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What are pseudo EMV-algebras?

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Abstract

In the paper, we present EMV-algebras as a common generalization of MV-algebras and generalized Boolean algebras where a top element is not assumed a priori. In addition, we present a non-commutative generalization of EMV-algebras, pseudo MV-algebras and of generalized Boolean algebras. We present main representation results showing a very close connection of pseudo EMV-algebra with pseudo MV-algebras, and we give a categorical representation of the category o pseudo EMV-algebras without top element. We study also states as analogs of finitely additive states, their topological properties, and we present an integral representation of states by σ-additive probability measures. The paper is a survey over papers [13]–[19].

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1 Introduction

G. Boole presented in [2] the logical foundations of mathematics which were based on a two-valued logic. In the twenties, J. Łukasiewicz presented in his short two-page paper, [30], a three-valued logic. Later in thirties this logic was generalized to an n-valued logic and as well as to an infinite-valued one. An algebraic counter part, an MV-algebra, of a Łukasiewicz many-valued logic was presented by C.C. Chang [4] more than sixty years ago. During the last six decades, MV-algebras entered deeply into mathematics and they have influenced it in many directions. A fundamental algebraic result was presented by D. Mundici [31] in 1986 showing that there is a categorical equivalence between the category of MV-algebras and the category of unital Abelian ℓ-groups consisting of couples (G, u), where G is an ℓ-group with a fixed strong unit $u \in G^+$. It is noteworthy of recalling that the class of MV-algebras forms a variety whereas the class of unital

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ℓ-groups is not a variety (it is not closed under direct products of unital ℓ-groups). This result have deeply influenced not only MV-algebras but it had impact to ℓ-groups, some classes of C^*-algebras, logic, states, observables, etc. as well as to their applications in probability theory, functional analysis, topology, fuzzy set theory, etc. There appeared also a whole family of different algebraic structures motivated by MV-algebras like BCK-algebras, BCI-algebras, Wajsberg algebras, bricks, effect algebras, residuated lattices, hoops, etc.

Fuzzy sets gave a new impulse to a many-valued logic, so that a fuzzy logic was born in the nineties together with its algebraic models, BL-algebras, [26], which generalize MV-algebras. Their basic models are so-called the unit intervals $[0,1]$ with t-norms.

In the late nineties and published in the beginning of the 21st century, G. Georgescu and A. Iorgulescu in [23] presented a non-commutative generalization of MV-algebras, called pseudo MV-algebras. Practically in the same time also J. Rachůnek, [34], presented independently an equivalent generalization of MV-algebras called generalized MV-algebras. A. Dvurečenskij, [12], presented a fundamental result on a representation of pseudo MV-algebras showing that the category of pseudo MV-algebras is categorically equivalent to the category of unital ℓ-groups, not necessarily Abelian ones. This representation is a natural non-commutative generalization of the Mundici's result [31]. These papers inspired also an introduction of non-commutative generalizations of BL-algebras, called pseudo BL-algebras, [8, 9], pseudo hoops [24], non-commutative residuated lattices, etc.

A non-commutative generalization of reasoning can be found, for example, in psychological processes, [12]: In clinical medicine, an experiment related with transplantation of human organs was performed in which the following two questions were posed to two groups of people: (1) Do you agree to dedicate your organs for medical transplantation after your death? (2) Do you agree to accept organs of a donor if you needed one? When the order of the two questions was changed for the second group, the number of positive answers to the first question was higher than that of the first group. Today there exists a programming language [1] based on a non-commutative logic.

Recently in [13], the authors introduced EMV-algebras which generalize both MV-algebras as well as generalized Boolean algebras. These algebras are distributive with a binary total operation $\oplus$, where top element is not guaranteed a priori. EMV-algebras locally resemble MV-algebras but globally, if there is not a top element, they are not MV-algebras. This means that idempotent elements play an important role, and every interval $[0,a]$ with restriction of $\oplus$, where a is an idempotent of an EMV-algebra M, can be naturally converted into an MV-algebras, so that $M = \bigcup_a [0,a]$. A basic representation result says that if an EMV-algebra has a top element, then it is equivalent to an MV-algebra, otherwise, if it has no top element, it can be embedded into an EMV-algebra with top element as its maximal ideal. This result generalizes an analogous statement for generalized Boolean algebras, see [6, Thm 2.2]. A variant of the Loomis–Sikorski theorem was establishes in [14], states on EMV-algebras as analogs of finitely additive measures were described in [15], free objects of EMV-algebras were studied in [16], and EMV-pairs were described in [21].

A non-commutative variant of EMV-algebras, pseudo EMV-algebras, was also introduced in [17, 18] and some applications of them for generalized pseudo EMV-effect algebras were presented in [20]. In this paper, we would like to present the basic properties of pseudo EMV-algebras and to show some perspectives for them.

The paper is organized as follows. In Section 2, we present elements of MV-algebras and their generalization, EMV-algebras, which is a commutative generalization of MV-algebras and of generalized Boolean algebras. Section 3 describes pseudo EMV-algebras as a non-commutative generalization of MV-algebras as well as of EMV-algebras. The basic representation theorem of pseudo EMV-algebras is presented in Section 4 together with its categorical equivalence. Section 5 is devoted to states as analogues of finitely additive measures on pseudo EMV-algebras. We show their important subfamily, state-morphisms, which correspond to extremal states. In addition, we present an integral representation of states by σ-additive probability measures on a Borel σ-algebra of some Hausdorff topological space. All present results are based on papers [13]–[19].

2 MV-algebras and EMV-algebras

The original Chang axioms [4] were a very long list and during the decades, they have been simplified and nowadays, we use a very short list of axioms. Let $M = (M;\oplus,^*,0,1)$ be an *MV-algebra*, i.e. an algebra of type $(2,1,0,0)$ such that $(M;\oplus,0)$ is a commutative monoid with a neutral element 0 and, for $x,y \in M$, we have

(i) $x^{**} = x$;

(ii) $x \oplus 1 = 1$;

(iii) $x \oplus (x \oplus y^*)^* = y \oplus (y \oplus x^*)^*$.

When we define $x \odot y = (x^* \oplus y^*)^*$, $x \vee y = x \oplus (x \oplus y^*)^*$, and $x \wedge y = x \odot (x^* \oplus y)$, then $(M;\vee,\wedge,0,1)$ is a bounded distributive lattice.

MV-algebras are intimately connected with Abelian unital ℓ-groups. We note that a group $(G;+,-,0)$ with a lattice order $\le$ is an ℓ-group if, for each $h_1,h_2 \in G$, $f \le g$ implies $h_1+f+h_2 \le h_1+g+h_2$. We denote by $G^+ = \{g \in G\colon g \ge 0\}$ the positive cone. An element $u \in G^+$ is a *strong unit* of an ℓ-group G if, given $g \in G$, there is an integer n such that $g \le nu$; the couple (G,u) is said to be a *unital ℓ-group*. For more information about ℓ-groups, see the books [7, 22, 25].

First assume that G is an Abelian ℓ-group and let $[0,u] := \{g \in G\colon 0 \le g \le u\}$. Then $\Gamma(G,u) := ([0,u];\oplus,',0,u)$, where $x \oplus y := \min\{x+y,u\}$ and $x' := u-x$ for all $x,y \in [0,u]$, is a prototypical example of an MV-algebra, and Γ defines a categorical equivalence of the category of MV-algebras and the category of Abelian unital ℓ-groups, [31]. For more information about MV-algebras we recommend to consult with the book [5].

An element $a \in M$ is a *Boolean element* of M or an *idempotent*, if $a \oplus a = a$ or equivalently, $a \vee a^* = 1$; then the set $B(M)$ of Boolean elements of M is a Boolean algebra that is also an MV-subalgebra of M. If a is a Boolean element of M, then the interval $M_a := [0,a]$ can be converted into an MV-algebra $([0,a];\oplus,^{*_a},0,a)$, where $x^{*_a} := a \odot x^*$ for each $x \in [0,a]$. Then we have

$$x^{*_a} = \min\{z \in [0,a]\colon z \oplus x = a\}.$$

In the paper, we will write also $\lambda_a(x) := x^{*_a}$, $x \in [0,a]$, i.e.

$$\lambda_a(x) = \min\{z \in [0,a]\colon z \oplus x = a\}, \tag{1}$$

and then $(M_a;\oplus,\lambda_a,0,a)$ is an MV-algebra. Equation (1) is a starting point for our definition of EMV-algebras.

