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# Relations between $L$-algebras and other logical algebras

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**Abstract**

In this paper, by considering the notion of $L$-algebra, we show that there are relations between $L$-algebras and some of other logical algebras such as residuated lattices, $MTL$-algebras, $BL$-algebras, $MV$-algebras, $BCK$-algebras, equality algebras, $EQ$-algebras and Hilbert algebras. The aim of this paper is to find under what conditions $L$-algebras are equivalent to these logical algebras.

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## 1 Introduction

The quantum Yang-Baxter equation (QYBE for short) was created by Z. Yang and R. J. Baxter in 1967 and 1972, respectively. QYBE is closely related to a series of mathematical structures, such as quantum binomial algebras [19, 20], I-type semigroups and Bieberbach groups [21, 39], plane curves and dyeing of bijective 1-type cocycles [17], semimunipolar small triangular Hopf algebra [41], dynamic system [16], geometric crystal [14], etc. Many early solutions of QYBE have been discovered and their related algebraic structures have been studied extensively, but these solutions are all variants of the common identity solutions, so it is necessary to find some non-trivial solutions. Drinfeld suggested the study of a class of &-quot: Solutions of set theory &-quot; that consider linear operators [14] whose vector spaces $V$ are generated by a set $A$ and are induced by a mapping $A \times A \to A \times A$. In 2005, W. Rump studied the algebraic solution of the quantum

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Yang-Baxter equation. He pointed out that $A$ is a set with a binary operation $\to$. The equation

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z), \qquad (L)$$

is a true statement of propositional logic and some of its generalizations [40]. On the other hand, (L) is closely related to the quantum Yang-Baxter equation [34]. In algebraic logic, new aspects were recently found in [35]. We say that an element $1 \in A$ is a logical unit [35] if $x \to x = x \to 1 = 1$ and $1 \to x = x$ holds for all $x \in A$. In the presence of (L), a logical unit defines a quasi-ordering $x \leq y$ if and only if $x \to y = 1$, and if this is a partial order, we call $A$ an $L$-algebra [35]. For the theory of $L$-algebras, the reader is referred to [35, 36].

In this paper, we investigate the relations among hoops and some logical algebras such as residuated lattices, $MTL$-algebras, $BL$-algebras, $MV$-algebras, $BCK$-algebras and Hilbert algebras.

## 2 Preliminaries

This section lists the known default contents that will be used later.

**Definition 2.1.** [35] *An algebraic structure* $(A, \to, 1)$ *is called an* $L$-algebra *if for any* $x, y, z \in A$ *satisfies in the following conditions:*
$(L_1)$ $x \to x = x \to 1 = 1$ *and* $1 \to x = x$,
$(L)$ $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,
$(L_3)$ *if* $x \to y = y \to x = 1$, *then* $x = y$.
*Condition* $(L_1)$ *satisfies that* $1$ *is a logical unit, while* $(L)$ *is related to the quantum Yang-Baxter equation.*

**Remark 2.2.** *A logical unit is always unique and it is the element of an $L$-algebra. You can see the proof in* [35].
*If the operation* $\to$ *is taken as logical implication, then there is a partial order on* $A$ *defined by*

$$x \leq y \text{ if and only if } x \to y = 1, \tag{1}$$

*for any* $x, y \in A$.

*If* $A$ *has a smallest element* $0$, *then* $A$ *is called a* bounded $L$-algebra.

**Remark 2.3.** *For an element $x$ of a bounded $L$-algebra* $(A, \to, 0, 1)$, *we define an operation* $'$ *on* $A$ *such that* $x' = x \to 0$. *We say that an $L$-algebra $A$ has a* negation *if $A$ admits a smallest element* $0$ *such that the map* $x \mapsto x'$ *is bijective. The inverse map will then be denoted by* $x \mapsto x^{\sim}$. *If* $x^{\sim} = x'$, *then $L$ is called an* $L$-algebra with double negation, *where* $(x')' = x$.
*For an $L$-algebra* $(A, \to, 0, 1)$ *with negation, for any* $x, y \in A$ *we set*

$$x \wedge y := ((x \to y) \to x')^{\sim}, \qquad x \vee y := (x^{\sim} \to y^{\sim}) \to x. \tag{2}$$

*Now, we introduce a second implication*

$$x \rightsquigarrow y := y^{\sim} \to x^{\sim}, \tag{3}$$

*for an $L$-algebra* $(A, \to, 0, 1)$ *with negation. In particular, this gives a second negation*

$$x \rightsquigarrow 0 = x^{\sim}. \tag{4}$$

**Definition 2.4.** [35] *An $L$-algebra $(A, \to, 1)$ is said to be* semi-regular *if the equation*

$$((x \to y) \to z) \to ((y \to x) \to z) = ((x \to y) \to z) \to z, \tag{5}$$

*holds.*

**Definition 2.5.** [35] *A semi-regular $L$-algebra $(A, \to, 1)$ is called a* regular $L$-algebra *if for any pair of element $x \leq y$ in $A$, there is an element $z \geq x$ in $A$ with $z \to x = y$.*

**Definition 2.6.** [35] *An $L$-algebra $(A, \to, 1)$ which satisfies the following condition*

$$x \to (y \to x) = 1, \qquad (K)$$

*for any $x, y \in A$ is called a $KL$-*algebra.

**Definition 2.7.** [35] *A $CL$-*algebra *is an $L$-algebra which satisfies the following condition*

$$(x \to (y \to z)) \to (y \to (x \to z)) = 1, \qquad (C)$$

*for any $x, y, z \in A$. It follows that in any $L$-algebra $(A, \to, 1)$ satisfying condition $(C)$ we have*

$$x \to (y \to z) = y \to (x \to z), \qquad (Ex)$$

*for all $x, y, z \in A$.*

**Proposition 2.8.** [9] *Any $CL$-algebra is a $KL$-algebra.*

# 3 Relation between $L$-algebras and other logical algebras

## 3.1 Relation with hoops

Hoops are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach in [2, 3]. In the last years, hoops theory was enriched with deep structure theorems. Many of these results have a strong impact in fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic, introduced by Hájek. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops.

In this section, we investigate the relation between hoops and $L$-algebras.

