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Some applications of maximal product in RL-graphs

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Abstract

This research targets the investigation of characteristics within the maximal product of two RL-graphs by scrutinizing particular types of RL-graphs. Our first step in this quest entails introducing RL-graph concepts, followed by defining what constitutes a strong RL-graph, further elucidated by a practical example. Subsequently, we lay out the connection between RL-graphs and their maximal products. In particular, a theorem establishes that two RL-graphs are regular if their maximal product maintains regularity, and a parallel rule applies to α-regular RL-graphs. Contrarily, the reverse is not inherently true, a claim supported by a specific example. Nonetheless, by incorporating an additional condition, we validate the converse. Lastly, we assert that two RL-graphs are connected only if their maximal product is also a connected RL-graph. In conclusion, the maximal product of two RL-graphs holds potential in modeling societal health metrics and road accident rates.

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1 Introduction

1.1 Fuzzy graph

Tracing back to 1736, Leonard Euler deployed graph theory to unravel the Paul Konigsberg conundrum [4]. Ever since, this tool has seen diverse applications in representing real-world dilemmas across myriad scientific disciplines [2, 3]. As graph theory evolved, unveiling its potential to tackle worldly issues, new concepts emerged to boost problem modeling accuracy.

A key innovation was Zadeh's introduction of the fuzzy set in 1965, designed to handle uncertain and ambiguous events [16, 17, 18]. Building on this, Kaufman pioneered the fuzzy graph concept [7], which researchers subsequently utilized to holistically model urban development [14]. Of late,

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fuzzy graph theory is increasingly deployed to model intricate subjects [9]. Graph and fuzzy graph theories alike have found extensive usage in diverse fields to address human-centric and everyday problems.

1.2 Zero forcing set

Consider a graph G with a vertex set V, where in vertices are either white or black. Let Z denote the initial set of black vertices within graph G. Following the color change rule, a vertex changes from white to black if it only has one neighbor - a black vertex. If the application of this color change rule results in all vertices within G turning black, then Z is deemed a zero forcing set of G. $Z(G)$ symbolizes the least $|Z|$ among all zero forcing sets. The zero forcing process, responsible for turning all white vertices black, serves as a model of a graph propagation process, a topic often broached in computer science and mathematics as referenced in [8]. Various graph processes find utility in modeling social or technical phenomena, such as social network analysis [6] and physics [2]. A comprehensive survey of different applications and models is available in [5]. The zero forcing process is employed in [1] to set a limit on the minimum rank or the maximum nullity of a graph G.

1.3 Some motivations to applications

In 2022, Zahedi and his team proposed the concept of an L-graph (also known as an RL-graph) as a tool to categorize and establish correlations between library books, identifying the minimal number of books required to encompass all pertinent subjects [13]. This graph concept also offers utility in areas such as drug classification and determining the minimum selection of drugs for effective disease treatment. Moreover, RL-graphs are invaluable in ascertaining the fewest number of companies necessary to fulfill all product demands. Subsequently, additional properties of these graphs were delineated in papers [12, 19]. In a further development, scholars introduced operations such as maximal and Kronecker products for these graphs, enabling the modeling of more intricate problems and thereby facilitating more precise and comprehensive decisions [10, 11]. For instance, consider an educational system where both personality traits and general conditions are crucial factors in determining system quality. We can assess these traits and conditions by utilizing an L-graph, consequently refining the educational system. Moreover, by applying the maximal product of two L-graphs, we can gauge the influence of each personality trait on the general conditions. Hence, this modeling strategy can guide us toward appropriate solutions for enhancing the educational system. Our initial review outlines this education system's most effective route for individual success. Also, considering two construction companies, their combined work efficiency can be assessed using the Kronecker product. With its diverse applicability, researchers from varying fields can exploit this tool for problem-solving. Moreover, an insightful relationship between graphs and automata was unveiled, offering mutual benefits for researchers across both domains.

1.4 Contribution

As we navigate an era of technological advancement and urban expansion, societal challenges have become increasingly multifaceted. To tackle such complex issues, we require broader and more intricate concepts that can accurately represent and propose solutions. The introduction of the L-graph (RL-graph) concept has facilitated the modeling of complex scenarios and provided viable solutions.

Our present focus is on modeling even more complex problems using the maximal product of two RL-graphs and devising corresponding solutions. In this research, we elaborate on the notion of the maximal product of two RL-graphs. This concept facilitates the formation of a relationship between two distinct structures and relates the influence of these structures. Furthermore, we illustrate the diverse applications of this operator, though we highlight only two. In addition, we scrutinize the relationships between these graphs and their operations through concepts such as the regular RL-graph, α-regular RL-graph, and connected RL-graph. Finally, leveraging the properties of RL-graphs and the maximal product of two RL-graphs, we present a model that incorporates factors impacting societal health and road accident frequency. We also provide examples of these model applications and encourage researchers in these fields to empirically validate the accuracy of these proposed solutions.

1.5 Framework

Section 2 will encompass definitions of the residuated lattice and the maximal product of two RL-graphs, in addition to introducing several notions that will be requisite in Sections 3 and 4. Section 3 will inaugurate a novel category of RL-graphs predicated on the memberships of vertices and edges. Subsequently, we will elucidate some relationships between two specific RL-graphs and their maximal product. Section 4 will then present practical applications, supplemented with examples.

2 Preliminaries

In this section, we give some definitions for L-graph (RL-graph), which should be used in the next section.

Definition 2.1. [15] *A residuated lattice is an algebra* $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *so that*

1. $L = (L, \wedge, \vee, 0, 1)$ *is a lattice (the corresponding order will be noted by* $\leq$*) with the smallest element* 0 *and the greatest element* 1*,*
2. $L = (L, \otimes, 1)$ *is a commutative monoid (i.e.,* $\otimes$ *is commutative, associative, and* $x \otimes 1 = x$ *holds),*
3. $x \otimes y \leq z$ *if and only if* $x \leq y \rightarrow z$ *holds (adjointness condition).*

Definition 2.2. [13] $G = (\alpha, \beta)$ *is called an RL-graph on a simple graph* $G^* = (V, E)$ *if* $\alpha : V \rightarrow L$ *and* $\beta : E \rightarrow L$ *are functions while* L *is a residuated lattice, with* $\beta(st) \leq \alpha(s) \otimes \alpha(t)$ *for every* $st \in E$*. Besides, if* G^* *is a path (cycle, bipartite, complete, complete bipartite, etc) graph, then* G *is called a path (cycle, bipartite, complete, complete bipartite, etc) L-graph on* G^**.*
If $G = (\alpha, \beta)$ *is an RL-graph on* $G^* = (V, E)$ *so that* $\beta(st) = \alpha(s) \otimes \alpha(t)$*, for every* $st \in E$*, then* G *is a strong RL-graph.*

