Journal of Algebraic Hyperstructures and Logical Algebras
Volume 4, Number 1, (2023), pp. 61-79

Zipped coherent quantales

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Abstract

The aim of this paper is to define an abstract quantale framework for extending some properties of the zip rings (studied by Faith, Zelmanowitz, etc.) and the weak zip rings (defined by Ouyang). By taking as prototype the quantale of ideals of a zip ring (resp. a weak zip ring) we introduce the notion of zipped quantale (resp. weakly zipped quantale). The zipped quantales also generalize the zipped frames, defined by Dube and Blose in a recent paper. We define the zip (bounded distributive) lattices and we prove that a coherent quantale A is weakly zipped iff the reticulation $L(A)$ of A is a zip lattice. From this result we obtain the following corollary: the coherent quantale A is weakly zipped iff the frame $R(A)$ of the radical elements of A is zipped. Such theorems allow us to extend to quantale framework a lot of results obtained by Dube and Blose for the zipped frames and for the weak zip rings.

Article Information

Corresponding Author:
G. Georgescu;
Received: May 2023;
Revised: ————;
Accepted: Invited paper;
Paper type: Original.

Keywords:

Coherent quantale, reticulation of a quantale, weak annihilator, zipped quantale, weakly zipped quantales.

1 Introduction

Abstract Ideal Theory is a branch of algebra concerned with some abstract structures that arise from the lattices of ideals (or filters) in rings, lattices, etc. The quantales and the frames are abstract structures that offer a framework in which important algebraic and topological properties of ideals, filters or congruences in rings, lattices or other algebras can be generalized (see [3, 7, 8, 18, 25]).

Now we shall shortly present some ideas regarding the way in which a quantale version of the zip rings and the weak zip rings can be developed.

Let R be a commutative (unital) ring. We denote by $Id(R)$ the set of ideals in R and by $R(Id(R))$ the set of radical ideals in R. We know that $Id(R)$ is a quantale [26] and $R(Id(R))$ is a frame [18].

https://doi.org/10.52547/HATEF.JAHLA.4.1.5

An ideal I of R is said to be a faithful ideal if the annihilator $Ann_R(I)$ of I is the zero ideal of R. According to [10, 11], R is said to be a zip ring if for any faithful ideal I of R there exists a finitely generated ideal J of R such that $J \subseteq I$. The terminology of zip ring was introduced by Faith in [10, 11] ("zip" is the acronym for "zero intersection property"). We mention that the zip rings were firstly studied by Zelmanowitz in [28] under other denomination. The papers [21, 22] concern the more general notion of weak zip ring. This new class of rings is defined by using the weak annihilators.

Dube and Blose observed in [7] that "a reduced ring R is a zip ring iff every dense ideal of $R(Id(R))$ is above a dense compact element". Starting from this remark, they introduced in [7] the notion of zipped frame: an algebraic frame L is said to be a zipped frame if for any dense element a of A there exists a compact element c of A such that $c \leq a$. In Section 6 of [7], the weak zip rings are studied in relationship with the properties of the frame $R(Id(R))$.

In this paper we shall define the notions of zipped quantale and weakly zipped quantale. The zipped quantales (resp. the weakly zipped quantales) provide a suitable abstract framework for the properties of zip rings (resp. weak zip rings). Of course, the zipped quantales generalize the zipped frames. The aim of this paper is to prove the quantale-style versions of some properties of the zip rings and the weak zip rings. We extend from ring theory to quantale theory some results obtained in [7].

Now we shall present the content of this paper. Section 2 contains some elementary matter on quantales and frames, with emphasis on the prime spectra and radical elements.

In Section 3 we recall from [5, 13] the construction and some basic properties of the reticulation $L(A)$ of a coherent quantale A. $L(A)$ is a bounded distributive lattice whose prime spectrum (endowed with the Stone topology) is homeomorphic with the m-prime spectrum of A (endowed with a Zarisky-style topology). The frame $R(A)$ of the radical elements of A is isomorphic with the frame $Id(L(A))$ of ideals of $L(A)$. We shall use some transfer properties of the reticulation in proving the main results of Section 5.

The notion of weak annihilator in an algebraic quantale A is introduced in Section 4 as an abstraction of the notion of weak annihilator in ring theory (see e. g. [21]). We study the connections between the weak annihilators of A and the annihilators of the frame $R(A)$ of radical elements in A. For example, we prove that the set $Pol_w(A)$ of weak annihilator elements of A is a Boolean algebra, isomorphic with the Booleanization of the frame $R(A)$.

Section 5 is concerned with the zipped quantales and the weakly zipped quantales as an abstraction of zip rings and the weakly zipped rings, respectively. The definition of zipped quantales (resp. weakly zipped quantales) uses the annihilators (resp. the weak annihilators). We define the zip (bounded distributive) lattices and prove that a coherent quantale A is weakly zipped if and only if the reticulation $L(A)$ of A is a zip lattice. Then A is a weakly zipped quantale if and only if $R(A)$ is a zipped frame. This last result is a quantale generalization of Theorem 5.4 of [7].

In Section 6 we study the following problem: given two coherent quantales A, B and a coherent quantale morphism $u : A \to B$, find sufficient conditions for the next equivalence to take place: A is weakly zipped iff B is weakly zipped. We mention that the main results of this section (Theorem 6.6 and Proposition 6.10) are quantale versions of the following results proven in [7, Theorem 4.3 and Proposition 5.11].

2 Preliminaries on quantales

This section contains some basic notions and results on quantales and frames (see [26, 8, 23] for the quantale theory and [18, 25] for the frame theory).

Recall from [26] that a quantale is a complete lattice A endowed with a multiplicative operation $\cdot$ such that for any subset S of A and for any $a \in A$ we have $a \cdot \bigvee S = \bigvee\{a \cdot s | s \in S\}$ and $(\bigvee S) \cdot a = \bigvee\{s \cdot a | s \in S\}$. Usually, for all $a, b \in A$ we shall write ab instead of $a \cdot b$. 0 is the first element of A and 1 is the last element of A. The quantale A is said to be *integral* if the structure $(A, \cdot, 1)$ is a monoid and *commutative*, if $\cdot$ is a commutative operation. A *frame* is a quantale in which the multiplication $\cdot$ coincides with the meet (see [18], [25]). An element $c \in A$ is compact if for any $S \subseteq A$, $c \leq \bigvee S$ implies $c \leq \bigvee T$, for some finite subset T of S. Let us denote by $K(A)$ the set of the compact elements of the quantale A. Then the quantale A is said to be *algebraic* if any element $a \in A$ has the form $a = \bigvee X$ for some subset X of $K(A)$. An algebraic quantale A is said to be *coherent* if $1 \in K(A)$ and the set $K(A)$ of compact elements is closed under multiplication. Coherent frames are defined in a similar way (see [18], [25]). The main example of a coherent quantale (resp. a coherent frame) is the set $Id(R)$ of ideals of a unital commutative ring R (resp. the set $Id(L)$ of ideals of a bounded distributive lattice L).

All the quantales that appear in this paper are assumed to be integral and commutative.

In any quantale A one can introduce the residuation operation (also named implication)

$$a \to b = \bigvee\{x | ax \leq b\},$$

and the annihilator operation $a^{\perp} = a^{\perp_A}$, defined by $a^{\perp} = a \to 0 = \bigvee\{x \in A | ax = 0\}$ ($a^{\perp}$ is also called the polar of a). We mention that $\to$ fulfills the following residuation rule: for all $a, b, c \in A$, $a \leq b \to c$ if and only if $ab \leq c$, hence $(A, \vee, \wedge, \cdot, \to, 0, 1)$ is a (commutative) residuated lattice. Particularly, the annihilator operation is characterized by the following equivalence: for any $a \in A$, $a \leq b^{\perp}$ if and only if $ab = 0$. $Pol(A)$ will denote the set of annihilators of the quantale A. An element $a \in A$ is dense if $a^{\perp} = 0$.

If L is a frame, then $Pol(L)$ is a complete Boolean algebra, named in [4] the Booleanization of L. We remind that $Pol(L)$ is closed under arbitrary meets and the join of a family $\{a_i\}_{i \in I} \subseteq Pol(L)$ is $(\bigvee\limits_{i \in I} a_i)^{\perp\perp}$.

The standard text for residuation lattices is the monograph [12]. The basic properties of residuation operation and annihilators will be used without mention.