**Definition 3.1.** [2, 3] *A* left hoop *is an algebraic structure $(A, \odot, \to, 1)$ of type $(2, 2, 0)$ such that, for all $x, y, z \in A$:*
$(LH1)$ $x \odot 1 = 1 \odot x = x$,
$(LH2)$ $x \to x = 1$,
$(LH3)$ $(x \odot y) \to z = x \to (y \to z)$,
$(LH4)$ $(x \to y) \odot x = (y \to x) \odot y$.

*Also, a right hoop is an algebraic structure $(A, \odot, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ such that, for all $x, y, z \in A$:*
$(RH1)$ $x \odot 1 = 1 \odot x = x$,
$(RH2)$ $x \rightsquigarrow x = 1$,
$(RH3)$ $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$,
$(RH4)$ $x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

*An algebraic structure* $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ *is called a pseudo-hoop if* $(A, \odot, \rightarrow, 1)$ *and* $(A, \odot, \rightsquigarrow, 1)$ *are a left hoop and a right hoop, respectively.*

*On pseudo-hoop* $A$*, we define* $x \leq y$ *if and only if* $x \rightarrow y = 1$ *if and only if* $x \rightsquigarrow y = 1$*. Obviously,* $\leq$ *is a partial order relation on* $A$*. If* $\odot$ *is commutative (or equivalently* $\rightarrow = \rightsquigarrow$*), then* $A$ *is called a hoop.*

*A pseudo-hoop* $A$ *is bounded if there is an element* $0 \in A$ *such that for all* $x \in A$*,* $x \geq 0$*. For any* $x \in A$*, we consider* $x' = x \rightarrow 0$ *and* $x^{\sim} = x \rightsquigarrow 0$*.*

**Proposition 3.2.** [38] *For any* $x, y$ *in a hoop* $(A, \odot, \rightarrow, 1)$*, define*

$$x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x).$$

*Then the next equivalent conditions hold:*
*(i)* $\vee$ *is an associative operation on* $A$*,*
*(ii)* $x \leq y$ *implies* $x \vee z \leq y \vee z$*,*
*(iii)* $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$*,*
*(iv)* $\vee$ *is the join operation on* $A$*.*

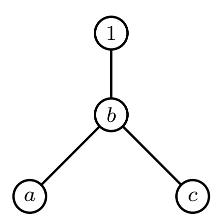
**Definition 3.3.** [38] $(A, \odot, \rightarrow, 1)$ *is called a* $\vee$-hoop *if* $\vee$ *is a join operation on* $A$*.*

**Theorem 3.4.** [38] *A semi-regular L-algebra* $(A, \rightarrow, 0, 1)$ *with negation is a left hoop, where for any* $x, y \in A$*,* $x \odot y = (x \rightarrow y')^{\sim}$*.*

**Corollary 3.5.** [38] *If* $(A, \rightarrow, 0, 1)$ *is a semi-regular L-algebra with negation, then* $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ *is a pseudo-hoop, where for any* $x, y \in A$*,* $x \odot y = (x \rightarrow y')^{\sim}$*.*

**Remark 3.6.** *Clearly, every L-algebra is not a hoop, since every L-algebra is not bounded and so we can not define negation on L-algebra* $(A, \rightarrow, 1)$*.*

**Example 3.7.** *Let* $(A = \{a, b, c, 1\}, \leq)$ *be a poset with the following Hasse diagram:*




*Define the operation* $\rightarrow$ *on* $A$ *as follows:*

| $\rightarrow$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|
| $a$ | 1 | 1 | $a$ | 1 |
| $b$ | $a$ | 1 | $c$ | 1 |
| $c$ | $a$ | 1 | 1 | 1 |
| $1$ | $a$ | $b$ | $c$ | 1 |

.

*Then* $(A, \rightarrow, 1)$ *is an L-algebra without zero element. So,* $(A, \rightarrow, 1)$ *does not have a negation operation.*

**Theorem 3.8.** [9] *Let* $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ *be a pseudo-hoop. Then* $(A, \rightarrow, 1)$ *and* $(A, \rightsquigarrow, 1)$ *are L-algebras.*

### 3.2 Relation with residuated lattices ($MTL$, $BL$ and $MV$-algebras)

A residuated lattice is an algebraic structure which is defined by Dilworth in 1939, [10, 11]. In recent years, due to the development of artificial intelligence and the use of logical algebraic structures in this field, the study of these structures has become particularly important and has attracted the attention of many mathematicians.

Among various non-classical logic systems, monoidal t-norm based logic (MTL for short) is one of the most significant of them. The point of departure in defining $MTL$ and the corresponding $MTL$-algebras is the structure introduced on $[0,1]$ by a left continuous t-norm [15].

$BL$-algebras have been introduced by Hájek [24] in order to investigate many valued logic by algebraic way. He provided an algebraic counterpart of a propositional logic, called Basic Logic, which typifies a portion common to some of the most important many-valued logics, namely, Łukasiewicz logic, Gödel logic and Product logic. This Basic Logic ($BL$ for short) is proposed as the most general many-valued logic with truth values in $[0,1]$ and $BL$-algebras are the corresponding Lindenbaum-Tarski algebras. Also, Hájek presented an algebraic mean for the study of continuous t-norms (or triangular norms) on the unit real interval $[0,1]$.

In 1958, C. C. Chang defined $MV$-algebras [8] as the algebraic counterpart of $\mathcal{N}_0$-valued Łukasiewicz logic, which allowed him to give another completeness proof for this logic. In fact, $MV$-algebras are $BL$-algebras but, the converse is not true. Proved by Höhle [25], a $BL$-algebra $A$ becomes an $MV$-algebra if we adjoin to the axioms the double negation law, i.e., $x'' = x$, for every $x \in A$. Thus, a $BL$-algebra is in some intuitive way, a non-double negation $MV$-algebra. Hence, the theory of $MV$-algebras becomes one of the guiding to the development of the theory of $BL$-algebras.

In this section, we investigate the relation between $L$-algebras and residuated lattice and some of their sub-algebras such as $MTL$, $BL$ and $MV$-algebras.

**Definition 3.9.** [43] *A* residuated lattice *is an algebraic structure* $(A, \vee, \wedge, \odot, \to, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *satisfying the following axioms:*
$(RL1)$ $(A, \vee, \wedge, 0, 1)$ *is a bounded lattice,*
$(RL2)$ $(A, \odot, 1)$ *is a commutative monoid,*
$(RL3)$ $x \odot y \leq z$ *if and only if* $x \leq y \to z$*, for all* $x, y, z \in A$.