Definition 2.3. [13] *Let* $G_1 = (\alpha_1, \beta_1)$ *and* $G_2 = (\alpha_2, \beta_2)$ *be two RL-graphs on* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$*, respectively, and* $c \in L \backslash \{1\}$*. Then* G_1 *and* G_2 *are isomorphic with threshold* c*, noted by* $G_1 \cong_c G_2$ *if there exists a bijection* h *from* V_1 *into* V_2 *such that the following conditions hold for all* $u, v \in V_1$*:*

(i) $uv \in E_1$ *if and only if* $h(u)h(v) \in E_2$*,*

(ii) $\alpha_1(u) > c$ *if and only if* $\alpha_2(h(u)) > c$*,* *(iii)* $\beta_1(uv) > c$ *if and only if* $\beta_2(h(u)h(v)) > c$.

h is an isomorphism ($\cong$) *if and only if h is an isomorphism with threshold c for every* $c \in L\backslash\{1\}$.

Definition 2.4. [11] *Let* $G = (\alpha, \beta)$ *on* $G^* = (V, E)$ *be an RL-graph that* G^* *is a* $(k-)$*regular graph. Then G is called the* $(k-)$*regular RL-graph. If* α *has the same value for all vertices of the regular RL-graph G, then G is* α*-regular RL-graph. Additionally, if* β *has the same value for all edges of the regular RL-graph G, then G is* β*-regular RL-graph. Besides, it is a totally regular RL-graph if G is* α*-regular and* β*-regular RL-graph.*

Definition 2.5. [10] *Let G and H be two RL-graphs. Then the maximal product of two RL-graphs G and H is defined by* $G \star H = (\alpha, \beta)$ *on* $(G \star H)^* = (V, E)$, *where*

(i) $V = V_1 \times V_2$,

(ii) $E = \{(q, q'_k)(q, q'_l) |\ q \in V_1, q'_k q'_l \in E_2\} \cup \{(q_i, q')(q_j, q') |\ q' \in V_2, q_i q_j \in E_1\}$,

(iii) $\alpha(q_i, q'_k) = \alpha_1(q_i) \vee \alpha_2(q'_k)$, *for every* $(q_i, q'_k) \in V$,

(iv) $\beta((q_i, q'_k)(q_j, q'_l)) = \begin{cases} \alpha_1(q_i) \otimes \beta_2(q'_k q'_l) \text{ if } q_i = q_j, q'_k q'_l \in E_2, \\ \alpha_2(q'_k) \otimes \beta_1(q_i q_j) \text{ if } q'_k = q'_l, q_i q_j \in E_1, \end{cases}$ *for every* $(q_i, q'_k)(q_j, q'_l) \in E$.

3 Some results on the maximal product of two RL-graphs

In the ensuing section, we will first expound on the concept of a totally strong RL-graph, employing an example to provide further clarity. Additionally, we establish, through a theorem, the regularity of two RL-graphs if and only if their maximal product is likewise regular. It is also demonstrated that α-regularity in two RL-graphs ensures the same characteristic in their maximal product. Conversely, we provide an example to highlight that the reverse of this statement is not inherently true. However, upon introducing an additional condition, we subsequently prove this converse. We also present that if two RL-graphs are deemed totally strong with identical membership vertices, then their maximal product achieves total regularity, but not total strength. A supporting example disproves the reverse of the aforementioned statement. Following this, we introduce a condition on two strong RL-graphs that guarantees their maximal product is also strong. Lastly, we establish that two RL-graphs are connected if and only if their maximal product is a connected RL-graph.

Definition 3.1. *Let* $G = (\alpha, \beta)$ *on* $G^* = (V, E)$ *be an RL-graph. Then it is a totally strong RL-graph if it is the strong and the totally regular RL-graph.*

Example 3.2. *Suppose* $L = (P(X), \cap, \cup, \otimes, \rightarrow, \emptyset, X)$, *where* $X = \{a, b, c\}$, $A \otimes B = A \cap B$ *and* $A \rightarrow B = \begin{cases} X & \text{otherwise}, \\ B & \text{if } B \subset A, \end{cases}$ *for every* $A, B \in P(X)$. *Then* $G = (\alpha_1, \beta_1)$ *on* $G^* = (V_1, E_1)$ *is a totally strong RL-graph, as in Figure 1, where* $V_1 = \{q_1, q_2, q_3, q_4\}$, $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_1q_4\}$, $\alpha_1(q_i) = \{a, b\}$ *and* $\beta_1(q_iq_j) = \{a, b\}$, *for every* $1 \le i, j \le 4$ *and* $q_iq_j \in E_2$.

Theorem 3.3. *Let* $G = (\alpha_1, \beta_1)$ *on* $G^* = (V_1, E_1)$ *and* $H = (\alpha_2, \beta_2)$ *on* $H^* = (V_2, E_2)$ *be two RL-graphs. Then:*

(i) If G is a k-regular RL-graph, and H is a k'*-regular RL-graph, then their maximal product is a* $k + k'$*-regular RL-graph.*
Generally, if G and H are two regular RL-graphs, then their maximal product is a regular RL-graph.

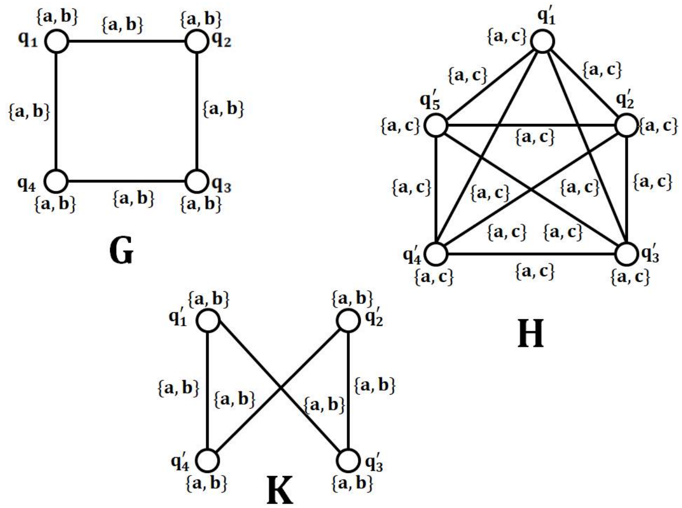

Figure 1: The RL-graphs G and H.

