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Extended ideals in residuated lattices

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Abstract

In this paper, we generalized the notion of extended ideals and stable ideals associated to a subset B of a residuated lattices L and, we discuss what kind of residuated lattices have extended ideals. Then we investigate their related properties. We show that if L is an involutive residuated lattice, then $E_I(B)$ is a stable ideal relative to B, and so $E_I(B)$ is the smallest stable ideal relative to B. We also give a characterization of this extended ideal in the complete Heyting algebra. We also prove that the class $S(B)$ of all stable ideals relative to B is a complete Heyting algebra. Finally, we prove that the set of extended ideals and the set of extended filters on involutive residuated lattices are one-to-one correspondence.

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1 Introduction

In modern fuzzy logic theory, residuated lattices and some related algebraic systems play an essential role, because they provided an algebraic frameworks to fuzzy logic and fuzzy reasoning. The residuated lattices were introduced by Ward and Dilworth in [23] as a generalization of ideal lattices of rings. More studies on residuated lattices were developed by Jipsen, Kowalski, Ono and Tsinakis, [12, 14]. By using the theory of residuated lattices, Pavelka built up a more generalized logic systems and proved the semantical completeness of the Łukasiewicz's axiom system [17]. Residuated lattices are very basic and an essential algebraic structure, because the other logical algebras are all particular cases of residuated lattices [1, 23], such as MV algebras by Chang [5], BL-algebras by Hájek [9], R0-algebras by Wang [21]. Therefore, studying the algebraic structure of residuated lattices is very meaningful.

Involutive residuated lattices have received considerable attention from the logic and algebra communities. From a logical perspective, they were the algebraic counterparts of the propositional non-commutative linear logic without exponentials. From an algebraic perspective, they provided a common framework within which a various of disparate structures, such as Boolean algebras, MV-algebras, lattice-ordered groups and relation algebras, can be studied. In the meantime, ideal theory is a very effectively tool for investigating these various algebraic and logic systems. The ideal notion has been introduced in many algebraic structure such as lattices, rings, MV-algebras, lattice implication algebras. In these algebraic structure, as filter, the ideal is in the center position. Several [15, 16, 20] have claimed that the notion of an ideal is missing in BL-algebras. This has been partially associated with the fact that there was no suitable algebraic addition in BL-algebras. It is known the importance of ideals and congruences in classification problems, data organization and formal concept analysis, so it is significant to make an intensive study of ideals in residuated lattices. In residuated lattices, the notion of ideal was introduced as a natural generalization of that of ideal in MV algebras. After many scholars' research, many good results have been obtained. Dana Piciu introduced minimal prime ideal in residuated lattices and they proved since $mP(L)$, the set of minimal prime ideals of L, and $M(L)$, the set of maximal ideals of L, are subsets of $P(L)$, they endowed $mP(L)$ and $M(L)$ with the topology induced by the Zariski topology on $P(L)$ and they characterized these topological spaces for residuated [18]. S. Khosravi Shoar surveyed the structure of various ideals and found some types of ideals, such as positive implicative ideals, strong ideals and MV-ideals in residuated lattices. They also introduced the concept of quasi ideals and showed that any ideal is a quasi ideal, but the converse does not hold in general [19]. Francis Woumfoe introduced the notion of state ideal in the framework of state residuated lattices and presented two types of state residuated lattices: state i-simple residuated lattices and state i-local residuated lattices, and characterized them. Moreover, they proved that the lattice of all state ideals of a given state residuated lattice is complete [24]. Holdon, Liviu, and Arsham Borumand Saeid investigated some related results between the obstinate ideals and other types of ideals of a residuated lattice, likeness Boolean, primary, prime, implicative, maximal and $\odot$-prime ideals, and they proved that an ideal is an $\odot$-prime ideal if and only if its quotient algebra is an $\odot$-residuated lattice [11]. Wei Wang, and Bin Zhao investigated the topologies constructed by some ideals on residuated lattices and some topologies induced by lattice ideals and distance functions on involutive residuated lattices [22].

The notion of extended ideals in MV-algebras is introduced by F. Forouzesh [6]. They also proved I is a stable ideal relative to $B \subseteq A$ such that $B \cap I = \emptyset$ if and only if A/I is a chain MV-algebra, and the class $S(B)$ of all stable ideals relative to $B \subseteq A$ is also a complete Heyting algebra, for an MV-algebra A. They also considered the quotient algebras induced by stable ideals and proved some related theorems.

The organization of this paper is as follows. In Section 2, we recall some definitions and results which will be used in the following. In Section 3, we introduce the notion of extended ideals and stable ideals associated to a subset B of a residuated lattices L and we discuss what kind of residuated lattices have extended ideals. Then, we investigate their related properties. We show that if L is an involutive residuated lattice, then $E_I(B)$ is a stable ideal relative to B, and so $E_I(B)= \bigcap \{J\colon\ J$ is stable relative to B, and $I \subseteq J\}$. Also, we consider the quotient algebras induced by extended ideals and prove some related theorems. Finally, for some conclusions that hold on MV-algebras, we prove corresponding counterexamples to prove that they do not hold on involutive residuated lattices. In Section 4, We give a characterization of this extended ideal: $E_I(B) = (B] \to I$ in the complete Heyting algebra $(\mathrm{Id}(L), \wedge, \vee, \to, \{0\}, L)$. We also prove that the class $S(B)$ of all stable ideals relative to $B \subseteq L$ is a complete Heyting algebra, and we prove that the set of all extended ideals and the set of all extended filters on involutive residuated lattices are one-to-one correspondence.

2 Preliminaries

Definition 2.1. [3] *A residuated lattice is an algebra* $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$*, where* $\vee$*,* $\wedge$*,* $\odot$*,* $\rightarrow$ *are binary operations on* L *and 0,1* $\in L$ *such that:*
(1) $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice;*
(2) $(L, \odot, 1)$ *is a commutative monoid;*
(3) *For* $x, y, z \in L$*,* $x \odot z \leq y$ *iff* $z \leq x \rightarrow y$*.*

In a residuated lattice L, for every $x, y \in L$, we consider the following identities:
(div) $x \odot (x \rightarrow y) = x \wedge y$ (divisibility);
(prel) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity);
(inv) $x^{**} = x$ (involutivity).
Note: We say that the element $x \in L$ has order n and we write $\mathrm{ord}(x) = n$, if n is the smallest natural number such that $nx = 1$, where $x \oplus y = (x^* \odot y^*)^* = x^* \rightarrow y$.