Recall from [26], an element $p < 1$ of a quantale A is *m-prime* if for all $a, b \in A$, $ab \leq p$ implies $a \leq p$ or $b \leq p$. The m-prime elements of a quantale extend the notions of prime ideals of a commutative ring and the prime ideals of a bounded distributive lattice. It is well-known that if A is an algebraic quantale, then $p < 1$ is m-prime if and only if for all $c, d \in K(A)$, $cd \leq p$ implies $c \leq p$ or $d \leq p$. Let us recall the following usual notations: $Spec(A)$ is the set of m-prime elements of A and $Max(A)$ is the set of maximal elements of A. If 1 is a compact element, then for any $a < 1$ there exists $m \in Max(A)$ such that $a \leq m$. The same hypothesis $1 \in K(A)$ implies that $Max(A) \subseteq Spec(A)$. We remark that the set $Spec(R)$ of prime ideals in a commutative ring R is the prime spectrum of the quantale $Id(R)$ and the set of prime ideals in a bounded distributive lattice L is the prime spectrum of the frame $Id(L)$. Keeping the terminology, we say that $Spec(A)$ is the m-prime spectrum of the quantale A (abbreviated, $Spec(A)$ is the prime spectrum of A).

If R is a commutative ring, then its nilradical is the ideal $Nil(A) = \bigcap Spec(A)$ (cf. [1]). If A is the quantale $Id(R)$ of ideals of R, then $Nil(A) = \rho_A(\{0\})$ ($\{0\}$ is here the zero ideal of R).

The paper [9] provides a deep analysis of various abstract theories of m–prime elements and of corresponding spectra developed in the last decades.

Recall from [26] that the radical $\rho(a) = \rho_A(a)$ of an element a of A is defined by

$$\rho(a) = \rho_A(a) = \bigwedge\{p \in Spec(A) | a \leq p\},$$

(this notion generalizes the radical of an ideal in a commutative ring). For all $a, b \in A$ we have $\rho(ab) = \rho(a) \wedge \rho(b)$ and for any subset S of A the following equality holds:

$$\rho(\bigvee S) = \rho(\bigvee\{\rho(s)|s \in S\}.$$

If $a = \rho(a)$, then a is said to be a radical element of A. An arbitrary meet of radical elements is a radical element. For any set S of radical elements we denote $\dot{\bigvee} S = \rho(\bigvee S)$. The set $R(A)$ of the radical elements of A is a frame in which the join of a subset S of $R(A)$ is $\rho(\bigvee S)$ (cf. [26], [27]). In [5] it is proven that $Spec(A) = Spec(R(A))$ and $Max(A) = Max(R(A))$. The quantale A is semiprime if the meet $\rho(0)$ of all m-prime elements in A is 0.

The following lemma describes the form of radical elements in a coherent quantale. It is a quantale version of Poposition 1.14 of [1] and will play an important role in the proofs of some results in this paper.

Lemma 2.1. [19] *Let A be a coherent quantale and $a \in A$. Then the following hold:*

(1) $\rho(a) = \bigvee\{c \in K(A)|c^k \leq a$ *for some integer* $k \geq 1\}$;

(2) *For any $c \in K(A), c \leq \rho(a)$ iff $c^k \leq a$ for some integer $k \geq 1$.*

(3) *The quantale A is semiprime if and only if for any integer $k \geq 1$, $c^k = 0$ implies $c = 0$.*

3 Reticulation of a coherent quantale

The reticulation $L(A)$ of a coherent quantale A was defined in [13] as a generalization of the notion of reticulation of a commutative ring [27], [18]. $L(A)$ is a bounded distributive lattice whose prime spectrum $Spec(L(A))$ is homeomorphic with the m-prime spectrum $Spec(A)$ of A. The reticulation $L(A)$ allows us to transfer results from quantales to bounded distributive lattices and vice-versa (see [5], [14], [15]). In this section we recall from [5], [13] the construction of reticulation and some elementary transfer results.

Let us fix a coherent quantale A. On the set $K(A)$ of compact elements of A we define the following equivalence relation: for all $c, d \in K(A)$, $c \equiv d$ iff $\rho(c) = \rho(d)$. Consider the quotient set $L(A) = K(A)/\equiv$. For any $c \in K(A)$ denote by $c/\equiv$ its equivalence class. Consider the canonical surjection $\lambda_A : K(A) \to L(A)$ defined by $\lambda_A(c) = c/\equiv$, for any $c \in K(A)$. For all $c, d \in K(A)$ define the following operations on $L(A)$: $\lambda_A(c) \vee \lambda_A(d) = \lambda_A(c \vee d)$ and $\lambda_A(c) \wedge \lambda_A(d) = \lambda_A(cd)$. Then $(L(A), \vee, \wedge, \lambda_A(0), \lambda_A(1))$ is a bounded distributive lattice. Of course, we shall denote $0 = \lambda_A(0)$ and $1 = \lambda_A(1)$ the first element and the last element of $L(A)$, respectively. We recall from [5] that for any $c \in K(A)$, $\lambda_A(c) = 1$ iff $c = 1$ and $\lambda_A(c) = 0$ iff $c \leq \rho(0)$.

The pair $(L(A), \lambda_A : K(A) \to L(A))$ (or shortly $L(A)$) will be called the reticulation of A. In [5], it was given an axiomatic definition of the reticulation. We observe that the reticulation $L(R)$ of a commutative ring R (defined in [18], [27]) is isomorphic with the reticulation $L(Id(R))$ of the quantale $Id(R)$.

For any $a \in A$ and $I \in Id(L(A))$ let us denote $a^* = \{\lambda_A(c)|c \in K(A), c \leq a\}$ and $I_* = \bigvee\{c \in K(A)|\lambda_A(c) \in I\}$. The assignments $a \mapsto a^*$ and $I \mapsto I_*$ define two order - preserving maps $(\cdot)^* : A \to Id(L(A))$ and $(\cdot)_* : Id(L(A)) \to A$. The following lemma collects the main properties of the maps $(\cdot)^*$ and $(\cdot)_*$.

Lemma 3.1. [5] *The following assertions hold:*

(1) *If $a \in A$, then a^* is an ideal of $L(A)$ and $a \leq (a^*)_*$;*

(2) *If* $I \in Id(L(A))$, *then* $(I_*)^* = I$;

(3) *If* $p \in Spec(A)$, *then* $(p^*)_* = p$ *and* $p^* \in Spec(L(A))$;

(4) *If* $P \in Spec((L(A))$, *then* $P_* \in Spec(A)$;

(5) *If* $p \in K(A)$, *then* $c^* = (\lambda_A(c)]$;

(6) *If* $c \in K(A)$ *and* $I \in Id(L(A))$, *then* $c \leq I_*$ *iff* $\lambda_A(c) \in I$;

(7) *If* $a \in A$ *and* $I \in Id(L(A))$, *then* $\rho(a) = (a^*)_*$, $a^* = (\rho(a))^*$ *and* $\rho(I_*) = I_*$;

(8) *If* $c \in K(A)$ *and* $p \in Spec(A)$, *then* $c \leq p$ *iff* $\lambda_A(c) \in p^*$.

By Lemma 3.1 one can consider the functions $u = u_A : Spec(A) \to Spec(L(A))$ and $v = v_A : Spec(L(A)) \to Spec(A)$, defined by $u(p) = p^*$ and $v(I) = I_*$, for all $p \in Spec(A)$ and $I \in Spec(L(A))$.

Lemma 3.2. [5, 15] *The functions* u *and* v *are homeomorphisms, inverse to one another.*

We also observe that u and v are also order - isomorphisms. Let us consider the maps $\Phi : R(A) \to Id(L(A))$ and $\Psi : Id(L(A)) \to R(A)$ defined by $\Phi(a) = a^*$ and $\Psi(I) = I_*$, for all $a \in R(A)$ and $I \in Id(L(A))$.

Lemma 3.3. [15] *The maps* Φ *and* Ψ *are frame isomorphisms, inverse to one another.*

4 Weak annihilator elements

Starting from the notion of weak annihilator of an ideal in a commutative ring [21],[22], we define the weak annihilator of an element of an algebraic quantale A. We prove a lot of properties of these abstract weak annihilators and we study their relationship with the annihilators of the frame $R(A)$.

Let R be a (unital) commutative ring and I an ideal of R. Recall from [1] that the annihilator of I is the ideal

$Ann_R(I) = \{a \in R | ax = 0$, for any $x \in I\}$.

If A is the quantale $Id(R)$ of ideals of R, then $I^{\perp_A} = Ann_R(I)$. Recall that $Nil(R)$ denotes the nilradical of R.

According to [21],[22], the weak annihilator of the ideal I is the following ideal of R:

$Ann_{R,w}(I) = \{x \in A | xu \in Nil(R)$ for each $u \in I\}$.

The notion of weak annihilator can be extended to an algebraic quantale A. The weak annihilator $a^{\perp_w}$ of an element a of A is defined by $a^{\perp_w} = a \to \rho(0)$. An element of the form $a^{\perp_w}$ is said to be a weak annihilator element (shortly, a weak annihilator). Let us denote by $Pol_w(A)$ the set of weak annihilators of the quantale A.

An element $a \in A$ is said to be weakly dense ($=$ w-dense) if $a^{\perp_w} = \rho(0)$. If A is a semiprime quantale, then $a^{\perp_w} = a^{\perp}$ and a is w-dense iff a is dense.