(ii) *If the maximal product $G \star H$ is a regular RL-graph, then G and H are two regular RL-graphs.*

Proof. (i) Suppose that their maximal product is $G \star\ H = \ (\alpha,\ \beta)$ on $(G \star\ H)^* = (V, E)$. We know that it is valid for every (q_i, q'_j) in their maximal connect to the vertices (q_i, q'_k), for every $q'_j q'_k \in E_2$ and (q_l, q'_j), for every $q_i q_i \in E_1$. Besides, we can say that this RL-graph is the regular RL-graph that is the $k + k'$-regular RL-graph.
(ii) We know that $d_{G\star H}(q_i, q_j) = d_G(q_i) + d_H(q_j)$. If consider that i is constant, then $d_{G\star H}(q_i, q_j) = d_G(q_i) + d_H(q_j) = k$, for every $1 \le j \le n$. So, $d_H(q_j) = k'$, for every $1 \le j \le n$. Thus, H is the regular RL-graph. Also, as with H, we prove that G is also a regular RL-graph. □

Theorem 3.4. *Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ be two α-regular RL-graphs. Then their maximal product is the α-regular RL-graph.*

Proof. By using the definitions of α, the proof becomes clear. □

Example 3.5. *Suppose L in Example 3.2 and two RL-graphs $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$, as in Figure 2, where $V_1 = \{q_1, q_2\}$, $E_1 = \{q_1 q_2\}$, $\alpha_1(q_1) = \{a, b\}$, $\alpha_1(q_2) = \{a, c\}$, $\beta_1(q_1 q_2) = \{a\}$, $V_2 = \{q'_1, q'_2\}$, $E_2 = \{q'_1 q'_2\}$, $\alpha_2(q'_1) = \{a, b, c\}$, $\alpha_2(q'_2) = \{b, c\}$ and $\beta_2(q'_1 q'_2) = \{b, c\}$. Hence, $G \star H = (\alpha, \beta)$ on $(G \star H)^* = (V, E)$ is the maximal product, as in Figure 2, where $V = \{q_1, q_2, q'_1, q'_2\}$, $E = \{q_1 q_2, q'_1 q'_2\}$, $\alpha((q_1, q'_1)) = \alpha((q_1, q'_2)) = \alpha((q_2, q'_1)) = \alpha((q_2, q'_2)) = \{a, b, c\}$, $\beta((q_1, q'_1)(q_1, q'_2)) = \{b\}$, $\beta((q_1, q'_1)(q_2, q'_1)) = \{a\}$, $\beta((q_1, q'_2)(q_2, q'_2)) = \emptyset$ and $\beta((q_1, q'_2)(q_2, q'_2)) = \{c\}$. Clearly, the RL-graph $G \star H$ is the α-regular.*

Note 3.6. *The aforementioned example indicates that the maximal product of two RL-graphs is α-regular; however, their components are α-regular RL-graphs.*

Theorem 3.7. *If $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ are two isomorphic RL-graphs and their maximal product is α-regular RL-graph, then G and H are α-regular RL-graphs.*

Proof. By using the definition of two isomorphic RL-graphs, the proof is straightforward. □

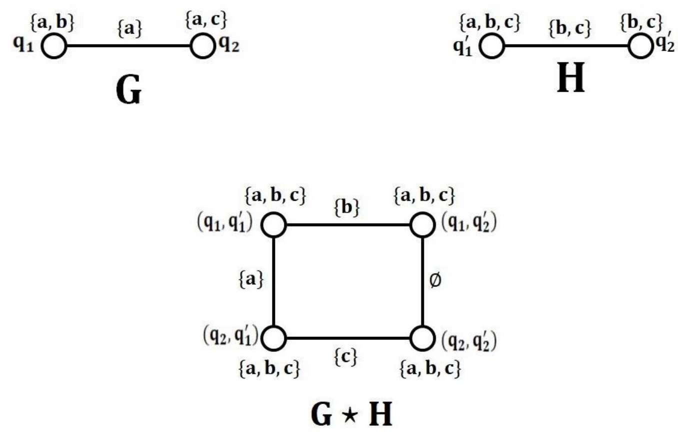

Figure 2: The RL-graphs G, H and $G \star H$.

Theorem 3.8. *Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ be totally strong RL-graphs such that $\alpha_1 = \alpha_2$. Then then their maximal product is a totally regular RL-graph.*

Proof. By using definitions of α and β, the proof is straightforward. □

Example 3.9. *Consider totally strong RL-graphs G in Example 3.2 and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$, as in Figure 1, where $V_2 = \{q'_1, q'_2, \ldots, q'_5\}$, $E_2 = \{q'_1q'_2, q'_1q'_5, q'_1q'_3, q'_1q'_4, q'_2q'_3, q'_2q'_4,$ $q'_2q'_5, q'_3q'_4, q'_3q'_5, q'_4q'_5\}$, $\alpha_2(q'_i) = \{a, c\}$, $\beta_2(q'_iq'_j) = \{a, c\}$, for every $1 \leq i \leq 5$ and $q'_iq'_j \in E_2$. Then their maximal product $G \star H = (\alpha, \beta)$ on $(G \star H)^* = (V, E)$, as in Figure 3, where $V = \{(q_i, q'_j) |\ 1 \leq\ i \leq 4, 1 \leq j \leq 5\}$, $E = \{(q_1, q'_j)(q_2, q'_j), (q_2, q'_j)(q_3, q'_j), (q_3, q'_j)(q_4, q'_j), (q_1, q'_j)(q_4, q'_j),$ $(q_i, q'_1)(q_i, q'_2), (q_i, q'_1)(q_i, q'_5), (q_i, q'_1)(q_i, q'_3), (q_i, q'_1)(q_i, q'_4), (q_i, q'_2)(q_i, q'_3), (q_i, q'_2)(q_i, q'_4), (q_i, q'_2)(q_i, q'_5),$ $(q_i, q'_3)(q_i, q'_4), (q_i, q'_3)(q_i, q'_5), (q_i, q'_4)(q_i, q'_5) |\ \ 1 \leq i \leq 4, 1 \leq j \leq 5\}$, $\alpha(q_i, q'_j) = \{a, b, c\}$, for every $(q_i, q'_j) \in E$, $\beta(q_i, q'_j)(q_l, q'_k) = \{a\}$, for every $(q_i, q'_j)(q_l, q'_k) \in E$. Thus, $G \star H$ is the totally regular RL-graph; however, this is not a strong RL-graph.*