Let $(M;\oplus,0)$ be a monoid (not necessarily commutative) with a neutral element 0. An element $a \in M$ is said to be an *idempotent* if $a \oplus a = a$. We denote by $\mathcal{I}(M)$ the set of idempotent elements of M; clearly $0 \in \mathcal{I}(M)$, and if $a,b \in I(M)$, then $a \oplus b \in \mathcal{I}(M)$. A monoid $(M;\oplus,0)$ endowed with a partial order $\le$ is *ordered* if $x \le y$ implies $z_1 \oplus x \oplus z_2 \le z_1 \oplus y \oplus z_2$ for all $z_1,z_2 \in M$.

According to [13], we say that an *EMV-algebra* is an algebra $(M;\vee,\wedge,\oplus,0)$ of type $(2,2,2,0)$ such that

(i) $(M;\oplus,0)$ is a commutative ordered monoid with a neutral element 0;

(ii) $(M;\vee,\wedge,0)$ is a distributive lattice with the bottom element 0;

(iii) for each idempotent $a\in\mathcal{I}(M)$, the element $\lambda_a(x)=\min\{z\in[0,a]\colon z\oplus x=a\}$ exists in M for each $x\in[0,a]$, and the algebra $([0,a];\oplus,\lambda_a,0,a)$ is an MV-algebra;

(iv) for each $x\in M$, there is an idempotent a of M such that $x\le a$.

We note that we do not assume that an EMV-algebra M has a top element. If it has no top element, M is said to be *proper*.

2.1 MV-algebras

If $(M;\oplus,',0,1)$ is an MV-algebra, then $(M;\vee,\wedge,\oplus,0)$ is an EMV-algebra with top element 1, and conversely, if $(M;\vee,\wedge,\oplus,0)$ is an EMV-algebra with top element 1, then $(M;\oplus,\lambda_1,0,1)$ is an EMV-algebra with top element. Moreover, every EMV-algebra with top element is termwise equivalent to an MV-algebra and vice-versa.

2.2 Finite EMV-algebras

Every finite EMV-algebra or every EMV-algebra with a finite set of idempotents is termwise equivalent to an MV-algebra.

2.3 Generalized Boolean algebras

If $(M;\vee,\wedge,0)$ is a generalized Boolean algebra, then $(M;\vee,\wedge,\oplus,0)$ is an EMV-algebra, where $\oplus=\vee$.

2.4 Rings of subsets

If $\mathcal{S}$ is a ring of subsets of a set $\Omega\neq\emptyset$, i.e. $\mathcal{S}$ is closed under under $\cup$, $\cap$, $\setminus$ (the set-theoretical difference) and containing $\emptyset$, then $(\mathcal{S};\cup,\cap,\cup,\emptyset)$ is an EMV-algebra if we set $\vee=\cup$, $\wedge=\cap$, $\oplus=\cup$, and $0=\emptyset$.

2.5 EMV-algebra $\mathcal{N}$ of integers

Let $\mathcal{N}$ be the system of all finite subsets of the set $\mathbb{N}$ of all natural numbers. Then $(\mathcal{N};\cup,\cap,\cup,\emptyset)$ is an EMV-algebra without top element; we put EMV-operation $\vee=\cup$, $\wedge=\cap$, $\oplus=\cup$, and $0=\emptyset$.

2.6 EMV-clan of fuzzy sets

A system $\mathcal{T}\subseteq[0,1]^\Omega$ of fuzzy sets of a set $\Omega\neq\emptyset$ is said to be an EMV-*clan* if

(i) $0_\Omega\in\mathcal{T}$ where $0_\Omega(\omega)=0$ for each $\omega\in\Omega$;

(ii) if $a\in\mathcal{T}$ is a characteristic function (i.e. a is a $0-1$-valued function), then (a) $a-f\in\mathcal{T}$ for each $f\in\mathcal{T}$ with $f(\omega)\le a(\omega)$ for each $\omega\in\Omega$, (b) if $f,g\in\mathcal{T}$ with $f(\omega),g(\omega)\le a(\omega)$ for each $\omega\in\Omega$, then $f\oplus g\in\mathcal{T}$, where $(f\oplus g)(\omega)=\min\{f(\omega)+g(\omega),a(\omega)\}$, $\omega\in\Omega$;

(iii) for each $f,g\in\mathcal{T}$, there is a characteristic function $a\in\mathcal{T}$ such that $f(\omega),g(\omega)\le a(\omega)$ for each $\omega\in\Omega$;

(iv) given $\omega \in \Omega$, there is $f \in \mathcal{T}$ such that $f(\omega) = 1$.

It is possible to show that the operation $\oplus$ is correctly defined and it does not depend on the choice of a characteristic function a. Indeed, by (iii), we have that if $f, g \in \mathcal{T}$, there is a characteristic function $a \in \mathcal{T}$ such that $f, g \leq a$. If $b \in \mathcal{T}$ is another characteristic function such that $a \leq b$, we have

$$(f \oplus_a g)(\omega) = \begin{cases} f(\omega) + g(\omega) & \text{if } f(\omega) + g(\omega) \leq a(\omega) \\ a(\omega) & \text{if } f(\omega) + g(\omega) > a(\omega), \end{cases} \quad \omega \in \Omega,$$

and

$$(f \oplus_b g)(\omega) = \begin{cases} f(\omega) + g(\omega) & \text{if } f(\omega) + g(\omega) \leq b(\omega) \\ b(\omega) & \text{if } f(\omega) + g(\omega) > b(\omega), \end{cases} \quad \omega \in \Omega.$$

If $a(\omega) = 0$ or $b(\omega) = 0$, then $f(\omega) = g(\omega) = 0$ and $(f \oplus_a g)(\omega) = 0 = (f \oplus_b g)(\omega)$. Otherwise, $a(\omega) = 1 = b(\omega) = 0$, then $(f \oplus_a g)(\omega) = (f \oplus_b g)(\omega)$. Hence, if $f, g \leq u, v$, where u, v are characteristic functions from $\mathcal{T}$, there is a characteristic function $c \in \mathcal{T}$ such that $u, v \leq c$. Then $f \oplus_u g = f \oplus_c g = f \oplus_v g$, and the binary operation $\oplus$ does not depend on the chosen characteristic functions $a, b, u, v, c \in \mathcal{T}$ dominating f, g, and $\oplus$ is a total binary operation such that $(\mathcal{T}; \oplus, 0_\Omega)$ is a commutative ordered monoid. It is easy to see that, for $f \in \mathcal{T}$, we have $f \oplus f = f$ iff f is a characteristic function. Finally $\lambda_a(f) = a - f$ whenever $f \leq a$ and $a \in \mathcal{T}$ is a characteristic function. So that $([0, a]; \oplus, \lambda_a, 0, a)$ is an MV-algebra of fuzzy sets, and $(\mathcal{T}; \vee, \wedge, \oplus, 0_\Omega)$, where $\vee = \max$ and $\wedge = \min$, is an EMV-algebra, where top element is not guaranteed.

2.7 Positive cone of an ℓ-group

Let G^+ be the positive cone of an ℓ-group $G \neq \{0\}$, then G^+ has only one Boolean element $a = 0$, so that G^+ cannot be transformed into an EMV-algebra.

2.8 Direct product of EMV-algebras

Let $\{M_t \colon t \in T\}$ be a system of EMV-algebras. Then the direct product $M = \prod_{t \in T} M_t$, where the operations in the product are defined by coordinates, is an EMV-algebra.

2.9 Sum of MV-algebras

Let $\{(M_i; \oplus, ', 0, 1)\}_{i \in I}$ be a family of MV-algebras and

$$S = \{f \in \prod_{i \in I} M_i \colon \operatorname{supp}(f) \text{ is finite}\},$$

where $\operatorname{supp}(f) = \{i \in f(i) \neq 0\}$. Then S can be converted into an EMV-algebra, we denote it by $\sum_{i \in I} M_i$. In general, it is not more an MV-algebra if I is infinite.

The basic representation theorem, [13, Thm 5.21], says the following:

Theorem 2.1. [Basic Representation Theorem] *Every EMV-algebra M is either an EMV-algebra with top element or M can be embedded into an EMV-algebra N_0 with top element as a maximal ideal of N_0. Moreover, N_0 can be chosen in such a way that every element of N_0 either belongs to the image of M in N_0 or it is a complement of an element from the image of M in N_0.*

It is noteworthy of recalling that the class EMV of EMV-algebras is not a variety because it is not closed under subalgebras. However, it is possible to study operators close to the operators of subspaces, direct products and of homomorphic images, respectively, so that as it was shown in [13, Thm 5.22], the family of subclasses closed under these three operators is countable. In addition, in [19], wEMV-algebras were introduced in such a way that they form a variety and naturally containing the class of EMV-algebras.