The following proposition generalizes to quantale theory some results obtained in Section 5 of [7] for rings (e.g. Lemma 5.3 and Theorem 5.14).

Proposition 4.1. *Let* A *be a coherent quantale and* $a \in A$. *Then the following hold*

(1) $\rho(a^{\perp_w}) = a^{\perp_w}$;

(2) $a^{\perp_w} = \bigvee\{c \in K(A) | \forall d \in K(A)[d \leq a \Rightarrow cd \leq \rho(0)]\}$;

(3) *If* $a \in R(A)$, *then* $a^{\perp_{R(A)}} = a^{\perp_w}$;

(4) *If* $c \in K(A)$, *then* $(\rho(c))^{\perp_{R(A)}} = \bigvee_{n=1}^{\infty} \rho((c^n)^{\perp_A})$;

(5) *If* $a \in R(A)$, *then* $a^{\perp_{R(A)}} = \bigwedge\{\bigvee_{n=1}^{\infty} \rho((c^n)^{\perp_A}) | c \in K(A), c \leq a\}$;

(6) *If* $a \in R(A)$, *then* $a^{\perp_w} = \bigwedge\{\bigvee_{n=1}^{\infty} \rho((c^n)^{\perp_A}) | c \in K(A), c \leq a\}$.

Proof. (1) We have to prove that $\rho(a \to \rho(0)) = a \to \rho(0)$. It suffices to check the inequality $\rho(a \to \rho(0)) \leq a \to \rho(0)$. Let c be a compact element of A such that $c \leq \rho(a \to \rho(0))$, hence, by using Lemma 2.1(2), there exists a natural number $n \geq 1$ such that $c^n \leq a \to \rho(0)$. By using the residuation rule we obtain $c^n a \leq \rho(0)$.

Now we want to prove that $ca \leq \rho(0)$. Assume that d is a compact element of A such that $d \leq ca$, so $d^n \leq c^n a^n \leq c^n a \leq \rho(0)$, hence there exists a natural number $k \leq 1$ such that $d^{nk} = 0$. By using Lemma 2.1(2) we get $d \leq \rho(0)$, therefore $ca \leq \rho(0)$. It follows that $c \leq a \to \rho(0)$. The desired inequality $\rho(a \to \rho(0)) \leq a \to \rho(0)$ is proven.

(2) Let us denote $x = \bigvee\{c \in K(A) | \forall d \in K(A)[d \leq a \Rightarrow cd \leq \rho(0)]\}$. In order to show that $x \leq a^{\perp_w}$, let c be a compact element of A such that for all $d \in K(A)$, $d \leq a$ implies $cd \leq \rho(0)$. Thus $ca = c \cdot \bigvee\{d \in K(A) | d \leq a\} = \bigvee\{cd | d \in K(A), d \leq a\} \leq \rho(0)$, therefore $c \leq a \to \rho(0)$. We proved that $x \leq a^{\perp_w}$.

Now we shall prove the converse inequality $a^{\perp_w} \leq x$. Let c be a compact element of A such that $c \leq a^{\perp_w}$, so $c \leq a \to \rho(0)$, hence $ca \leq \rho(0)$. Then for each $d \in K(A)$, $d \leq a$ implies $cd \leq ca \leq \rho(0)$, hence $c \leq x$. It results that $a^{\perp_w} \leq x$, therefore $a^{\perp_w} = x$.

(3) In accordance with (1), $a^{\perp_w}$ is a radical element of A. Let x be an arbitrary element of A. If $x\rho(a) \leq \rho(0)$, then $xa \leq x\rho(a) \leq \rho(0)$. Conversely, if $xa \leq \rho(0)$, then $x\rho(0) \leq \rho(x) \wedge \rho(a) = \rho(xa) \leq \rho(\rho(0)) = \rho(0)$. It follows that $x\rho(a) \leq \rho(0)$ if and only if $xa \leq \rho(0)$. Then for each $x \in R(A)$ the following equivalences hold: $x \leq a^{\perp_w}$ iff $x \leq \rho(a) \to \rho(0)$ iff $x\rho(a) \leq \rho(0)$ iff $xa \leq \rho(0)$ iff $x \wedge a \leq \rho(0)$ iff $x \wedge a = \rho(0)$ iff $x \leq a^{\perp_{R(A}}$. Therefore, we conclude that $a^{\perp_{R(A}} = a^{\perp_w}$.

(4) Let us denote $x = \bigvee_{n=1}^{\infty} \rho((c^n)^{\perp_A})$. The proof of the equality $(\rho(c))^{\perp_{R(A)}} = x$ consists in four steps.

Step 1. $x \in R(A)$.

Let m, n be two positive integers such that $m \leq n$. Then $c^n \leq c^m$, so $(c^m)^{\perp_A} \leq (c^n)^{\perp_A}$, and so $\rho((c^m)^{\perp_A}) \leq \rho((c^n)^{\perp_A})$. It follows that $(\rho((c^n)^{\perp_A}))_{n\geq 1}$ is an increasing sequence in the frame $R(A)$.

Assume that d is a compact element of A such that $d \leq \rho(x)$, hence there exists an integer $n \geq 1$ such that $c^k \leq \bigvee_{n=1}^{\infty} \rho((c^n)^{\perp_A})$. But $c^k \in K(A)$, so there exists an integer $n \geq 1$ such that $c^k \leq \bigvee_{i=1}^{n} \rho((c^i)^{\perp_A}) = \rho((c^n)^{\perp_A})$. Then one can find an integer $l \geq 1$ such that $c^{kl} \leq (c^n)^{\perp_A}$, so $c^{kl+n} = 0$. If $m = kl + n$, then $c \leq (c^{m-1})^{\perp_A} \leq \rho((c^{m-1})^{\perp_A}) \leq x$. We proved that $\rho(x) \leq x$, so $\rho(x) = x$. Thus, we conclude that $x \in R(A)$.

Step 2. $\rho(c) \wedge x = \rho(0)$.

Since $x \in R(A)$ (by *Step*1) we get the inequality $\rho(0) \leq \rho(c) \wedge x$. In order to prove the converse inequality $\rho(c) \wedge x \leq \rho(0)$, let d be a compact element of A such that $d \leq \rho(c) \wedge x$, so there exist the integers $m, n \geq 1$ such that $d \leq \rho((c^n)^{\perp_A})$ and $d^m \leq c$. One can find an integer $k \geq 1$ such that $d^k \leq (c^n)^{\perp_A}$, hence $d^k c^n = 0$. We remark that $d^{mn} \leq c^n$, so $d^{k+mn} \leq d^k c^n = 0$. By using Lemma 2.1(2), from $d^{k+mn} = 0$ we obtain $d \leq \rho(0)$, therefore $\rho(c) \wedge x \leq \rho(0)$. It follows that $\rho(c) \wedge x = \rho(0)$.

Step 3. If $d \in K(A)$ and $\rho(c) \wedge \rho(d) = \rho(0)$, then $\rho(d) \leq x$.

Since $\rho(cd) = \rho(c) \wedge \rho(d) = \rho(0)$ one can find an integer $k \geq 1$ such that $c^k d^k = 0$, so $d^k \leq (c^k)^{\perp_A} \leq \rho((c^k)^{\perp_A}) \leq x$. According to Lemma 2.1(2) and Step 1, it follows $d \leq \rho(x) = x$, therefore $\rho(d) \leq \rho(x) = x$.

Step 4. If $b \in R(A)$ and $\rho(c) \wedge b = \rho(0)$, then $b \leq x$.

Let d be a compact element of A such that $d \leq b$, so $\rho(c) \wedge \rho(d) \leq \rho(c) \wedge b = \rho(0)$. By using Step 3 it follows that $\rho(d) \leq x$, so $d \leq x$. It results that $b \leq x$.

In conclusion, x is the largest element of the set $\{b \in R(A) | \rho(c) \wedge b = \rho(0)\}$, hence $a^{\perp_{R(A}} = x$.

(5) Assume that a is a radical element of A. We know from Lemma 8 of [5] that $K(R(A)) = \{\rho(c) | c \in K(A)\}$, so $a = \dot{\bigvee}\{\rho(c) | c \in K(A), c \leq a\}$. By using Step 4 and residuation theory in the frame $R(A)$, we obtain the following equalities:

$a^{\perp_{R(A)}} = (\dot{\bigvee}\{\rho(c) | c \in K(A), c \leq a\})^{\perp_{R(A)}} = \bigwedge\{(\rho(c))^{\perp_{R(A)}} | c \in K(A), c \leq a\} = \bigwedge\{\bigvee_{n=1}^{\infty} \rho((c^n)^{\perp_A}) | c \in K(A), c \leq a\}$.

(6) By (3) and (5). $\square$

The following result extends [22, Proposition 2.1] (see also [7, Lemma 5.2]).