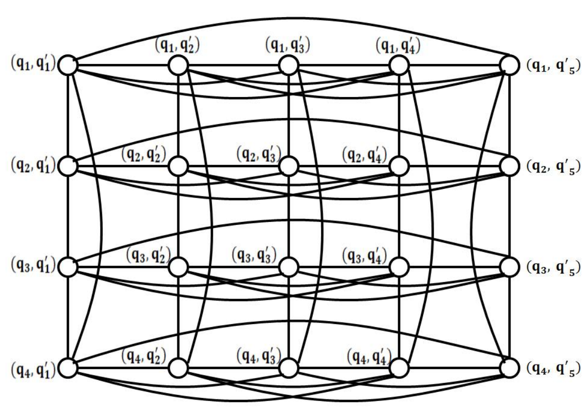

Figure 3: The maximal product $(G \star H)^*$.

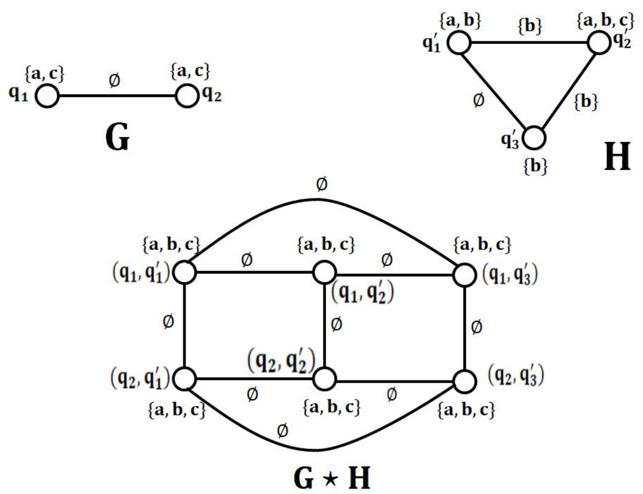

Figure 4: The RL-graphs G, H and $G \star H$.

Note 3.10. *According to the above example, the maximal product of two totally strong RL-graphs can not be a totally strong RL-graph. As a result, the maximal product of two totally strong RL-graphs is not necessarily a totally strong RL-graph.*

Example 3.11. *Suppose L in Example 3.2 and two RL-graphs* $G = (\alpha_1, \beta_1)$ *on* $G^* = (V_1, E_1)$ *and* $H = (\alpha_2, \beta_2)$ *on* $H^* = (V_2, E_2)$*, as in Figure 4, where* $V_1 = \{q_1, q_2\}$, $E_1 = \{q_1q_2\}$, $\alpha_1(q_1) = \{a, c\}$, $\alpha_1(q_2) = \{a, c\}$, $\beta_1(q_1q_2) = \emptyset$, $V_2 = \{q_1', q_2', q_3'\}$, $E_2 = \{q_1'q_2', q_1'q_3', q_2'q_3'\}$, $\alpha_2(q_1') = \{a, b\}$, $\alpha_2(q_2') = \{a, b, c\}$, $\alpha_2(q_3') = \{b\}$, $\beta_2(q_1'q_2') = \{b\}$, $\beta_2(q_2'q_3') = \{b\}$ *and* $\beta_2(q_1'q_3') = \emptyset$*. Hence,* $G \star H = (\alpha, \beta)$ *on* $(G \star H)^* = (V, E)$ *is the maximal product, as in Figure 4, where* $V = \{(q_1, q_i'), (q_2, q_i') | \ 1 \leq i \leq 3\}$, $E = \{(q_1, q_1')(q_1, q_2'), (q_1, q_1')(q_1, q_3'), (q_1, q_2')(q_1, q_3'), (q_2, q_1')(q_2, q_2'), (q_2, q_1')(q_2, q_3'), (q_2, q_2')(q_2, q_3'),$ $(q_1, q_1')(q_2, q_1'), (q_1, q_2')(q_2, q_2'), (q_1, q_3')(q_2, q_3'), \}$, $\alpha((q_i, q_j')) = \{a, b, c\}$*, for every* $(q_i, q_j') \in V$, $\beta((q_i, q_j')(q_k, q_l')) = \emptyset$*, for every* $(q_i, q_j')(q_k, q_l') \in E$*. Clearly, the RL-graph* $G \star H$ *is the totally regular RL-graphs.*

Note 3.12. *The aforementioned example indicates that the maximal product of two RL-graphs is a totally regular; however, their components are not totally regular RL-graphs.*

Theorem 3.13. *Let* $G = (\alpha_1, \beta_1)$ *on* $G^* = (V_1, E_1)$ *and* $H = (\alpha_2, \beta_2)$ *on* $H^* = (V_2, E_2)$ *be two isomorphic totally strong RL-graphs. Then their maximal product is a totally strong RL-graphs.*

Proof. The proof is straightforward, using Theorem 3.3 and the definition of two isomorphic RL-graphs. □

Example 3.14. *Suppose two totally strong RL-graphs G, as in Example 3.2 and* $K = (\alpha_2, \beta_2)$ *on* $K^* = (V_2, E_2)$*, as in Figure 1, where* $V_2 = \{q_1', q_2', q_3', q_4'\}$, $E_2 = \{q_1'q_3', q_1'q_4', q_2'q_3', q_2'q_4'\}$, $\alpha_2(q_i') = \{a, b\}$ *and* $\beta_2(q_i'q_j') = \{a, b\}$*, for every* $q_i' \in V_2$ *and for every* $q_iq_j \in E_2$*. Clearly, these RL-graphs are isomorphic. Then their maximal product is* $G\star K = (\alpha, \beta)$ *on* $(G\star K)^* = (V, E)$*, as in Figure 5, where* $V = \{(q_i, q_j') | \ 1 \leq i, j \leq 4\}$, $E = \{(q_1, q_j')(q_2, q_j'), (q_2, q_j')(q_3, q_j'), (q_3, q_j')(q_4, q_j'),$ $(q_1, q_j')(q_4, q_j'), (q_i, q_1')(q_i, q_3'), (q_i, q_1')(q_i, q_4'), (q_i, q_2')(q_i, q_3'), (q_i, q_2')(q_i, q_4') | \ 1 \leq i, j \leq 4\}$, $\alpha(q_i, q_j') = \{a, b\}$*, for every* $(q_i, q_j') \in E$, $\beta(q_i, q_j')(q_l, q_k') = \{a, b\}$*, for every* $(q_i, q_j')(q_l, q_k') \in E$*. Clearly, it is a totally strong RL-graph.*