Proposition 2.2. [2, 9, 23] *Let* L *be a residuated lattice, then, for any x, y, z,* $\in L$*, we have:*
(1) $1 \rightarrow x = x, x \rightarrow x = 1$;
(2) $x \odot y \leq x, y,$ *hence* $x \odot y \leq x \wedge y, x \leq y \rightarrow x$ *and* $x \odot 0 = 0$;
(3) $x \leq y$ *if and only if* $x \rightarrow y = 1$;
(4) $x \rightarrow 1 = 1, 0 \rightarrow x = 1$;
(5) $x \leq (x \rightarrow y) \rightarrow y$;
(6) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y) \leq z \rightarrow (x \rightarrow y)$;
(7) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ *and* $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$;
(8) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z, z \rightarrow x \leq z \rightarrow y, x \odot z \leq y \odot z, y^* \leq x^*,$ *and* $x^{**} \leq y^{**}$;
(9) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z),$ *in particular* $x \rightarrow y^* = y \rightarrow x^* = (x \odot y)^*$;
(10) $x \leq x^{**}, x^{***} = x^*$ *and* $x \leq x^* \rightarrow y$;
(11) $x \odot x^* = 0, x \odot y = 0$ *iff* $x \leq y^*$;
(12) $x^* \odot y^* \leq (x \odot y)^*$ *so,* $(x^*)^n \leq (x^n)^*$*, for every* $n \geq 1$;
(13) $x^{**} \odot y^{**} \leq (x \odot y)^{**}$ *so,* $(x^{**})^n \leq (x^n)^{**}$*, for every* $n \geq 1$;
(14) $(x \vee y)^* = x^* \wedge y^*$;
(15) $(x \rightarrow y^{**})^{**} = x \rightarrow y^{**}$;
(16) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
(17) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

Here we define $B(L) = \{a \in L \mid a \wedge b = 0$ and $a \vee b = 1, \exists b \in L\}$, $B^{\perp} = \{a \in L \mid a \wedge b = 0, \forall b \in B\}$, If $f : L_1 \rightarrow L_2$ is an morphism of residuated lattices, then $\mathrm{Ker}(f) = \{x \in L_1 : f(x) = 0\}$.

Proposition 2.3. [10] *Let* L *be an involutive residuated lattice, then, for any x, y* $\in L$*, we have:*
(1) $x \rightarrow y = [x \odot (y \rightarrow 0)] \rightarrow 0 = (y^* \odot x)^*$*;*
(2) $(x \wedge y) \rightarrow 0 = (x \rightarrow 0) \vee (y \rightarrow 0)$*;*
(3) $(x \rightarrow y) \vee (y \rightarrow x) = 1 \Leftrightarrow x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$*.*

Definition 2.4. [18] *A residuated lattice L is an MV-algebra iff it satisfies the additional condition:* $(x \to y) \to y = (y \to x) \to x$ *for every* $x,\ y \in L$.

Theorem 2.5. [18] *A residuated lattice L is an MV-algebra iff it satisfies the additional condition:*
(1) $(x \to y) \vee (y \to x) = 1$;
(2) $x \odot (x \to y) = x \wedge y$;
(3) $x^{**} = x$, *for every* $x,\ y \in L$.

Definition 2.6. [3] *A nonempty subset* I *of a residuated lattice* L *is called an ideal of* L *if :*
(1) *If* $x \leq y$ *and* $y \in I$, *then* $x \in I$;
(2) *If* $x, y \in I$, *then* $x^* \to y \in I$.

We denote by $Id(L)$ the set of ideals of a residuated lattice L. An ideal I of L is proper iff $I \neq L$. For a nonempty subset S of L, we denote by $(S]$ the ideal of L generated by S and for $x \in L$ we denote $(\{x\}]$ by $(x]$. Also, for $I \in Id(L)$ and $x \in L$, $I(x) = (I \cup \{x\}]$.

Proposition 2.7. [3, 4] *Let* L *be a residuated lattice,* $S \subseteq L$ *a nonempty subset,* $x, y \in L$ *and* $I \in Id(L)$, *then:*
(1) $(S] = \{x \in L : x \leq s_1 \oplus \ldots \oplus s_n, for\ some\ n \geq 1\ and\ \ s_1, \ldots, s_n \in S\}$;
(2) $(x] = \{z \in L : z \leq nx, for\ some\ n \geq 1\}$;
(3) $I(x) = \{z \in L : z \leq i \oplus nx, for\ some\ i \in I\ and\ n \geq 1\}$;
(4) $(\mathrm{Id}(L), \subseteq)$ *is a complete lattice, where for* $I_1, I_2 \in Id(L)$, $I_1 \wedge I_2 = I_1 \cap I_2$ *and* $I_1 \vee I_2 = (I_1 \cup I_2]$=$\{x \in L : x \leq i_1 \oplus i_2$, *with* $i_1 \in I_1$ *and* $i_2 \in I_2\}$.

Definition 2.8. [8] *A Heyting algebra is a lattice* $(A, \vee, \wedge)$ *with 0 such that for every* $a, b \in A$, *there exists an element* $a \to b \in A$ *such that for every* $x \in A$, $a \wedge x \leq b$ *if and only if* $x \leq a \to b$.

Note that in a Heyting algebra A, $x \odot x = x$, for $x \in A$, hence $x \odot y = x \wedge y = x \odot (x \to y)$, for all $x, y \in A$.

Proposition 2.9. [8] *Let* A *be a Heyting algebra, then, for any* $x,\ y,\ z \in A$, *we have:*
(1) $1 \to x = x$, $x \to x = 1$;
(2) $x \leq y$ *implies* $z \to x \leq z \to y$;
(3) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;
(4) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Proposition 2.10. [18] *Let* L *be a residuated lattice, then* $(Id(L), \vee, \wedge, \to, \{0\})$ *is a Heyting algebra, where for* $I, J \in Id(L)$, $I \wedge J = I \cap J$, $I \vee J = (I \cup J]$ *and* $I \to J = \{x \in A \mid I \cap (x] \subseteq J\}$.

Proposition 2.11. [18] *Let* L *be a residuated lattice and* $I_1, I_2 \in Id(L)$, *then:*
(1) $I_1 \to I_2 \in Id(L)$;
(2) *If* $I \in Id(L)$, *then* $I_1 \cap I \subseteq I_2$ *iff* $I \subseteq I_1 \to I_2$, *that is,* $I_1 \to I_2 = \sup\{I \in Id(L) : I_1 \cap I \subseteq I_2\}$.

Definition 2.12. [15] *Let* I *be a proper ideal of a residuated lattice* L. I *is said to be a prime ideal of the first kind if for any* $x, y \in L$, $(x \to y)^* \in I$ *or* $(y \to x)^* \in I$.

Definition 2.13. [15] *Let* I *be a proper ideal of a residuated lattice* L. I *is said to be a prime ideal of the second kind if for any* $x, y \in L$, $x \wedge y \in I$ *implies* $x \in I$ *or* $y \in I$.

