y) \), that is, \( d_i := \left( \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right) \land (\pi_i \circ \hat{\varphi})(y) \) for \( i = 1, 2, \ldots, k \). Then \( U(\hat{\varphi}, \hat{d}) \) is a strong hyper BCK-ideal of \( \mathcal{H} \) by hypothesis. The condition (24) implies that there exists \( a_0 \in x \circ y \) such that \( \hat{\varphi}(a_0) = \bigvee_{a \in x \circ y} \hat{\varphi}(a) \), i.e.,

\[
(\pi_i \circ \hat{\varphi})(a_0) = \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \text{ for } i = 1, 2, \ldots, k.
\]

Hence

\[
(\pi_i \circ \hat{\varphi})(a_0) = \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \geq \left( \bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a) \right) \land (\pi_i \circ \hat{\varphi})(y) = d_i
\]
for \(i = 1, 2, \ldots, k\), which implies that \(a_0 \in \bigcap_{i=1}^{k} U(\hat{\varphi}, \hat{d})^i = U(\hat{\varphi}, \hat{d})\). Hence \((x \circ y) \cap U(\hat{\varphi}, \hat{d}) \neq \emptyset\), and thus \(x \in U(\hat{\varphi}, \hat{d})\). It follows that

\[
(\pi_i \circ \hat{\varphi})(x) \geq d_i = \left( \bigvee_{a \in \text{ex o y}} (\pi_i \circ \hat{\varphi})(a) \right) \wedge (\pi_i \circ \hat{\varphi})(y)
\]

for \(i = 1, 2, \ldots, k\). Therefore \(\hat{\varphi}\) is a \(k\)-polar fuzzy strong hyper BCK-ideal of \(\mathcal{H}\). \[\square\]

**Definition 3.18.** A \(k\)-polar fuzzy set \(\hat{\varphi}\) on a hyper BCK-algebra \(\mathcal{H}\) is called a \(k\)-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra \(\mathcal{H}\) if it satisfies:

\[
(\forall x, y \in \mathcal{H}) \left( \hat{\varphi}(y) \leq \bigwedge_{a \in \text{ex o x}} \hat{\varphi}(a) \right), \tag{25}
\]

\[
(\forall x, y \in \mathcal{H}) \left( \hat{\varphi}(x) \geq \left( \bigvee_{a \in \text{ex o y}} \hat{\varphi}(a) \right) \wedge \hat{\varphi}(y) \right), \tag{26}
\]

that is, \((\pi_i \circ \hat{\varphi})(y) \leq \bigwedge_{a \in \text{ex o x}} (\pi_i \circ \hat{\varphi})(a)\) and \((\pi_i \circ \hat{\varphi})(x) \geq \left( \bigvee_{a \in \text{ex o y}} (\pi_i \circ \hat{\varphi})(a) \right) \wedge (\pi_i \circ \hat{\varphi})(y)\) for all \(x, y \in \mathcal{H}\) and \(i = 1, 2, \ldots, k\).

The following theorem is straightforward.

**Theorem 3.19.** Every \(k\)-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra \(\mathcal{H}\) is a \(k\)-polar fuzzy strong hyper BCK-ideal of \(\mathcal{H}\).

**Theorem 3.20.** If \(\hat{\varphi}\) is a \(k\)-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra \(\mathcal{H}\), then the \(k\)-polar level set \(U(\hat{\varphi}, \hat{t})\) is a reflexive hyper BCK-ideal of \(\mathcal{H}\) for all \(\hat{t} \in [0,1]^k\).

**Proof.** Assume that \(\hat{\varphi}\) is a \(k\)-polar fuzzy reflexive hyper BCK-ideal of a hyper BCK-algebra \(\mathcal{H}\). Then \(\hat{\varphi}\) is a \(k\)-polar fuzzy strong hyper BCK-ideal of \(\mathcal{H}\) by Theorem 3.19, and so \(\hat{\varphi}\) is a \(k\)-polar fuzzy hyper BCK-ideal of \(\mathcal{H}\). It follows from Theorem 3.4 that the \(k\)-polar level set \(U(\hat{\varphi}, \hat{t})\) is a hyper BCK-ideal of \(\mathcal{H}\) for all \(\hat{t} \in [0,1]^k\). Let \(\hat{t} \in [0,1]^k\) be such that \(U(\hat{\varphi}, \hat{t})\) is nonempty. Then \(\hat{\varphi}(c) \geq \hat{t}\) for some \(c \in \mathcal{H}\). For any \(x \in \mathcal{H}\), let \(b \in x \circ x\). The condition (25) implies that \(\hat{\varphi}(b) \geq \bigwedge_{a \in \text{ex o x}} \hat{\varphi}(a) \geq \hat{\varphi}(c) \geq \hat{t}\), that is, \(b \in U(\hat{\varphi}, \hat{t})\). Thus \(x \circ x \subseteq U(\hat{\varphi}, \hat{t})\) for all \(x \in \mathcal{H}\), and therefore \(U(\hat{\varphi}, \hat{t})\) is a reflexive hyper BCK-ideal of \(\mathcal{H}\) for all \(\hat{t} \in [0,1]^k\). \[\square\]

**Lemma 3.21.** (12) Every reflexive hyper BCK-ideal of a hyper BCK-algebra \(\mathcal{H}\) is a strong hyper BCK-ideal of \(\mathcal{H}\).

In order to consider the converse of Theorem 3.20, we need an additional condition.