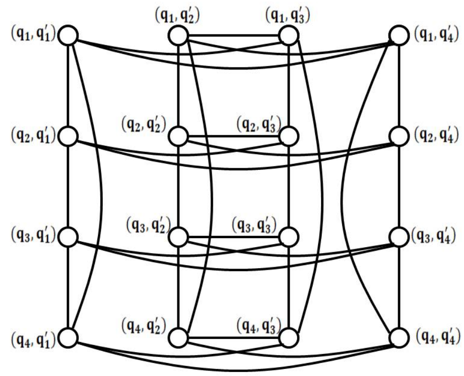

Figure 5: The maximal product $(G \star K)^*$.

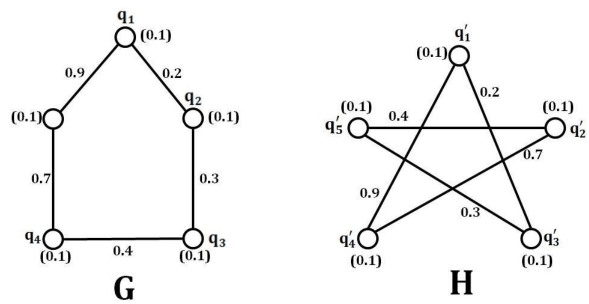

Figure 6: Two isomorphic RL-graphs G and H.

Example 3.15. *Suppose $L = ([0,1], \wedge, \vee, \otimes, \rightarrow, 0, 1)$, where*

$$a \otimes b = \begin{cases} (a+b-1) & \text{if } a+b \geq 1, \\ 0 & \text{if } a+b < 1, \end{cases} \qquad \text{and} \qquad a \rightarrow b = \begin{cases} 1 & \text{if } b-a \geq 0, \\ (1-a+b) & \text{if } b-a < 0. \end{cases}$$

Also, suppose that two isomorphic RL-graphs $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$, as in Figure 6, where $V_1 = \{q_1, q_2, q_3, q_4, q_5\}$, $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_5, q_1q_5\}$, $\alpha_1(q_i) = 0.1$, for every $1 \leq i \leq 5$, $\beta_1(q_1q_2) = 0.2$, $\beta_1(q_2q_3) = 0.3$, $\beta_1(q_3q_4) = 0.4$, $\beta_1(q_4q_5) = 0.7$, $\beta_1(q_1q_5) = 0.9$, $V_2 = \{q'_1, q'_2, q'_3, q'_4, q'_5\}$, $E_2 = \{q'_1q'_3, q'_1q'_4, q'_2q'_4, q'_2q'_5, q'_3q'_5\}$, $\alpha_2(q'_i) = 0.1$, for every $1 \leq i \leq 5$, $\beta_2(q'_1q'_3) = 0.2$, $\beta_2(q'_3q'_5) = 0.3$, $\beta_2(q'_5q'_2) = 0.4$, $\beta_2(q'_2q'_4) = 0.7$, and $\beta_2(q'_1q'_4) = 0.9$. Hence, their maximal product is $G\star K = (\alpha, \beta)$ on $(G\star K)^* = (V, E)$, as in Figure 7, where $V = \{(q_i, q_j)|\ 1 \leq i, j \leq 5\}$, $E = \{(q_i, q'_1)(q_i, q'_3), (q_i, q'_3)(q_i, q'_5), (q_i, q'_5)(q_i, q'_2), (q_i, q'_2)(q_i, q'_4), (q_i, q'_1)(q_i, q'_4), (q_1, q'_i)(q_2, q'_i), (q_2, q'_i)(q_3, q'_i), (q_3, q'_i)(q_4, q'_i), (q_4, q'_i)(q_5, q'_i), (q_1, q'_i)(q_5, q'_i)|\ \ 1 \leq i \leq 5\}$, $\alpha((q_i, q_j)) = 0.1$, for every $(q_i, q_j) \in V$ and $\beta((q_i, q'_j)(q_l, q'_k)) = 0$, for every $(q_i, q'_j)(q_l, q'_k) \in E$. Clearly, it is a totally strong RL-graphs.

Note 3.16. *The aforementioned example indicates that the maximal product of two isomorphic RL-graphs is a totally strong; however, their components are not totally strong RL-graphs.*

Theorem 3.17. *If $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ are two connected RL-graphs if and only if their maximal product, $G\star K = (\alpha, \beta)$ on $(G\star K)^* = (V, E)$, is also a connected RL-graphs.*

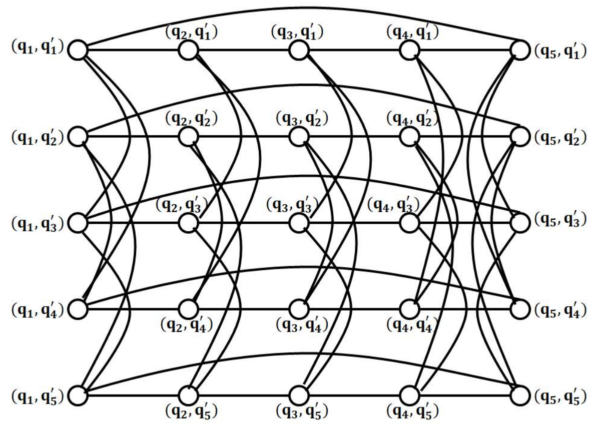

Figure 7: The graph $(G \star H)^*$.