Proposition 2.14. [15] *Let* L *be a residuated lattice. Every prime ideal of the first kind of* L *is also a prime ideal of the second kind of* L. *If* L *satisfies prelinearity, then every prime ideal of the second kind of* L *is also a prime ideal of the first kind of* L.

Definition 2.15. [7, 13] *Let L be a residuated lattice, F be a filter of L and $B \subseteq L$. We define the extended filter of F associated with B as follows: $E_F(B) = \{x \in L \mid x \vee b \in F, \forall b \in B\}$.*

Theorem 2.16. [6] *Let A be an MV-algebra, $B \subseteq A$ and I be an ideal of A, then I is a stable ideal relative to B such that $B \cap I = \emptyset$ if and only if A/I is a chain.*

Theorem 2.17. [6] *Let A be an MV-algebra, then A is a chain if and only if for $x, y \in A$, $x \wedge y = 0$, implies $x = 0$ or $y = 0$.*

3 Extended ideals in residuated lattices

In this Section, we introduce the notion of extended ideals and stable ideals associated to a subset B of a residuated lattices L and, we discuss what kind of residuated lattices have extended ideals. Then, we investigate their related properties. We show that if L is an involutive residuated lattice, then $E_I(B)$ is a stable ideal relative to B and so $E_I(B)$ is the smallest stable ideal relative to B. Also, we consider the quotient algebras induced by extended ideals and prove some related theorems.

Definition 3.1. *Let L be a residuated lattice, I be an ideal of L and $B \subseteq L$. We denote the set of I associated with B as follows: $E_I(B) = \{x \in L \mid x \wedge b \in I, \forall b \in B\}$.*
If $E_I(B)$ is an ideal of L, then $E_I(B)$ is called an extended ideal of L.

Example 3.2. *Let $L = \{0, a, b, c, d, 1\}$ with $0 \leq a,\ b,\ c \leq d \leq 1$, where a,b and c are incomparable. Define operations $\odot$ and $\rightarrow$ on L as follows.*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	d	1	1
b	d	d	1	d	1	1
c	d	d	d	1	1	1
d	d	d	d	d	1	1
1	0	a	b	c	d	1

Then L is a residuated lattice, and clearly, $I = \{0\}$ is an ideal of L. Let $B = \{c\}$. Then $E_I(B) = \{0, a, b\}$ is not an ideal of L. Thus, $E_I(B) = \{0, a, b\}$ is not an extended ideal of L.

Remark 3.3. *Example 3.2 shows that $E_I(B)$ is not an extended ideal of L. Since $a^* \rightarrow b = d \notin I$, $E_I(B) = \{0, a, b\}$ is not an ideal of L. This means that Definition 3.1 is reasonable.*

Example 3.4. *Let $L = \{0, a, b, c, 1\}$ with $0 \leq a,\ b \leq c \leq 1$, where a and b are incomparable. Define operations $\odot$ and $\rightarrow$ on L as follows.*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $L = \{0, a, b, c, 1\}$ is a residuated lattice, and it is clear $I = \{0, a\}$ is an ideal of L. Let $B_1 = \{c\}$ and $B_2 = \{a\}$. Then $E_I(B_1) = \{0, a\}$ and $E_I(B_2) = L$. Thus, $E_I(B_1) = \{0, a\}$ and $E_I(B_2) = L$ is are extended ideals of L.

Example 3.5. *Let $L = \{0, n, a, b, c, d, m, 1\}$ with $0 \leq n \leq a$, $b \leq c \leq m \leq 1$, where a and b be incomparable and $b \leq c$, $d \leq m \leq 1$, where c and d are incomparable. Define operations $\odot$ and $\rightarrow$ on L as follows.*

$\odot$	0	n	a	b	c	d	m	1
0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	a
b	0	0	0	0	0	b	b	b
c	0	0	a	0	a	b	c	c
d	0	0	0	b	b	d	d	d
m	0	0	a	b	c	d	m	m
1	0	n	a	b	c	d	m	1

$\rightarrow$	0	n	a	b	c	d	m	1
0	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1
a	d	d	1	d	1	d	1	1
b	c	c	c	1	1	1	1	1
c	b	b	c	d	1	d	1	1
d	a	a	a	c	c	1	1	1
m	n	n	a	b	c	d	1	1
1	0	n	a	b	c	d	m	1

Then $L = \{0, n, a, b, c, d, m, 1\}$ is a residuated lattice and it is clear $I_1 = \{0, n, a\}$ and $I_2 = \{0, n, b\}$ are ideals of L. Let $B = \{c\}$. Then $E_{I_1}(B) = \{0, n, a\}$ and $E_{I_2}(B) = \{0, n, b, d\}$. Thus, $E_{I_1}(B) = \{0, n, a\}$ and $E_{I_2}(B) = \{0, n, b, d\}$ is an extended ideal of L.

From Example 3.2, we can see that there may be no extended ideal on a residuated lattice. Now, we discuss what kind of residuated lattices have extended ideals.

Theorem 3.6. *Let L be an involutive residuated lattice, I be an ideal of L and $B \subseteq L$, then $E_I(B)$ is an extended ideal of L.*

Proof. We have $0 \wedge b = 0 \in I$, for all $b \in B$ and so $0 \in E_I(B)$. For $x, y, z \in L$, we have

$$\begin{aligned}
(x \odot y) \vee z &= (x \odot y) \vee z \vee z \vee z \\
&\geq (x \odot y) \vee (z \odot y) \vee (z \odot x) \vee (z \odot z) \\
&= ((x \vee z) \odot y) \vee ((x \vee z) \odot z) \\
&= (x \vee z) \odot (y \vee z).
\end{aligned}$$

So $(x \odot y) \vee z \geq (x \vee z) \odot (y \vee z)$. Suppose that $x, y \in E_I(B)$. Then $x \wedge b, y \wedge b \in I$, for all $b \in B$. Since I is an ideal, so $(x \wedge b)^* \rightarrow (y \wedge b) \in I$. Then, we have

$$\begin{aligned}
& (x^* \odot y^*) \vee b^* \geq (x^* \vee b^*) \odot (y^* \vee b^*) \\
\Leftrightarrow\ & (x^* \odot y^*) \vee b^* \geq (x \wedge b)^* \odot (y \wedge b)^* \\
\Leftrightarrow\ & (x^* \odot y^*)^* \wedge b \leq ((x \wedge b)^* \odot (y \wedge b)^*)^* \\
\Leftrightarrow\ & (x^* \rightarrow y) \wedge b \leq (x \wedge b)^* \rightarrow (y \wedge b).
\end{aligned}$$