Lemma 4.2. *If A is a coherent quantale and $a, b \in A$, then the following hold:*

(1) *$a \leq b$ implies $b^{\perp_w} \leq a^{\perp_w}$;*

(2) *$a \leq a^{\perp_w \perp_w}$; $a^{\perp_w} = a^{\perp_w \perp_w \perp_w}$;*

(3) *$a^{\perp_w} = (\rho(a))^{\perp_w}$.*

Proof. (1) If $a \leq b$, then $b^{\perp_w} = b \to \rho(0) \leq a \to \rho(0) = a^{\perp_w}$.

(2) From $a \to \rho(0) \leq a \to \rho(0)$ we get $a(a \to \rho(0)) \leq \rho(0)$, therefore $a \leq (a \to \rho(0)) \to \rho(0) = a^{\perp_w \perp_w}$. Thus we obtain $a^{\perp_w} \leq a^{\perp_w \perp_w \perp_w}$. By using (1), from $a \leq a^{\perp_w \perp_w}$ we get $a^{\perp_w \perp_w \perp_w} \leq a^{\perp_w}$, therefore the equality $a^{\perp_w} = a^{\perp_w \perp_w \perp_w}$ follows.

(3) According to (1), $a \leq \rho(a)$ implies $(\rho(a))^{\perp_w} \leq a^{\perp_w}$. In order to prove that $a^{\perp_w} \leq (\rho(a))^{\perp_w}$, consider a compact element c such that $c \leq a^{\perp_w} = a \to \rho(0)$. By Lemma 2.1(1) we have

$\rho(a) = \bigvee\{d \in K(A) | d^k \leq a$ for some integer $k \geq 1\}$

therefore the following equality holds:

$\rho(a) \to \rho(0) = (\bigvee\{d \in K(A) | d^k \leq a$ for some integer $k \geq 1\}) \to \rho(a)$.

By using the elementary residuation theory one obtains

$\rho(a) \to \rho(0) = \bigwedge\{d \to \rho(0) \ | d \in K(A), d^k \leq a$ for some integer $k \geq 1\}$.

Denoting by x the second member of the previous equality we want to prove that $c \leq x$. Let d be a compact element of A such that $d^n \leq a$, for some integer $n \geq 1$. Then $c \leq a \to \rho(0) \leq d^n \to \rho(0)$, so $cd^n \leq \rho(0)$. A new application of Lemma 2.1(2) gives $(cd^n)^k = 0$, for some integer $k \geq 1$, so $(cd)^{nk} = 0$. It follows that $cd \leq \rho(0)$, so $c \leq d \to \rho(a)$. According to the previous form of $\rho(a) \to \rho(0)$ we get $c \leq x = \rho(a) \to \rho(0)$. Thus we obtain $a^{\perp_w} \leq \rho(a) \to \rho(0) = (\rho(a))^{\perp_w}$. $\square$

The following result provides a description of $Pol_w(A)$ as the Booleanization of the frame $R(A)$.

Proposition 4.3. $Pol_w(A) = Pol(R(A))$.

Proof. Recall that the Booleanization $Pol(R(A))$ of the frame $R(A)$ is defined by

$Pol(R(A)) = \{a^{\perp_{R(A)}} | a \in R(A)\}$.

In accordance with Proposition 4.1(3), for any $a \in R(A)$ we have $a^{\perp_{R(A)}} = a^{\perp_w}$, therefore

$Pol(R(A)) = \{a^{\perp_w} | a \in R(A)\} = Pol_w(A)$. $\square$

For any family $\{a_i\}_{i \in I} \subseteq Pol_w(A)$ we denote

$\bigsqcup_{i \in I} a_i = (\bigvee_{i \in I} a_i)^{\perp_w \perp_w}$.

Lemma 4.4. *$Pol_w(A)$ is closed under the infinite operations $\bigsqcup$ and $\bigwedge$.*

Proof. Consider an arbitrary family $\{a_i\}_{i\in I} \subseteq Pol_w(A)$. It is clear that $\bigsqcup_{i\in I} a_i = (\bigvee_{i\in I} a_i)^{\perp_w\perp_w} \in Pol_w(A)$. In order to prove that $Pol_w(A)$ is closed under $\bigwedge$ we consider a family $\{b_i\}_{i\in I} \subseteq A$ such that $a_i = b_i^{\perp_w}$, for all $i \in I$. Then $\bigwedge_{i\in I} a_i = \bigwedge_{i\in I} b_i^{\perp_w} = (\bigvee_{i\in I} b_i)^{\perp_w}$, so we get $\bigwedge_{i\in I} a_i \in Pol_w(A)$. $\square$

Proposition 4.5. *$Pol_w(A)$ has a canonical structure of complete Boolean algebra in which the following hold:*

- *the meet of a family $\{a_i\}_{i\in I} \subseteq Pol_w(A)$ is $\bigwedge_{i\in I} a_i$;*
- *the join of a family $\{a_i\}_{i\in I} \subseteq Pol_w(A)$ is $\bigsqcup_{i\in I} a_i$.*

Proof. We remind that $Pol(R(A))$ is a complete Boolean algebra in which the following hold:

- the meet of a family $\{a_i\}_{i\in I} \subseteq Pol(R(A))$ is $\bigwedge_{i\in I} a_i$;
- the join of a family $\{a_i\}_{i\in I} \subseteq Pol(R(A))$ is $(\dot{\bigvee}_{i\in I} a_i)^{\perp_{R(A)}\perp_{R(A)}}$.

Therefore, in accordance with Proposition 4.3 and Lemma 4.4, $Pol_w(A)$ is endowed with a structure of complete Boolean algebra in which the following hold:

- the meet of a family $\{a_i\}_{i\in I} \subseteq Pol_w(A)$ is $\bigwedge_{i\in I} a_i$;
- the join of a family $\{a_i\}_{i\in I} \subseteq Pol_(A)$ can be calculated by using Proposition 4.1(3) and Lemma 4.2(3):

$(\dot{\bigvee}_{i\in I} a_i)^{\perp_{R(A)}\perp_{R(A)}} = (\dot{\bigvee}_{i\in I} a_i)^{\perp_w\perp_w} = (\rho(\bigvee_{i\in I} a_i))^{\perp_w\perp_w} = (\bigvee_{i\in I} a_i)^{\perp_w\perp_w} = \bigsqcup_{i\in I} a_i$. $\square$

5 Zipped and weakly zipped coherent quantales

This section concerns the zipped quantales and the weakly zipped quantales. The definitions of these two notions are inspired by the zip rings ([10, 11, 28]) and the weak zip rings [21]. We mention that the zipped quantales generalize the zipped frames, introduced in [7]. The main result of section is a transfer property: a coherent quantale A is weakly zipped if and only if the reticulation $L(A)$ is a zip lattice. From this theorem we obtain as corollaries the quantale versions of some results established in [7].

Let R be a commutative (unital) ring. An ideal I of R is faithful if $Ann_R(I)$ is the zero ideal of R. We remark the an ideal I of R is faithful if and only if I is a dense element of the quantale $Id(R)$. According to [21], [22] R is said to be a zip ring if for each faithful ideal I of R, one can find a finitely generated faithful ideal J such that $J \subseteq I$.

Let us recall from [7] the following characterization of the reduced zip rings:

Theorem 5.1. *If R is a reduced ring, then the following are equivalent:*

(1) *R is a zip ring.*

(2) *Each dense element of the frame $RId(A)$ is above a dense compact element.*

Inspired by this theorem, in [7] is introduced the notion of zipped frame: an algebraic frame L is a zipped frame if for each dense element $a \in L$ there exists a dense compact element c of L such that $c \leq a$.

In an obvious way the notion of zipped frame can be generalized to quantales: an algebraic quantale A is a zipped quantale if for each dense element $a \in A$ there exists a dense compact element c of A such that $c \leq a$.

Lemma 5.2. *A commutative ring R is a zip ring if and only if the quantale $Id(R)$ of ideals of R is a zipped quantale.*

Then the zipped quantales constitute an abstraction of the zip rings as well as a generalization of the zipped frames.

Recall from [21] that an ideal I of R is weakly faithful if $Ann_{R,w}(I) = Nil(R)$. In accordance with [21], R is said to be a weak zip ring if for each weakly faithful ideal I of R there exists a finitely generated weakly faithful ideal J of R such that $J \subseteq I$.

Let us fix an algebraic quantale A. Recall that the weak annihilator $a^{\perp_w}$ of an element a of A is defined by $a^{\perp_w} = a \to \rho(0)$. An element $a \in A$ is said to be weakly dense (= w-dense) if $a^{\perp_w} = \rho(0)$. We observe that an ideal I of a ring R is weakly faithful if and only if it is a weakly dense element of the quantale $Id(R)$.