Proof. ($\Rightarrow$): Suppose the vertex $(q_i, q'_j) \in V$ such that $q_i \in V_1$ and $q'_j \in V_2$. Hence, sine G is connected, there exists at least one path so that this vertex is connected to the $(q_k, q'_j) \in V$ for every $q_k \in V_1$. Also, H is connected, there exists at least one path so that this vertex is connected to the $(q_i, q'_l) \in V$ for every $q_l \in V_2$. Therefore, their maximal product is connected RL-graphs.
($\Leftarrow$): The proof same as above with some modifications. $\square$

Example 3.18. *Consider two connected RL-graphs G and H in Example 3.9. We can see that their maximal product is connected RL-graph.*

4 Applications

In today's era, marked by rapid technological progression and urban expansion, social issues have evolved to exhibit unprecedented complexity. Hence, addressing these challenges necessitates the implementation of sophisticated, broad-ranging concepts capable of engineering and proposing solutions. The advent of L-graphs (or RL-graphs) empowered us to model such complex issues and devise problem-solving strategies. Currently, we aim to model even more intricate scenarios employing the concept of the maximal product of two RL-graphs and subsequently propose solutions.

Initially, we introduce a theorem related to the maximal product of two RL-graphs. Leveraging this notion alongside the concepts of RL-graphs and their maximal products, we construct a model of societal health. Moreover, utilizing Theorem 4.1, we propose an expedited pathway to enhance societal health, elucidated further through a practical example. This section also delves into factors influencing road accident rates, bifurcated into personal and societal elements. Through our modeling, we advocate for governmental emphasis on ameliorating public factors to mitigate road accident frequency.

Theorem 4.1. [10] *Let G and H be two RL-graphs. Then*

(i) *If $|Z(G)| \times |V_2| < |Z(H)| \times |V_1|$, then $Z(GH) = \{(x, y) | \; x \in Z(G), \; y \in V_2\}$, $|Z(GH)| = |Z(G)| \times |V_2|$, and $S(Z(GH)) = S(Z(G))$, which $S(Z(G))$ is the number of steps in which all vertices of G are black.*

(ii) *If $|Z(H)| \times |V_1| < |Z(G)| \times |V_2|$, then*

$$Z(GH) = \{(x, y) | \; x \in V_1, \; y \in Z(H)\},$$

$|Z(GH)| = |Z(H)| \times |V_1|$, *and* $S(Z(GH)) = S(Z(H))$.

(*iii*) *If* $|Z(H)| \times |V_1| = |Z(G)| \times |V_2|$, *then the maximal product of two RL-graphs has at least two zero forcing sets, where* $Z_1(GH) = \{(x, y)|\ x \in Z(G),\ y \in V_2\}$ *with* $S(Z_1(GH)) = S(Z(G))$ *and* $Z_2(GH) = \{(x, y)|\ x \in V_1,\ y \in Z(H)\}$ *with* $S(Z_2(GH)) = S(Z(H))$.

Application 4.2. *Societal health comprises two facets: personal hygiene and public health, both crucial to determining overall health quality. Personal hygiene encompasses elements like individual grooming, consumption of nutritious food, regular exercise, and consistent medical check-ups. Conversely, public health is influenced by the quality of healthcare facilities, medical expertise, and the degree of disease awareness dissemination.*

To model societal health, we use two RL-graphs, G *and* H*, as personal and public health proxies, respectively. Every individual factor within these categories is symbolized as a vertex in these RL-graphs. An edge connecting the corresponding vertices represents a meaningful correlation between factors. For instance, in RL-graph* G*, which models personal health, the mutual dependency between personal grooming and regular medical consultations is indicated by an edge linking their respective vertices. Utilizing RL-graphs* G *and* H*, we analyze personal and societal factors to improve overall health conditions. Furthermore, we harness the maximal product of these RL-graphs to quantify the influence of each personal health factor on broader societal conditions. To ascertain this influence, we incorporate* α *and* β*, which encapsulate the impact of each new condition (a fusion of personality traits and societal conditions) on one another.*
Let $L = (\{1, 2, \ldots, 10\}, \vee, \wedge, \otimes, \rightarrow, 1, 10)$*, where*

$$a \otimes b = \begin{cases} (a + b - 10) & \text{if } a + b > 10, \\ 1 & \text{if } a + b \leq 10, \end{cases} \quad \text{and} \quad a \rightarrow b = \begin{cases} 10 & \text{if } b - a \geq 0, \\ (10 - a + b) & \text{if } b - a < 0, \end{cases}$$

and let $G = (\alpha_1, \beta_1)$ *on* $G^* = (V_1, E_1)$ *and* $H = (\alpha_2, \beta_2)$ *on* $H^* = (V_2, E_2)$ *be two RL-graphs, as in Figure 8, where* $V_1 = \{$*personal grooming*(q_1)*, use of healthy foods*(q_2)*, regular exercise* (q_3)*, regular visits to the doctor*$(q_4)\}$, $E_1 = \{q_1q_2, q_1q_4, q_2q_4, q_3q_4\}$, $\alpha_1(q_i) =$ *the amount of* q_i, $\beta_1(q_iq_j) = \alpha_1(q_i) \otimes \alpha_1(q_j)$*, for every* $q_iq_j \in E_1$, $V_2 = \{$*treatment centers* (q'_1)*, doctors' expertise*(q'_2)*, notification for all types of diseases*$(q'_3)\}$, $E_2 = \{q'_1q'_2, q'_1q'_3\}$, $\alpha_2(q'_k) =$ *quality of* q'_k*, for every* $k = 1, 2, 3$ *and* $\beta_2(q'_iq'_j) = \alpha_2(q'_i) \otimes \alpha_2(q'_j)$*, for every* $q'_iq'_j \in E_2$*. Besides, their maximal product is* $G \star H = (\alpha, \beta)$ *on* $(G \star H)^* = (V, E)$*, as in Figure 8, where* $V = \{(q_i, q'_1), (q_i, q'_2), (q_i, q'_3)|1 \leq i \leq 4\}$, $E = \{(q_1, q'_i)(q_2, q'_i), (q_1, q'_i)(q_4, q'_i), (q_2, q'_i)(q_4, q'_i), (q_3, q'_i)(q_4, q'_i), (q_j, q'_1)(q_j, q'_2), (q_j, q'_1)(q_j, q'_3)\}$, $\alpha((q_i, q'_j)) = \alpha_1(q_i) \vee \alpha_2(q'_j)$*, for every* $(q_i, q'_j) \in V$, $\beta((q_i, q'_j)(q_i, q'_k)) = \alpha_1(q_i) \otimes \beta_2(q'_jq'_k)$*, for every* $q'_jq'_k \in E_2$ *and* $1 \leq i \leq 4$*, and* $\beta((q_i, q'_j)(q_l, q'_j)) = \alpha_2(q'_j) \otimes \beta_1(q_iq_l)$*, for every* $q_iq_l) \in E_1$ *and* $1 \leq j \leq 3$*. We can calculated that the zero forcing sets of* G *are five sets* $Z_1(G) = \{q_1, q_2\}$, $Z_2(G) = \{q_1, q_3\}$, $Z_3(G) = \{q_1, q_4\}$, $Z_4(G) = \{q_2, q_3\}$*, and* $Z_5(G) = \{q_2, q_4\}$*, and the zero forcing sets of* H *are two sets* $Z_1(H) = \{q'_1\}$ *and* $Z_2(H) = \{q'_3\}$*. Also, we know that* $S(Z(G)) = 3$*, for every zero forcing set and* $S(Z(H)) = 3$*, for every zero forcing set. Also, since*