Since $(x \wedge b)^* \rightarrow (y \wedge b) \in I$, we conclude $(x^* \rightarrow y) \wedge b \in I$. This results $x^* \rightarrow y \in E_I(B)$. Now, let $x \leq y, y \in E_I(B)$. It is clear that $x \in E_I(B)$. Thus. $E_I(B)$ is an ideal of L. Also, let $x \in I$. Since $x \wedge b \leq x$, for all $b \in B$. Since I is an ideal, we obtain $x \wedge b \in I$, for all $b \in B$. This means $x \in E_I(B)$. So $I \subseteq E_I(B)$. □

Definition 3.7. *An ideal I is a stable relative to a subset B of L if $I = E_I(B)$.*

Example 3.8. *In Example 3.4, $I = \{0, a\}$ is a stable ideal relative to $B = \{c\}$ and in Example 3.5, $I_1 = \{0, n, a\}$ is a stable ideal relative to $B = \{c\}$.*

Theorem 3.9. *Let L be an involutive residuated lattice, I and J be ideals of L, and $B \subseteq L$. Then*
(1) $E_I(B) = L$ if and only if $B \subseteq I$;
(2) If $E_I(B)$ is an extended ideal of L, $x \in L$ is a finite order and $x \in E_I(B)$, then $E_I(B) = L$;
(3) If $I \subseteq J$, then $E_I(B) \subseteq E_J(B)$;
(4) If $x, y \in E_I(B)$, then $x^* \to y \in E_I(B)$;
(5) $B \subseteq E_I(E_I(B))$;
(6) If $I \subseteq J$, then $E_I(J) \cap J = I$;
(7) $E_I(E_I(B)) \cap E_I(B) = I$;
(8) If $B \subseteq C$, then $E_I(C) \subseteq E_J(B)$;
(9) $B^\perp \subseteq E_I(B)$;
(10) If $1 \in B$, then I is stable relative to B;
(11) $E_I(B) \to E_J(B) \subseteq I \to E_J(B)$;
(12) $E_{I \to J}(B) \subseteq E_{I \to E_J(B)}(B)$.

Proof. (1) Let $E_I(B) = L$ and $b \in B$. Since $b \in B$, $B \subseteq L$ and $E_I(B) = L$, so $b \in E_I(B)$, then $b = b \wedge b \in I$. Thus, $B \subseteq I$. Conversely, $B \subseteq I$, $x \in L$ and $b \in B$. Since $x \wedge b \leq b$, $B \subseteq I$ and I is an ideal, we obtain $x \wedge b \in I$, for all $b \in B$. This means $x \in E_I(B)$. Thus, $E_I(B) = L$.
(2) Let $x \in L$ be a finite order, then there exists $n \in N$ such that $nx = 1$. Suppose that $x \in E_I(B)$. Since $E_I(B)$ is an ideal, $nx = 1 \in E_I(B)$. Hence, $b = 1 \wedge b \in I$, for all $b \in B$ and so $B \subseteq I$. Using (1), we get $E_I(B) = L$.
(3) Let $I \subseteq J$ and $x \in E_I(B)$. Then $x \wedge b \in I \subseteq J$, for all $b \in B$. So, $x \in E_J(B)$. Thus, $E_I(B) \subseteq E_J(B)$.
(4) Let $x, y \in E_I(B)$. Then $x \wedge b \in I$ and $y \wedge b \in I$, for all $b \in B$. Since I is an ideal of L. So $(x \wedge b)^* \to (y \wedge b) \in I$. By the proof of Theorem 3.6, we have $(x^* \to y) \wedge b \leq (x \wedge b)^* \to (y \wedge b)$. Hence, $(x^* \to y) \wedge b \in I$. Thus, $x^* \to y \in E_I(B)$.
(5) Let $x \in B$ and $y \in E_I(B)$. Then $x \wedge y = y \wedge x \in I$. This means $x \in E_I(E_I(B))$.
(6) Let $x \in E_I(J) \cap J$. Hence $x = x \wedge x \in I$. Conversely, let $x \in I$ and $y \in J$. Then, we have $x \wedge y \leq x$. Since I is an ideal and $I \subseteq J$, we get $x \in E_I(J) \cap J$. Thus, $E_I(J) \cap J = I$.
(7) Let $x \in E_I(E_I(B)) \cap E_I(B)$. Then $x = x \wedge x \in I$. Conversely, let $x \in I$ and $y \in E_I(B)$. Clearly, $x \in I \subseteq E_I(B)$. Since I is an ideal and $x \wedge y \leq x$, we have $x \wedge y \in I$. Hence, $x \in E_I(E_I(B)) \cap E_I(B)$.
(8) Let $x \in E_I(C)$ and $B \subseteq C$. Then $x \wedge c \in I$, for all $c \in C$ and so $x \wedge c \in I$ for all $c \in B$. Hence, $x \in E_I(B)$.
(9) Let $a \in B^\perp$. Then $a \wedge b = 0 \in I$, for all $b \in B$. Hence $a \in E_I(B)$.
(10) Let $1 \in B$ and $x \in E_I(B)$. Then $x = x \wedge 1 \in I$. Hence, $E_I(B) \subseteq I$. Since $I \subseteq E_I(B)$. Thus, $I = E_I(B)$.
(11) Let $x \in E_I(B) \to E_J(B)$. Hence, $(x] \cap I \subseteq (x] \cap E_I(B) \subseteq E_J(B)$. Thus, $x \in I \to E_J(B)$.
(12) Let $x \in E_{I \to J}(B)$. Then $x \wedge b \in I \to J$, for all $b \in B$. It follows that $(x \wedge b] \cap I \subseteq J \subseteq E_J(B)$. Hence, $x \wedge b \in I \to E_J(B)$, for all $b \in B$. Thus, $x \in E_{I \to E_J(B)}(B)$. □

Proposition 3.10. *Let $f : L \to M$ be an onto residuated lattice-homomorphism, then $E_{Ker(f)}(B) = f^{-1}(E_{\{0\}}(C))$, for $B \subseteq L$ and $C \subseteq M$ such that $f(B) = C$.*

Proof. Clearly. □

Theorem 3.11. *Let L be a residuated lattice, I be a proper ideal of L and $B \subseteq L$. Then $E_I(B) \cap B \subseteq I$.*

Proof. Let $x \in E_I(B) \cap B$. Then $x \in E_I(B)$ and $x \in B$. So, $x \wedge b \in I$, for all $b \in B$. Now, let $b = x \in B$. Hence, $x = x \wedge x \in I$. Thus $E_I(B) \cap B \subseteq I$. □

Corollary 3.12. *Let $0 \in B$. Then $E_{\{0\}}(B) \cap B = \{0\}$.*

Theorem 3.13. *Let L be an involutive residuated lattice, I be a proper ideal of L and $B \subseteq L$. Then the following statements hold:*
(1) $E_{\{0\}/I}(B/I) = E_I(B)/I$,
(2) If $E_I(B(L)) \subseteq B(L)$, then $E_I(B(L)) = I$ and $B(L/E_I(B(L)) = B(L)/E_I(B(L))$.