**Definition 3.10.** [8, 10, 11, 15, 24, 34] *Let* $(A, \wedge, \vee, \odot, \to, 0, 1)$ *be a residuated lattice. Then L is called;*
$(i)$ *a* divisible residuated lattice*, if for any* $x, y \in A$*,*

$$x \wedge y = x \odot (x \to y), \qquad (div)$$

$(ii)$ *an* $MTL$-algebra*, if for any* $x, y \in A$*,*

$$(x \to y) \vee (y \to x) = 1, \quad (prel)$$

$(iii)$ *a* $BL$-algebra*, if* $A$ *is an* $MTL$*-algebra and, for any* $x, y \in A$*,*

$$x \wedge y = x \odot (x \to y), \quad (div)$$

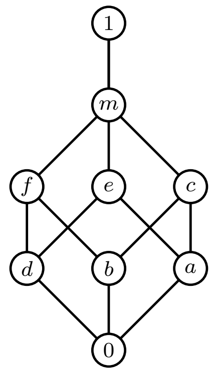
$(iv)$ *an* $MV$-algebra*, if* $A$ *is a* $BL$*-algebra and, for any* $x \in A$*,*

$$x'' = x, \quad (DN)$$

*where* $x' = x \to 0$.

Let us remark that residuated lattices and $L$-algebras are incomparable. Indeed, clearly, not all $L$-algebras are residuated lattices. The following counterexample indicates that any residuated lattices is not an $L$-algebra, in general.

**Example 3.11.** *Consider the formed lattice* $(A = \{0, a, b, c, d, e, f, m, 1\}, \leq, 0, 1)$ *with the following Hasse diagram.*




*Define the operations* $\rightarrow$ *and* $\odot$ *on* $A$ *as follows:*

| $\rightarrow$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $m$ | 1 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $m$ | 1 | $m$ | 1 | $m$ | 1 | $m$ | 1 | 1 |
| $b$ | $m$ | $m$ | 1 | 1 | $m$ | $m$ | 1 | 1 | 1 |
| $c$ | $m$ | $m$ | $m$ | 1 | $m$ | $m$ | $m$ | 1 | 1 |
| $d$ | $m$ | $m$ | $m$ | $m$ | 1 | 1 | 1 | 1 | 1 |
| $e$ | $m$ | $m$ | $m$ | $m$ | $m$ | 1 | $m$ | 1 | 1 |
| $f$ | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ | 1 | 1 | 1 |
| $m$ | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $m$ | 1 |

| $\odot$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $m$ | 1 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a$ |
| $b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $b$ |
| $c$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $c$ |
| $d$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $d$ |
| $e$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $e$ |
| $f$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $f$ |
| $m$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $m$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $m$ | 1 |

.

*Then* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *is a residuated lattice which is not an* $L$*-algebra, since* $(L)$ *is not satisfied*

$$(e \rightarrow a) \rightarrow (e \rightarrow f) = m \rightarrow m = 1 \neq m = 1 \rightarrow m = (a \rightarrow e) \rightarrow (a \rightarrow f).$$

**Proposition 3.12.** *Let* $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *be a divisible residuated lattice. Then* $(L, \rightarrow, 1)$ *is an* $L$*-algebra.*

*Proof.* Assume $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a divisible residuated lattice, then by [1, Proposition 3.4], $(A, \odot, \rightarrow, 1)$ is a hoop. Thus, by Theorem 3.8, $(A, \rightarrow, 1)$ is an $L$-algebra. □

**Corollary 3.13.** *Any* $BL$*-algebra is an* $L$*-algebra.*

*Proof.* Since every $BL$-algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a divisible residuated lattice, by Proposition 3.12, $(A, \rightarrow, 1)$ is an $L$-algebra. □

**Theorem 3.14.** *Assume* $(A, \rightarrow, 0, 1)$ *is a bounded semi-regular* $L$*-algebra with negation. Then* $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *is a divisible residuated lattice, where for any* $x, y \in A$, $x \odot y = (x \rightarrow y')^{\sim}$, $x \odot y = y \odot x$ *and* $x \vee y = (x^{\sim} \rightarrow y^{\sim}) \rightarrow x$.

*Proof.* Assume $(A, \to, 0, 1)$ is a bounded semi-regular $L$-algebra with negation. By Corollary 3.5, $(A, \vee, \odot, \to, 0, 1)$ is a bounded $\vee$-hoop. Then by [1, Theorem 3.5], $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a divisible residuated lattice. $\square$

The following counterexample indicates that any $MTL$-algebra is not an $L$-algebra, in general and conversely, does not hold, clearly.

**Example 3.15.** *Let $(A = \{0, a, b, 1\}, \leq)$ be a chain. Define the operations $\odot$ and $\to$ on $A$ as follows:*

| $\odot$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | $a$ |
| $b$ | 0 | 0 | $b$ | $b$ |
| 1 | 0 | $a$ | $b$ | 1 |

| $\to$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | 1 | 1 |
| $b$ | $a$ | $a$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | 1 |

.

*Then $(A, \vee, \wedge, \odot, \to, 0, 1)$ is an $MTL$-algebra which is not an $L$-algebra since $(L)$ is not satisfied,*

$$(b \to a) \to (b \to 0) = a \to a = 1 \neq b = 1 \to b = (a \to b) \to (a \to 0).$$

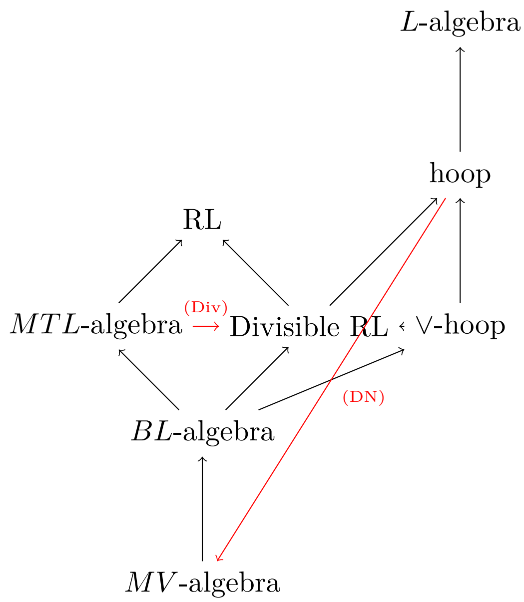
**Remark 3.16.** *Clearly, a divisible $MTL$-algebra $(A, \vee, \wedge, \odot, \to, 0, 1)$ is a $BL$-algebra and by Corollary 3.13, $(A, \to, 1)$ is an $L$-algebra.*