$$4 = |Z(H)| \times |V_1| < |Z(G)| \times |V_2| = 6,$$

by according Theorem 4.1, we have

$$Z(G \star H) = Z(GH) = \{(x,\ y)|\ x \in V_1,\ y \in Z(H)\}, |Z(GH)| = |Z(H)| \times |V_1| = 4,$$

and

$$S(Z(GH)) = S(Z(H)) = 3.$$

In general, improving the quality of a society's system increases the $Z(H)$ *rate. Therefore, it can be seen that to improve the society's system, how to implement public health is important.*

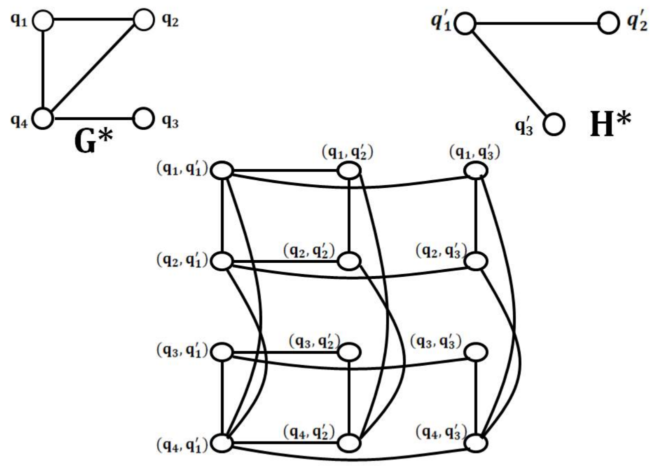

Figure 8: Graphs G^* and H^*, and the maximal product of RL-graphs G and H.

Remark 4.3. *Significant resources are allocated in various societies towards enhancing healthcare infrastructure, advancing medical knowledge and expertise, and promoting disease awareness. Our method enables the visualization of the justification for this considerable investment, underscoring its importance. Moreover, it highlights that enhancing public health constitutes one of the fastest pathways to overall societal health improvement.*

Example 4.4. *Let G and H be two RL-graphs that are modeling of personal and general hygiene, where $\alpha_1(q_i) = 6$, for every $1 \leq i \leq 4$, $\beta_1(q_iq_j) = 2$, for every $q_iq_j \in E_1$, $\alpha_2(q'_i) = 8$, for every $1 \leq i \leq 3$, $\beta_2(q'_iq'_j) = 6$, for every $q'_iq'_j \in E_2$. Then their maximal product is $G \star H = (\alpha, \beta)$ on $(G \star H)^* = (V, E)$, as in Application 4.2, where $\alpha((q_i, q'_j)) = 8$, for every $(q_i, q'_j) \in V$, $\beta((q_i, q'_j)(q_i, q'_k)) = 2$, for every $q'_jq'_k \in E_2$ and $1 \leq i \leq 4$, and $\beta((q_i, q'_j)(q_l, q'_j)) = 1$, for every $q_iq_l \in E_1$ and $1 \leq j \leq 3$.*

Remark 4.5. *The aforementioned example underscores how enhancing public health can positively influence societal health at large. We thereby urge communities to focus their efforts on uplifting the overall well-being of their members.*

Application 4.6. *Road accidents are influenced by a multitude of factors, which can be categorized into two categories: personal and societal. Personal factors encompass an individual's driving skills, their understanding of traffic regulations, adherence to these rules, focus while driving, and the pre-driving technical inspection of the vehicle. In contrast, societal factors include the instillation of obedience to traffic rules, law enforcement's surveillance of traffic regulation compliance, technical inspection of market vehicles, and measures to ensure road safety.*

To evaluate the factors contributing to road accidents, we utilize two RL-graphs, G and H, which represent personal and societal factors, respectively. Every individual factor is symbolized as a vertex within these RL-graphs, and meaningful relationships between factors are represented by edges connecting their respective vertices.

For example, in RL-graph G, which models personal factors, an edge connects the vertex symbolizing a person's driving ability and the one representing their understanding of traffic rules, illustrating their interdependence. Using RL-graphs G and H, we can assess personal and societal

conditions with a view to reducing road accidents. Moreover, by using the maximal product of these RL-graphs, we can discern the influence of each personal factor on broader societal conditions. We employ α and β to indicate the impact of each new condition, derived from a mix of personal traits and societal conditions, on one another.
Let $L = (\{1, 2, \ldots, 10\}, \vee, \wedge, \otimes, \rightarrow, 1, 10)$, *where*

$$a \otimes b = \begin{cases} (a+b-10) & \text{if } a+b > 10, \\ 1 & \text{if } a+b \leq 10, \end{cases} \quad \text{and} \quad a \rightarrow b = \begin{cases} 10 & \text{if } b-a \geq 0, \\ (10-a+b) & \text{if } b-a < 0, \end{cases}$$