We shall generalize the notion of weak zip ring to quantale theory: an algebraic quantale A is a weakly zipped quantale if for any w-dense element a of A there exists a w-dense compact element c of A such that $c \leq a$. We remark that a semiprime quantale is weakly zipped if and only if it is zipped. In particular, this equivalence holds for algebraic frames.

Lemma 5.3. *Let R be a commutative ring. Then R is a weak zip ring if and only if $Id(R)$ is a weakly zipped quantale.*

Lemma 5.4. *Let A be a coherent quantale. Then the following hold:*

(1) *If $a \in A$, then $Ann(a^*) = (a^{\perp_w})^*$;*

(2) *If $I \in Id(L(A))$, then $(Ann(I))_* = (I_*)^{\perp_w}$.*

Proof. (1) See [15, Proposition 4.5].

(2) See [15, Proposition 4.6]. □

Recall that an ideal of a bounded distributive lattice L is a dense ideal if the annihilator $Ann(I)$ of I is the zero ideal of L.

The following two lemmas show that the reticulation transforms the w-dense elements of A into the dense ideals of $L(A)$ and vice-versa.

Lemma 5.5. *Let A be a coherent quantale and $a \in A$. Then the following are equivalent:*

(1) *a is a w-dense element of A;*

(2) *a^* is a dense ideal of $L(A)$.*

Proof. (1) ⇒ (2) Assume that a is w-dense, so $a^{\perp_w} = \rho(0)$. By Lemmas 5.4(1) and 3.1(7), we have $Ann(a^*) = (a^{\perp_w})^* = (\rho(0))^* = 0^* = \{0\}$. Then a^* is a dense ideal of $L(A)$.

(2) ⇒ (1) Assume that a^* is a dense ideal of $L(A)$, so $Ann(a^*) = \{0\}$. In virtue of Proposition 4.1(1), 3.1(7) and 5.4(1), the following equalities hold:

$a^{\perp_w} = \rho(a^{\perp_w}) = ((a^{\perp_w})^*)_* = (Ann(a^*))_* = (\{0\})_* = \rho(0)$.

Then a is a w-dense element of A.

□

Lemma 5.6. *Let A be a coherent quantale and I an ideal of $L(A)$. Then the following are equivalent:*

(1) *I is a dense ideal of $L(A)$;*

(2) *I_* is a w-dense element of A.*

Proof. (1) $\Rightarrow$ (2) Assume that I is a dense ideal of $L(A)$, i.e $Ann(I) = \{0\}$. By Lemma 5.4(2), we have $(I_*)^{\perp_w} = (Ann(I))_* = \{0\}_* = \rho(0)$, so I_* is a w-dense element of A.

(2) $\Rightarrow$ (1) Assume that I_* is a w-dense element of A, i.e $(I_*)^{\perp_w} = \rho(0)$. According to Lemmas 3.1(7) and 5.4(2), the following equalities hold:

$Ann(I) = ((Ann(I))_*)^* = ((I_*)^{\perp_w})^* = (\rho(0))^* = \{0\}$.

Therefore, I is a dense ideal of $L(A)$. □

Let L be a bounded distributive lattice. L is said to be a zip lattice if for each dense ideal I of L there exists an element $x \in I$ such that the principal ideal $(x]$ is a dense ideal of L. It is easy to see that L is a zip lattice if and only if the frame $Id(L)$ of ideals of L is zipped.

Theorem 5.7. *Let A be a coherent quantale. Then the following are equivalent:*

(1) *A is a weakly zipped quantale;*

(2) *The reticulation $L(A)$ of A is a zip lattice.*

Proof. (1) $\Rightarrow$ (2) Assume that A is a weakly zipped quantale. Let I be a dense ideal of the lattice $L(A)$. By Lemma 5.6, I_* is a w-dense element of the quantale A, so there exists a w-dense compact element c of A such that $c \leq I_*$. According to Lemma 5.5, c^* is a dense ideal of $L(A)$. By applying Lemma 3.1(6), $c \leq I_*$ implies $\lambda_A(c) \in I$, and so $(\lambda_A(c)] = a^*$ is dense ideal of $L(A)$ and $(\lambda_A(c)] \subseteq I$. Then $L(A)$ is a zip lattice.

(2) $\Rightarrow$ (1) Suppose that $L(A)$ is a zip lattice. Let a be a w-dense element of A, so a^* is a dense ideal of A (cf. Lemma 5.5). Thus there exists $x \in a^*$ such that $(x]$ is a dense ideal of $L(A)$. According to the definition of the ideal a^* there exists a compact element c of A such that $c \leq a$ and $x = \lambda_A(c)$. By Lemma 3.1(5), we have $c^* = (\lambda_A(c)] = (x]$. Due to Lemma 5.5, c is a w-dense element of A, so A is a weakly zipped quantale. □

Corollary 5.8. *Let A be a coherent quantale. Then the following are equivalent:*

(1) *A is a weakly zipped quantale;*

(2) *$R(A)$ is a zipped frame.*

Proof. According to Lemma 3.3, $R(A)$ and $Id(L(A))$ are isomorphic frames. Then, by applying Theorem 5.7, we get the following equivalences: A is a weakly zipped quantale iff $L(A)$ is a zip lattice iff $Id(L(A))$ is a zipped frame iff $R(A)$ is a zipped frame. □

Corollary 5.9. *Let A be a semiprime coherent quantale. Then the following are equivalent:*

(1) *A is a zipped quantale;*

(2) *$R(A)$ is a zipped frame.*

Remark 5.10. *If we apply Corollaries 5.8 and 5.9 to the quantale $Id(R)$ of ideals of a commutative ring R then we obtain [7, Theorems 5.4 and 2.1].*

Let S be a subset of the quantale A. Recall that the quantale A satisfies the ascending chain condition on the elements of S if for any ascending chain $a_1 \leq a_2 \leq \cdots \leq a_n \cdots$ in S there exists an integer $n \geq 1$ such that $a_n = a_m$, for all integers $m \geq n$.

Proposition 5.11. *If the coherent quantale A satisfies the ascending chain condition on the weak annihilators elements, then A is weakly zipped.*

Proof. Assume that A satisfies the ascending chain condition on the weak annihilators elements of A. By Proposition 4.3 we have $Pol_w(A) = Pol(R(A))$, so the frame $R(A)$ satisfies the ascending chain condition on the annihilators elements. By applying [7, Corollary 3.11] it follows that $R(A)$ is a zipped frame. Therefore, in virtue of Corollary 5.8, we conclude that A is weakly zipped. □

Let A be a coherent quantale. For any element a of A, let us consider the interval $[a)_A = \{x \in A | a \leq x\}$ of A. We observe that $[a)_A$ is closed under joins of A. We introduce a new operation $\cdot_a$ on the set $[a)_A$: for all $x, y \in [a)_A$, denote $x \cdot_a y = x \cdot y \vee a$ (it is easy to see that $[a)_A$ is closed under the new multiplication $\cdot_a$).

Lemma 5.12. [5] $([a)_A, \bigvee, \wedge, \cdot_a, a, 1)$ *is a coherent quantale.*

Lemma 5.13. *The reticulations $L(A)$ and $L([\rho(0))_A)$ of the quantales A and $[\rho(0))_A$, respectively, are isomorphic bounded distributive lattices.*

Proof. We know that $(\rho(0))^* = \{0\}$. Then, by applying Proposition 8 of [5], we get the following isomorphisms in the category of bounded distributive lattices:
$L([\rho(0))_A) \simeq L(A)/(\rho(0))^* = L(A)/\{0\} \simeq L(A)$. □

Corollary 5.14. *Let A be a coherent quantale. Then the following are equivalent:*

(1) *A is a weakly zipped quantale;*

(2) *$[\rho(0))_A$ is a zipped quantale.*

Proof. We observe that $Spec(A) = Spec([\rho(0))_A)$, hence $\bigwedge Spec([\rho(0))_A) = \bigwedge Spec(A) = \rho(0)$, therefore the quantale $[\rho(0))_A$ is semiprime. By using Theorem 5.7 and Lemma 5.13, the following equivalences hold: A is a weakly zipped quantale iff the reticulation $L(A)$ of A is a zip lattice iff the reticulation $L([\rho(0))_A)$ of $[\rho(0))_A$ is a zip lattice iff $[\rho(0))_A$ is a zipped quantale. □

A lattice L with 0 is said to be irreducible if for all $x, y \in L$, $x \wedge y = 0$ implies $x = 0$ or $y = 0$. A quantale A is $\cdot$ -irreducible if for all $x, y \in L$, $xy = 0$ implies $x = 0$ or $y = 0$. We observe that a frame L is irreducible iff it is $\wedge$ -irreducible as a quantale.

The following lemma is well-known (for sake of completeness we present its proof).