**Corollary 3.17.** *Let $(A, \to, 0, 1)$ be a bounded semi-regular $L$-algebra with negation.*
*Then $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a/an*
*$(i)$ $MTL$-algebra if condition $(prel)$ holds.*
*$(ii)$ $BL$-algebra if conditions $(prel)$ and $(div)$ hold,*
*where for any $x, y \in A$, $x \odot y = (x \to y')^{\sim}$ and $(2)$ holds.*

**Remark 3.18.** *Since every $MV$-algebra $(A, \vee, \wedge, \odot, \to, 0, 1)$ is a $BL$-algebra, we get $(A, \to, 1)$ is an $L$-algebra, by Corollary 3.13.*

**Theorem 3.19.** *Let $(A, \to, 0, 1)$ be a bounded semi-regular $L$-algebra with double negation $(x'' = x)$. Then $(A, \odot, \to, ', 0, 1)$ is an $MV$-algebra, where for any $x, y \in A$, $x \odot y = (x \to y')'$ and $x \odot y = y \odot x$.*

*Proof.* Let $(A, \to, 0, 1)$ be a bounded semi-regular $L$-algebra with double negation. Then by Corollary 3.5, $(A, \odot, \to, 0, 1)$ is a bounded hoop with (DNP). Thus by [1, Theorem 3.12], $(A, \odot, \to, ', 0, 1)$ is an $MV$-algebra. $\square$




In the following, according to Subsections 3.1 and 3.2, we add some results about pseudo-MV-algebras and pseudo $BL$-algebars:

**Definition 3.20.** [13, 12] *A* pseudo-BL algebra *is a structure* $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ *where* $A$ *is a non-empty set,* $\wedge$, $\vee$, $\odot$,$\to$, $\rightsquigarrow$ *are binary operations and* $0, 1$ *are constant satisfying:*
$(PBL_1)$ $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice;*
$(PBL_2)$ $(A, \odot, 1)$ *is a monoid;*
$(PBL_3)$ $x \odot y \leq z$ *if and only if* $x \leq y \to z$ *if and only if* $y \leq x \rightsquigarrow z$*;*
$(PBL_4)$ $x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y)$*;*
$(PBL_5)$ $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$,
*for all* $x, y \in A$.

**Theorem 3.21.** *Every pseudo* $BL$*-algebra is a pseudo-hoop.*

**Theorem 3.22.** *For any pseudo-*$BL$*-algebra* $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$*,* $(A, \to, 1)$ *and* $(A, \rightsquigarrow, 1)$ *are* $L$*-algebras.*

*Proof.* By Theorem 3.21 and Theorem 3.8, the proof is clear. □

**Definition 3.23.** [22] *A* pseudo $MV$-algebra *is an algebra* $(A; \oplus, ^-, ^\sim, 0, 1)$ *of type* $(2, 1, 1, 0, 0)$ *such that the following axioms hold for all* $x, y, z \in A$,
$(PMV_1)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
$(PMV_2)$ $x \oplus 0 = 0 \oplus x = x$,
$(PMV_3)$ $x \oplus 1 = 1 \oplus x = 1$,
$(PMV_4)$ $1^- = 1^\sim = 0$,
$(PMV_5)$ $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,
$(PMV_6)$ $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$,
$(PMV_7)$ $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$,
$(PMV_8)$ $(x^-)^\sim = x$,
*where* $x \odot y = (x^- \odot y^-)^\sim$.

**Theorem 3.24.** *Every pseudo $MV$-algebra is a pseudo $BL$-algebra.*

**Theorem 3.25.** *For any pseudo $MV$-algebra $(A;\oplus,-,\sim,0,1)$, $(A,\rightarrow,1)$ and $(A,\rightsquigarrow,1)$ are L-algebras, where $x\rightarrow y=x^{-}\oplus y$ and $x\rightsquigarrow y=x^{\sim}\oplus y$.*

*Proof.* By Theorems 3.24, 3.21 and 3.8, the proof is clear. □

Chang introduced $MV$-algebras as an algebraic counterpart of many-valued reasoning. It was recognized that every interval in a unital Abelian $\ell$-group gives an example of $MV$-algebras. Moreover, in [23, Example 5.1] we can see that:
Let $G=(G,+,-,0,\vee,\wedge)$ be an arbitrary $\ell$-group. For an arbitrary element $u\in G$, $u\geq 0$ define on the set $G[u]=[0,u]$ the following operations:

$$x\odot y=(x-u+y)\wedge 0,\ \ x\rightarrow y=(y-x+u)\wedge u,\ \text{and}\ x\rightsquigarrow y=(u-x+y)\wedge u.$$

Then $G[u]=(G[u],\odot,\rightarrow,\rightsquigarrow,u)$ is a bounded Wajsberg pseudo-hoop that is term-wise equivalent to a pseudo $MV$-algebra. So, we conclude the following proposition:

**Proposition 3.26.** *Let $G=(G,+,-,0,\vee,\wedge)$ be an arbitrary $\ell$-group. For an arbitrary element $u\in G$, $u\geq 0$ define on the set $G[u]=[0,u]$ the following operations:*

$$x\rightarrow y=(y-x+u)\wedge u,$$

*for $x,y\in G[u]$. Then $(G[u],\rightarrow,u)$ is an L-algebra.*

## 3.3 Relation with $BCK(BCI)$-algebras

In 1966, Imai and Iseki [27] introduced two classes of abstract algebras, $BCK$-algebras and $BCI$-algebras. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. It is well known that the class of $MV$-algebras is a proper subclass of the class of $BCK$-algebras. Therefore, both $BCK$-algebras and $MV$-algebras are important for the study of fuzzy logic. These algebras have been intensively studied by many authors.

In this section, we investigate the relation between $L$-algebras and $BCK$-algebras. Since every $BCK$-algebra is a $BCI$-algebra, the relation between $L$-algebra and $BCI$-algebra is clear.