We recall that an algebra $(M;\vee,\wedge,\oplus,\ominus,0)$ of type $(2,2,2,2,0)$ is called a *wEMV-algebra* (w means weak) if it satisfies the following conditions:

(i) $(M,\vee,\wedge,0)$ is a distributive lattice with the least element 0;

(ii) $(M;\oplus,0)$ is a commutative monoid;

(iii) $(x\oplus y)\ominus x\le y$;

(iv) $x\oplus(y\ominus x)=x\vee y$;

(v) $x\ominus(x\wedge y)=x\ominus y$;

(vi) $z\ominus(z\ominus x)=x\wedge z$;

(vii) $z\ominus(x\vee y)=(z\ominus x)\wedge(z\ominus y)$;

(viii) $(x\wedge y)\ominus z=(x\ominus z)\wedge(y\ominus z)$;

(ix) $x\ominus(y\oplus z)=(x\ominus y)\ominus z$;

(x) $x\oplus(y\vee z)=(x\oplus y)\vee(x\oplus z)$.

2.10 Examples of wEMV-algebras

(1) If $M=G^+$ is the positive cone of an Abelian ℓ-group G, and if we define on G^+ two operations $x\oplus y:=x+y$ and $x\ominus y:=(x-y)\vee 0$, $x,y\in G^+$, then $(M;\vee,\wedge,\oplus,\ominus,0)$ is an example of a wEMV-algebra, called also a *wEMV-algebra of a positive cone*. We note that $(M;\vee,\wedge,\oplus,0)$ is not an EMV-algebra if $G\neq\{0\}$.

(2) Let $(M;\oplus,',0,1)$ be an MV-algebra. If we set $x\ominus y:=x\odot y'$, $x,y\in M$, then $(M;\vee,\wedge,\oplus,\ominus,0)$ is a wEMV-algebra with a top element 1.

(3) Consider an arbitrary proper EMV-algebra $(M;\vee,\wedge,\oplus,0)$. By Theorem 2.1, M can be embedded into an EMV-algebra N_0 with top element as a maximal ideal of N_0. Then $(N_0;\oplus,\lambda_1,0,1)$ is an MV-algebra. For simplicity, we use x' instead of $\lambda_1(x)$, for all $x\in N_0$. Let $\ominus$ be the well-known operation on N_0, that is $x\ominus y=(x'\oplus y)'$ for all $x,y\in N_0$. Since M is an ideal of N_0, then M is closed under $\ominus$. So, $(M;\vee,\wedge,\oplus,\ominus,0)$ is an example of a wEMV-algebra without top element.

Therefore, if $(M;\vee,\wedge,\oplus,0)$ is an arbitrary EMV-algebra, extending its language with a binary operation $\ominus$, we obtain a wEMV-algebra $(M;\vee,\wedge,\oplus,\ominus,0)$; it is said to be a *wEMV-algebra associated* with the EMV-algebra $(M;\vee,\wedge,\oplus,0)$; simply we say M is an *associated wEMV-algebra*. We denote by EMV_a the class of associated wEMV-algebras $(M;\vee,\wedge,\oplus,\ominus,0)$, where $(M;\vee,\wedge,\oplus,0)$ is any EMV-algebra.

For wEMV-algebras, we have the following important results showing a relationship between EMV-algebras and wEMV-algebras, for details see [19]:

Theorem 2.2. *Each wEMV-algebra is a subalgebra of an associated wEMV-algebra with top element.*

A representation theorem similar to Theorem 2.1 is as follows:

Theorem 2.3. *Every wEMV-algebra M either has a top element and so it is an associated wEMV-algebra or it can be embedded into an associated wEMV-algebra N with top element as a maximal ideal of N. Moreover, every element of N is either the image of $x \in M$ or is a complement of the image of some element $x \in M$.*

The following result says that the class of EMV-algebras is in some sense the least one in the variety of wEMV-algebras.

Theorem 2.4. *The class* wEMV *is the least subvariety of the variety* wEMV *containing* EMV_a. *Moreover,* $\mathsf{wEMV} = HSP(C)$*, where C is the class of all linearly ordered wEMV-algebras, and the lattice of subvarieties of* wEMV *is countably infinite.*

3 Pseudo MV-algebras and pseudo EMV-algebras

In the section, we present elements of pseudo MV-algebras and we introduce pseudo EMV-algebras showing their most important properties.

Pseudo MV-algebras, as a non-commutative MV-algebras, were introduced in [23] and independently in [34] as generalized MV-algebras. A *pseudo MV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2,1,1,\,0,0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$; $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

We note that we assume that $\odot$ has higher binding priority than $\wedge$ and $\oplus$, and $\oplus$ is higher than $\vee$. We denote by PMV the variety of pseudo MV-algebras.

A partial order $\le$ on M is defined by $x \le y$ iff $x^- \oplus y = 1$. Then M is a distributive lattice with $x \vee y = x \oplus (x^\sim \odot y)$ and $x \wedge y = x \odot (x^- \oplus y)$. In addition, $0 \le x \le 1$ for each $x \in M$ and $x \le y$ iff $y \oplus x^\sim = 1$.

Clearly, a pseudo MV-algebra is an MV-algebra iff $\oplus$ is a commutative binary operation.

Also pseudo EMV-algebras are intimately connected with unital ℓ-groups. A prototypical example of pseudo MV-algebras is from ℓ-groups: If u is a strong unit of a (not necessarily Abelian) ℓ-group G, set

$$\Gamma(G, u) := [0, u]$$

and

$$\begin{aligned} x \oplus y &:= (x+y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then $\Gamma(G,u) := ([0,u];\oplus,^-,^\sim,0,u)$ is a pseudo MV-algebra [23]. The converse statement is also true as it follows from the basic representation of pseudo MV-algebras by unital ℓ-groups, see [12], which generalizes a famous result by Mundici [31] for MV-algebras:

Theorem 3.1. *For any pseudo MV-algebra M, there exists a unique (up to isomorphism of unital ℓ-groups) unital ℓ-group (G,u) with a strong unit u such that $M \cong \Gamma(G,u)$. The functor Γ defines a categorical equivalence of the category of pseudo MV-algebras with the category of unital ℓ-groups.*

We note that we have $x^{-\sim} = x = x^{\sim-}$, however if $x^- = x^\sim$, then it does not mean that M is an MV-algebra. Indeed, let G be an ℓ-group that is not Abelian, and define $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0))$, where $\mathbb{Z}$ is the group of integers and $\overrightarrow{\times}$ denotes the lexicographic product on $\mathbb{Z} \times G$, i.e. $(a,g) \le (b,h)$, $(a,g),(b,h) \in \mathbb{Z} \times G$, iff either $a < b$ or $a = b$ and $g \le h$. Then in this pseudo MV-algebra, which is not commutative, we have $x^- = x^\sim$, $x \in M$.

In every pseudo MV-algebra, we have

$$x^- = \min\{z \in M \colon z \oplus x = 1\}, \tag{2}$$

$$x^\sim = \min\{z \in M \colon x \oplus z = 1\}. \tag{3}$$

Let $a \in M$ be a Boolean element of a pseudo MV-algebra M, i.e. $a \oplus a = a$. The interval $[0,a] := \{x \in M \colon 0 \le x \le a\}$ can be converted into a pseudo MV-algebra as follows: For $x \in [0,a]$, we set $x^{-a} = a - x$ and $x^{\sim a} = -x + a$, and since $a \oplus a = a$, we put $\oplus_a = \oplus|_{[0,a]\times[0,a]}$. Then $M_a = ([0,a];\oplus_a,^{-a},^{\sim a},0,a)$ is a pseudo MV-algebra. We define two unary operators λ_a and ρ_a on $M_a = [0,a]$ by

$$\lambda_a(x) := \min\{z \in [0,a] \colon z \oplus x = a\}, \quad x \in [0,a], \tag{4}$$

and

$$\rho_a(x) := \min\{z \in [0,a] \colon x \oplus z = a\}, \quad x \in [0,a]. \tag{5}$$

Hence, we have for each $x \in [0,a]$

$$x^{-a} = \lambda_a(x), \quad x^{\sim a} = \rho_a(x).$$

Therefore, we are ready to define pseudo EMV-algebras as a non-commutative generalization of EMV-algebras as it was presented in [17].

Definition 3.2. An algebra $(M;\vee,\wedge,\oplus,0)$ of type $(2,2,2,0)$ is called a *pseudo EMV-algebra* if it satisfies the following conditions:

(E1) $(M;\vee,\wedge,0)$ is a distributive lattice with the least element 0;

(E2) $(M;\oplus,0)$ is an ordered monoid with a neutral element 0;

(E3) for each $a \in \mathcal{I}(M)$, the elements

$$\lambda_a(x) := \min\{z \in [0,a] \colon z \oplus x = a\}, \quad \rho_a(x) := \min\{z \in [0,a] \colon x \oplus z = a\}$$

exist in M for all $x \in [0,a]$, and the algebra $([0,a];\oplus,\lambda_a,\rho_a,0,a)$ is a pseudo MV-algebra;

(E4) for each $x \in M$, there is $a \in \mathcal{I}(M)$ such that $x \le a$.