Proof. (1) $x/I \wedge b/I = 0/I \Leftrightarrow x \wedge b \in I \Leftrightarrow x \in E_I(B) \Leftrightarrow x/I \in E_I(B)/I$.
(2) Let $E_I(B(L)) \subseteq B(L)$. By Theorem 3.11, we have $E_I(B(L)) = I$ and

$$\begin{aligned} B(L)/E_I(B(L)) &= \{e/E_I(B(L)) : e \in B(L)\}, \\ &= \{e/E_I(B(L)) : e \vee e^* = 1\}, \\ &= \{e/E_I(B(L)) : e/E_I(B(L)) \vee e^*/E_I(B(L)) = 1/E_I(B(L))\}, \\ &= B\left(L/E_I(B(L)\right). \end{aligned}$$

□

Proposition 3.14. *Let A, B be involutive residuated lattices, $A' \subseteq A$ and $B' \subseteq B$. If $f : A \to B$ be a homomorphism such that $f(A') = B'$, then the following statements hold:*
(1) If I is a stable ideal relative to a subset B', then $f^{-1}(I)$ is a stable ideal relative to A'.
(2) If f is onto, I is a stable ideal relative to A' and $Ker(f) \subseteq I$, then $f(I)$ is a stable ideal relative to B'.

Proof. (1) Let I be a stable relative to B'. Then $I = E_I(B')$. We show that $f^{-1}(I) = f^{-1}\left(E_I\left(B'\right)\right) = E_{f^{-1}(I)}\left(A'\right)$

$$\begin{aligned} x \in E_{f^{-1}(I)}\left(A'\right) &\Leftrightarrow x \wedge a \in f^{-1}(I), \quad \text{for all } a \in A' \\ &\Leftrightarrow f(x) \wedge f(a) \in I, \quad \text{for all } a \in A', \\ &\Leftrightarrow f(x) \wedge b \in I, \quad \text{for all } b = f(a) \in B', \\ &\Leftrightarrow f(x) \in E_I\left(B'\right), \\ &\Leftrightarrow x \in f^{-1}\left(E_I\left(B'\right)\right), \\ &\Leftrightarrow x \in f^{-1}(I). \end{aligned}$$

(2) Let I be a stable relative to A'. Then $I = E_I(A')$. We show that $E_{f(I)}\left(B'\right) = f\left(E_I\left(A'\right)\right) = f(I)$.
Let $x \in f\left(E_I\left(A'\right)\right)$. Then there exists $t \in E_I\left(A'\right)$ such that $x = f(t)$. So, $t \wedge a \in I$, for all $a \in A'$. Hence, $x \wedge b = f(t) \wedge f(a) = f(t \wedge a) \in f(I)$, for all $b = f(a) \in B'$. Thus, $x \in E_{f(I)}\left(B'\right)$.
Conversely, let $x \in E_{f(I)}\left(B'\right)$, since f is onto, there exists $s \in A$ such that $f(s) = x$. We have

$$\begin{aligned} & x \in E_{f(I)}\left(B'\right) \\ \Leftrightarrow\ & x \wedge b \in f(I), \quad \text{for all } b \in B', \\ \Leftrightarrow\ & f(s) \wedge f(c) \in f(I), \quad \text{for all } b = f(c), c \in A', \\ \Leftrightarrow\ & f(s \wedge c) = f(t), \quad \exists t \in I, \quad \text{for all } c \in A', \\ \Leftrightarrow\ & (s \wedge c) \odot t^* \in \mathrm{Kerf} \subseteq I, \quad \exists t \in I, \quad \text{for all } c \in A', \\ \Leftrightarrow\ & s \wedge c \leq t \vee (s \wedge c) = t^* \to (t^* \odot (s \wedge c)) \in I, \quad \text{for all } c \in A', \end{aligned}$$

$$\begin{aligned}&\Leftrightarrow s\wedge c\in I, \quad \text{for all } c\in A',\\&\Leftrightarrow s\in E_I\left(A'\right),\\&\Leftrightarrow x\in f\left(E_I\left(A'\right)\right).\end{aligned}$$

Thus, $f(I)$ is a stable relative to B'. □

Theorem 3.15. *Let L be an involutive residuated lattice, I be an ideal of L and $B \subseteq L$. Then $E_I(B)$ is stable relative to B and so $E_I(B)=\bigcap\{J\colon J$ is stable relative to B and $I\subseteq J\}$.*

Proof. By Theorem 3.6, we have $E_I(B) \subseteq E_{E_I(B)}(B)$. Conversely, let $x \in E_{E_I(B)}(B)$. Then $x \wedge b \in E_I(B)$, for all $b \in B$. Hence, $(x \wedge b) \wedge b \in I$, for all $b \in B$. So $x \wedge b \in I$, for all $b \in B$. Thus, $x \in E_I(B)$. Hence $E_{E_I(B)}(B) \subseteq E_I(B)$. So $E_{E_I(B)}(B) = E_I(B)$. Therefore, $E_I(B)$ is stable relative to B.
Now, let J be a stable relative to B such that $I \subseteq J$. By Theorem 3.9(3), we have $E_I(B) \subseteq E_J(B) = J$. So $E_I(B)=\bigcap\{J\colon J$ is stable relative to B and $I\subseteq J\}$. □

Theorem 3.16. *Let L be an involutive residuated lattice, I be an ideal of L and $B \subseteq L$, then $E_{E_I(B)/I}(B/I) = \left(E_{E_I(B)}(B)\right)/I = E_I(B)/I$.*

Proof. Let $a/I \in E_{E_I(B)/I}(B/I)$. Then $a/I \wedge b/I \in E_I(B)/I$, for all $b/I \in B/I$, so $(a\wedge b)/I = t/I$, for some $t \in E_I(B)$. Since $t \in E_I(B)$, hence $a \wedge b \le t \vee (a \wedge b) = t^* \to (t^* \odot (a \wedge b)) \in E_I(B)$, so $a \wedge b \in E_I(B)$, for all $b \in B$. Thus, $a \in E_{E_I(B)}(B)$. Therefore, $a/I \in \left(E_{E_I(B)}(B)\right)/I$.

