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Some results on graded prime and primary hyperideals

P. Ghiasvand[1]

[1]*Department of Mathematics, Payame Noor University, P.O.BOX 19395-3697 Tehran, Iran*

p_ghiasvand@pnu.ac.ir

Abstract

Let G be a group with identity e and R be a multiplicative hyperring. We introduce the concept of G-graded multiplicative hyperring R and present some new results and examples. This article aim is to introduce and study graded prime and graded primary hyperideals which are different generalizations of prime and primary hyperideals. Several basic properties, examples and characterizations of graded prime (graded primary) hyperideals of a graded multiplicative hyperring R are presented such as investigating of this structure under homogeneous components, graded hyperring homomorphisms, quotient graded hyperrings and fundamental relations.

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1 Introduction

The first publications on algebraic hyperstructures, a natural suitable generalization of classical algebraic structures, are first encountered in 1934. The hypergroup notion was introduced by a French mathematician F. Marty [14], at the 8th Congress of Scandinavian Mathematicians. The notion of hyperrings was introduced by M. Krasner in 1983, where the addition is a hyperoperation, while the multiplication is an operation [11]. The notion of multiplicative hyperrings are an important class of algebraic hyperstructures which generalize rings, initiated the study by Rota in 1982, where the multiplication is a hyperoperation, while the addition is an operation [21]. Procesi and Rota introduced and studied in brief the prime hyperideals of multiplicative hyperrings [17, 18, 19] and this idea is further generalized in a paper by Dasgupta [7]. R. Ameri *et al*. in [3] described multiplicative hyperring of fractions and coprime hyperideals. Later on, many researches have observed that generalizations of prime hyperideals in multiplicative hyperrings [23, 25]. The principal notions of algebraic hyperstructure theory can be found in [5, 6, 8, 22]. Furthermore, the study of graded rings arises naturally out of the study of affine schemes and allows them to formalize and unify arguments by induction [24]. In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied (see [1, 2, 9, 10, 13, 15, 16, 20]). In this article, we define the notions of G-graded multiplicative hyperrings and graded hyperideals. In the third section, we introduce and study graded prime hyperideals of a graded multiplicative hyperring $(R, +, \circ)$. For example, we prove that every graded maximal hyperideal of a commutative graded multiplicative hyperring with an i-set, is a graded prime hyperideal. Also, we discuss that if R is a graded multiplicative hyperring and P is a proper graded hyperideal of R. Then P is graded prime if and only if P/γ^* is a graded prime ideal of R/γ^*. In the last section, we define the notion of graded radical of a graded hyperideal of a graded multiplicative hyperring R and introduce the concept of graded primary hyperideals of R. We give some results and basic properties of them.

2 Preliminaries

First of all let us remember of basic definitions and terms of hypertheory.

Definition 2.1. [21] *Let R be a nonempty set and $P^*(R) = \{H \mid \emptyset \neq H \subseteq R\}$. Let $\circ : R \times R \to P^*(R)$ be a hyperoperation. A triple $(R, +, \circ)$ is called a multiplicative hyperring, if*

(i) *$(R, +)$ is an abelian group;*

(ii) *$(R, \circ)$ is a semihypergroup;*

(iii) *For all $x, y, z \in R$, we have $x \circ (y + z) \subseteq x \circ y + x \circ z$ and $(y + z) \circ x \subseteq y \circ x + z \circ x$;*

(iv) *For all $x, y \in R$, we have $x \circ (-y) = (-x) \circ y = -(x \circ y)$.*

If in (iii) we have equalities instead of inclusions, then we say that the multiplicative hyperring is strongly distributive.

Let A, B be two subsets of R and $x \in R$, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ and $A \circ x = A \circ \{x\}$.

Moreover, A multiplicative hyperring $(R, +, \circ)$ is called *commutative* if for any $x, y \in R$ we have $x \circ y = y \circ x$.

Example 2.2. [17] *Let $(R, +, \circ)$ be a ring and I be an ideal of R. We define the following hyperoperation on R: For all $x, y \in R$, $x \circ y = x \cdot y + I$. Then $(R, +, \circ)$ is a multiplicative hyperring.*

Definition 2.3. [17] *(a) Let $(R, +, \circ)$ be a multiplicative hyperring and S be a nonempty subset of R. Then S is said to be a subhyperring of R if $(S, +, \circ)$ is itself a multiplicative hyperring.*

(b) We say that S is a hyperideal of $(R, +, \circ)$ if $S - S \subseteq S$ and for all $x \in S$, $r \in R$; $x \circ r \cup r \circ x \subseteq S$.

Definition 2.4. [7] *Let $(R, +, \circ)$ be a multiplicative hyperring and A_l (respectively A_r) $= \{a_1, a_2, \ldots, a_n\}$ be a A nonempty finite subset of R. We say that A_l (respectively A_r) is a left (respectively right) identity set (or i-set, in short) of R if*

(a) *$a_i \neq 0$ for at least one $i = 1, 2, \ldots, n$,*

(b) *for any $r \in R$, $r \in \sum_{i=1}^{n} r_i \circ a$ (respectively $r \in \sum_{i=1}^{n} a \circ r_i$).*

A nonempty finite subset A of a multiplicative hyperring R is called an i-set of R, if it is both a left i-set and a right i-set of R.

Definition 2.5. [8] (a) *A proper hyperideal M of a multiplicative hyperring R is maximal in R, if for any hyperideal I of R, $M \subset I \subseteq R$, then $I = R$.*

(b) *Let P be a proper hyperideal of R. We say that P is a prime hyperideal of R, if for all $x, y \in R$, $x \circ y \subseteq P$, then $x \in P$ or $y \in P$.*

(c) *A proper hyperideal Q of a multiplicative hyperring R is said to be a primary hyperideal of R, if for any $a, b \in R$, $a \circ b \subseteq Q$, then $a \in Q$ or $b^n \subseteq Q$ for some $n \in \mathbb{N}$.*

Definition 2.6. [8] *A homomorphism (good homomorphism) between two multiplicative hyperrings $(R, +, \circ)$ and $(T, +, \circ)$ is a map $\varphi : R \to T$ such that for all x, y of R, we have*

(a) $\varphi(x + y) = \varphi(x) + \varphi(y)$,

(b) $\varphi(x \circ y) \subseteq \varphi(x) \circ \varphi(y)(\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$, *respectively*).

Throughout this article, R is a commutative graded multiplicative hyperring.

3 Graded prime hyperideals

Definition 3.1. *Let G be a group with identity e and T be a multiplicative hyperring. Then T is called a G-graded if $T = \bigoplus_{g \in G} T_g$ with $T_g T_h \subseteq T_{gh}$ for all $g, h \in G$, where T_g is an additive subgroup of T for all $g \in G$, such that $T_g T_h = \bigcup\{x_g \circ y_h : x_g \in T_g,\ y_h \in T_h\}$. An element of T is called homogeneous if it belongs to $\bigcup_{g \in G} T_g$ and this set of homogeneous elements is denoted by $h(T)$. The elements of T_g are called homogeneous of degree g. If $x \in T$, then there exist unique elements $x_g \in h(T)$ such that $x = \sum_{g \in G} x_g$.*

In fact, every multiplicative hyperring is trivially a G-graded by letting $T_e = T$ and $T_g = 0$ for all $g \neq e$.