It is noteworthy of recalling that the orders following from (E1) and (E2), respectively, are the same.

Now, we present some important examples of pseudo EMV-algebras.

3.1 EMV-algebras

We note that a pseudo EMV-algebra is an EMV-algebra iff $\oplus$ is commutative.

3.2 Pseudo MV-algebras

If $(M;\oplus,^-,^\sim,0,1)$ is a pseudo MV-algebra, then $(M;\vee,\wedge,\oplus,0)$ is a pseudo EMV-algebra with a top element 1. Conversely, if a pseudo EMV-algebra $(M;\vee,\wedge,\oplus,0)$ has a top element 1, then M can be converted into a pseudo MV-algebra whose pseudo MV-structure is compatible with the pseudo EMV-algebraic structure of M. In addition, it is easy to show that pseudo MV-algebras are termwise equivalent to pseudo EMV-algebras with top element; the equivalence is given by $(M;\vee,\wedge,\oplus,0)$ with a top element 1 is equivalent to $(M;\oplus,\lambda_1,\rho_1,0,1)$ and vice-versa.

3.3 Sum of pseudo EMV-algebras

Let $\{M_i\colon i \in I\}$ be a family of pseudo EMV-algebras. We can easily show that $\sum_{i\in I} M_i := \{(x_i)_{i\in I} \in \prod_{i\in I} M_i\colon x_i = 0$ for all but a finite number of $i \in I\}$ is a pseudo EMV-algebra with componentwise operations. For example, let $A_1 = M$ be a pseudo EMV-algebra with top element and $A_i = \{0,1\}$ for all $i \in \mathbb{N}$, then $\sum_{i\in I} A_i$ is a pseudo EMV-algebra.

We note that if we take the positive cone G^+ of an ℓ-group $G \ne \{0\}$, then G^+ with $\oplus = +$ is not a pseudo EMV-algebras because it has only the zero idempotent which does not dominate any non-zero element of G^+.

Now, we describe some of basic properties of pseudo EMV-algebras.

Proposition 3.3. *Let $(M;\vee,\wedge,\oplus,0)$ be a pseudo EMV-algebra, $a,b \in \mathcal{I}(M)$ such that $a \le b$. Then for each $x \in [0,a]$, we have*

(i) *$\lambda_b(a) = \rho_b(a)$ is an idempotent, and $\lambda_a(a) = 0 = \rho_a(a)$;*

(ii) *$\lambda_a(x) = \lambda_b(x) \wedge a$ and $\rho_a(x) = \rho_b(x) \wedge a$;*

(iii) *$\lambda_b(x) = \lambda_a(x) \oplus \lambda_b(a) = \lambda_b(a) \oplus \lambda_a(x)$ and $\rho_b(x) = \rho_a(x) \oplus \rho_b(a) = \rho_b(a) \oplus \rho_a(x)$;*

(iv) *$\rho_a(\lambda_a(x)) = x = \lambda_a(\rho_a(x))$;*

(v) *$\lambda_a(x) \le \lambda_b(x)$ and $\rho_a(x) \le \rho_b(x)$.*

In an analogy with pseudo MV-algebras, we can define a total binary operation $\odot$ in the following way: For all $x, y \in M$, we define

$$x \odot y = \rho_a(\lambda_a(y) \oplus \lambda_a(x)),$$

where $a \in \mathcal{I}(M)$ and $x, y \in [0,a]$. Then $x \odot y$ is correctly defined and it does not depend on $a \in \mathcal{I}(M)$, and

$$x \odot y = \lambda_a(\rho_a(y) \oplus \rho_a(x)).$$

In addition, if $x, y \in M$, $x \le y$, then

$$y \odot \lambda_a(x) = y \odot \lambda_b(x), \quad \rho_a(x) \odot y = \rho_b(x) \odot y \tag{6}$$

for all idempotents a, b of M with $x, y \le a, b$, and

$$y = (y \odot \lambda_a(x)) \oplus x = x \oplus (\rho_a(x) \odot y). \tag{7}$$

If $x, y \in [0, a]$ for some idempotent $a \in M$, then

$$x \odot \lambda_a(y) = x \odot \lambda_a(x \wedge y), \quad \rho_a(y) \odot x = \rho_a(x \wedge y) \odot x, \tag{8}$$

and

$$(x \odot \lambda_a(y)) \oplus (x \wedge y) = x = (x \wedge y) \oplus (\rho_a(y) \odot x). \tag{9}$$

Finally, if $x, y \le a \in \mathcal{I}(M)$, then

$$((x \oplus y) \odot \lambda_a(x)) \oplus x = x \oplus y = y \oplus (\rho_a(y) \odot (x \oplus y)) \tag{10}$$

and if $x \le a \in \mathcal{I}(M)$, then

$$x \odot \lambda_a(x) = 0 = \rho_a(x) \odot x. \tag{11}$$

An *ideal* of a pseudo EMV-algebra M is any subset I of M such that (i) $x, y \in I$ implies $x \oplus y \in I$, and (ii) if $x \le y \in I$, then $x \in I$. An ideal I of M is (i) *maximal* if I is a proper subset of M and it cannot be a proper subset of any other proper ideal of M, (ii) *normal* if, for each $x \in M$, we have $x \oplus I := \{x \oplus y \colon y \in M\} = \{y \oplus x \colon y \in M\} = I \oplus x$. Equivalently, an ideal I is normal if, for each $x, y \in M$, we have $y \odot \lambda_a(x) \in I$ iff $\rho_a(x) \odot y \in I$.

If M is an EMV-algebra, according to [13, Thm 5.6], M contains at least one maximal ideal. If M is a pseudo EMV-algebra, then it can happen that it does not possess a maximal ideal which is also normal. This is true even for pseudo EMV-algebras, see [**?**, Prop 7.3]. In any rate, we have the following result:

Proposition 3.4. *Every linearly ordered pseudo EMV-algebra $M \neq \{0\}$ has a unique maximal ideal, it is normal, and M possesses a top element.*

Let M_1 and M_2 be pseudo EMV-algebras. A mapping $f : M_1 \to M_2$ is said to be a *homomorphism* of pseudo EMV-algebras if f preserves $\vee, \wedge, \oplus, 0$ and, for each idempotent $a \in \mathcal{I}(M)$ and for each $x \in [0, a]$, $f(\lambda_a(x)) = \lambda_{f(a)}(f(x))$ and $f(\rho_a(x)) = \rho_{f(a)}(f(x))$.

4 Representation of pseudo EMV-algebras and categorical equivalence

In the section, we present a basic representation result which is parallel to Theorem 2.1 for EMV-algebras which was established in [18, Thm 6.4]. Here we present a main steps of the proof from [18].

Theorem 4.1. [Basic Representation Theorem] *Every pseudo EMV-algebra M is either a pseudo EMV-algebra with top element or M can be embedded into a pseudo EMV-algebra N_0 with top element as a maximal and normal ideal of N_0. In the second case, every element $x \in N_0$ is either from the image of M or there is a unique x_0 from the image of M such that $x = \rho_1(x_0)$.*

Proof. For each idempotent $a \in \mathcal{I}(M)$, $([0,a];\vee,\wedge,\oplus,0)$ is a pseudo EMV-algebra with top element. Therefore, $N=\prod_{a\in\mathcal{I}(M)}[0,a]$ is a pseudo EMV-algebra with top element, see Proposition 3.4. It is possible to show that the mapping $\varphi: M\to N$ defined by $\varphi(x)=(x\wedge a)_{a\in\mathcal{I}(M)}[0,a]$, $x\in M$, is an embedding of M into the pseudo EMV-algebra N with top element.

We define

$$N_0=\{x\in N\colon \text{ either } x=x_0\in M \text{ or } x=\rho_1(x_0) \text{ for some } x_0\in M\}.$$

Claim. If $x_0\in M$ and $x_0\le a\in\mathcal{I}(M)$, then $\lambda_1^2(x_0),\rho_1^2(x_0)\in M$. Moreover, $\lambda_1^2(x_0)=\lambda_a^2(x_0)=:\varphi_\lambda(x_0)$, $\rho_1^2(x_0)=\rho_a^2(x_0)=:\varphi_\rho(x_0)$, and $\varphi_\lambda(x_0)$ and $\varphi_\rho(x_0)$ do not depend on $a\ge x_0$.