Conversely, let $a/I \in \left(E_{E_I(B)}(B)\right)/I$. Then $a \in E_{E_I(B)}(B)$, and so $a\wedge b \in E_I(B)$, for all $b \in B$. Hence, $a/I \wedge b/I = a \wedge b/I \in E_I(B)/I$, for all $b/I \in B/I$. This results $a/I \in E_{E_I(B)/I}(B/I)$, so $E_{E_I(B)/I}(B/I) = \left(E_{E_I(B)}(B)\right)/I$. It follows from Theorem 3.15 that $E_{E_I(B)/I}(B/I) = \left(E_{E_I(B)}(B)\right)/I = E_I(B)/I$. □

Theorem 3.17. *Let L be a residuated lattice, I be a prime ideal of the first kind of L and $B \subseteq L$. Then extended ideal $E_I(B)$ is a prime ideal of L.*

Proof. Let I be a prime ideal of the first kind of L. Then $(x \to y)^* \in I$ or $(y \to x)^* \in I$, for all $x, y \in L$. Since $(x \to y)^* \wedge b \le (x \to y)^* \in I$ or $(y \to x)^* \wedge b \le (y \to x)^* \in I$, for all $x, y \in L$, $b \in B$. So $(x \to y)^* \in E_I(B)$ or $(y \to x)^* \in E_I(B)$, for all $x, y \in L$. □

Theorem 3.18. *Let L be a residuated lattice, I be a prime ideal of the second kind of L and $B \subseteq L$, then so is the extended ideal $E_I(B)$.*

Proof. Let I be a prime ideal of the second kind of L and $x \wedge y \in E_I(B)$. Then $(x \wedge y) \wedge b \in I$, for all $b \in B$. Since I is a prime ideal of the second kind of L. So $x \in I$ or $y \wedge b \in I$, hence $x \wedge b \le x \in I$ or $y \wedge b \in I$. Thus $x \in E_I(B)$ or $y \in E_I(B)$. □

Theorem 3.19. *Let L be an involutive residuated lattice, I be a prime ideal of the second kind of L and $B \subseteq L$. Then I is stable relative to B.*

Proof. Assume I is not a stable relative to B. Then $E_I(B) \neq I$. There exists $x \in E_I(B)$ such that $x \notin I$. Hence $x \wedge b \in I$, for all $b \in B$. Since I is a prime ideal of the second kind of L, so $b \in I$. Hence $B \subseteq I$. By Theorem 3.9(1), we have $E_I(B) = L$, which is a contradiction. □

Remark 3.20. *Let L be an involutive residuated lattice, $B \subseteq L$, I be a maximal ideal and $E_I(B)$ be a proper ideal of L. Then $E_I(B)$ is a maximal ideal and I is a stable relative to B.*

Theorem 3.21. *Let L be an involutive residuated lattice, $B \subseteq C \subseteq L$ and I be a stable ideal relative to B. Then I is a stable ideal relative to C.*

Proof. Since I is a stable ideal relative to B, we get $I = E_I(B)$. By Theorem 3.6 and Theorem 3.9(8), we have $I \subseteq E_I(C) \subseteq E_I(B) = I$. So $I = E_I(C)$. Thus, I is a stable ideal relative to C. $\square$

Theorem 3.22. *Let L be a residuated lattice, I be an ideal of L and $B \subseteq L$. Then*
(1) $E_I(B) = E_I((B])$*, where* $(B]$ *is the ideal generated by* B.
(2) *If* $B \subseteq I$*, then* $(B] \cap E_I((B]) = (B]$.
(3) $a \in E_I(B)$ *if and only if* $a/I \in (B/I)^{\perp}$.

Proof. (1) Since $B \subseteq (B]$, by using Theorem 3.9(8), we obtain $E_I((B]) \subseteq E_I(B)$. Suppose that $x \in E_I(B)$ and $z \in (B]$. Then $x \wedge b \in I$, for all $b \in B$ and there exists $b_1, b_2, \cdots, b_n \in B$ such that $z \leq b_1 \oplus \cdots \oplus b_n$. Thus, $x \wedge z \leq x \wedge (b_1 \oplus \cdots \oplus b_n) \leq (x \wedge b_1) \oplus \cdots \oplus (x \wedge b_n)$, since $x \wedge b_i \in I$, for all $1 \leq i \leq n$ and I is an ideal of L, we get $x \wedge z \in I$. Thus, $x \in E_I((B])$.
(2) Since $B \subseteq I$, by using Theorem 3.9(1), we have $E_I(B) = L$. Using (1), we get $E_I(B) = E_I((B])$, so $E_I((B]) = L$. Thus, $(B] \cap E_I((B]) = (B]$.
(3) We have

$$\begin{aligned} a \in E_I(B) &\Leftrightarrow a \wedge x \in I, \quad \forall x \in B \\ &\Leftrightarrow (a \wedge x)/I = 0/I, \quad \forall x \in B \\ &\Leftrightarrow a/I \wedge x/I = 0/I, \quad \forall x/I \in B/I \\ &\Leftrightarrow a/I \in (B/I)^{\perp} \end{aligned}$$

$\square$

Theorem 3.23. *Let L be an involutive residuated lattice, $B, C \subseteq L$ and I be an ideal of L. Then $E_{E_I(B)}(C) = E_{E_I(C)}(B)$.*

Proof. Let L be an involutive residuated lattice, $B, C \subseteq L$ and I be an ideal of L. Then, we have

$$\begin{aligned} & x \in E_{E_I(B)}(C) \\ \Leftrightarrow\ & x \wedge c \in E_I(B), \quad \text{for all } c \in C, \\ \Leftrightarrow\ & (x \wedge c) \wedge b \in I, \quad \text{for all } c \in C, b \in B, \\ \Leftrightarrow\ & (x \wedge b) \wedge c \in I, \quad \text{for all } c \in C, b \in B, \\ \Leftrightarrow\ & x \wedge b \in E_I(C), \quad \text{for all } b \in B, \\ \Leftrightarrow\ & x \in E_{E_I(C)}(B). \end{aligned}$$

$\square$

Theorem 3.24. *Let L be an involutive residuated lattice, $\{I_j\}_{j \in J}$ be a family of ideals of L and $B \subseteq L$. Then*
(1) $\bigcap_{j \in J} E_{I_j}(B) = E_{\cap I_j}(B)$.
(2) *If* $\{I_j\}_{j \in J}$ *is a chain of ideals, then* $\bigcup_{j \in J} E_{I_j}(B) = E_{\cup I_j}(B)$.