Lemma 3.2. *If $T = g \in G \bigoplus T_g$ is a graded multiplicative hyperring, then T_e is a subhyperring of T where e is the identity element of group G.*

Proof. As $T_e T_e \subseteq T_e$, so for any $a_e, b_e \in T_e$ we have $a_e \circ b_e \subseteq T_e T_e \subseteq T_e$. Therefore, T_e is closed under multiplicative and so it is a subhyperring of T. □

Example 3.3. *Suppose that $G = (\mathbb{Z}_2, +)$ is the integers modulo 2 and $T = \{0, 1, 2, 3\}$. Consider the multiplicative hyperring $(T, +, \circ)$, where operation $+$ and hyperoperation $\circ$ defined on T as follow:*

$+$	0	1	2	3
0	0	1	2	3
1	1	2	1	0
2	2	3	0	1
3	3	0	3	2

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0, 3\}$	$\{0, 2\}$	$\{0, 1\}$
2	$\{0\}$	$\{0, 2\}$	$\{0\}$	$\{0, 2\}$
3	$\{0\}$	$\{0, 1\}$	$\{0, 2\}$	$\{0, 3\}$

It is easy to verify that $T_0 = \{0, 1\}$ and $T_1 = \{0, 3\}$ are subgroups of $(T, +)$. We have $0 = 0+0$, $3 = 0 + 3$, $2 = 1 + 3$ and $1 = 1 + 0$ and these forms are unique. Hence, $T = T_0 \bigoplus T_1$. We obtain that $T_0 T_0 \subseteq T_0$, $T_0 T_1 \subseteq T_1$, $T_1 T_0 \subseteq T_1$ and $T_1 T_1 \subseteq T_0$. Thus T is a G-graded hyperring and $h(T) = \{0, 1, 3\}$.

Example 3.4. *Suppose that $G = (\mathbb{Z}_3, +)$ is the integers modulo 3 and $S = \{0, 1, 2, 3\}$. Consider the multiplicative hyperring $(S, +, \circ)$, where operation $+$ and hyperoperation $\circ$ defined on R as follow:*

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1, 3\}$	$\{2\}$	$\{1, 3\}$
2	$\{0\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{0\}$	$\{1, 3\}$	$\{2\}$	$\{2\}$

We know that $S_0 = \{0, 3\}$, $S_1 = \{0, 1\}$ and $S_2 = \{0, 2\}$ are all non trivial subgroups of $(S, +)$. We obtain that S is not a G-graded hyperring.

Example 3.5. *Let $T = (\mathbb{Z}[i], +, \cdot)$ where $\mathbb{Z}[i] = \{x + iy \mid x, y \in \mathbb{Z}\}$. Suppose that $B \in P^*(T)$ such that $|B| \geq 2$. Then there exists a multiplicative hyperring with absorbing zero $(T_B, +, \circ)$ and*

$$x \circ y = \{x \cdot b \cdot y : \ \forall x, y \in T, \, b \in B\}.$$

(a) Let $B = \{3, 4\}$ and $G = \mathbb{Z}_2$. Then $T_B = T_0 \bigoplus T_1$ is a G- graded multiplicative hyperring with $T_0 = \mathbb{Z}$ and $T_1 = i\mathbb{Z}$.
(b) Let $B = \{2, i\}$ and $G = \mathbb{Z}_2$. Then $T_0 = \mathbb{Z}$ and $T_1 = i\mathbb{Z}$ are the only subgroups of $(T_B, +)$. It is clear that $(T_B, +, \circ)$ is not a G-graded multiplicative hyperring because $T_1T_1 \nsubseteq T_0$.

Definition 3.6. *A subhyperring S of R is called a graded subhyperring of $R = \bigoplus_{g\in G} R_g$, if $S = \bigoplus_{g\in G} (S \cap R_g)$. Equivalently, S is graded if for every element $x \in S$, all the homogeneous components of x (as an element of R) are in S.*

Definition 3.7. *Let I be a hyperideal of R. Then I is a graded hyperideal, if $I = \bigoplus_{g\in G} (I \cap R_g)$. For any $a \in I$ and for some $r_g \in h(R)$ that $a = \sum_{g\in G} r_g$, then $r_g \in I \cap R_g$ for all $g \in G$.*

Lemma 3.8. *Let J_1 and J_2 be graded hyperideals of R. Then*

(i) *$J_1 \cap J_2$ is a graded hyperideal of R.*

(ii) *$J_1J_2 = \bigcup\{\sum_{i=1}^n x_i \circ y_i : x_i \in J_1, y_i \in J_2 \text{and } n \in \mathbb{N}\}$ is a graded hyperideal of R.*

(iii) *$J_1 \cup J_2$ is a graded hyperideal of R if and only if $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$.*

(iv) *$J_1 + J_2$ is a graded hyperideal of R.*

Proof. (i) We know that $J_1 \cap J_2$ is a hyperideal of R. Now, we show that it is a graded hyperideal. Let $x \in J_1 \cap J_2$. So, $x = \sum_{g\in G} x_g$ where $x_g \in h(R)$. It is enough to show that $x_g \in J_1 \cap J_2$ for any $g \in G$. We have $x \in J_1$ and $x \in J_2$, and so for any $g \in G$, $x_g \in J_1$ and $x_g \in J_2$ because J_1, J_2 are graded hyperideals. Hence $x_g \in J_1 \cap J_2$ for any $g \in G$.

(ii) By [7, Lemma 2.11], J_1J_2 is a hyperrideal of R. Now, we show that grading. Suppose that $a = \sum_{g\in G} a_g \in J_1J_2$, so $\sum_{g\in G} a_g \in \sum_{i=1}^n x_i \circ y_i$ where $x_i \in J_1$ and $y_i \in J_2$. Therefore, for any $i = 1, 2, \ldots, n$; $x_i = \sum_{g\in G} x_{ig}$ and $y_i = \sum_{g\in G} y_{ig}$ where $x_{ig} \in J_1 \cap R_g$ and $y_{ig} \in J_2 \cap R_g$. Hence, $a_g \in \sum_{i=1}^n x_i \circ y_i = \sum_{i=1}^n (\sum_{g\in G} x_{ig} \circ y_{ig})$, and so $\sum_{g\in G} a_g = \sum_{g\in G} t_g$ for $t_g \in \sum_{i=1}^n x_i \circ y_i \subseteq J_1J_2$, by $\sum$

comparing degrees, we have for any $g \in G$, $a_g = t_g \in J_1 J_2$, therefore $J_1 J_2$ is a graded hyperideal of R.

(iii) and (iv) are straightforward. □

Definition 3.9. *Let C be the class of all finite products of homogeneous elements of R i.e.,*

$$C = \{c_1 \circ c_2 \circ \cdots \circ c_t : c_i \in h(R), t \in \mathbb{N}\} \subseteq P^*(h(R)).$$

A graded hyperideal J of R is called a C^{gr}-ideal of R if for any $B \in C$, $B \cap J \neq \emptyset$, then $B \subseteq J$.