Let $a\in\mathcal{I}(M)$ be an idempotent such that $x_0\le a$. By Proposition 3.3(iii), we have $\lambda_1(x_0)=\lambda_a(x_0)\oplus\lambda_1(a)$, so that by (8) and (6), we have $\lambda_1^2(x_0)=\lambda_1(\lambda_a(x_0)\oplus\lambda_1(a))=a\odot\lambda_1(\lambda_a(x_0))=a\odot(\lambda_a^2(x_0)\oplus\lambda_1(a))=a\wedge(\lambda_a^2(x_0)\vee\lambda_1(a))=(a\wedge\lambda_a^2(x_0))\vee(a\odot\lambda_1(a))=a\wedge\lambda_a^2(x_0)=\lambda_a^2(x_0)=\varphi_\lambda(x_0)\in M$. In the same way we establish $\rho_1^2(x_0)=\varphi_\rho(x_0)\in M$, and $\varphi_\lambda(x_0)$ and $\varphi_\rho(x_0)$ do not depend on $a\in\mathcal{I}(M)$ such that $x_0\le a$.

In what follows, we show that N_0 is a pseudo EMV-algebra with top element. Since $(N;\oplus,\lambda_1,\rho_1,0,1)$ is a pseudo MV-algebra that is termwise equivalent to the pseudo EMV-algebra N, we will use $x^-:=\lambda_1(x)$ and $x^\sim:=\rho_1(x)$, $x\in N$. First we note that according to Claim, we have $x_0^-=(x_0^{--})^\sim=\varphi_\lambda(x_0)^\sim\in N_0$ for each $x_0\in M$, so that, N_0 is closed under $^-$ and $^\sim$.

Clearly, $M\subseteq N_0$ and $1\in N_0$. Let $x,y\in N_0$. Using ideas from the proof of Theorem 5.11, we show that $x\oplus y\in N_0$. There are four cases. Case (i): $x=x_0, y=y_0\in M$. Then $x\vee y, x\wedge y, x\oplus y\in N_0$.

Case (ii): $x=x_0^\sim$, $y=y_0^\sim$ for some $x_0,y_0\in M$. Then $x\vee y=x_0^\sim\vee y_0^\sim=(x_0\wedge y_0)^\sim$, $x\wedge y=(x_0\vee y_0)^\sim$ and $x\oplus y=x_0^\sim\oplus y_0^\sim=(y_0\odot x_0)^\sim\in N_0$.

Case (iii): $x=x_0$ and $y=y_0^\sim$ for some $x_0,y_0\in M$. Then

$$x\oplus y=x_0\oplus y_0^\sim=(y_0\odot x_0^-)^\sim=(y_0\odot(x_0\wedge y_0)^-)^\sim=(y_0\odot\lambda_b(x_0\wedge y_0))^\sim,$$

where b is an idempotent of M such that $x_0,y_0\le b$; for the last equality we use equality (6). Hence, we have $y_0\odot\lambda_b(x_0\wedge y_0)\in M$ so that $x\oplus y\in N_0$.

Case (iv): $x=x_0^\sim$ and $y=y_0$ for some $x_0,y_0\in M$. There is $b\in\mathcal{I}(M)$ such that $x_0,y_0\le b$. Check

$$\begin{aligned}x\oplus y&=x_0^\sim\oplus y_0=(y_0^\sim\odot x_0^{\sim\sim})^-=(y_0^\sim\odot\varphi_\rho(x_0))^-=\big((y_0\wedge\varphi_\rho(x_0))^\sim\odot\varphi_\rho(x_0)\big)^-\\&=\big((\rho_b(y_0\wedge\varphi_\rho(x_0))\vee b^\sim)\odot\varphi_\rho(x_0)\big)^{--\sim}=\big(\rho_b(y_0\wedge\varphi_\rho(x_0))\odot\varphi_\rho(x_0)\big)^{--\sim}\\&=\big(\varphi_\lambda(\rho_b(y_0\wedge\varphi_\rho(x_0))\odot\varphi_\rho(x_0))\big)^\sim\in N_0.\end{aligned}$$

Now, let $x_0\in M$. Then $x_0^\sim\in N_0\setminus M$ and if we set $y_0=x_0^{--}=\varphi_\lambda(x_0)\in M$, then $x_0^-=y_0^\sim\in N_0\setminus M$. Claim implies $x_0^{\sim\sim}=\varphi_\rho(x_0)\in M$ and $x_0^{--}=\varphi_\lambda(x_0)\in M$. Hence, $(N_0;\oplus,^-,^\sim,0,1)$ is a pseudo MV-algebra, so that $(N_0;\vee,\wedge,\oplus,0)$ is its termwise equivalent pseudo EMV-algebra with top element.

The set M is a proper subset of N_0 and M is closed under $\oplus$. Let $y\le x_0\in M$. Then y cannot be from $N_0\setminus M$, otherwise $1\in M$. Hence $y\in M$ and M is an ideal of N_0. If we take $y\in N_0\setminus M$, then the ideal I of N_0 generated by $M\cup\{y\}$ contains 1, so that $I=N_0$ which establishes M is a maximal ideal of N_0. To prove normality of M in N_0, let $y\oplus x_0\in y\oplus M$, $y\in N_0$. It is sufficient to assume $y=y_0^\sim$ for some $y_0\in M$. Then $y_0^\sim\oplus x_0=(y_0^\sim\oplus x_0)\vee y_0^\sim=((y_0^\sim\oplus x_0)\odot y_0)\oplus y_0^\sim$. Since $(y_0^\sim\oplus x_0)\odot y_0\le y_0\in M$, we have $(y_0^\sim\oplus x_0)\odot y_0\in M$, so that $y_0^\sim\oplus M\subseteq M\oplus y_0^\sim$. In a similar way, we prove the opposite inclusion. □

The pseudo EMV-algebra N_0 with top element from Theorem 4.1 is said to be the pseudo EMV-algebra *representing* M. It can be shown that all pseudo EMV-algebras with top element representing M are isomorphic.

Similarly as the class of EMV-algebras, the class PEMV of pseudo EMV-algebras is not a variety because it is not closed under forming subalgebras with respect to the original operations $\vee, \wedge, \oplus$, and 0, see [13, Thm 3.11]. Indeed, let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1,0))$ (it is also known as the Chang MV-algebra). It defines an EMV-algebra with top element having a unique maximal ideal I, namely $I = \{(0,n)\colon n \geq 0\}$. The set I is closed under $\vee, \wedge, \oplus, 0$, so it is a subalgebra of M, but I is not an EMV-subalgebra of the EMV-algebra M because it does not have enough idempotent elements.

Therefore, similarly as in [13, Thm 3.11], instead of the classical operators H, S, P, we define on the class of pseudo EMV-algebras new operators, qH, qS, qP, mapping classes of pseudo EMV-algebras to classes of pseudo EMV-algebras, which are analogues of H, S, P, as follows: Let $\mathcal{V}$ be a class of pseudo EMV-algebras and M be a pseudo EMV-algebra:

qH: $M \in qH(\mathcal{V})$ if there are a pseudo EMV-algebra $N \in \mathcal{V}$ and a surjective pEMV-homomorphism $h : N \to M$;

qS: $M \in qS(\mathcal{V})$ if there is $N \in \mathcal{V}$ such that M is a pEMV-subalgebra of N;

qP: $M \in qP(\mathcal{V})$ if $M = \prod_t M_t$, where $\{M_t\}$ is a system of pseudo EMV-algebras of $\mathcal{V}$,

and the class $\mathcal{V}$ of pseudo EMV-algebras is said to be a *q-variety* of pseudo EMV-algebras if it is closed under qH, qS, and qP operators. In the same way we define a *q-subvariety* of pseudo EMV-algebras.

We note that if $\mathcal{K}$ is a family of pseudo EMV-algebras, then there is the least q-subvariety $\mathbb{V}^q_0(\mathcal{K})$ of pseudo EMV-algebras containing $\mathcal{K}$. Using the same ideas as those used in the proof of the Tarski theorem, [3, Thm 9.5], we can show that $\mathbb{V}^q_0(\mathcal{K}) = qHqSqP(\mathcal{K})$.