Proof. (1) Let $x \in \bigcap_{j\in J} E_{I_j}(B)$. Then

$$x \in \bigcap_{j\in J} E_{I_j}(B) \Leftrightarrow x \in E_{I_j}(B), \quad \forall j \in J,$$
$$\Leftrightarrow x \wedge b \in I_j, \quad \forall b \in B, \quad \forall j \in J,$$
$$\Leftrightarrow x \wedge b \in \bigcap I_j, \quad \forall b \in B,$$
$$\Leftrightarrow x \in E_{\cap I_j}(B).$$

(2) Let $\{I_j\}_{j\in J}$ be a chain of ideals. It is easy to see that $\bigcup_{j\in J} I_j$ is an ideal of L. Let $x \in \bigcup_{j\in J} E_{I_j}(B)$. Then there exists $j \in J$ such that $x \in E_{I_j}(B)$, there exists $j \in J$ such that $x \wedge b \in I_j$, for all $b \in B$. Thus, $x \wedge b \in \cup I_j$, for all $b \in B$, and so $x \in E_{\cup I_j}(B)$. Therefore, $\bigcup_{j\in J} E_{I_j}(B) \subseteq E_{\cup I_j}(B)$.
Conversely, let $x \in E_{\cup I_j}(B)$. Then $x \wedge b \in \cup I_j$, for all $b \in B$. Since $\{I_j\}_{j\in J}$ a chain of ideals, there exists $j \in J$ such that $x \wedge b \in I_j$, for all $b \in B$. Hence, $x \in E_{I_j}(B)$, for some $j \in J$ and so $x \in \bigcup_{j\in J} E_{I_j}(B)$. Therefore, $E_{\cup I_j}(B) \subseteq \bigcup_{j\in J} E_{I_j}(B)$. □

The Theorem 2.16 is not true in involutive residuated lattices, now we provide a counterexample to illustrate it.

Example 3.25. *Let $L = \{0, a, b, c, d, 1\}$ with $0 \le a \le b$, $c \le d \le 1$, where b and c are incomparable. Define operations $\odot$ and $\to$ on L as follows.*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	a	a	b
c	0	0	a	0	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	b	d	1	d	1	1
c	c	d	d	1	1	1
d	a	d	d	d	1	1
1	0	a	b	c	d	1

We have verified that L is an involutive residuated lattice, but it is not an MV-algebra. Since $(b \to c) \vee (c \to b) = d \neq 1$, it is not satisfied prelinearity. Now we have $I = \{0\}$ is a stable ideal relative to $B = \{d\}$ such that $B \cap I = \emptyset$. However, L/I is not a chain. Since $[b]_{\approx_I} \to [c]_{\approx_I} = [b \to c]_{\approx_I} = [d]_{\approx_I} \neq [1]_{\approx_I}$, $[c]_{\approx_I} \to [b]_{\approx_I} = [c \to b]_{\approx_I} = [d]_{\approx_I} \neq [1]_{\approx_I}$, where $[b]_{\approx_I}$ and $[c]_{\approx_I}$ are not incomparable. So L/I is not a chain.
In Theorem 2.16, the sufficiency of the conclusion about MV-algebra is correct. We now provide the following proof.

Proposition 3.26. *Let L be an involutive residuated lattice , $B \subseteq L$ and I be an ideal of L such that $B \cap I = \emptyset$. If L/I be a chain, then I is a stable ideal relative to B.*

Proof. Let L/I be a chain and $a \in E_I(B)$. Hence, $a \wedge x \in I$, for $x \in B$. This results $a/I \wedge x/I = (a \wedge x)/I = 0/I$. Since L/I is chain, $a/I = 0/I$ or $x/I = 0/I$ and so $a \in I$ or $x \in I$. By assumption, since $B \cap I = \emptyset$, hence, $a \in I$. Thus, $E_I(B) \subseteq I$. Also, by Theorem 3.6, we have $I \subseteq E_I(B)$. Therefore $E_I(B) = I$ and so I is a stable ideal relative to B. □

The theory in Theorem 2.17 is not trulutive residuated lattices, now we provide a counterexample to illustrate it.

Example 3.27. *In Example 3.25, let $x \wedge y = 0$, for $x, y \in L$. We will discuss the following two situations:* (1) *if $x = 0$ or $y = 0$, then proposition is true. However, L is not a chain.* (2) *if $x, y \neq 0$, then $x, y \in \{a, b, c, d, 1\}$ such that $x \wedge y = 0$, hence, $x = 0$ or $y = 0$, however L is not a chain.*

In Theorem 2.17, the necessity of the conclusion about MV-algebra is correct. We now provide the following proof.

Proposition 3.28. *Let L be an involutive residuated lattice. If L is a chain. Then $x \wedge y = 0$ implies $x = 0$ or $y = 0$.*

Proof. Clearly. □

4 The structure of extended ideals

In this Section, We give a characterization of this extended ideal: $E_I(B) = (B] \to I$ in the complete Heyting algebra $(\mathrm{Id}(L), \wedge, \vee, \to, \{0\}, L)$. We also prove that the class $S(B)$ of all stable ideals relative to $B \subseteq L$ is a complete Heyting algebra, and we prove that the set of extended ideals and the set of extended filters on involutive residuated lattices are one-to-one correspondence.

Theorem 4.1. *Let L be an involutive residuated lattice, I be an ideal of L and $B \subseteq L$. Then $(S(B), \sqcap, \sqcup)$ is a lattice, where $S(B) = \{E_I(B) \mid I \in Id(L)\}$.*

Proof. By Theorem 3.15, we get $E_I(B)$ is a stable ideal relative to B. For all elements $E_I(B)$, $E_J(B) \in S(B)$, we define two operations $\sqcap$ and $\sqcup$ as follows: $E_I(B) \sqcap E_J(B) = E_{I\wedge J}(B)$ and $E_I(B) \sqcup E_J(B) = E_{I\vee J}(B)$, where $E_{I\wedge J}(B)$ (or $E_{I\vee J}(B)$) is infimum (supremum) of $E_I(B)$, $E_J(B)$ in $S(B)$. It is easy to show that $E_I(B) \sqcap E_J(B) = E_{I\wedge J}(B)$. We show that $E_{I\vee J}(B)$ is a supremum of $E_I(B)$, $E_J(B)$ in $S(B)$. By Theorem 3.9(3), we get $E_I(B)$, $E_J(B) \subseteq E_{I\vee J}(B)$. For any stable ideal relative to B, $E_K(B)$, such that $E_I(B)$, $E_J(B) \subseteq E_K(B)$, we prove that $E_{I\vee J}(B) \subseteq E_K(B)$. Let $x \in E_{I\vee J}(B)$. Then $x \wedge b \in I \vee J$, for all $b \in B$. Hence, $x \wedge b \leq a \oplus c$, for some $a \in I \subseteq E_I(B)$, $c \in J \subseteq E_J(B)$. We obtain $x \wedge b \in E_I(B) \vee E_J(B) \subseteq E_K(B)$, for all $b \in B$. Thus, $x \in E_{E_K(B)}(B) = E_K(B)$. This means that $E_{I\vee J}(B)$ is the supremum of $\{E_I(B), E_J(B)\}$ in $S(B)$. Thus, $(S(B), \sqcap, \sqcup)$ is a lattice. □

Theorem 4.2. *Let L be an involutive residuated lattice, $B \subseteq L$ and I be an ideal of L. Then, we have $E_I(B) = (B] \to I$ in the complete Heyting algebra* $(\mathrm{Id}(L), \wedge, \vee, \to, \{0\}, L)$.