Definition 3.10. *Let P be a proper graded hyperideal of R. We say that P is graded prime, if $a_g \circ b_h \subseteq I$ for some $a_g, b_h \in h(R)$, then $a_g \in I$ or $b_h \in I$.*

Example 3.11. *Let $T = (\mathbb{Z}[i], +, \cdot)$ and $G = (\mathbb{Z}_2, +)$ be the integers modulo 2. Consider the multiplicative hyperring $(T_B, +, \circ) = (\mathbb{Z}[i], +, \circ) = \{x + yi \mid x, y \in \mathbb{Z}\}$ with $B = \{1, 3\}$. Then, $(T_B, +, \circ)$ is a G-graded multiplicative hyperring with $T_0 = \mathbb{Z}$ and $T_1 = i\mathbb{Z}$ and $T_B = T_0 \bigoplus T_1$. We set $J' = 2T = \{2x + 2yi, 6x + 6yi : x, y \in \mathbb{Z}\}$. Then J' becomes a graded hyperideal. One can easily show that J' is a graded prime hyperideal of T.*

Definition 3.12. *Let $(R, +, \circ)$ be a graded multiplicative hyperring and S be a nonempty subset of $h(R)$. Then S is said to be multiplicative close subset, briefly, m.c.s of R, if $s_g, t_h \in S$, then $(s_g \circ t_h) \cap S \neq \emptyset$.*

Proposition 3.13. *Let P be a proper graded hyperideal of R. Then P is graded prime if and only if $h(R) - P$ is a m.c.s of R.*

Proof. Let P be a graded hyperideal such that $h(R) - P$ be a m.c.s of R. Assume that $x_g \circ y_h \subseteq P$ for $x_g, y_h \in h(R)$. Therefore, $(x_g \circ y_h) \cap (h(R) - P) = \emptyset$. Hence, $x_g \notin h(R) - P$ or $y_h \notin h(R) - P$ since $h(R) - P$ is a m.c.s of R. Hence, $x_g \in P$ or $y_h \in P$. Then P is a graded prime hyperideal of R. Conversely, let P be a graded prime hyperideal and $x_g, y_h \in h(R) - P$. Thus, $x_g \circ y_h \nsubseteq P$ and $(x_g \circ y_h) \cap (h(R) - P) \neq \emptyset$, i.e., $h(R) - P$ is a m.c.s of R. □

Proposition 3.14. *Let P be a graded prime hyperideal of R. Then if $IJ \subseteq P$ for some graded hyperideals I, J of R, then $I \subseteq P$ or $J \subseteq P$.*

Proof. Suppose that $IJ \subseteq P$ and $I \nsubseteq P$. Let $y \in J$ and so $y = \sum\limits_{g' \in G} y_{g'}$ where $y_{g'} \in J \cap h(R)$. Since $I \nsubseteq P$, there exists $x \in I$ such that $x \notin P$. Hence we have $x = \sum\limits_{g' \in G} x_{g'}$ where $x_{g'} \in I \cap h(R)$, so $x_{h'} \in I - P$ for some $h' \in G$. We have for any $g' \in G$, $x_{h'} \circ y_{g'} \subseteq IJ \subseteq P$, then $y_{g'} \in P$ for any $g' \in G$ since P is a graded prime hyperideal of R. Clearly we have $y = \sum\limits_{g' \in G} y_{g'} \in I$. □

Proposition 3.15. *Let R be a commutative graded multiplicative hyperring. Then $\langle \alpha_g \rangle \circ \langle \beta_h \rangle \subseteq \langle \alpha_g \circ \beta_h \rangle$ for each $\alpha_g, \beta_h \in h(R)$.*

Proof. Let $t \in \langle \alpha_g \rangle$ and $s \in \langle \beta_h \rangle$. Thus $t = \sum_{i=1}^{n_i} x_i + n'_t \alpha_g$ for some $n'_t \in \mathbb{Z}$ and $x_i \in r_i \circ \alpha_g$ and also $s = \sum_{i=1}^{s_t} y_i + s'_t \beta_h$ for some $s'_t \in \mathbb{Z}$ and $y_i \in r'_i \circ \beta_h$. This implies that

$$\begin{aligned} t \circ s &= (\sum_{i=1}^{n_i} x_i + n'_t \alpha_g) \circ (\sum_{i=1}^{s_t} y_i + s'_t \beta_h) \\ &\subseteq \sum_{i=1}^{n_i} \sum_{i=1}^{s_t} x_i \circ y_i + n'_t \sum_{i=1}^{s_t} \alpha_g \circ y_i + s'_t \sum_{i=1}^{n_i} x_i \circ \beta_h + n'_t s'_t (\alpha_g \circ \beta_h) \\ &\subseteq \langle \alpha_g \circ \beta_h \rangle, \end{aligned}$$

which completes the proof of the proposition. □

Proposition 3.16. *Let P be a proper graded hyperideal of R such that for each graded hyperideals I, J of R, $IJ \subseteq P$, we conclude $I \subseteq P$ or $J \subseteq P$. Then P is a graded prime hyperideal of R.*

Proof. Let $x_{g'} \circ y_{h'} \subseteq P$ where $x_{g'}, y_{h'} \in h(R)$. $\langle x_{g'} \circ y_{h'} \rangle \subseteq P$. Then by Proposition 3.15, we have $\langle x_{g'} \rangle \circ \langle y_{h'} \rangle \subseteq P$. Thus $\langle x_{g'} \rangle \subseteq P$ or $\langle y_{h'} \rangle \subseteq P$, so $x_{g'} \in P$ or $y_{h'} \in P$. Hence P is a graded prime hyperideal of R. □

Proposition 3.17. *Let $S \subseteq h(R)$ be a m.c.s of R and I be a graded hyperideal of R with $I \cap S = \emptyset$. Then there exists a graded hyperideal M which is maximal in the set of all graded hyperideals of R disjoint from S, containing I. In particular, M is a graded prime hyperideal of R.*

Proof. Let Ω be the set of all graded hyperideals of R disjoint from S, containing I. Then $\Omega \neq \emptyset$ because $I \in \Omega$. Consider $(\Omega, \subseteq)$. By Zorn's Lemma, there is a graded hyperideal M which is maximal in Ω. Let $IJ \subseteq M$ for graded hyperideals I, J of R. If $I \nsubseteq M$ and $J \nsubseteq M$, then $I \subset M + I$ and $J \subset M + J$. Thus by maximality of M in Ω, we have $(M + I) \cap S \neq \emptyset$ and $(M + J) \cap S \neq \emptyset$. Then there exist $m_g, m'_h \in M \cap h(R)$, $a_g \in I \cap h(R)$ and $b_h \in J \cap h(R)$ such that $m_g + a_g \in S$ and $m'_h + b_h \in S$. Hence,

$$(m_g + a_g) \circ (m'_h + b_h) \subseteq m_g \circ m'_h + a_g \circ m'_h + m_g \circ b_h + a_g \circ b_h \subseteq M + IJ \subseteq M \ (IJ \subseteq M).$$

Therefore, $M \cap S \neq \emptyset$ which is a contradiction with $M \in \Omega$. The second part follows from Proposition 3.16. □

Proposition 3.18. *If M is a graded maximal hyperideal of R with an i-set $A = \{a_1, a_2, \ldots, a_n\}$, then M is a graded prime hyperideal.*

Proof. Assume that M is a graded maximal hyperideal of R. Let I and J be graded hyperideals of R such that $IJ \subseteq M$, but $I \nsubseteq M$. Then $M \subset M + I$ and so by maximality of M, $M + I = R$. Hence $A \subseteq M + I$. Thus for each $a_i \in A$, there exist $m_i \in M$ and $x_i \in I$ such that $a_i = m_i + x_i$. Then for each $i = 1, 2, \ldots, n$, and for any $y \in J$,

$$a_i \circ y \subseteq (m_i + x_i) \circ y \subseteq m_i \circ y + x_i \circ y \subseteq M.$$

Then $y \in \sum_{i=1}^{n} a_i \circ y \subseteq M$, so $J \subseteq M$. Therefore by Proposition 3.16, M is a graded prime hyperideal of R. □

Let $S = \bigoplus_{g \in G} S_g$ and $T = \bigoplus_{g \in G} T_g$ be graded multiplicative hyperrings. The map $\varphi : S \to T$ is a graded homomorphism, if

(i) for every $x, y \in S$, $\varphi(x + y) = \varphi(x) + \varphi(y)$,

(ii) for every $x, y \in S$, $\varphi(x \circ y) \subseteq \varphi(x) \circ \varphi(y)$,

(iii) for every $g' \in G$, $\varphi(S_{g'}) \subseteq T_{g'}$.