Lemma 5.15. *A bounded distributive lattice L is irreducible if and only if $Id(L)$ is an irreducible frame.*

Proof. Assume that the bounded distributive lattice L is irreducible. Let I, J be two ideals of L such that $I \cap J = 0$. Assume that J is a nonzero ideal so there exists a nonzero element $x \in I$. Let y be an arbitrary element of J. Then $x \wedge y \in I \cap J$, hence $x \wedge y = 0$. But L is irreducible, so $x \wedge y = 0$ and $x \neq 0$ imply $y = 0$, therefore J is the zero ideal. Then we conclude that $Id(L)$ is an irreducible frame. The converse implication is obvious. □

Lemma 5.16. *Let A be an algebraic quantale. Then the following are equivalent:*

(1) *A is $\cdot$ -irreducible;*

(2) *For all $c, d \in K(A)$, $cd = 0$ implies $c = 0$ or $d = 0$.*

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) Let x, y be two elements of A such that $xy = 0$ and $x \neq 0$. Since A is algebraic we have $x = \bigvee_{k\in K} c_k$ and $y = \bigvee_{l\in K} d_l$, for some families $(c_k)_{k\in K}$ and $(d_l)_{l\in L}$ of compact elements of A. We remark that the following equality holds: $xy = \bigvee\{c_k d_l | k \in K, l \in L\}$. It follows that $c_k d_l = 0$ for all $k \in K$ and $l \in L$. Since $x \neq 0$ one can find an element $i \in K$ such that $c_i \neq 0$. By using the hypothesis (2), from $c_i d_l = 0$, for $l \in L$ we obtain $d_l = 0$, for each $l \in L$. Therefore, we conclude that $y = 0$. $\square$

Proposition 5.17. *If A is a coherent quantale, then the following are equivalent:*

(1) *$R(A)$ is an irreducible frame;*

(2) *$L(A)$ is an irreducible lattice;*

(3) *$[\rho(0))_A$ is a $\cdot$ -irreducible quantale;*

(4) *$\rho(0)$ is an m-prime element of A.*

Proof. (1) $\Leftrightarrow$ (2) We know from Lemma 3.3 that $R(A)$ and $Id(L(A))$ are isomorphic frames. Therefore, by using Lemma 5.15, the following properties are equivalent: $R(A)$ is an irreducible frame iff $Id(L(A))$ is an irreducible frame iff $L(A)$ is an irreducible lattice.

(1) $\Rightarrow$ (3) Let x, y be two elements of $[\rho(0))_A$ such that $x \cdot_{\rho(0)} y = \rho(0)$. But $x \cdot_{\rho(0)} y = xy \vee \rho(0)$, so $xy \leq \rho(0)$. Thus $\rho(x) \wedge \rho(y) = \rho(xy) \leq \rho(0)$, hence $\rho(x) \wedge \rho(y) = \rho(0)$. Since $R(A)$ is an irreducible frame we get $\rho(x) = \rho(0)$ or $\rho(y) = \rho(0)$, so $x \leq \rho(0)$ or $y \leq \rho(0)$. It follows that $x = \rho(0)$ or $y = \rho(0)$, so $[\rho(0))_A$ is a $\cdot$ -irreducible quantale.

(3) $\Rightarrow$ (4) Let x, y be two elements of A such that $xy \leq \rho(0)$. Then we have $x \vee \rho(0), y \vee \rho(0) \in [\rho(0))_A$ and $(x \vee \rho(0)) \cdot_{\rho(0)} (y \vee \rho(0)) = xy \vee \rho(0) = \rho(0)$, therefore $x \vee \rho(0)) = \rho(0)$ or $y \vee \rho(0) = \rho(0)$ (because $[\rho(0))_A$ is a $\cdot$ -irreducible quantale). It follows that $x \leq \rho(0)$ or $y \leq \rho(0)$, so $\rho(0)$ is an m-prime element of A.

(4) $\Rightarrow$ (2) Let x, y be two elements of the reticulation $L(A)$ such that $x \wedge y = 0$. Then $x = \lambda_A(c)$, $y = \lambda_A(d)$ for some compact elements c, d of A. Therefore $\lambda_A(cd) = \lambda_A(c) \wedge \lambda_A(d) = 0$, so $cd \leq \rho(0)$ (cf. the definition of reticulation). Since $\rho(0)$ is an m-prime element of A, it follows that $c \leq \rho(0)$ or $d \leq \rho(0)$. By Lemma 2.1(2) there exists an integer $n \geq 1$ such that $c^n = 0$ or $d^n = 0$. Then $x = \lambda_A(c) = \lambda_A(c^n) = 0$ or $y = \lambda_A(d) = \lambda_A(d^n) = 0$. Therefore we conclude that $L(A)$ is an irreducible lattice.

$\square$

Proposition 5.18. *Let A be a semiprime zipped quantale. If any non-zero m-prime element of A is dense, then $0 \in Spec(A)$.*

Proof. If A is a semiprime zipped quantale, then $R(A)$ is a zipped frame (cf. Corollary 5.9). Since $Spec(A) = Spec(R(A))$ it results that any non-zero prime element of $R(A)$ is dense (i.e. $p \in Spec(R(A)) - \{\rho(0)\}$ implies $p^{\perp_{R(A)}} = \rho(0)$). By using [7, Corollary 3.13] it follows that the frame $R(A)$ is irreducible. In virtue of the equivalence (1) $\Leftrightarrow$ (4) of Proposition 5.17 we conclude that $0 = \rho(0)$ is an m-prime element of A. $\square$

6 Quantale morphisms and zippedeness

A result of [7, Theorem 4.3] concerns the coherent frame morphisms "with the property that their domain is zipped if and only if the codomain is zipped". Inspired by this theorem, in this section we study the coherent quantale morphisms having the following properties: the domain is a weakly zipped (resp. zipped) quantale if and only if the codomain is a weakly zipped (resp. zipped) quantale.

Let A, B be two quantales. A function $u : A \to B$ is a quantale morphism if it preserves the arbitrary joins and the multiplication (in this case we have $u(0) = 0$); f is an integral morphism if $f(1) = 1$. If $u(K(A)) \subseteq K(B)$ then we say that u preserves the compacts elements. If u is an integral quantale morphism that preserves the compacts elements then it is called a coherent quantale morphism. In a similar manner one defines the frame morphisms, the integral frame morphisms, the coherent frame morphism, etc. (cf. [18], [25]).

Let us fix two coherent quantales A, B and a coherent quantale morphism $u : A \to B$. According to Lemma 8 of [5], $R(A)$ and $R(B)$ are coherent frames. Consider the right adjoint $u_* : B \to A$ of the quantale morphism u, defined by $u_*(b) = \bigvee\{a \in A | u(a) \leq b\}$, for any $b \in A$. Then for all $a \in A, b \in B$, the following equivalence holds: $u(a) \leq b$ if and only if $a \leq u_*(b)$.

The quantale morphism $u : A \to B$ is said to be dense (resp. $\star$-dense) if for any $a \in A, b \in B$, $u(a) = 0$ (resp. $u_*(b) = 0$) implies $a = 0$ (resp. $b = 0$). According to [20], p. 568, the quantale morphism $u : A \to B$ is dense if and only if $u_\star(0) = 0$.

The following lemma is an elementary result in the quantale theory (a proof can be found in [14]).

Lemma 6.1. *If $u : A \to B$ is a surjective coherent quantale morphism, then $u(K(A)) = K(B)$.*

Let us consider the function $u^\rho : R(A) \to R(B)$ defined by $u^\rho(a) = \rho_B(u(a))$, for any $a \in R(A)$.

Lemma 6.2. *The following hold:*

(1) *u^ρ is a coherent frame morphism;*

(2) *The following diagram is commutative:*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\rho_A"'] & B \arrow[d, "\rho_B"] \\ R(A) \arrow[r, "u^\rho"] & R(B) \end{tikzcd}$$

Proof. Firstly, we shall prove the commutativity of the diagram of (2). Let a be an arbitrary element of A. We shall verify that $\rho_B(u(\rho_A(a))) = \rho_B(u(a))$.

Since $a \leq \rho_A(a)$ it follows that $\rho_B(u(a)) \leq \rho_B(u(\rho_A(a)))$. In order to establish the converse inequality $\rho_B(u(\rho_A(a))) \leq \rho_B(u(a))$, let us consider a compact element d of B such that $d \leq \rho_B(u(\rho_A(a)))$. In accordance with Lemma 2.1(2), there exists an integer $n \geq 1$ such that $d^n \leq u(\rho_A(a))$. By using Lemma 2.1(1) the following equalities hold:

$u(\rho_A(a)) = u(\bigvee\{c \in K(A) | c \leq a\}) = \bigvee\{u(c) | c \in K(A), c \leq a\}$.