The Basic Representation Theorem imply the following results showing a close connection between q-subvarieties of pseudo EMV-algebras and subvarieties of pseudo MV-algebras, for more details see [17, Thm 5.16]:

Theorem 4.2. *Given a variety $\mathcal{V}$ of pseudo MV-algebras, let $\mathbb{V}_0(\mathcal{V})$ be the class of pseudo EMV-algebras $M \in \mathsf{PEMV}$ such that, for each $a \in \mathcal{I}(M)$, the pseudo MV-algebra $([0,a]; \oplus, \lambda_a, \rho_a, 0, a)$ belongs to $\mathcal{V}$. Then $\mathbb{V}_0(\mathcal{V})$ is a q-subvariety of pseudo EMV-algebras containing $\mathcal{V}^*$. Conversely, given a q-subvariety $\mathbb{V}$ of pseudo EMV-algebras, let $\mathcal{V}_0(\mathbb{V})$ be the class of pseudo MV-algebras $(M; \oplus, ^-, ^\sim, 0, 1)$ such that $(M; \vee, \wedge, \oplus, 0)$ belongs to $\mathbb{V}$. Then $\mathcal{V}_0(\mathbb{V})$ is a variety of pseudo MV-algebras. The mappings $\mathcal{V} \mapsto \mathbb{V}_0(\mathcal{V})$ and $\mathbb{V} \mapsto \mathcal{V}_0(\mathbb{V})$ are bijective mappings which are mutually invertible and preserving the set-theoretical inclusion.*

In particular, the q-variety PEMV has uncountably many q-subvarieties.

We note that due to a famous result by Komori [28], the lattice of subvarieties of the variety MV is countable, this is not case for all subvarieties of PMV, the variety of pseudo MV-algebras, and consequently also the family of all q-subvarieties of PEMV as we have seen in the latter Theorem, is uncountable. We note that in [10] it was shown that, for any subvariety of the variety MV, there is a finite equational base which consists of finitely many MV-equations using only $\oplus$ and $\odot$. Hence, as it was shown in [13], these equations define any q-subvariety of EMV-algebras. Therefore, if we take an arbitrary finite base of pseudo EMV-algebras using only $\oplus$ and $\odot$, we can obtain a q-subvariety of pseudo EMV-algebras.

Also the latter theorem gives a way how to define a q-subvariety of pseudo EMV-algebras. We note that a pseudo EMV-algebra M is:

- *Representable* if it is a subdirect product of linearly ordered pseudo MV-algebras; let RPEMV be the class of representable pseudo EMV-algebras.
- *Normal-valued* if every value V of M is normal in its cover V^*, that is $y \oplus V = V \oplus y$ for each $y \in V^*$ (for unknown notions, see for them in [17]); let NVPEMV be the class of normal-valued pseudo EMV-algebras.
- *Boolean* if $M = \mathcal{I}(M)$, we denote by B the class of Boolean pseudo MV-algebras.
- The class MPEMV of pseudo EMV-algebras M such that either $M = \{0\}$ or every maximal ideal of M is normal.
- O is the singleton consisting only of the zero pseudo EMV-algebra.

Then O, B, RPEMV, NVPEMV, MPEMV are q-subvarieties of pseudo MV-algebras such that

$$\mathsf{O} \subsetneq \mathsf{B} \subsetneq \mathsf{EMV} \subsetneq \mathsf{RPEMV} \subsetneq \mathsf{NVPEMV} \subsetneq \mathsf{MPEMV} \subsetneq \mathsf{PEMV}. \tag{12}$$

Let $\mathcal{PPEMV}$ be the category of proper pseudo EMV-algebras whose objects are proper pseudo EMV-algebras and morphisms are homomorphisms of pseudo EMV-algebras, i.e. pseudo MV-algebras which have no top element. On the other hand, let $\mathcal{PPMV}$ be the category whose objects are pairs (M, I), where M is a pseudo MV-algebra with a fixed maximal and normal ideal I of M such that (i) I has enough idempotents, i.e. for each $x \in I$, there is an idempotent $a \in \mathcal{I}(I)$ such that $x \le a$, (ii) no $x \in I$ is a top element of I, and (iii) $I \cup I^\sim = M$, where $I^\sim := \{x^\sim \colon x \in I\}$. Morphisms $\phi : (M_1, I_1) \to (M_2, I_2)$ in the category $\mathcal{PPMV}$ are homomorphisms of pseudo MV-algebras $\phi : M_1 \to M_2$ such that $\phi(I_1) \subseteq I_2$. Then $I \cap I^\sim = \emptyset$: If $x = y^\sim$ for $x, y \in I$, then $1 = x^- \oplus x = y \oplus x \in I$.

For example, if N is the Chang MV-algebra, i.e. isomorphic to $\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$, then $I = \{(0, n) \colon n \ge 0\}$ is the unique maximal ideal of N but I does not have enough idempotents. So (N, I) does not belong to $\mathcal{PPMV}$.

Define a mapping $\Phi : \mathcal{PPMV} \to \mathcal{PPEMV}$ as follows: For any object $(N, I) \in \mathcal{PPMV}$, let

$$\Phi(N, I) := I$$

and if (N_1, I_1) and (N_2, I_2) are objects of $\mathcal{PPMV}$ and $\phi : (N_1, I_1) \to (N_2, I_2)$ is a morphism, then

$$\Phi(\phi)(x) := \phi(x), \quad x \in I_1.$$

It is possible to show that Φ is a well-defined functor that is faithful and full from the category $\mathcal{PPMV}$ into the category $\mathcal{PPEMV}$. In addition, using results from [18, Sec 7], the following result holds:

Theorem 4.3. *The functor Φ defines a categorical equivalence between the category $\mathcal{PPMV}$ and the category of proper pseudo EMV-algebras $\mathcal{PPEMV}$.*

In addition, if $h : \Phi(N, I) \to \Phi(N', I')$ is a morphism of proper pseudo EMV-algebras, then there is a unique homomorphism $\phi : N \to N'$ of pseudo MV-algebras with $\phi(I) \subseteq I'$ such that we have $h = \Phi(\phi)$, and

(i) *if h is surjective, so is ϕ;*

(ii) *if h is injective, so is ϕ.*

5 States and state-morphisms on pseudo EMV-algebras

In the section, we define states, state-morphisms and extremal states on pseudo EMV-algebras. We present their basic properties, exhibit the weak topology of states, and we show an integral representation of a state by a σ-additive probability measure.

A state on an MV-algebra is a mapping $s : M \to [0,1]$ such that (i) $s(1) = 1$, and $s(s \oplus b) = s(a) + s(b)$ whenever $a \odot b = 0$, see [32]. It is an analogue of a finitely additive probability measure for MV-algebras. It is important to note that every non-trivial MV-algebra possesses at least one state. If M is a pseudo MV-algebra, we can define a partial operation $+$ on M by $a + b$ is defined iff $y \odot x = 0$, and then $a + b = a \oplus b$. It is possible to show that $+$ is an associative operation. Then a *state* on a pseudo MV-algebra M is a mapping $s : M \to [0,1]$ such that (i) $s(1) = 1$ and (ii) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in M. In contrast to MV-algebras, there are non-trivial pseudo MV-algebras which have no state, see [**?**]. A criterion when a pseudo EMV-algebra admits at least one state is the following result:

Theorem 5.1. *The state space $\mathcal{S}(M)$ of a pseudo EMV-algebra M is non-empty if and only if M possesses at least one maximal and normal ideal.*

As a corollary, we have that every non-trivial pseudo EMV-algebra from q-varieties in (12) has at least one state.

Also for pseudo EMV-algebras we can define a partial operation $+$ as follows: The element $x + y$ is defined in a pseudo EMV-algebra M if $y \odot x = 0$, and then we define $x + y = x \oplus y$. An absence of a top element in an EMV-algebra or in a pseudo EMV-algebra can produce a problem in defining a state on a pseudo EMV-algebra in the demand $s(1) = 1$. Therefore, a *state* on a pseudo EMV-algebra is defined as a mapping $s : M \to [0,1]$ such that (i) there is $x \in M$ such that $s(x) = 1$, and (ii) $s(x + y) = s(x) + s(y)$ whenever $x + y$ is defined in M. If M admits a top element, i.e. M is equivalent to a pseudo MV-algebra, then both notions of states coincide. In [15], it was shown that every non-trivial EMV-algebra possesses at least one state. We denote by $\mathcal{S}(M)$ the set of states on M. In contrast to EMV-algebras, it can happen that $\mathcal{S}(M)$ is empty; this is true even for pseudo MV-algebras, see [13].