In particular, φ is called a graded good homomorphism in case $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$. The kernel of a graded homomorphism is defined as $\ker(\varphi) = \varphi^{-1}(\langle 0 \rangle) = \{r \in R : \varphi(r) \in \langle 0 \rangle\}$.

Proposition 3.19. *Let S and T be graded multiplicative hyperrings and $\varphi : S \to T$ be a graded good homomorphism. Suppose that I, J are graded hyperideals of S and T, respectively. Then the following statements hold:*

(i) *If I is a graded prime hyperideal containing* $\ker(\varphi)$ *and φ is onto, then $\varphi(I)$ is a graded prime hyperideal of T.*

(ii) *If J is a graded prime hyperideal of T, then $\varphi^{-1}(J)$ is a graded prime hyperideal of S.*

Proof. (i) Let $I = \bigoplus_{g\in G}(I \cap S_g)$. It is clear that $\varphi(I) = \bigoplus_{g\in G}(\varphi(I) \cap T_g)$ and since φ is onto, so $\varphi(I)$ is a graded hyperideal of T. Let $\varphi(a_g) \circ \varphi(b_h) \subseteq \varphi(I)$ where $a_g, b_h \in h(S)$. Since φ is a graded homomorphism, $\varphi(a_g \circ b_h) \subseteq \varphi(a_g) \circ \varphi(b_h) \subseteq \varphi(I)$. Assume that $p \in a_g \circ b_h$. Then $\varphi(p) \in \varphi(a_g \circ b_h) \subseteq \varphi(I)$ and so $\varphi(p) = \varphi(q)$ for some $q \in I$. Thus $\varphi(p - q) = 0 \in \langle 0\rangle$, that is, $p - q \in \ker(\varphi) \subseteq I$ and so $p \in I$. Hence $a_g \circ b_h \subseteq I$. Since I is a graded prime hyperideal of R, we obtain $a_g \in I$ or $b_h \in I$ and so $\varphi(a_g) \in \varphi(I)$ or $\varphi(b_h) \in \varphi(I)$. Therefore, $\varphi(I)$ is a graded prime hyperideal of T.

(ii) Let $J = \bigoplus_{g\in G}(J \cap T_g)$ be a graded hyperideal of T. Then it is easy to see that $\varphi^{-1}(J) = \bigoplus_{g\in G}(\varphi^{-1}(J) \cap S_g)$ is a graded hyperideal of S. Let $a_g \circ b_h \subseteq \varphi^{-1}(J)$ for some $a_g, b_h \in h(S)$. Then $\varphi(a_g \circ b_h) = \varphi(a_g) \circ \varphi(b_h) \subseteq J$. Since J is a graded prime hyperideal of T, then $\varphi(a_g) \in J$ or $\varphi(b_h) \in J$ and so $a_g \in \varphi^{-1}(J)$ or $b_h \in \varphi^{-1}(J)$. Therefore, $\varphi^{-1}(J)$ is a graded prime hyperideal of S. □

Proposition 3.20. *Let R and T be graded multiplicative hyperrings and $\varphi : R \to T$ be a graded good homomorphism. Suppose that I, J are graded hyperideals of R and T, respectively. Then the following assertions hold:*

(i) *If I is a C^{gr}-graded hyperideal containing* $\ker(\varphi)$ *and φ is onto, then $\varphi(I)$ is a C^{gr}-graded hyperideal of T.*

(ii) *If J is a C^{gr}-graded hyperideal of T, then $\varphi^{-1}(J)$ is a C^{gr}-graded hyperideal of R.*

Proof. (i) Let $c_1 \circ c_2 \circ \cdots \circ c_n \cap \varphi(I) \neq \emptyset$ for some $c_1, c_2, \ldots, c_n \in h(T)$. Since φ is onto, we have $\varphi(a_i) = c_i$ for some $a_i \in h(R)$, $1 \leq i \leq n$. Then $(\varphi(a_1) \circ \varphi(a_2) \circ \cdots \circ \varphi(a_n)) \cap \varphi(I) = \varphi(a_1 \circ a_2 \circ \cdots \circ a_n) \cap I \neq \emptyset$ because φ is a graded good homomorphism. Thus there exists $t \in a_1 \circ a_2 \circ \cdots \circ a_n$ such that $\varphi(t) \in \varphi(I)$. Since $\ker(\varphi) \subseteq I$, we have $t \in I$, so $a_1 \circ a_2 \circ \cdots \circ a_n \cap I \neq \emptyset$. As I is a C^{gr}-ideal of R, $a_1 \circ a_2 \circ \cdots \circ a_n \subseteq I$, hence $\varphi(a_1) \circ \varphi(a_2) \circ \cdots \circ \varphi(a_n) = \varphi(a_1 \circ a_2 \circ \cdots \circ a_n) \subseteq \varphi(I)$. Therefore $c_1 \circ c_2 \circ \cdots \circ c_n \subseteq \varphi(I)$, so $\varphi(I)$ is a C^{gr}-ideal of R.

(ii) Let $a_1 \circ a_2 \circ \cdots \circ a_n \cap \varphi^{-1}(I) \neq \emptyset$ for some $a_1, a_2, \ldots, a_n \in h(R)$. This implies that $p \in \varphi^{-1}(J)$ for some $p \in a_1 \circ a_2 \circ \cdots \circ a_n$, hence $\varphi(t) \in J \cap \varphi(a_1 \circ a_2 \circ \cdots \circ a_n)$. Then we have $J \cap \varphi(a_1) \circ \varphi(a_2) \circ \cdots \circ \varphi(a_n) \neq \emptyset$. Since J is a C^{gr}-ideal of T,

$$\varphi(a_1) \circ \varphi(a_2) \circ \cdots \circ \varphi(a_n) = \varphi(a_1 \circ a_2 \circ \cdots \circ a_n) \subseteq J.$$

Thus $a_1 \circ a_2 \circ \cdots \circ a_n \subseteq \varphi^{-1}(J)$. □

Assume that J is a graded hyperideal of $R = \bigoplus_{g\in G} R_g$. Then quotient group $R/J = \{a + J : a \in R\}$ becomes a multiplicative hyperring with the multiplication $(a+J) \circ (b+J) = \{r + J : r \in a \circ b\}$ ([7]). It is easy to see that R/J is a graded hyperring with $R/J = \bigoplus_{g\in G}(R/J)_g$ where for all $g \in G$, $(R/J)_g = (R_g + J)/J$. Moreover, all graded hyperideals of R/J is of the form I/J, where I is a graded hyperideal of R containing J since the natural graded homomorphism $\varphi : R \to R/J$ is a graded good epimorphism.