Proof. Let $x \in E_I(B)$. We prove that $x \in (B] \to I$. It is sufficient to show that $(x] \cap (B] \subseteq I$. Suppose that $t \in (B] \cap (x]$. Then there exists $b_i \in B$ and $n \in \mathbb{N}$ such that $t \leq b_1 \oplus \cdots \oplus b_k$ and $t \leq nx$. We have $t = t \wedge t \leq (b_1 \oplus \cdots \oplus b_k) \wedge nx \leq (nx \wedge b_1) \oplus \cdots \oplus (nx \wedge b_k)$. Since $x \in E_I(B) \in Id(L)$, we have $nx \in E_I(B)$. This results $(nx) \wedge b_i \in I, (1 \leq i \leq k)$. This means $t \leq (nx \wedge b_1) \oplus \cdots \oplus (nx \wedge b_k) \in I$. Hence, $t \in I$. Thus, $(B] \cap (x] \subseteq I$ and we get $E_I(B) \subseteq (B] \to I$.

Conversely, let $x \in (B] \to I$. Hence, $(b \wedge x] = (b] \cap (x] \subseteq (B] \cap (x] \subseteq I$, for all $b \in B$. We obtain that $x \wedge b \in I$, for all $b \in B$ and thus $x \in E_I(B)$. So $(B] \to I \subseteq E_I(B)$. Therefore, we have $E_I(B) = (B] \to I$ in the Heyting algebra $Id(L)$. □

Theorem 4.3. *Let L be an involutive residuated lattice, I be an ideal of L and $B \subseteq L$. Then $\big(S(B), \wedge, \sqcup, \to, E_{\{0\}}(B), L\big)$ is a complete Heyting algebra.*

Proof. By Theorem 3.24, we have $\bigwedge_{j\in J} E_{I_j} = E_{\wedge I_j}(B)$ and so $S(B)$ is complete. We only show that for all $E_I(B), E_J(B), E_K(B) \in S(B)$,
(1) $E_I(B) \to E_J(B) \in S(B)$,
(2) $E_I(B) \sqcap E_J(B) \subseteq E_K(B) \Leftrightarrow E_I(B) \subseteq E_J(B) \to E_K(B)$.
By Theorem 4.2, we obtain $E_I(B) \to E_J(B) = ((B] \to I) \to ((B] \to J) = (B] \to [((B] \to I) \to J] = [(B] \odot ((B] \to I)] \to J = ((B] \wedge I)) \to J = ((B] \odot I)) \to J = (B] \to (I \to J) \in S(B)$ For the case of (2), it follows from Theorem 4.2 that

$$\begin{aligned}
E_I(B) \sqcap E_J(B) \subseteq E_K(B) &\Leftrightarrow E_{I\wedge J}(B) \subseteq E_K(B),\\
&\Leftrightarrow (B] \to I \wedge J \subseteq (B] \to K\\
&\Leftrightarrow (B] \wedge ((B] \to I \wedge J) \subseteq K,\\
&\Leftrightarrow (B] \wedge I \wedge J \subseteq K,\\
&\Leftrightarrow (B] \wedge I \subseteq J \to K,\\
&\Leftrightarrow (B] \to ((B] \wedge I) \subseteq (B] \to (J \to K),\\
&\Leftrightarrow ((B] \to (B]) \wedge ((B] \to I) \subseteq (B] \to (J \to K),\\
&\Leftrightarrow (B] \to I \subseteq (B] \to (J \to K),\\
&\Leftrightarrow E_I(B) \subseteq E_J(B) \to E_K(B).
\end{aligned}$$

Thus, $S(B)$ of all stable ideals relative to B is the Heyting algebra. □

Theorem 4.4. *If L is an involutive residuated lattice and $B \subseteq L$, then $\{E_I(B) \mid I \in I(L)\}$ and $\{E_F(B) \mid F \in F(L)\}$ are one-to-one correspondence.*

Proof. Let $\phi : E_I(B) \to E_F(B)$ be a map such that $\phi\ (E_I(B)) = E_{I^*}(B)$, and

$$I^* = \{a^* \mid a^* = a \to 0, \forall a \in I\}.$$

Now, we prove that I^* is a filter of L. Let $a \in I^*$ and $a \le b$. Then $a^* \in I$. Since L is an involutive residuated lattice, so $b^* \le a^*$. Since I is an ideal of L, so $b^* \in I$. Thus, $b \in I^*$. Let $a, b \in I^*$. Then $a^*, b^* \in I$. Since I is an ideal of L, so $(a^*)^* \to b^* = a \to b^* \in I$. Thus $(a \to b^*)^* = a \odot b \in I^*$. So I^* is a filter of L. Let $\varphi : E_F(B) \to E_I(B)$ be a map such that $\varphi\ (E_F(B))=E_{F^*}(B)$, where $F^* = \{b^* \mid b^* = b \to 0, \forall b \in F\}$. We can prove that F^* is an ideal of L, similarly. Then

$$\begin{aligned}
\varphi(\phi(E_I(B))) = \varphi(E_{I^*}(B)) = E_{I^{**}}(B) = E_I(B),\\
\phi(\varphi(E_F(B))) = \phi(E_{F^*}(B)) = E_{F^{**}}(B) = E_F(B).
\end{aligned}$$

So $\{E_I(B) \mid I \in I(L)\}$ and $\{E_F(B) \mid F \in F(L)\}$ are one-to-one correspondence. □

5 Conclusion

In this paper, motivating by the previous research on MV-algebras, we extend the concept of extended ideals and stable ideals associated to a subset B to the more general fuzzy structures, namely residuated lattices.This provides us with a great way to find ideals of residuated lattices, and we can characterize some special residuated lattices by extended ideals. We have proven the existence of extended ideals on involutive residuated lattices. Moreover, we have studied some of their related properties. We show that if L is an involutive residuated lattice, then $E_I(B)$ is a stable ideal relative to B and So $E_I(B)= \bigcap\ \{J\colon\ J$ is stable relative to B and $I \subseteq J\}$. We also give a characterization of this extended ideal: $E_I(B) = (B] \to I$ in the complete Heyting algebra $(\mathrm{Id}(L), \wedge, \vee, \to, \{0\}, L)$. We also prove that the class $S(B)$ of all stable ideals relative to $B \subseteq L$ is a complete Heyting algebra. And we prove that the set of extended ideals and the set of extended filters on involutive residuated lattices are one-to-one correspondence. Finally, for some conclusions that hold on MV-algebras, we prove corresponding counterexamples to prove that they do not hold on involutive residuated lattices. This also makes our work meaningful. We attempt to fuzzify the extended ideals on residuated lattices and study their properties in the future.

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Declarations

This article does not contain any studies with human participants or animals performed by any of the authors.
The authors declare that there is no conflict of interest.
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