Therefore we obtain the inequality $d^n \leq \bigvee\{u(c) | c \in K(A), c \leq a\}$. Since $d^n \in K(B)$, there exists an integer $k \leq 1$ and $c_a \cdots c_k \in K(A)$ such that $d^n \leq \bigvee_{i=1}^k u(c_i)$ and $c_i \leq a$, for any $i = 1 \cdots k$. Denoting $c = \bigvee_{i=1}^k u(c_i)$ we get $c \in K(A)$ and $c \leq a$. Then $d^n \leq u(c) \leq u(a)$, hence, by using Lemma 2.1(1) we get $d \leq \rho_B(u(a))$. Therefore, it results that $\rho_B(u(\rho_A(a))) \leq \rho_B(u(a))$.

We conclude that $u^\rho(\rho_A(a)) = \rho_B(u(\rho_A(a))) = \rho_B(u(a))$, for any $a \in A$, so the diagram is commutative.

Now we shall prove that u^ρ preserves the arbitrary joins. Consider a family $(a_i)_{i\in I}$ of elements of $R(A)$. Due the commutativity of the diagram, the following equalities hold:

$u^\rho(\dot{\bigvee}_{i\in I} a_i) = u^\rho(\rho_A(\bigvee_{i\in I} a_i)) = \rho_B(u(\bigvee_{i\in I} a_i)) = \rho_B(\bigvee_{i\in I} u(a_i)) = \dot{\bigvee}_{i\in I} u(a_i)$.

For all $a, b \in R(A)$ we have $a \wedge b = ab$, therefore $u^\rho(a \wedge b) = \rho_B(u(ab)) = \rho_B(u(a)u(b)) = \rho_B(u(a)) \wedge \rho_B(u(b)) = u^\rho(a) \wedge u^\rho(b)$.

Then u^ρ preserves the arbitrary joins and the finite meets, so it is a frame morphism.

Let x be a compact element of $R(A)$, so $x = \rho_A(c)$, for some $c \in K(A)$. But u preserves the compact elements, so $u(c) \in K(B)$, therefore $u^\rho(x) = u^\rho(\rho_A(c)) = \rho_B(u(c)) \in K(R(B))$.

□

Let A, B, C be three coherent quantales and $u : A \to B, v : B \to C$ two coherent quantale morphisms. According to Lemma 6.2(2) we have $\rho_B \circ u = u^\rho \circ \rho_A$ and $\rho_C \circ v = v^\rho \circ \rho_B$. By using these two equalities it is easy to obtain $(v \circ u)^\rho = v^\rho \circ u^\rho$.

Therefore, we conclude that the assignments $A \mapsto R(A)$ and $u \mapsto u^\rho$ define a covariant functor from the category of coherent quantales and coherent quantale morphisms to the category of coherent frames and coherent frame morphisms.

Lemma 6.3. *If $b \in R(B)$, then $u_\star(b) \in R(A)$.*

Proof. Let c be a compact element of A such that $c \leq \rho_A(u_\star(b))$. In accordance with Lemma 2.1(2) there exists an integer $n \geq 1$ such that $c^n \leq u_\star(b)$, hence $u(c^n) \leq b$ (cf. the adjointness property). It follows that $u(c) \in K(B)$ and $(u(c))^n \leq b$, hence, by a new application of Lemma 2.1(2), we get $u(c) \leq \rho_B(b) = b$. Thus $c \leq u_\star(b)$, so we get $\rho_A(u_\star(b)) \leq u_\star(b)$. It results that $\rho_A(u_\star(b)) = u_\star(b)$, so $u_\star(b) \in R(A)$. □

By Lemma 6.3 we can consider the map $u_\star|_{R(B)} : R(B) \to R(A)$. The following proposition shows that $u_\star|_{R(B)}$ is exactly the right adjoint of the frame morphism u^ρ.

Proposition 6.4. $(u^\rho)_\star = u_\star|_{R(B)}$.

Proof. Let b be an arbitrary element of the frame $R(B)$. By Lemma 6.3 we have $u_\star(b) \in R(A)$ and for any $a \in R(A)$ the following equivalences hold: $u^\rho(a) \leq b$ iff $\rho_B(u(a)) \leq b$ iff $u(a) \leq b$ iff $a \leq u_\star(b)$. Then $u_\star|_{R(B)}$ is the right adjoint of the frame morphism u^ρ. □

Proposition 6.5. *If A, B are coherent quantales and $u : A \to B$ is a coherent quantale morphism, then the following hold:*

(1) *If u is dense, then u^ρ is a dense frame morphism;*

(2) *If u is $\star$-dense, then u^ρ is a $\star$-dense frame morphism.*

Proof. (1) Assume that u is a dense quantale morphism. Recall that $\rho_A(0)$ (resp. $\rho_B(0)$) is the first element of the frame $R(A)$ (resp. $R(B)$).

Let x be an element of $R(A)$ such that $u^\rho(x) = \rho_B(0)$. We have to prove that $x = \rho_A(0)$. Let c be a compact element of A such that $c \leq x$, hence $u(c) \in K(B)$ and $u(c) \leq \rho_B(u(c)) \leq \rho_B(u(x)) = u^\rho(x) = \rho_B(0)$.

According to Lemma 2.1(2) there exists an integer $n \geq 1$ such that $(u(c))^n = 0$, so $u(c^n) = 0$. But u is dense, so $c^n = 0$. A new application of Lemma 2.1(2) gives $c \leq \rho(0)$. We proved that $x \leq \rho_A(0)$, hence $x = \rho_A(0)$.

(2) Let y be an element of $R(B)$ such that $(u^\rho)_\star(y) = \rho_A(0)$. By applying Proposition 6.4 we get $u_\star(y) = \rho_A(0)$, therefore $x = \rho_B(0)$ (because u is $\star$-dense). It follows that u^ρ is a $\star$-dense frame morphism. □

Theorem 6.6. *Assume that A and B are coherent quantales. If $u : A \to B$ is a dense and $\star$-dense coherent quantale morphism, then the following are equivalent:*

(1) *A is a weakly zipped quantale;*

(2) *B is a weakly zipped quantale.*

Proof. We know from [5, Lemma 8] that $R(A)$ and $R(B)$ are coherent frames. By using Proposition 6.5 it follows that $u^\rho : R(A) \to R(B)$ is a dense and $\star$-dense coherent frame morphism. Thus one can apply [7, Theorem 4.13] to u^ρ, resulting that the frame $R(A)$ is zipped if and only if the frame $R(B)$ is zipped. According to Corollary 5.8 the following properties are equivalent:

- A is a weakly zipped quantale;
- $R(A)$ is a zipped frame;
- $R(B)$ is a zipped frame;
- B is a weakly zipped quantale.

□

Corollary 6.7. *Assume that A and B are semiprime coherent quantales. If $u : A \to B$ is a dense and $\star$-dense coherent quantale morphism, then the following are equivalent:*

(1) *A is a zipped quantale;*

(2) *B is a zipped quantale.*

Corollary 6.8. *Suppose that A and B are semiprime coherent quantales. If $u : A \to B$ is a surjective and dense coherent quantale morhism, then the following are equivalent:*

(1) *A is a zipped quantale;*

(2) *B is a zipped quantale.*

Proof. Assume that $u : A \to B$ is a surjective and dense coherent quantale morphism. Due to Proposition 6.5(1) it follows that u^ρ is a dense frame morphism. Let b be an element of $R(B)$, so $b = u(a)$ for some element $a \in A$. According to Lemma 6.2(2) we have $u^\rho(\rho_A(a)) = \rho_B(u(a)) = \rho_B(b) = b$. We observe that $\rho_A(a) \in R(A)$, so u^ρ is a surjective map. Therefore, by applying [7, Corollary 4.14], it follows that $R(A)$ is a zipped frame if and only if $R(B)$ is a zipped frame. Due to Corollary 5.9, we get the equivalence of the properties (1) and (2). □

For an arbitrary element a of a coherent quantale A, let us consider the function $u_a^A : A \to [a)_A$, defined by $u_a^A(x) = x \vee a$, for any $x \in A$. Recall from Lemma 5.12 that $[a)_A$ is a coherent quantale.

Lemma 6.9. [5] *Let A, B be two coherent quantales. For any $a \in A$ the following hold:*

(1) *u_a^A is an integral quantale morphism;*

(2) *If $c \in K(A)$, then $u_a^A(c) \in K([a)_A)$;*

(3) *$u_a^A(K(A)) = K([a)_A)$.*

The previous lemma asserts that the map u_a^A is a coherent quantale morphism.

Let A, B be two coherent quantales and $u : A \to B$ a coherent quantale morphism. Let us consider the map $\tilde{u} : [\rho(0))_A \to [\rho_B(0))_B$, defined by $\tilde{u}(a) = u(a) \vee \rho_B(0)$, for any $a \in [\rho_A(0))_A$.