Theorem 3.21. *Let $J \subseteq P$ be graded hyperideals of R. Then the following assertions hold:*

(i) *P is a graded prime hyperideal of R if and only if P/J is a graded prime hyperideal of R/J. In particular, all graded prime hyperideals of R/J is of the form P/J where P is a graded prime hyperideal of R containing J.*

(ii) *P is a C^{gr}-graded hyperideal of R if and only if P/J is a C^{gr}-graded hyperideal of R/J. In particular, all C^{gr}-graded hyperideals of R/J is of the form P/J where P is a C^{gr}-graded hyperideal of R containing J.*

Proof. (i) Consider the natural graded homomorphism $\varphi : R \to R/J$ defined by $\varphi(r) = r + J$. Since φ is a graded good homomorphism, the proof holds by Proposition 3.19.

(ii) Apply Proposition 3.20. □

Consider the fundamental relation γ^* defined in [8]. In the following theorem, we show that if R is a graded multiplicative hyperring, then R/γ^* is a graded ring.

Theorem 3.22. *Let $\gamma^*(0)$ be a graded hyperideal of R. Then R/γ^* is a G-graded ring such that $(R/\gamma^*)_{h'} = \{\gamma^*(z_{h'}) \mid z_{h'} \in R_{h'}\}$.*

Proof. Let $R = \bigoplus_{h' \in G} R_{h'}$ be a G-graded multiplicative hyperring. Assume that $z \in R/\gamma^*$, so there exists $r \in R$ where $z = \gamma^*(r')$. Thus $r' = \sum_{h' \in G} r'_{h'}$ for some $r_{h'} \in R_g$ and hence $z = \gamma^*(r') = \gamma^*(\sum_{h' \in G} r'_{h'}) = \sum_{h' \in G} \gamma^*(r'_{h'})$. Therefore $R/\gamma^* = \sum_{h' \in G} (R/\gamma^*)_{h'}$. Assume that $\sum_{g \in G} \gamma^*(r'_{h'}) = \gamma^*(0)$ where $r'_{h'} \in R_g$. Then $\gamma^*(\sum_{h' \in G} r'_{h'}) = \gamma^*(0)$ and so $\sum_{h' \in G} r'_{h'}) \in \gamma^*(0)$. Since $\gamma^*(0)$ is a graded hyperideal of R, then for any $h' \in G$, $r'_{h'} \in \gamma^*(0)$. Hence for all $h' \in G$, $\gamma^*(r'_{h'}) = \gamma^*(0)$. We have $R/\gamma^* = \sum_{h' \in G} (R/\gamma^*)_{h'}$ is an internal direct sum. Consequently, $(R/\gamma^*)_{h'}(R/\gamma^*)_{g'} \subseteq (R/\gamma^*)_{h'g'}$ for any $g', h' \in G$ and so R/γ^* is a graded ring. □

Theorem 3.23. *Let R be with identity 1 and P be a proper graded hyperideal of R. Then P is graded prime if and only if P/γ^* is a graded prime ideal of R/γ^*.*

Proof. $(\Rightarrow)$ Let $c_g \circ d_h \in P/\gamma^*$ where $c_g, d_h \in h(R/\gamma^*)$. Then there exist $a_g, b_h \in h(R)$ such that $c_g = \gamma^*(a_g)$ and $d_h = \gamma^*(b_h)$. Thus $c_g \odot d_h = \gamma^*(a_g) \odot \gamma^*(b_h) = \gamma^*(a_g \circ b_h)$. Hence $a_g \circ b_h \subseteq P$, so $a_g \in P$ or $b_h \in P$ since P is a graded prime hyperideal of R. Hence $c_g = \gamma^*(a_g) \in P/\gamma^*$ or $d_h = \gamma^*(b_h) \in P/\gamma^*$. Therefore P/γ^* is a graded prime ideal of R/γ^*.

$(\Leftarrow)$ Let $a_g \circ b_h \subseteq P$ for $a_g, b_h \in h(R)$. Then we have $\gamma^*(a_g), \gamma^*(b_h) \in R/\gamma^*$ and $\gamma^*(a_g) \odot \gamma^*(b_h) = \gamma^*(a_g \circ b_h) \in P/\gamma^*$. Thus $\gamma^*(a_g) \in P/\gamma^*$ or $\gamma^*(b_h) \in P/\gamma^*$ since P/γ^* is a graded prime ideal of R/γ^*. Hence $a_g \in I$ or $b_h \in P$. Therefore P is a graded prime hyperideal of R. □

Let R be a multiplicative hyperring. Then $M_n(R)$ denotes the set of all hypermatixes of R. Also, for all $A = (A_{ij})_{nn}$, $B = (B_{ij})_{nn} \in P^*(M_n(R))$, $A \subseteq B$ if and only if $A_{ij} \subseteq B_{ij}$.

If $R = \bigoplus_{g \in G} R_g$ be a graded multiplicative hyperring, then $M_n(R)$ is a graded hypermatixes of R with g-component $(M_n(R))_g = M_n(R_g)$.

Theorem 3.24. *Let R be with identity 1 and I be a graded hyperideal of R. If $M_n(I)$ is a graded prime hyperideal of $M_n(R)$, then I is a graded prime hyperideal of R.*

Proof. Let $x_{g'} \circ y_{h'} \subseteq I$ where $x_{g'}, y_{h'} \in h(R)$. Then

$$A = \begin{pmatrix} x_{g'} \circ y_{h'} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \subseteq M_n(I).$$

We have

$$\begin{pmatrix} x_{g'} \circ y_{h'} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} x_{g'} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} y_{h'} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Since $M_n(I)$ is a graded prime hyperideal of $M_n(R)$ then

$$\begin{pmatrix} x_{g'} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M_n(I) \text{ or } \begin{pmatrix} y_{h'} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M_n(I).$$

Therefore, $x_{g'} \in I$ or $y_{g'} \in I$. Hence I is a graded prime hyperideal of R. □

4 Graded primary hyperideals

In this section, we define and study graded primary hyperideals of a graded multiplicative hyperring $(R, +, \circ)$.

For any element $z \in R$, we mean $z^k = z \circ z \circ \cdots \circ z$ (k times) for any positive integer $k > 1$ and $z^1 = \{z\}$.

We begin this section by the following definition.

Definition 4.1. (*a*) *Let I be a graded hyperideal of $(R, +, \circ)$. The intersection of all graded prime hyperideals of R containing I is called the graded radical of I, denoted by $Grad(I)$.*

(*b*) *Let J be a graded hyperideal of R. We define*

$$D(J) = \{r \in R : \text{ for any } g' \in G,\ r_{g'}^{n_{g'}} \subseteq J \text{ for some } n_{g'} \in \mathbb{N}\}.$$

Clearly, $D(J)$ is a graded hyperideal of R.

Theorem 4.2. *Let I be a graded hyperideal of R. Then $D(I) \subseteq Grad(I)$. The equality holds when I is a C^{gr}-ideal of R.*

Proof. If $Grad(I) = R$, then $D(I) \subseteq Grad(I)$. Assume that $Grad(I) \neq R$. Let $x \in D(I)$. Then for any $g \in G$, $x_g^{n_g} \subseteq I$ for some $n_g \in \mathbb{N}$. Thus for any graded prime hyperideal P of R, containing I, $x_g^{n_g} \subseteq P$. Hence for any $g \in G$, $x_g \in P$ because P is a graded prime hyperideal of R, and so $x = \sum_{g \in G} x_g \in P$, so $x \in Grad(I)$. Therefore $D(I) \subseteq Grad(I)$.