**Definition 3.27.** [27] *An algebraic structure $(A,\rightarrow,1)$ of type $(2,0)$ is called a $BCK$-*algebra *if for any $x,y,z\in A$ the following conditions hold:*
*$(BCK1)$ $(x\rightarrow y)\rightarrow((y\rightarrow z)\rightarrow(x\rightarrow z))=1$,*
*$(BCK2)$ $1\rightarrow x=x$,*
*$(BCK3)$ $x\rightarrow 1=1$,*
*$(L_3)$ if $x\rightarrow y=y\rightarrow x=1$, then $x=y$.*

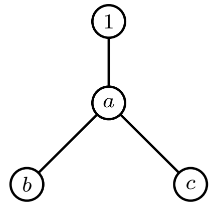
**Remark 3.28.** *Any $BCK$-algebra $(A,\rightarrow,1)$ satisfies condition $(Ex)$, that is $x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$, for any $x,y,z\in A$ (See [27]).*

By the following examples we show that every $L$-algebra is not a $BCK$-algebra and vice versa, in general.

**Example 3.29.** *According to Example 3.7, $(A,\rightarrow,1)$ is not a $BCK$-algebra, since $(BCK1)$ is not satisfied*

$$(a\rightarrow b)\rightarrow((b\rightarrow c)\rightarrow(a\rightarrow c))=1\rightarrow(c\rightarrow a)=1\rightarrow a=a\neq 1.$$

**Example 3.30.** *Let $(A = \{a, b, c, 1\}, \leq)$ be a poset with the following Hasse diagram;*




*Define the operation $\to$ on $A$ as follows:*

| $\to$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|
| $a$ | $1$ | $a$ | $a$ | $1$ |
| $b$ | $1$ | $1$ | $a$ | $1$ |
| $c$ | $1$ | $a$ | $1$ | $1$ |
| $1$ | $a$ | $b$ | $c$ | $1$ |

.

*Then $(A, \to, 1)$ is a BCK-algebra which is not an L-algebra, since*

$$(c \to a) \to (c \to b) = 1 \to a = a \neq 1 = a \to a = (a \to c) \to (a \to b).$$

**Theorem 3.31.** [9] *Any CL-algebra is a BCK-algebra.*

Let $(A, \to, 1)$ be an $L$-algebra. Then, for any $x, y \in A$, the following condition is called *commutative.*

$$(x \to y) \to y = (y \to x) \to x. \qquad (Com)$$

**Theorem 3.32.** [9] *Any commutative KL-algebra is a BCK-algebra.*

**Definition 3.33.** [27] *A BCK-algebra $(A, \to, 1)$ is called* positive implicative *if it satisfies condition*

$$x \to (y \to z) = (x \to y) \to (x \to z), \qquad (pimpl)$$

*and it is said to be implicative if*

$$(x \to y) \to x = x. \qquad (impl)$$

**Proposition 3.34.** [27] *Let $(A, \to, 1)$ be a BCK-algebra. Then the following statements are equivalent, for all $x, y \in A$:*
*$(i)$ $A$ is positive implicative,*
*$(ii)$ $x \to (x \to y) = x \to y$,*
*$(iii)$ $((x \to y) \to y) \to (x \to y) = x \to y$.*

**Proposition 3.35.** [27] *A BCK-algebra is implicative if and only if it is both positive implicative and commutative.*

**Proposition 3.36.** [9] *Any positive implicative BCK-algebra is a CL-algebra.*

**Corollary 3.37.** [9] *Any implicative BCK-algebra is a CL-algebra.*

**Definition 3.38.** [26] *A $BCK$-algebra $(A, \to, 1)$ satisfying condition*

$$x \odot y = \min\{z \mid x \leq y \to z\} \text{ exists for all } x, y \in A,$$

*is called a $BCK$-*algebra with product *($BCK(P)$-algebra, for short) and it is denoted by $(A, \odot, \to, 1)$.*

*A $BCK(P)$-algebra $(A, \odot, \to, 1)$ is of Gödel type if $x \odot x = x$, for $x \in A$.*

**Proposition 3.39.** [9] *Let $(A, \odot, \to, 1)$ be a $BCK(P)$-algebra of Gödel type. Then $(A, \to, 1)$ is an $L$-algebra.*

## 3.4 Relation with $BE$, Hilbert and quantum $B$-algebras

In 2007, Kim and Kim [31] introduced the notion of $BE$-algebras, which is another generalization of $BCK$-algebras. In recent years, the study of these structures has become particularly important and has attracted the attention of many mathematicians.

In this section, we investigate the relation between $L$-algebras and $BE$, Hilbert and quantum $B$-algebras.

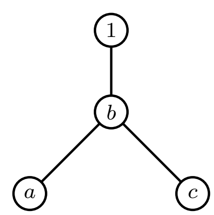
**Definition 3.40.** [31] *An algebraic structure $(A, \to, 1)$ is called a $BE$-*algebra *where 1 is a logical unit and for any $x, y, z \in A$,*

$$x \to (y \to z) = y \to (x \to z). \qquad (Ex) \tag{6}$$

*A $BE$-algebra $(A, \to, 1)$ is called a positive implicative if it satisfies condition*

$$x \to (y \to z) = (x \to y) \to (x \to z). \qquad (pimpl)$$

**Example 3.41.** *Let $(A = \{a, b, c, 1\}, \leq)$ be a poset with the following Hasse diagram:*




*Define the operation $\to$ on $A$ as follows:*

| $\to$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|
| $a$ | $1$ | $1$ | $a$ | $1$ |
| $b$ | $a$ | $1$ | $c$ | $1$ |
| $c$ | $c$ | $1$ | $1$ | $1$ |
| $1$ | $a$ | $b$ | $c$ | $1$ |

.