A mapping $s : M \to [0,1]$ is said to be a *state-morphism* if s is a pEMV-homomorphism such that there is an element $x \in M$ such that $s(x) = 1$. Let $\mathcal{SM}(M)$ be the set of state-morphisms. We note that every state-morphism is a state. A state s is *extremal* if from the equality $s = \lambda s_1 + (1-\lambda)s_2$, where $\lambda \in (0,1)$ and $s_1, s_2 \in \mathcal{S}(M)$, we conclude $s = s_1 = s_2$. We denote by $\partial S(M)$ the set of extremal states on M. The basic properties of states are described in [18, Prop 8.1]:

Proposition 5.2. *Let s be a state on a pseudo EMV-algebra M. For all $x, y \in M$, we have*

(i) $s(0) = 0$;

(ii) *if $x \le y \le a \in \mathcal{I}(M)$, then $s(x) \le s(y)$ and $s(y \odot \lambda_a(x)) = s(y) - s(x) = s(\rho_a(y) \odot x)$; in particular, $s(\lambda_a(x)) = s(a) - s(x) = s(\rho_a(x))$;*

(iii) $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$;

(iv) $s(x \oplus y) + s(x \odot y) = s(x) + s(y)$;

(v) $\mathrm{Ker}(s) = \{x \in M : s(x) = 0\}$ *is a normal ideal of M and* $\mathrm{Ker}_1(s) = \{x \in M : s(x) = 1\}$ *is a normal filter of the pseudo EMV-algebra M;*

(vi) *if* $s_1, s_2 \in \mathcal{S}(M)$ *and* $\lambda \in [0,1]$ *is a real number, then the convex combination* $s = \lambda s_1 + (1-\lambda)s_2$ *of states* s_1, s_2 *is a state on* M*;*

(vii) *if we define a mapping* $\hat{s}$ *on the quotient pseudo EMV-algebra* $M/\mathrm{Ker}(s)$ *by* $\hat{s}(x/\mathrm{Ker}(s)) := s(x)$, $(x \in M)$*, then* $\hat{s}$ *is a state on* $M/\mathrm{Ker}(s)$*, and* $M/\mathrm{Ker}(s)$ *is a pseudo EMV-algebra with top element;*

(viii) $s(x \oplus y) = s(y \oplus x)$ *and* $M/\mathrm{Ker}(s)$ *is an Archimedean EMV-algebra.*

State-morphisms and extremal states are equivalent as the following result shows:

Theorem 5.3. *Let* s *be a state on a pseudo EMV-algebra* M*. The following statements are equivalent:*

(i) s *is a state-morphism.*

(ii) $\mathrm{Ker}(s)$ *is a maximal and normal ideal of* M*.*

(iii) s *is an extremal state on* M*.*

In addition,

$$\partial S(M) = \mathcal{SM}(M).$$

Moreover, a state s is a state-morphism iff $s(x \wedge y) = \min\{s(x), s(y)\}$, $x, y \in M$, equivalently, $s(x \oplus y) = \min\{s(x) + s(y), 1\}$, $x, y \in M$.

Now, we exhibit topological properties of states. We say that a net $\{s_\alpha\}_\alpha$ of states on a pseudo EMV-algebra M *converges weakly* to a state s on M, and we write $\{s_\alpha\}_\alpha \stackrel{w}{\to} s$, if $\lim_\alpha s_\alpha(x) = s(x)$ for every $x \in M$. Hence, $\mathcal{S}(M)$ is a subset of $[0,1]^M$. If M has a top element, then $\mathcal{S}(M)$ is either empty or compact non-void. If we endow $[0,1]^M$ with the product topology which is a compact Hausdorff space, we see that the weak topology, which is in fact the relative topology of the product topology of $[0,1]^M$, yields a Hausdorff topological space. The case $\mathcal{S}(M) = \emptyset$ is not excluded. In addition, the system of subsets of $\mathcal{S}(M)$ of the form $S(x)_{\alpha,\beta} = \{s \in \mathcal{S}(M) \colon \alpha < s(x) < \beta\}$, where $x \in M$ and $\alpha < \beta$ are real numbers, forms a subbase of the weak topology of states.

In the same way as we have defined the weak topology of states, we define the weak topology also for the set of state-morphisms. Then $\mathcal{SM}(M)$ is a closed subset of $\mathcal{S}(M)$, and $\mathcal{SM}(M)$ is also a Hausdorff space. The spaces $\mathcal{S}(M)$ and $\mathcal{SM}(M)$ are not necessarily compact sets because if, for a net $\{s_\alpha\}$ of states (state-morphisms), there is a limit $s(x) = \lim_\alpha s_\alpha(x)$, $x \in M$, s preserves $+$ $(\oplus, \wedge, \vee)$, but there is no guarantee that there is an element $x \in M$ with $s(x) = 1$ as the following example from [15, Ex 4.8] shows.

Example 5.4. *Let* $\mathcal{T}$ *be the set of all finite subsets of the set* $\mathbb{N}$ *of natural numbers. Then* $\mathcal{T} = (\mathcal{T}; \vee, \wedge, \oplus, 0)$ *is an EMV-algebra with respect to* $\vee = \cup$, $\wedge = \cap$, $\oplus = \vee$, *and* $0 = \emptyset$, *and* $\mathcal{SM}(\mathcal{T}) = \{s_n \colon n \in \mathbb{N}\}$, *where* $s_n(A) = \chi_A(n)$, $A \in \mathcal{T}$*. Given* $A \in \mathcal{T}$*, there is* $s(A) = \lim_n s_n(A) = 0$*, but* s *is not a state on* $\mathcal{T}$*.*

The weak topology of states gives also a criterion when a pseudo EMV-algebras is with top element:

Theorem 5.5. *Let* M *be a pseudo EMV-algebra. Define the following statements.*

(i) M *has a top element.*

(ii) *The space* $\mathcal{S}(M)$ *is compact.*

(iii) *The space* $\mathcal{SM}(M)$ *is compact.*

Then (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii). *If* M *has the property that every maximal ideal of* M *is normal, then all statements are equivalent.*

Using Basic Representation Theorem 4.1, we can describe states on a pseudo EMV-algebra M and on its representing pseudo EMV-algebra N with top element, for details see [18, Prop 8.11]:

Proposition 5.6. *Let* M *be a pseudo EMV-algebra without top element. For each* $x \in M$, *we put* $x^- = \lambda_1(x)$ *and* $x^\sim = \rho_1(x)$, *where* 1 *is the top element of the representing pseudo EMV-algebra* N. *Given a state* s *on* M, *the mapping* $\tilde{s} : N \to [0,1]$, *defined by*

$$\tilde{s}(x) = \begin{cases} s(x) & \text{if } x \in M, \\ 1 - s(x_0) & \text{if } x = x_0^\sim,\ x_0 \in M, \end{cases} \qquad x \in N, \tag{13}$$

is a state on N, *and the mapping* $s_\infty : N \to [0,1]$ *defined by* $s_\infty(x) = 0$ *if* $x \in M$ *and* $s_\infty(x) = 1$ *if* $x \in N \setminus M$, *is a state-morphism on* N. *If* s *is a state-morphism on* M, *then* $\tilde{s}$ *is a state-morphism on* N. *Moreover,* $\mathcal{SM}(N) = \{\tilde{s} \colon s \in \mathcal{SM}(M)\} \cup \{s_\infty\}$ *and* $\mathrm{Ker}(\tilde{s}) = \mathrm{Ker}(s) \cup \mathrm{Ker}_1^*(s)$, $s \in \mathcal{SM}(M)$, *where* $\mathrm{Ker}_1^*(s) = \{\rho_1(x) \colon x \in \mathrm{Ker}_1(s)\}$ *and* $\mathrm{Ker}_1(s) = \{x \in M \colon s(x) = 1\}$.

A net $\{s_\alpha\}_\alpha$ *of states on* M *converges weakly to a state* s *on* M *if and only if* $\{\tilde{s}_\alpha\}_\alpha$ *converges weakly to* $\tilde{s}$ *on* N, *and the mapping* $\phi : \mathcal{S}(M) \to \mathcal{S}(N)$ *defined by* $\phi(s) = \tilde{s}$, $s \in \mathcal{S}(M)$, *is injective, continuous and affine.*

We have also the following topological characterizations:

Theorem 5.7. *If a pseudo EMV-algebra* M *does not have a top element, then* $\mathcal{SM}(M)$ *is either an empty set or is a locally compact non-empty Hausdorff space in the weak topology such that if* a *is an idempotent, then* $S(a) = \{s \in \mathcal{SM}(M) \colon s(a) > 0\}$ *is a compact clopen subset.*

Theorem 5.8. *Let* M *be a pseudo EMV-algebra with the property that every maximal ideal of* M *is normal. If the state space of a pseudo EMV-algebra* M *is non-empty and locally compact, then* M *has a top element.*

The next result describes state-morphisms on a pseudo EMV-algebra without top element and on a representing pseudo EMV-algebra with top element. According to Theorem 5.7, $\mathcal{SM}(M)$ is a locally compact Hausdorff topological space. Due to the Alexandroff theorem, see [27, Thm 4.21], we can do a one-point compactification of the set of state-morphisms:

Theorem 5.9. *Let* M *be a proper pseudo EMV-algebra and* N *be its representing pseudo EMV-algebra with top element. If* $\mathcal{SM}(M)$ *is non-empty, then* $\mathcal{SM}(N)$ *is the one-point compactifications of the space* $\mathcal{SM}(M)$.