**Theorem 3.22.** Let \(\hat{\varphi}\) be a \(k\)-polar fuzzy set on a hyper BCK-algebra \(\mathcal{H}\) which satisfies the condition (24). If the \(k\)-polar level set \(U(\hat{\varphi}, \hat{t})\) is a reflexive hyper BCK-ideal of \(\mathcal{H}\) for all \(\hat{t} \in [0,1]^k\), then \(\hat{\varphi}\) is a \(k\)-polar fuzzy reflexive hyper BCK-ideal of \(\mathcal{H}\).

**Proof.** Assume that the \(k\)-polar level set \(U(\hat{\varphi}, \hat{t})\) is a reflexive hyper BCK-ideal of \(\mathcal{H}\) for all \(\hat{t} \in [0,1]^k\). Then \(U(\hat{\varphi}, \hat{t})\) is a strong hyper BCK-ideal of \(\mathcal{H}\) by Lemma 3.21. Using Theorem 3.17, we know that \(\hat{\varphi}\) is a \(k\)-polar fuzzy strong hyper BCK-ideal of \(\mathcal{H}\) and so (26) is valid. Let \(x, y \in \mathcal{H}\)
and \((\pi_i \circ \hat{\varphi})(y) = t_i\) for \(i = 1, 2, \ldots, k\). Since \(U(\hat{\varphi}, \hat{t})\) is a reflexive hyper BCK-ideal of \(\mathcal{H}\), we get \(x \circ x \subseteq U(\hat{\varphi}, \hat{t})\) and thus \(c \in U(\hat{\varphi}, \hat{t})\) for all \(c \in x \circ x\). Hence \((\pi_i \circ \hat{\varphi})(c) \geq t_i\) which implies that

\[
\bigwedge_{c \in x \circ x} (\pi_i \circ \hat{\varphi})(c) \geq t_i = (\pi_i \circ \hat{\varphi})(y)
\]

for all \(i = 1, 2, \ldots, k\). Therefore \(\hat{\varphi}\) is a \(k\)-polar fuzzy reflexive hyper BCK-ideal of \(\mathcal{H}\).

**Theorem 3.23.** Let \(\hat{\varphi}\) be a \(k\)-polar fuzzy strong hyper BCK-ideal of \(\mathcal{H}\) which satisfies the condition (24). Then \(\hat{\varphi}\) is a \(k\)-polar fuzzy reflexive hyper BCK-ideal of \(\mathcal{H}\) if and only if \(\bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(0)\) for all \(x \in \mathcal{H}\) and \(i = 1, 2, \ldots, k\).

**Proof.** Assume that \(\hat{\varphi}\) is a \(k\)-polar fuzzy strong hyper BCK-ideal of \(\mathcal{H}\) which satisfies the condition (24). The necessity is clear. Assume that \(\bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(0)\) for all \(x \in \mathcal{H}\) and \(i = 1, 2, \ldots, k\). Since \(\hat{\varphi}\) is a \(k\)-polar fuzzy hyper BCK-ideal of \(\mathcal{H}\) by Corollary 3.13, we have \((\pi_i \circ \hat{\varphi})(0) \geq (\pi_i \circ \hat{\varphi})(y)\) for all \(y \in \mathcal{H}\) and \(i = 1, 2, \ldots, k\). It follows that \(\bigwedge_{a \in x \circ x} (\pi_i \circ \hat{\varphi})(a) \geq (\pi_i \circ \hat{\varphi})(y)\) for all \(x, y \in \mathcal{H}\) and \(i = 1, 2, \ldots, k\). For any \(x, y \in \mathcal{H}\) and \(i = 1, 2, \ldots, k\), let \(t_i := (\pi_i \circ \hat{\varphi})(y) \land \left(\bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a)\right)\). The condition (24) implies that there exists \(a_0 \in x \circ y\) such that \(\hat{\varphi}(a_0) = \bigvee_{a \in x \circ y} \hat{\varphi}(a)\) and so \(\hat{\varphi}(a_0) \geq \hat{t}\), i.e., \(a_0 \in U(\hat{\varphi}, \hat{t})\). Hence \((x \circ y) \cap U(\hat{\varphi}, \hat{t}) \neq \emptyset\). Since \(U(\hat{\varphi}, \hat{t})\) is a strong hyper BCK-ideal of \(\mathcal{H}\) by Theorem 3.16, it follows that \(x \in U(\hat{\varphi}, \hat{t})\). Hence \((\pi_i \circ \hat{\varphi})(x) \geq t_i = (\pi_i \circ \hat{\varphi})(y) \land \left(\bigvee_{a \in x \circ y} (\pi_i \circ \hat{\varphi})(a)\right)\). Therefore \(\hat{\varphi}\) is a \(k\)-polar fuzzy reflexive hyper BCK-ideal of \(\mathcal{H}\).

### 4 Conclusion

We have applied the \(m\)-polar fuzzy set to hyper BCK-algebra. We have introduced the notions of \(k\)-polar fuzzy hyper BCK-ideal, \(k\)-polar fuzzy weak hyper BCK-ideal, \(k\)-polar fuzzy \(s\)-weak hyper BCK-ideal, \(k\)-polar fuzzy strong hyper BCK-ideal and \(k\)-polar fuzzy reflexive hyper BCK-ideal, and have investigated related properties and their relations. We have discussed \(k\)-polar fuzzy (weak, \(s\)-weak, strong, reflexive) hyper BCK-ideal in relation to \(k\)-polar level set. In the future work, we will use the idea and results in this paper to study hyper MV-algebra, hyper hoop, hyper equality algebra, hyper BCI-algebra etc.

### References