Assume that I is a C^{gr}-ideal. Let $t \notin D(I)$. Then there exists $g \in G$, $t_g^n \nsubseteq I$ for any $n \in \mathbb{N}$. Hence $t_g^n \bigcap I = \emptyset$ for all $n \in \mathbb{N}$. Let $S = \bigcup \{t_g^n + I_{g^n} : n \in \mathbb{N}\}$. It is clear that $S \subseteq h(R)$. Let $x, y \in S$, then $x \in t_g^n + I_{g^n}$ and $y \in t_g^m + I_{g^m}$ for some $n, m \in \mathbb{N}$, and so $x = c_{g^n} + a_{g^n}$ and $y = d_{g^m} + b_{g^m}$ for some $c_{g^n} \in t_g^n$, $d_{g^m} \in t_g^m$, $a_{g^n} \in I_{g^n}$ and $b_{g^m} \in I_{g^m}$. Thus

$$x \circ y = (c_{g^n} + a_{g^n}) \circ (d_{g^m} + b_{g^m}) \subseteq c_{g^n} \circ d_{g^m} + c_{g^n} \circ b_{g^m} + a_{g^n} \circ d_{g^m} + a_{g^n} \circ b_{g^m} \subseteq t_g^{n+m} + I_{g^{n+m}} \subseteq S.$$

It concludes that S is a multiplicative close subset. We have $S \cap I = \emptyset$, because if $z \in S \cap I$, then there exist $x \in I_{g^n} \subseteq I$ and $y \in t_g^n$ for some $n \in \mathbb{N}$, such that $z = x + y$, so $y \in I$ which is contradictory to the fact that $t_g^n \bigcap I = \emptyset$ for all $n \in \mathbb{N}$. Thus $t_g \notin P$ and $t \notin P$ because P is a graded hyperideal. Therefore $t \notin Grad(I)$, and so $Grad(I) \subseteq D(I)$. □

Proposition 4.3. *Let J and $J_1, J_2, \ldots, J_n$ be graded C^{gr}-hyperideals of R. The following statements hold:*

(i) $Grad(Grad(J)) = Grad(J)$.

(ii) $Grad(J_1 J_2 \ldots J_n) = Grad(\bigcap_{i=1}^{n} J_i) = \bigcap_{i=1}^{n} Grad(J_i)$.

Proof. (i) Let $x \in Grad(Grad(J))$. Then for any $g \in G$, there exists $n_g \in \mathbb{N}$ such that $x_g^{n_g} \subseteq Grad(J)$. Hence for any $t \in x_g^{n^g}$, there exists $m \in \mathbb{N}$ such that $t^m \subseteq J$. Since $t \in x_g^n$, then $t^m \subseteq (x_g^n)^m = x_g^{nm}$. Hence $x_g^{nm} \bigcap J \neq \emptyset$. Thus $x_g^{nm} \subseteq J$ for any $g \in G$ (since J is a C^{gr}-ideal). Therefore, $Grad(Grad(J)) \subseteq Grad(J)$. Clearly, $Grad(J) \subseteq Grad(Grad(J))$, so $Grad(Grad(J)) = Grad(J)$.

(ii) We have $J_1 J_2 \ldots J_n \subseteq \bigcap_{i=1}^{n} J_i$. So $Grad(J_1 J_2 \ldots J_n) \subseteq Grad(\bigcap_{i=1}^{n} J_i)$. It is clear that, if J_i is a C^{gr}-ideal, then $\bigcap_{i=1}^{n} J_i$ is also a C^{gr}-ideal. Thus for any $x \in Grad(\bigcap_{i=1}^{n} J_i)$, we have for any $g \in G$, $x_g^m \subseteq \bigcap_{i=1}^{n} J_i$. So $x_g^m \subseteq J_i$ for all $i = 1, 2, \ldots, n$, then $x \in Grad(J_i)$, and so $x \in \bigcap_{i=1}^{n} Grad(J_i)$. Therefore, $Grad(\bigcap_{i=1}^{n} J_i) \subseteq \bigcap_{i=1}^{n} Grad(J_i)$. Finally, let $x \in \bigcap_{i=1}^{n} Grad(J_i)$. Hence for each $i = 1, 2, \ldots, n$, there exists $m_i \in \mathbb{N}$ such that $x_g^{m_i} \subseteq J_i$ for all $g \in G$. Thus

$$x_g^{\sum_{i=1}^{n}(m_i)} \subseteq J_1 J_2 \ldots J_n.$$

Thus $x \in Grad(J_1 \ldots J_n)$. Consequently, $\bigcap_{i=1}^{n} Grad(J_i) \subseteq Grad(J_1 J_2 \ldots J_n)$. □

Definition 4.4. *A proper graded hyperideal Q of $(R, +, \circ)$ is said to be graded primary, if for any $a_g, b_h \in h(R)$ such that $a_g \circ b_h \subseteq Q$, then $a_g \in Q$ or $b_h^n \subseteq Q$ for some $n \in \mathbb{N}$.*

Proposition 4.5. *If Q is graded primary C^{gr}-ideal of R, then $Grad(Q)$ is a graded prime C^{gr}-ideal of R.*

Proof. First, we show that $Grad(Q)$ is a C^{gr}-ideal of R. Let $a_1 \circ a_2 \circ \cdots \circ a_n \bigcap Grad(Q) \neq \emptyset$ for some $a_1, a_2, \ldots, a_n \in h(R)$. Then we have $x \in a_1 \circ a_2 \circ \cdots \circ a_n$ such that $x \in Grad(Q)$. This implies that for any $g \in G$, $x_g^t \subseteq (a_1 \circ a_2 \circ \cdots \circ a_n)^t$ and $x_g^t \subseteq Q$ for some $t \in \mathbb{N}$. Since Q is a C^{gr}-ideal and $(a_1 \circ a_2 \circ \cdots \circ a_n)^t \bigcap Q \neq \emptyset$, we get $(a_1 \circ a_2 \circ \cdots \circ a_n)^t \subseteq Q$ and so $(a_1 \circ a_2 \circ \cdots \circ a_n) \subseteq Grad(Q)$. Therefore $Grad(Q)$ is a C^{gr}-ideal of R. Let $a_g \circ b_h \subseteq Grad(Q)$ and $a_g \notin Grad(Q)$ where $a_g, b_h \in h(R)$. Then for any $x_{gh} \in a_g \circ b_h$, there exists $n \in \mathbb{N}$ such that $x_{gh}^n \subseteq Q$. We have $x_{gh}^n \subseteq (a_g \circ b_h)^n = a_g^n \circ b_h^n$ (since R is commutative). So $(a_g^n \circ b_h^n) \cap Q \neq \emptyset$ and thus $a_g^n \circ b_h^n \subseteq Q$ (since Q is C^{gr}-ideal). Now $a_g \notin Grad(Q)$, then $a_g^n \nsubseteq Q$, and so $a_g^n \cap Q = \emptyset$. Thus for any $p \in a_g^n$ and $q \in b_h^n$ we have $p \notin Q$ and $p \circ q \subseteq a_g^n \circ b_h^n \subseteq Q$. Therefore $q^m \subseteq Q$ for some $m \in \mathbb{N}$ since Q is a graded primary hyperideal of R. Again, $q \in b_h^n$, then $q^m \subseteq (b_h^n)^m = b_h^{nm}$. Hence $b_h^{nm} \cap Q \neq \emptyset$ and so $b_h^{nm} \subseteq Q$, whence $b_h \in Grad(Q)$. Therefore $Grad(Q)$ is a graded prime hyperideal. □

Proposition 4.6. *Let Q be a C^{gr}-ideal and P be a graded hyperideal of R. Then Q is a P-graded primary C^{gr}-ideal of R if and only if*

(i) $Q \subseteq P \subseteq Grad(Q)$.