Proposition 6.10. *The map $\tilde{u} : [\rho_A(0))_A \to [\rho_B(0))_B$ is a coherent quantale morphism.*

Proof. Firstly, we shall prove that $u(\rho_A(0)) \leq \rho_B(0)$. By the adjointness of u and $u_\star$, it suffices to show that $\rho_A(0) \leq u_\star(\rho_B(0))$. Let c be a compact element of A such that $c \leq \rho_A(0)$. By Lemma 2.1(2), there exists an integer $n \geq 1$ such that $c^n = 0$, hence $(u(c))^n = u(c^n) = u(0) = 0$. But $u(c) \in K(B)$ (because u is a coherent quantale morphism), therefore, a new application of Lemma 2.1(2) gives $u(c) \leq \rho_B(0)$. By adjointness we obtain $c \leq u_\star(\rho_B(0))$. Then the inequality $\rho_A(0) \leq u_\star(\rho_B(0))$ follows.

Now we shall prove that $\tilde{u}$ preserves the finite meets. For all elements $a, b \in [\rho_A(0))_A$ the following equalities hold:

$\tilde{u}(a \cdot_{\rho_A(0)} b) = u(ab \vee \rho_A(0)) \vee \rho_B(0) = u(a)u(b) \vee u(\rho_A(0)) \vee \rho_B(0) = u(a)u(b) \vee \rho_B(0) = (u(a) \vee \rho_B(0))(u(b) \vee \rho_B(0)) \vee \rho_B(0) = \tilde{u}(a) \cdot_{\rho_B(0)} \tilde{u}(b)$.

It is easy to see that $\tilde{u}$ preserves the joins, so $\tilde{u}$ is a quantale morphism. We remark that $\tilde{u}(1) = u(1) \vee \rho_B(0) = 1$ (because $u(1) = 1$), so $\tilde{u}$ is an integral quantale morphism.

Now we shall prove that $\tilde{u}$ is coherent. According to Lemma 6.9(3) we have:

$K([\rho_A(0))_A) = \{c \vee \rho_A(0)) | c \in K(A)\}$;

$K([\rho_B(0))_B) = \{d \vee \rho_B(0)) | c \in K(B)\}$.

Let c be a compact element of A. Then $u(c)$ is a compact element of B. Observing that $\tilde{u}(c \vee \rho_A(0)) = u(c) \vee u(\rho_A(0)) \vee \rho_B(0) = u(c) \vee \rho_B(0)$ (because $u(\rho_A(0)) \leq \rho_B(0)$) it follows that $\tilde{u}$ preserves the compact elements. Therefore, we conclude that $\tilde{u}$ is a coherent quantale morphism. □

Lemma 6.11. *If A is a coherent quantale, then $R(A) = R([\rho_A(0))_A)$.*

Proof. We observe that $Spec(A) = Spec([\rho_A(0))_A)$. By using this equality it is easy to show that $\rho_A(x) = \rho_{[\rho_A(0))_A}(x)$, for any $x \in [\rho_A(0))_A$, therefore the frames $R(A)$ and $R([\rho_A(0))_A)$ coincide. □

Following [7], we say that a ring morphism $g : R \to T$ is inverse-dense if for each radical ideal J of T, $g^{-1}(J) = Nil(R)$ implies $J = Nil(T)$. This notion can be generalized to the quantale theory framework as follows: a coherent quantale morphism $u : A \to B$ is said to be inverse-dense if for any $b \in R(B)$, $u_\star(b) = \rho_A(0)$ implies $b = \rho_B(0)$.

Proposition 6.12. *If A, B are two coherent quantales and $u : A \to B$ is a coherent quantale morphism, then the following are equivalent:*

(1) *u is inverse-dense;*

(2) *u^ρ is a $\star$-dense frame morphism;*

(3) *$(\tilde{u})^\rho : R([\rho(_A(0))_A) \to R([\rho(_B(0))_B)$ is a $\star$-dense frame morphism.*

Proof. (1) ⇔ (2) By Lemma 6.2(1), u^ρ is a coherent frame morphism. According to Proposition 6.4 we have $(u^\rho)_\star = u_\star|_{R(B)}$, so the equivalence of properties (1) and (2) follows immediately.

(2) ⇔ (3) According to Lemma 6.11, we have $R(A) = R([\rho_A(0))_A)$ and $R(B) = R([\rho_B(0))_B)$. Therefore, it is straightforward to prove the frame morphisms $u^\rho : R(A) \to R(B)$ and $(\tilde{u})^\rho : R([\rho(_A(0))_A) \to R([\rho(_B(0))_B)$ coincide. □

Theorem 6.13. *If the inverse-inverse quantale morphism $u : A \to B$ satisfies the condition $u_\star(\rho_B(0)) = \rho_A(0)$, then the following are equivalent:*

(1) *A is a weakly zipped quantale;*

(2) $[\rho_A(0))_A$ *is a zipped quantale;*

(3) $[\rho_B(0))_B$ *is a zipped quantale;*

(4) B *is a weakly zipped quantale.*

Proof. (1) $\Leftrightarrow$ (2) By Corollary 5.14.

(2) $\Leftrightarrow$ (3) We know that $[\rho_A(0))_A$, $[\rho_B(0))_B$ are semiprime coherent quantales. Firstly, we shall prove that the hypothesis $u_\star(\rho_B(0)) = \rho_A(0)$ implies that $\tilde{u}$ is dense. Let a be an element of $[\rho_A(0))_A$ such that $\tilde{u}(a) = \rho_B(0)$, hence $u(a) \vee \rho_B(0) = \rho_B(0)$. Thus $u(a) \leq \rho_B(0)$, so $\rho_B(u(a)) \leq \rho_B(0)$, therefore we get $u^\rho(a) = \rho_B(u(a)) = \rho_B(0)$. According to Proposition 6.4 we have $(u^\rho)_\star(\rho_B(0)) = u_\star(\rho_B(0)) = \rho_A(0)$. We know from Lemma 6.2(1) that $u^\rho : R(A) \to R(B)$ is a coherent frame morphism, therefore, by using [20, Remark 1.2], equality $(u^\rho)_\star(\rho_B(0)) = \rho_A(0)$ implies that u^ρ is a dense frame morphism. Therefore, $u^\rho(a) = \rho_B(0)$ implies $a = \rho_A(0)$, so we conclude that $\tilde{u}$ is a dense quantale morphism.

In virtue of Proposition 6.5(1), $(\tilde{u})^\rho$ is a dense frame morphism.

In accordance with the hypothesis that the coherent quantale morphism u is inverse-dense we get that $(\tilde{u})^\rho$ is a $\star$-dense frame morphism (cf. Proposition 6.12).

By applying [7, Theorem 4.13] to the dense and $\star$-dense coherent frame morphism $(\tilde{u})^\rho$: $R([\rho_A(0))_A) \to R([\rho_B(0))_B)$ it follows that $R([\rho_A(0))_A)$ is a zipped frame if and only if $R([\rho_B(0))_B)$ is a zipped frame. Due this equivalence and Corollary 5.9, the following properties are equivalent:

- $[\rho_A(0))_A$ is a zipped quantale;
- $R([\rho_A(0))_A)$ is a zipped frame;
- $R([\rho_B(0))_B)$ is a zipped frame;
- $[\rho_B(0))_B$ is a zipped quantale.

(3) $\Leftrightarrow$ (4) By Corollary 5.14. □

7 Final remarks

An important theorem proven by Hochster in [16] asserts that for each bounded distributive lattice L there exists a commutative ring Q such that the reticulation $L(Q)$ of Q is isomorphic with the lattice L.

By applying Hochster's theorem, it follows that for any coherent quantale A there exists a commutative ring Q such that the reticulations $L(A)$ and $L(Q)$ are isomorphic lattices (we can identify $L(A)$ and $L(Q)$). Due to this observation one can find a strong relationship between the quantale A and the ring Q: via the maps $(\cdot)^*$ and $(\cdot)_*$, some properties of A can be transferred to the ideals of Q and vice-versa. Firstly, by using Lemma 3.2, it follows that the m-prime spectrum $Spec(A)$ of A is homeomorphic with the prime spectrum $Spec(Q)$ of Q.

According to Lemma 3.3 we get the following isomorphisms in the category of coherent frames: $R(A) \simeq Id(L(A)) \simeq Id(L(Q)) \simeq R(Id(Q))$ (here $R(Id(Q))$ is the frame of radical ideals of Q). Thus the complete Boolean algebras $Pol(R(A))$ and $Pol(R(Id(Q)))$ are isomorphic.

Let us denote by $Pol_w(Q)$ the complete Boolean algebra of weak ideals of Q (in fact, we have $Pol_w(Q) = Pol_w(Id(Q))$). By a twice application of Proposition 4.3 we get $Pol_w(A) = Pol(R(A)) = Pol(R(Id(Q))) = Pol_w(Q)$.

By using Theorem 5.5 it is easy to see that the coherent quantale A is weakly zipped if and only if the ring Q is weak zip.

Based on the above remarks, it would be interesting to investigate how some results on weak zip rings (resp. zip rings) can be converted in new results on weakly zipped quantales (resp. on zipped quantales).

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