and let $G = (\alpha_1, \beta_1)$ *on* $G^* = (V_1, E_1)$ *and* $H = (\alpha_2, \beta_2)$ *on* $H^* = (V_2, E_2)$ *be two RL-graphs, as in Figure 9, where* $V_1 = \{$*person's driving ability*(q_1), *person'sknowledge regarding traffic rules*(q_2), *compliance with traffic rules*(q_3), *person's consciousness while driving*(q_4), *technically checking the car before driving*$(q_5)\}$, $E_1 = \{q_1q_2, q_1q_3, q_1q_4, q_1q_5, q_2q_3, q_2q_4, q_2q_5, q_3q_4, q_3q_5\}$, $\alpha_1(q_i) =$ *the amount of* q_i, $\beta_1(q_iq_j) = \alpha_1(q_i) \otimes \alpha_1(q_j)$, *for every* $q_iq_j \in E_1$, $V_2 = \{$*cultivation to obey traffic rules*(q'_1), *police monitoring of traffic rules*(q'_2), *technically checkingthe cars in the market*(q'_3), *road safety*$(q'_4)\}$, $E_2 = \{q'_1q'_2, q'_2q'_3, q'_2q'_4\}$, $\alpha_2(q'_k) =$ *quality of* q'_k, *for every* $k = 1, 2, 3$ *and* $\beta_2(q'_iq'_j) = \alpha_2(q'_i) \otimes \alpha_2(q'_j)$, *for every* $q'_iq'_j \in E_2$. *Besides, their maximal product is* $G \star H = (\alpha, \beta)$ *on* $(G \star H)^* = (V, E)$, *as in Figure 10, where* $V = \{(q_i, q'_1), (q_i, q'_2), (q_i, q'_3), (q_i, q'_4) |\ 1 \leq i \leq 5\}$, $E = \{(q_1, q'_i)(q_2, q'_i), (q_1, q'_i)(q_3, q'_i), (q_1, q'_i)(q_4, q'_i), (q_1, q'_i)(q_5, q'_i), (q_2, q'_i)(q_3, q'_i), (q_2, q'_i)(q_4, q'_i), (q_2, q'_i)(q_5, q'_i), (q_3, q'_i)(q_4, q'_i), (q_4, q'_i)(q_5, q'_i), (q_j, q'_1)(q_j, q'_2), (q_j, q'_2)(q_j, q'_3), (q_j, q'_2)(q_j, q'_4)\}$, $\alpha((q_i, q'_j)) = \alpha_1(q_i) \vee \alpha_2(q'_j)$, *for every* $(q_i, q'_j) \in V$,

$$\beta((q_i, q'_j)(q_i, q'_k)) = \alpha_1(q_i) \otimes \beta_2(q'_jq'_k),$$

for every $q'_jq'_k \in E_2$ *and* $1 \leq i \leq 5$, *and* $\beta((q_i, q'_j)(q_l, q'_j)) = \alpha_2(q'_j) \otimes \beta_1(q_iq_l)$, *for every* $q_iq_l) \in E_1$ *and* $1 \leq j \leq 4$. *We can calculated that one of zero forcing sets of* G *is* $Z_1(G) = \{q_1, q_4, q_5\}$, *and one of zero forcing sets of* H *is* $Z_1(H) = \{q'_1, q'_2\}$.

Also, we know that $S(Z(G)) = 3$, *for every zero forcing set and* $S(Z(H)) = 3$, *for every zero forcing set. Also, since*

$$10 = |Z(H)| \times |V_1| < |Z(G)| \times |V_2| = 12,$$

by according Theorem 4.1, we have

$$Z(G \star H) = Z(GH) = \{(x,\ y)|\ x \in V_1,\ y \in Z(H)\}, |Z(GH)| = |Z(H)| \times |V_1| = 10,$$

and

$$S(Z(GH)) = S(Z(H)) = 3.$$

In general, the rate of road accidents can be reduced by increasing the $Z(H)$ *rate.*

Therefore, it is important to implement public factors to reduce the number of road accidents.

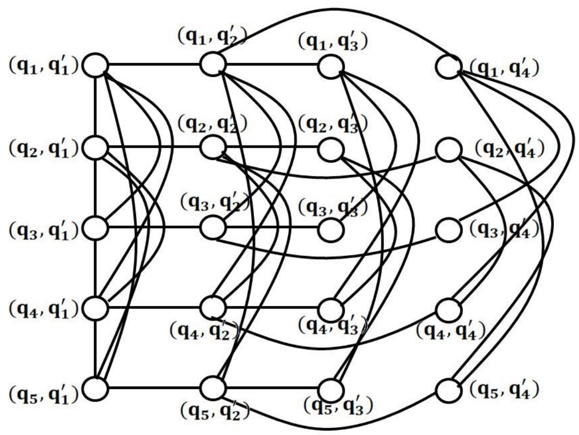

Figure 10: The graph $(G \star H)^*$.

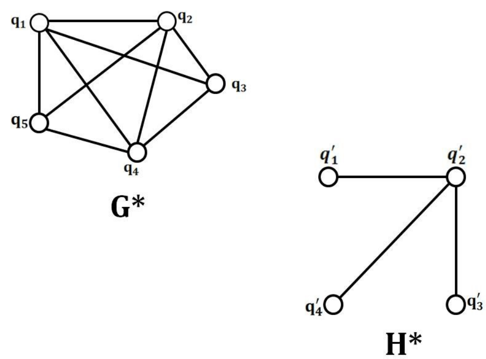

Figure 9: Graphs G^* and H^*.

Remark 4.7. *We often observe individuals who possess adequate personal driving skills yet are involved in road accidents, resulting in injuries. Our model demonstrates that to mitigate the incidence of road accidents, governments should prioritize the enhancement of societal factors.*

Example 4.8. *Let G and H be two RL-graphs that are modeling the amount of road accidents, where $\alpha_1(q_i) = 9$, for every $1 \leq i \leq 5$, $\beta_1(q_iq_j) = 8$, for every $q_iq_j \in E_1$, $\alpha_2(q'_i) = 5$, for every $1 \leq i \leq 4$, $\beta_2(q'_iq'_j) = 1$, for every $q'_iq'_j \in E_2$. Then their maximal product is $G \star H = (\alpha, \beta)$ on $(G \star H)^* = (V, E)$, as in Application 10, where $\alpha((q_i, q'_j)) = 9$, for every $(q_i, q'_j) \in V$, $\beta((q_i, q'_j)(q_i, q'_k)) = 1$, for every $q'_jq'_k \in E_2$ and $1 \leq i \leq 5$, and $\beta((q_i, q'_j)(q_l, q'_j)) = 3$, for every $q_iq_l \in E_1$ and $1 \leq j \leq 4$.*

5 Conclusion

This study endeavored to define and elucidate the maximal product of two specific types of *RL*-graphs. It utilized the maximal product of two *RL*-graphs to model societal health and the rate of road accidents, proposing the most expedient way to improve societal health. Moreover, we suggest that governments concentrate their efforts on improving societal factors to diminish road accident rates. Mirroring the way scientists explore biological issues to identify optimal treatments for specific diseases, we used the RL graph structure to produce comprehensive insights. Ultimately, our aim was to identify the most efficient solution to a given problem using the available data.

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