(ii) *For any $a_g, b_h \in h(R)$; $a_g \circ b_h \subseteq Q$ and $a_g \notin Q$, then $b_h \in P$.*

Proof. Suppose that (i) and (ii) hold. Let $a_g \circ b_h \subseteq Q$ and $a_g \notin Q$ where $a_g, b_h \in h(R)$. Thus by (ii), $b_h \in P$ and by (i), $b_h \in P \subseteq Grad(Q)$. Therefore, $b_h^n \subseteq Q$ for some $n \in \mathbb{N}$, and so Q is a graded primary hyperideal of R. Now, we show that $P = Grad(Q)$. Let $c = \sum_{g \in G} c_g \in Grad(Q)$. Suppose $g \in G$ and let n be the least positive integer such that $c_g^n \subseteq Q$. If $n = 1$, then $c_g \in \{c_g\} = c_g^1 \subseteq Q \subseteq P$ and so $c_g \in P$. If $n > 1$, $c_g^{n-1} \nsubseteq Q$ by the minimality of n and thus $c_g^n \cap Q = \emptyset$ since Q is a C^{gr}-ideal. Then for any $x_{g^{n-1}} \in c_g^{n-1}$; $x_{g^{n-1}} \circ c_g \subseteq c_g^{n-1} \circ c_g = c_g^n \subseteq Q$. Hence by (ii), $c_g \in P$ since $x_{g^{n-1}} \notin Q$. Thus $Grad(Q) \subseteq P$, whence $P = Grad(Q)$ by (i). Hence Q is a P-graded primary C^{gr}-ideal of R. The converse part is immediate. □

Proposition 4.7. *Let Q be a graded hyperideal of R. Then Q is a graded primary hyperideal of R if and only if $I \circ J \subseteq Q$ implies that $I \subseteq Q$ or $J \subseteq D(Q)$ where I, J are graded hyperideals of R.*

Proof. Let Q be a graded primary hyperideal of R such that $I \circ J \subseteq Q$ and $I \nsubseteq Q$. Then there exists $a \in I$ such that $a \notin Q$. Hence, $a = \sum_g a_g$ where $a_g \in I \cap R_g$, and so $a_h \notin Q$ for some $h \in G$. Let $b = \sum_{g \in G} b_g \in J$. Then for any $g \in G$, $b_g \in J$ since J is graded, thus $a_h \circ b_g \subseteq I \circ J \subseteq Q$. Since Q is a graded primary hyperideal, we have $b_g^n \subseteq Q$ for some $n \in \mathbb{N}$, and so $b \in D(Q)$. This implies that $J \subseteq D(Q)$. Conversely, let $a_g \circ b_h \subseteq Q$ for some $a_g, b_h \in h(R)$. Then, we have $\langle a_g \circ b_h \rangle \subseteq Q$. Hence, by Proposition 3.15, $\langle a_g \rangle \circ \langle b_h \rangle \subseteq \langle a_g \circ b_h \rangle$, so $\langle a_g \rangle \circ \langle b_h \rangle \subseteq Q$. Thus $\langle a_g \rangle \subseteq Q$ or $\langle b_h \rangle \subseteq D(Q)$, and so $a_g \in Q$ or $b_h^n \subseteq Q$ for some $n \in \mathbb{N}$. □

By induction hypothesis one can easily obtain following result:

Corollary 4.8. *If Q is a graded primary hyperideal of R such that $J_1 \circ J_2 \circ \cdots \circ J_n \subseteq Q$, then either $J_1 \subseteq Q$ or $J_i \subseteq D(Q)$ for some $2 \leq j \leq n$.*

It is clear that every graded prime hyperideal of R is a graded primary hyperideal of R, but the converse is not true in general. Consider the following example:

Example 4.9. *Suppose that $R_B = (\mathbb{Z}[i], +, \circ)$ where $R_B = \mathbb{Z} \bigoplus i\mathbb{Z}$ and $B = \{2, 3\} \in P^*(R)$. Take $Q = \langle 2 \rangle$. Then Q is a graded primary hyperideal of R_B, but Q is not a graded prime hyperideal of R_B. Because, $2 \circ 2i = \{2 \cdot 2 \cdot 2i, 2 \cdot 3 \cdot 2i\} \subseteq Q$, but $2 \notin Q$ and $2i \notin Q$.*

Let $I \subseteq Q_1 \cup Q_2 \cup \cdots \cup Q_n$ be a covering of graded hyperideals of R. Then this covering is called efficient if none of the Q_is are superfluous. Note that a covering by two graded hyperideals can not be efficient.

Proposition 4.10. *Suppose that $I \subseteq Q_1 \cup Q_2 \cup \cdots \cup Q_n$ is an efficient covering of graded hyperideals of R where I is a graded hyperideal of R. If $Grad(Q_i) \nsubseteq Grad(Q_j)$ for each $i \neq j$, then any of Q_is are not graded primary hyperideals of R.*

Proof. First we show that $Grad(J) = Grad(D(J))$ for any graded hyperideal J of R. Since $J \subseteq D(J)$, then we have $Grad(J) \subseteq Grad(D(J))$. Let P be a graded prime hyperideal of R containing J. Then it is sufficient to show that P contains $D(J)$. Let $x \in D(J)$. Then for any $g \in G$, $x_g^n = x_g \circ \cdots \circ x_g \subseteq J \subseteq P$ for some $n \in \mathbb{N}$. Thus $x_g \in P$ for any $g \in G$ (P is a graded prime hyperideal), then $x \in P$. Hence $Grad(D(J)) \subseteq Grad(J)$. Since covering is efficient, we have $n > 2$. Assume that Q_1 is a graded primary hyperideal of R. Since the covering is efficient, we have $I \cap Q_2 \cap Q_3 \cap \cdots \cap Q_n \subseteq I \cap Q_1 \subseteq Q_1$ (see [12]). As $I \nsubseteq Q_1$ and $I \circ Q_2 \circ \cdots Q_n \subseteq Q_1$, by Corollary 4.8, there exists $2 \leq j \leq n$ such that $Q_j \subseteq D(Q_1)$ and so $Grad(Q_j) \subseteq Grad(D(Q_1)) = Grad(Q_1)$ which is a contradiction. □

Theorem 4.11. *Suppose that $I \subseteq Q_1 \cup Q_2 \cup \cdots \cup Q_n$ is a covering and at most two of Q_is are not graded primary hyperideals of R. If $Grad(Q_i) \nsubseteq Grad(Q_j)$ for each $i \neq j$, then $I \subseteq Q_i$ for some $1 \leq i \leq n$.*

Proof. If $n = 2$, then the result is valid. Also, we may assume that the covering is efficient so $n \neq 2$. Assume that $n > 2$. But in this case, there exists a graded primary hyperideal Q_j of covering and this contradicts by Proposition 4.10. Thus, we have $n = 1$ and this completes the proof. $\square$

Corollary 4.12. *Suppose that $I \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$ is an efficient covering and at most two of P_is are not graded prime hyperideals of R, then $I \subseteq P_i$ for some $1 \leq i \leq n$.*

Proof. It follows from Theorem 4.11. $\square$

The proof of the following theorem is straightforward and it is left to the reader.