*Then $(A, \to, 1)$ is an $L$-algebra which is not a $BE$-algebra, since $(Ex)$ is not satisfied*

$$a \to (c \to a) = a \to c = a \neq 1 = c \to 1 = c \to (a \to a).$$

**Example 3.42.** *Assume $(A, \to, 1)$ is an algebra as in Example 3.30. Then $A$ is a $BE$-algebra which is not an $L$-algebra, since $(L)$ is not satisfied*

$$(a \to b) \to (a \to c) = a \to a = 1 \neq a = 1 \to a = (b \to a) \to (b \to c).$$

**Proposition 3.43.** [9] *Any commutative positive implicative $BE$-algebra $(A, \to, 1)$ is an $L$-algebra.*

**Definition 3.44.** [33] *An algebraic structure $(A, \to, 1)$ satisfying the axioms:*
$(H1)$ $x \to (y \to x) = 1$,
$(H2)$ $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,
$(H3)$ *if* $x \to y = y \to x = 1$, *then* $x = y$,
*is called a* Hilbert algebra*.*

**Proposition 3.45.** [26] *Any Hilbert algebra and positive implicative $BCK$-algebra are equivalent.*

**Proposition 3.46.** [33] *Any Hilbert algebra and commutative positive implicative $BE$-algebra are equivalent.*

**Example 3.47.** *According to Example 3.41, $(A, \to, 1)$ is not a Hilbert algebra, since*

$$a \to (c \to a) = a \to c = a \neq 1.$$

**Corollary 3.48.** [9] *Any Hilbert algebra is a $KL$-algebra.*

**Corollary 3.49.** [9] *Any Hilbert algebra is a $CL$-algebra.*

*Proof.* By Corollary 3.48 and Proposition 3.45(iii) the proof is clear. □

**Definition 3.50.** [37] *The* commutative quantum $B$-algebra *is an algebraic structure $(A, \leq, \to)$, where $(A, \leq)$ is a partial order set with a binary operation $\to$ satisfying the following conditions, for all $x, y, z \in A$:*
$(QB1)$ $y \to z \leq (x \to y) \to (x \to z)$,
$(QB2)$ $y \leq z$ *implies* $x \to y \leq x \to z$,
$(QB3)$ $x \leq y \to z$ *iff* $y \leq x \to z$.
*If there is an element $1 \in A$ such that $1 \to x = x$, for all $x \in A$, then $A$ is called unital.*

*If $A$ satisfies condition $x \to x = 1$, then it is a commutative normal quantum $B$-algebra. If $x \to 1 = 1$, for all $x \in A$, then $A$ is a commutative integral quantum $B$-algebra. An integral quantum $B$-algebra is normal. Note that in the case of quantum $B$-algebras the notion of commutativity differs from the one used in the present paper.*

**Proposition 3.51.** [9] *Any $CL$-algebra is a commutative integral quantum $B$-algebra.*

## 3.5 Relation with equality and EQ-algebras

A new structure, called equality algebra, is introduced by Jenei in [28]. The basic idea for examining equations of equality algebra is taken from EQ-algebras of Novák et al. [32]. The equality algebra has two connectives, a meet operation and an equivalence, and a constant. Given that equality algebra can be considered as corresponding algebras with fuzzy type theory, it is important to study this field. Other mathematicians studied the relation between equality algebra and $BCK$-meet semilattice and they proved that every $BCK(D)$-meet semilattice and equality algebra are equivalent.

In this section, we study the relation among $L$-algebras, equality algebras and EQ-algebras.

**Definition 3.52.** [29] *An algebraic structure* $(A, \wedge, \sim, 1)$ *of type* $(2, 2, 0)$ *is called an* equality algebra *if for any* $x, y, z \in A$ *the following statements hold:*
$(E1)$ $(A, \wedge, 1)$ *is a commutative idempotent monoid (i.e. meet semilattice with top element 1).*
$(E2)$ $x \sim y = y \sim x$.
$(E3)$ $x \sim x = 1$.
$(E4)$ $x \sim 1 = x$.
$(E5)$ $x \leq y \leq z$ *implies* $x \sim z \leq y \sim z$ *and* $x \sim z \leq x \sim y$.
$(E6)$ $x \sim y \leq (x \wedge z) \sim (y \wedge z)$.
$(E7)$ $x \sim y \leq (x \sim z) \sim (y \sim z)$.

*The operation* $\wedge$ *is called* meet *(infimum) and* $\sim$ *is an* equality operation. *We write* $x \leq y$ *if and only if* $x \wedge y = x$*, as usual. Define the following two derived operations, the implication and the equivalence operation of the equality algebra by,*

$$x \to y = x \sim (x \wedge y) \quad \text{and} \quad x \leftrightarrow y = (x \to y) \wedge (y \to x).$$

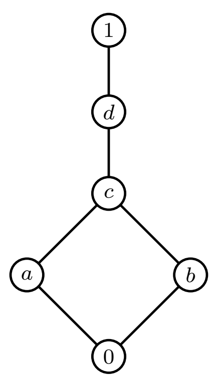
**Theorem 3.53.** [29, 46] *The following two statements hold:*
$(i)$ *For any equality algebra* $\mathcal{A} = (A, \wedge, \sim, 1)$, $\psi(\mathcal{A}) = (A, \wedge, \to, 1)$ *is a BCK-meet-semilattice, where* $x \to y = (x \wedge y) \sim x$*, for any* $x, y \in A$.
$(ii)$ *For any BCK-meet-semilattice* $\mathcal{L} = (A, \wedge, \to, 1)$ *such that, for any* $x, y, z \in A$, $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$*, we get that* $\varphi(\mathcal{A}) = (A, \wedge, \leftrightarrow, 1)$ *is an equality algebra, where* $x \leftrightarrow y = (x \to y) \wedge (y \to x)$*. Moreover, the implication of* $\varphi(\mathcal{A})$ *coincides with* $\to$*, that is* $x \to y = x \leftrightarrow (x \wedge y)$.

**Theorem 3.54.** *Let* $(A, \to, 0, 1)$ *be a bounded semi-regular* $L$*-algebra with negation. Then the structure* $\psi(\mathcal{A}) = (A, \wedge, \leftrightarrow, 1)$ *is an equality algebra, where* $x \leftrightarrow y = (x \to y) \wedge (y \to x)$ *and* $x \wedge y = ((x \to y) \to x')^{\sim}$*, for* $x, y \in A$.

*Proof.* Let $(A, \to, 0, 1)$ be a bounded semi-regular $L$-algebra with negation. Then by Corollary 3.5, $(A, \to, \odot, 0, 1)$ is a hoop and by [1, Theorem 3.28], $\psi(\mathcal{A}) = (A, \wedge, \leftrightarrow, 1)$ is an equality algebra. □

Clearly, any $L$-algebra is not an equality algebra, in general. The following example shows that every equality algebra is not an $L$-algebra, too.