State-morphisms which are in fact extremal states generate due to the Krein-Mil'man theorem all states on a pseudo EMV-algebra M in the following way, for details see [18, Thm 8.18]:

Theorem 5.10. *Let* M *be a pseudo EMV-algebra. Then*

$$\mathcal{S}(M) = (\mathrm{Con}(\mathcal{SM}(M)))^{-_M}, \tag{14}$$

where $^{-_M}$ *and* Con *denote the closure in the weak topology of states on* M *and the convex hull, respectively.*

If we denote by $\mathcal{MN}(M)$ the set of maximal ideals of a pseudo MV-algebra M which are also normal, we can introduce the hull-kernel topology on $\mathcal{MN}(M)$, i.e. a topology, where all closed subsets in $\mathcal{MN}(M)$ are of the form $C(J)=\{I\in\mathcal{MN}(M)\colon I\supseteq J\}$. Then this topology yields a Hausdorff topology on $\mathcal{MN}(M)$. Moreover, we have the following characterization:

Theorem 5.11. *Let M be a pseudo EMV-algebra. The mapping $\theta:\mathcal{SM}(M)\to\mathcal{MN}(M)$ given by $\theta(s)=\mathrm{Ker}(s)$, $s\in\mathcal{SM}(M)$, is a homeomorphism.*

In the last part, we concentrate on integral representation of states. It generates an analogous result for states on MV-algebras, which was established in [29, 33] saying that every state is in a one-to-one correspondence with a σ-additive states on a Borel σ-algebra of an appropriate compact Hausdorff topological space.

In what follows, we use the notion of a simplex in a linear space V which is affinely isomorphic to a base for a lattice cone in some real linear space. A simplex K in a locally convex Hausdorff space is said to be (i) *Choquet* if K is compact, and (ii) *Bauer* if K and ∂K are compact, where ∂K is the set of extreme points of K. For more info about simplices, see [25]. Then we have the following characterization of the state space $\mathcal{S}(M)$ of a pseudo EMV-algebra M:

Theorem 5.12. *Let M be a pseudo EMV-algebra. The state space $\mathcal{S}(M)$ is either empty or it is a non-empty simplex. In addition, if M has the property that every maximal ideal of M is normal, the following statements are equivalent:*

(i) *M has a top element.*

(ii) *$\mathcal{S}(M)$ is a Choquet simplex.*

(iii) *$\mathcal{S}(M)$ is a Bauer simplex.*

We need the following notions. Let $\mathcal{B}(K)$ be the Borel σ-algebra of a Hausdorff topological space K generated by all open subsets of K. Every element of $\mathcal{B}(K)$ is said to be a *Borel set* and each σ-additive (signed) measure on it is said to be a *Borel measure*. We recall that a Borel measure μ on $\mathcal{B}(K)$ is called *regular* if

$$\inf\{\mu(O):\ Y\subseteq O,\ O\ \text{open}\}=\mu(Y)=\sup\{\mu(C):\ C\subseteq Y,\ C\ \text{compact}\} \tag{15}$$

for any $Y\in\mathcal{B}(K)$. For example, let δ_x be the Dirac measure concentrated at the point $x\in K$, i.e., $\delta_x(Y)=1$ iff $x\in Y$, otherwise $\delta_x(Y)=0$, then every Dirac measure is a regular Borel probability measure whenever K is compact, see e.g. [25, Prop 5.24].

The following principal result on representing states is from [18, Thm 9.2]:

Theorem 5.13. [Integral Representation of States] *Let M be a pseudo EMV-algebra such that either M has a top element or M does not have a top element but M has the property that each maximal ideal of M is normal. Given a state s on M, there is a unique regular Borel probability measure μ_s on the Borel σ-algebra $\mathcal{B}(\mathcal{S}(M))$ with $\mu_s(\mathcal{SM}(M))=1$ such that*

$$s(x)=\int_{\mathcal{SM}(M)}\hat{x}(t)\,\mathrm{d}\mu_s(t),\quad x\in M, \tag{16}$$

where $\hat{x}$ ($x\in M$) is a continuous affine mapping from $\mathcal{S}(M)$ into the interval $[0,1]$ such that $\hat{x}(s):=s(x)$, $s\in\mathcal{S}(M)$.

Moreover, if M has no top element, there is a one-to-one correspondence between the set of regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(M))$ and the set of regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(N))$ vanishing at $\{s_\infty\}$, where N is its representing pseudo EMV-algebra with top element.

We note that uniqueness of μ_s in the latter theorem is guaranteed by the condition $\mu_s(\mathcal{SM}(M)) = 1$. For example, if s is a state on M, then the Dirac measure δ_s on $\mathcal{S}(M)$ concentrated in the point s is a regular Borel probability measure on $\mathcal{B}(\mathcal{S}(M))$ such that $s(x) = \int \hat{x}(t)\,\mathrm{d}\delta_s(t)$, $x \in M$. But if s is not an extremal state, then $\delta_s(\mathcal{SM}(M)) = 0 \neq 1$.

Remark 5.14. *It is worthy of note that the first part of Theorem* 5.13 *can be reformulated in the following equivalent way: For every state s on a pseudo EMV-algebra M which has the property that every maximal ideal of M is normal, there is a unique regular Borel probability measure μ_s on $\mathcal{B}(\mathcal{SM}(M))$ such that* (16) *holds for each $x \in M$, where $\hat{x}$ is a continuous function from $\mathcal{SM}(M)$ into the real interval* $[0, 1]$ *such that $\hat{x}(s) = s(x)$, $s \in \mathcal{SM}(M)$.*

We note that due to Theorem 5.11, the topological spaces $\mathcal{SM}(M)$ and $\mathcal{MN}(M)$ are homeomorphic under the mapping $\theta(s) = \mathrm{Ker}(s)$. In view of Remark 5.14, we can rewrite the integral representation of a state in the form that, for a state s on M, there is a unique σ-additive probability measure ν_s on $\mathcal{B}(\mathcal{NM}(M))$ such that

$$s(x) = \int_{\mathcal{NM}(M)} \tilde{x}(I)\,\mathrm{d}\nu_s(I), \quad x \in M,$$

where $\tilde{x} : \mathcal{MN}(M) \to [0, 1]$ is given by $\tilde{x}(I) = \theta^{-1}(I)(x)$, $I \in \mathcal{MN}(M)$.

6 Conclusions

In the paper we gave a survey on the latest research on two generalizations of MV-algebras and generalized Boolean algebras such that a top element is not assumed a priori: One generalization is through commutative EMV-algebras introduced in [13] and the second one is through a non-commutative EMV-algebras called pseudo EMV-algebras studied firstly in [17, 18]. In Section 2, we presented EMV-algebras where a main result is Theorem 2.1 showing that every EMV-algebra M either has a top element and then is equivalent to an MV-algebra or it has no top element and then it can be embedded into an EMV-algebra N with top element as a maximal ideal of N. We have shown that we have countably many analogs of subvarieties. Unfortunately, the class of EMV-algebras is not a variety because it is not closed under subalgebras. Therefore, we have introduced wEMV-algebras which form a variety and the class of EMV-algebras can be naturally embedded into this variety, Theorem 2.4.

Pseudo EMV-algebras are a non-commutative generalization of pseudo MV-algebras, generalized Boolean algebras and EMV-algebras. For them we have also presented a representation in an analogous way as Theorem 2.1, see Theorem 4.1. We also represented the category of proper pseudo EMV-algebras, i.e. with no top element, by a special category of pseudo MV-algebras with a fixed maximal and normal ideal, Theorem 4.3. We have defined also q-subvarieties of pseudo EMV-algebras. We have showed that we have uncountably many of q-subvarieties, Theorem 4.2, and some of them were presented in (12).

Finally, we have defined states, which correspond to finitely additive measures. We showed that state-morphisms are precisely extremal states, Theorem 5.3. We studied the weak topology of states and we showed that in contrast to EMV-algebras, there are pseudo EMV-algebras without any state. State-morphisms on M can be uniquely extended to state-morphisms on the representing pseudo EMV-algebra N, and if M is proper, the one-point compactification of the set of state-morphisms on M is precisely the set of state-morphisms of N, see Theorem 5.9. Finally, an integral representation of states by σ-additive probability measures on the Borel σ-algebra of $\mathcal{S}(M)$ is presented in Theorem 16.

We note that the class PEMV of pseudo EMV-algebras is not a variety. Therefore, it would be nice to have an analogue of the variety of non-commutative wEMV-algebras such that PEMV can be naturally embedded into the variety in the same way as in the case of EMV-algebras.

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