Theorem 4.13. *If $Q_1, Q_2, \ldots, Q_n$ are graded primary C^{gr}-ideal of R, all of which are P-graded primary for a graded prime hyperideal P, then $\bigcap_{i=1}^{n} Q_i$ is also a P-graded primary C^{gr}-ideal of R.*

Proposition 4.14. *Let $P \subseteq Q$ are graded hyperideals of R. Then the following are satisfied:*

(i) *$Grad(Q/P) = Grad(Q)/P$.*

(ii) *$D(Q/P) = D(Q)/P$.*

(iii) *If Q is a C^{gr}-graded hyperideal of R, then $D(Q/P) = Grad(Q/P)$.*

Proof. (i) If follows from Proposition 3.21.

(ii) Let $x+P \in D(Q/P)$ for some $x \in R$. Hence we have for all $g' \in G$, $(x_{g'}+P)^{m_{g'}} \subseteq Q/P$ for some $m_{g'} \in \mathbb{N}$. Let $t \in x_{g'}^{m_{g'}}$. Since $t+P \in (x_{g'}+P)^{m_{g'}}$, thus $t+P \in Q/P$ and $t \in Q$. Therefore for any $g' \in G$, $x_{g'}^{m_{g'}} \subseteq Q$, then $x \in D(Q)$, and hence $x+P \in D(Q)/P$. Conversely, assume that $x+P \in D(Q)/P$ for $x \in R$. Then $x \in D(Q)$ and so for any $g' \in G$, $x_{g'}^{m_{g'}} \subseteq Q$ for some $m_{g'} \in \mathbb{N}$. Take any $t+P \in (x_{g'}+P)^{m_{g'}}$, then we have $t+P = s+P$ for some $s \in x_{g'}^{m_{g'}}$, which means that $t-s \in P \subseteq Q$. Thus $t = (t-s)+s \in P + x_{g'}^{m_{g'}} \subseteq Q$ and we obtain $t+P \in Q/P$. Hence we conclude that for any $g' \in G$, $(x_{g'}+P)^{m_{g'}} \subseteq (Q/P)$ that is $x+P \in D(Q/P)$.

(iii) It follows from Proposition 4.3 and Proposition 3.21. $\square$

Proposition 4.15. *Let $\varphi : R \to T$ be a graded good homomorphism of graded multiplicative hyperrings. Suppose that P, Q are graded hyperideals of R and T, respectively. Then the followings hold:*

(i) *If P is a graded primary hyperideal containing $\ker(\varphi)$ and φ is onto, then $\varphi(P)$ is a graded primary hyperideal of T.*

(ii) *If Q is a graded primary hyperideal of T, then $\varphi^{-1}(Q)$ is a graded primary hyperideal of R.*

Proof. The proofs are similar to the proof of Proposition 3.19. $\square$

Corollary 4.16. *Let $J \subseteq Q$ be graded hyperideals of R. Then*

(i) *Q is a graded primary hyperideal of R if and only if Q/J is a graded primary hyperideal of R/J.*

(ii) *Let Q be a graded primary C^{gr}-hyperideal of R. Then $Grad(Q/J)$ is a graded primary hyperideal of R/J.*

Proof. (i) The proof follows easily from Proposition 4.15.
(ii) This follows from (i), Propositions 4.5, 3.21 and 4.14. □

Theorem 4.17. *Let R be a graded multiplicative hyperring with identity 1 and I be a graded hyperideal of R.*

(i) *I is a graded primary if and only if I/γ^* is a graded primary ideal of R/γ^*.*

(ii) *If $M_n(I)$ is a graded primary hyperideal of $M_n(R)$, then I is a graded primary hyperideal of R.*

Proof. (i) It follows from Theorem 3.23.
(ii) It is similar to Theorem 3.24. □

References

[1] A.S. Alshehry, R. Abu-Dawwas, *Graded weakly prime ideals of non-commutative rings*, Communications in Algebra, 49(11) (2021), 4712–4723.

[2] K. Al-Zoubi, R. Abu-Dawwas, S. Çeken, *On graded 2-absorbing and graded weakly 2-absorbing ideals*, Hacettepe Journal of Mathematics and Statistics, 48(3) (2019), 724–731.

[3] R. Ameri, A. Kordi, S. Hoskova-Mayerova, *Multiplicative hyperring of fractions and coprime hyperideals*, Analele stiintifice ale Universitatii Ovidius Constanta, 25(1) (2017), 5–23.

[4] M. Cohen, S. Montgomery, *Group-graded rings, smash products, and group actions*, Transactions of American Mathematical Society, 282(1) (1984), 237–258.

[5] P. Corsini, *Prolegomena of hypergroup theory*, Second ed., Aviani Editore, 1993.

[6] P. Corsini, V. Leoreanu, *Applications of hyperstructures theory*, Advances in Mathematics, Kluwer Academic Publishers, 2013.

[7] U. Dasgupta, *On prime and primary hyperideals of a multiplicative hyperring*, Annals of the Alexandru Ioan Cuza University-Mathematices, 58(1) (2012), 19–36.

[8] B. Davvaz, V. Leoreanu-Fotea, *Hyperring theory and applications*, International Academic Press, USA, 2007.

[9] F. Farzalipour, P. Ghiasvand, *On the union of graded prime submodules*, Thai Journal of Mathematics, 9(1) (2011), 49–55.

[10] F. Farzalipour, P. Ghiasvand, *On graded hyperrings and graded hypermodules*, Algebraic Structures and Their Applications, 7(2) (2020), 15–28.

[11] M. Krasner, *A class of hyperrings and hyperfield*, International Journal of Mathematics and Mathematical Sciences, 6(2) (1983), 307–312.

[12] N.H. MacCoy, *A note on finite union of ideals and subgroups*, Proceedings of the American Mathematical Society, 8(4) (1969), 633–637.

[13] A. Macdonal, *Commutative algebra*, Addison Wesley Publishing Company, 1969.

[14] F. Marty, *Sur une generalization de la notion de groupe*, in: 8iem Congres Math. Scandinaves, Stockholm, (1934), 45–49.

[15] N. Nastasescu, F. Van Oystaeyen, *Graded rings theory*, Mathematical Library 28, North Holland, Amsterdam, 1937.

[16] K.H. Oral, Ü. Tekir, A.G. Ağargün, *On graded prime and primary submodules*, Turkish Journal of Mathematics, 35 (2011), 159–167.

[17] R. Procesi-Ciampi, R. Rota, *On some classes of hyperstructures*, Combinatories Discrete Mathematics, 1 (1987), 71–80.

[18] R. Procesi-Ciampi, R. Rota, *The hyperring spectrum*, Riv. Mat. Pura Appl., 1 (1987), 71–80.

[19] R. Procesi-Ciampi, R. Rota, *Complementary multiplicative hyperrings*, Discrete Mathematics, 208/209 (1999), 485–497.

[20] M. Refaei, K. Al-Zoubi, *On graded primary ideals*, Turkish Journal of Mathematics, 28(3) (2004), 217–229.

[21] R. Rota, *Sugli iperanelli moltiplicativi*, Rend. Di Mat, Series 7(2) (1982), 711–724.

[22] R. Rota, *Strongly distributive multiplicative hyperrings*, Journal of Geometry, 39 (1990), 130–138.

[23] E. Sengelen Servim, B.A. Ersoy, B. Davvaz, *Primary hyperideals of multiplicative hyperrings*, International Balkan Journal of Mathematics, 1(1) (2018), 43–49.

[24] J.P. Serre, *Local algebra*, Springer Verlag, 2000.

[25] G. Ulucak, *On expansions of prime and 2-absorbing hyperideals in multiplicative hyperrings*, Turkish Journal of Mathematics, 43 (2019), 1504–1517.