**Example 3.55.** *Let* $(A = \{0, a, b, c, d, 1\}, \leq, 0, 1)$ *with the next Hasse diagram. Define two opera-*



*tions $\sim$ and $\rightarrow$ on $A$ as follows:*

| $\sim$ | $0$ | $a$ | $b$ | $c$ | $d$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $1$ | $d$ | $d$ | $d$ | $c$ | $0$ |
| $a$ | $d$ | $1$ | $c$ | $d$ | $c$ | $a$ |
| $b$ | $d$ | $c$ | $1$ | $d$ | $c$ | $b$ |
| $c$ | $d$ | $d$ | $d$ | $1$ | $d$ | $c$ |
| $d$ | $c$ | $c$ | $c$ | $d$ | $1$ | $d$ |
| $1$ | $0$ | $a$ | $b$ | $c$ | $d$ | $1$ |

| $\rightarrow$ | $0$ | $a$ | $b$ | $c$ | $d$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $d$ | $1$ | $d$ | $1$ | $1$ | $1$ |
| $b$ | $d$ | $d$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $d$ | $d$ | $d$ | $1$ | $1$ | $1$ |
| $d$ | $c$ | $c$ | $c$ | $d$ | $1$ | $1$ |
| $1$ | $0$ | $a$ | $b$ | $c$ | $d$ | $1$ |

*Then $(A, \wedge, \sim, 1)$ is an equality algebra which is not an $L$-algebra, since $(L)$ is not satisfies*

$$(a \rightarrow d) \rightarrow (a \rightarrow b) = 1 \rightarrow d = d \neq 1 = c \rightarrow c = (d \rightarrow a) \rightarrow (d \rightarrow b).$$

**Theorem 3.56.** *Consider $(A, \wedge, \sim, 1)$ be an involutive equality algebra and $x \odot y = (x \rightarrow y')'$ and $x \wedge y = x \odot (x \rightarrow y)$, for any $x, y \in A$. Then $(A, \odot, \rightarrow, 1)$ is a hoop and so $(A, \rightarrow, 1)$ is an $L$-algebra.*

*Proof.* By [1, Theorem 3.29] and Theorem 3.8, the proof is clear. □

**Theorem 3.57.** *Let $(A, \wedge, \sim, 1)$ be an equality algebra and $x \odot y = x \wedge y$, for any $x, y \in A$. If $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$, then $(A, \rightarrow, 1)$ is an $L$-algebra.*

*Proof.* By [1, Theorem 3.30] and Theorem 3.8, the proof is clear. □

**Definition 3.58.** [29] *An algebraic structure $(A, \wedge, \odot, \sim, 1)$ of type $(2, 2, 2, 0)$ is called an* EQ-algebra *if for any $x, y, z, w \in A$ the following statements hold:*
$(EQ1)$ *$(A, \wedge, 1)$ is a commutative idempotent monoid(i.e. meet semilattice with top element 1).*
$(EQ2)$ *$(A, \odot, 1)$ is a monoid such that the operation $\odot$ is isotone.*
$(EQ3)$ $x \sim x = 1$.
$(EQ4)$ $((x \wedge y) \sim z) \odot (w \sim x) \leq z \sim (w \wedge y)$.
$(EQ5)$ $(x \sim y) \odot (z \sim w) \leq (x \sim z) \sim (y \sim w)$.
$(EQ6)$ $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$.
$(EQ7)$ $x \odot y \leq x \sim y$.

*The operations $\wedge$, $\odot$, and $\sim$ are called* meet, multiplication, *and* fuzzy equality, *respectively. For any $a, b \in A$, we set $a \leq b$ if and only if $a \wedge b = a$ and we defined the binary operation* implication *on $A$ by, $a \rightarrow b = (a \wedge b) \sim a$. Also, in particular $1 \rightarrow a = 1 \sim a = \tilde{a}$. If $A$ contains a bottom element $0$, we denote it by $BEQ$-algebra and then an unary operation $'$ is defined on $A$ by $a' = a \sim 0$.*

*An EQ-algebra is* good *if $x \sim 1 = x$.*

**Theorem 3.59.** [46] *Every good EQ-algebra $(A, \wedge, \odot, \sim, 1)$ is an equality algebra.*

**Example 3.60.** *Let $(A = \{0, a, b, 1\}, \leq)$ be a chain where $0 \leq a \leq b \leq 1$. Define the operations $\odot$, $\sim$ and $\rightarrow$ on $A$ as follows:*

| $\odot$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $0$ | $a$ | $a$ |
| $b$ | $0$ | $a$ | $b$ | $b$ |
| $1$ | $0$ | $a$ | $b$ | $1$ |

| $\sim$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $1$ | $a$ | $a$ | $a$ |
| $a$ | $a$ | $1$ | $b$ | $b$ |
| $b$ | $a$ | $b$ | $1$ | $1$ |
| $1$ | $0$ | $a$ | $b$ | $1$ |

| $\rightarrow$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $a$ | $1$ | $1$ | $1$ |
| $b$ | $a$ | $b$ | $1$ | $1$ |
| $1$ | $0$ | $a$ | $b$ | $1$ |

Then $(A, \wedge, \odot, \sim, 0, 1)$ is an $EQ$-algebra which is not an $L$-algebra, since $(L)$ is not satisfied

$$(b \to a) \to (b \to 0) = b \to a = b \neq a = 1 \to a = (a \to b) \to (a \to 0).$$

**Corollary 3.61.** *Assume $(A, \to, 0, 1)$ is a bounded semi-regular $L$-algebra with negation. Then the structure $\psi(\mathcal{A}) = (A, \wedge, \odot, \leftrightarrow, 1)$ is an EQ-algebra, where $x \leftrightarrow y = (x \to y) \wedge (y \to x)$ and $x \leftrightarrow 1 = x$, for $x, y \in A$.*

*Proof.* By Corollary 3.5 and [1, Corollary 3.33], the proof is clear. □

**Corollary 3.62.** *Consider $(A, \wedge, \odot, \sim, 1)$ be an involutive good EQ-algebra where $x \odot y = (x \to y')'$ and $x \wedge y = x \odot (x \to y)$, for any $x, y \in A$. Then $(A, \odot, \to, 1)$ is a hoop and so $(A, \to, 1)$ is an $L$-algebra.*

**Corollary 3.63.** *Let $(A, \wedge, \odot, \sim, 1)$ be a good EQ-algebra where $x \odot y = x \wedge y$, for any $x, y \in A$. If $(x \odot y) \to z = x \to (y \to z)$, then $(A, \to, 1)$ is an $L$-algebra, where $x \to y = (x \wedge y) \sim x$.*

## 3.6 Relation with basic algebra

*Basic algebras*, which generalize both $MV$-algebras and orthomodular lattices, were introduced by Chajda in [7, 6] as a common base for axiomatization of many-valued propositional logics as well as of the logic of quantum mechanics. The relationship between basic algebras, $MV$-algebras, orthomodular lattices and lattice-ordered effect algebras was considered in Botur [4].

**Definition 3.64.** [42] *A* basic algebra *is an algebraic structure $A = (A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities:*
$(BA_1)$ $x \oplus 0 = x$,
$(BA_2)$ $\neg\neg x = x$,
$(BA_3)$ $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,
$(BA_4)$ $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$,
*for any $x, y, z \in A$. For the sake of brevity, we denote by $1 := \neg 0$.*

*Let $A = (A, \oplus, \neg, 0)$ be a basic algebra. The relation $\leq$ defined by*

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1$$

*is a partial order on $A$ such that $0$ and $1$ are the least and the greatest element of $A$, respectively.*

**Definition 3.65.** [18] *An* effect algebra *is a system $(A, +, 0, 1)$ consisting of a set $A$ with two special elements $0, 1 \in A$, called the zero and the unit, and with a partially defined binary operation $+$ satisfying the following conditions for all $x, y, z \in A$:*
$(E_1)$ *(Commutative law) If $x + y$ is defined, then $y + x$ is defined and $x + y = y + x$.*
$(E_2)$ *(Associative law) If $x + y$ and $(x + y) + z$ are defined, then $y + z$ and $x + (y + z)$ are defined and $(x + y) + z = x + (y + z)$.*
$(E_3)$ *(Orthosupplement law) For every $x \in A$, there exists a unique $y \in A$ such that $x + y$ is defined and $x + y = 1$. The unique element $y$ is written as $x'$ and called the orthosupplement of $x$.*
$(E_4)$ *(Zero-one law) If $x + 1$ is defined, then $x = 0$.*
*Let $(A, +, 0, 1)$ be an effect algebra. Define a binary relation on $A$ by $a \leq b$ if for some $c \in A$, $c + a = b$, which is a partial order on $A$ such that $0$ and $1$ are the smallest element and the greatest element of $A$, respectively. If the poset $(A, \leq)$ is a lattice, then $A$ is called a lattice-ordered effect algebra (LO-effect algebra for short).*

**Theorem 3.66.** [5] *Let $A = (A, +, 0, 1)$ be a lattice-ordered effect algebra. Define*

$$x \oplus y := (x \wedge y') + y \text{ and } \neg x := x'.$$

*Then, $B(A) = (A, \oplus, \neg, 0)$ is a basic algebra (whose lattice order coincides with the original one).*

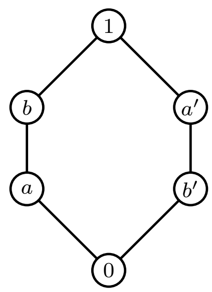
Define $x \to y := (x \wedge y) + x'$.

**Proposition 3.67.** [42] *Let $(A, +, 0, 1)$ be a lattice-ordered effect algebra. Then $(A, \to, 0, 1)$ is an $L$-algebra with negation, where*

$$x \to y := (x \wedge y) + x'.$$

The following example shows that every basic algebra is not an $L$-algebra, in general.

**Example 3.68.** *Let $(A = \{0, a, b, a', b', 1\}, \leq, 0, 1)$ be a bounded lattice with the next Hasse diagram.*




*Consider the operation $\oplus$ as follows:*

| $\oplus$ | $0$ | $a$ | $b$ | $b'$ | $a'$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $b'$ | $a'$ | $1$ |
| $a$ | $a$ | $a$ | $b$ | $a'$ | $1$ | $1$ |
| $b$ | $b$ | $a$ | $b$ | $1$ | $1$ | $1$ |
| $b'$ | $b'$ | $b$ | $1$ | $b'$ | $a'$ | $1$ |
| $a'$ | $a'$ | $1$ | $1$ | $b'$ | $a'$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

*Then, $(A, \oplus, \neg, 0)$ is a basic algebra, where $\neg x = x'$, but $(A, \to, 1)$ is not an $L$-algebra since*

$$(b \to a) \to (b \to 0) = a' \to a' = 1 \neq a' = 1 \to a' = (a \to b) \to (a \to 0),$$

*where $x \to y := (x \wedge y) \oplus x'$.*

**Theorem 3.69.** [42] *Let $(A, \oplus, \neg, 0)$ be a basic algebra which satisfies the following condition:*

$$(z \oplus \neg x) \oplus \neg(y \oplus \neg x) = (z \oplus \neg y) \oplus \neg(x \oplus \neg y).$$

*Then, $(A, \to, \neg 0)$ is an $L$-algebra, where $x \to y := (x \wedge y) \oplus \neg x$.*

**Theorem 3.70.** [42] *Let $(A, \to, 0, 1)$ be a bounded $L$-algebra, where $x \leq y$ if and only if $x \to y = 1$ such that $(A, \leq)$ is a bounded lattice, where $x' = x \to 0$ and $x'' = x$. Define $x \oplus y := y' \to x$ and $\neg x := x'$. If $(A, \oplus, \neg, 0)$ is a basic algebra, then $A$ must be a lattice ordered effect algebra.*

**Proposition 3.71.** [45] *Let $(A, \to, 0, 1)$ be a bounded $L$-algebra with negation and $x'$ denote $x \to 0$. Then the following conditions are equivalent for all $x, y, z \in A$, whenever the meets and joins exist (The condition (2) holds).*
$(i)$ *If $x \leq y$, then $y \to x = x' \to y'$,*
$(ii)$ $((y \to x) \to y') \to x' = x \to y$,
$(iii)$ $(y' \vee x') \to x' = x \to y$ *and* $x \to (x \wedge y) = x \to y$,
$(iv)$ $(x' \to y') \to x = x' \to (x' \to y')'$,
$(v)$ $((y \vee z) \to z)' \to (x' \wedge z') = z' \to ((y \vee z) \to (x' \wedge z'))$ *and if* $x' = 0$, *then* $x = 1$,
$(vi)$ *if* $x \leq y'$ *and* $y' \to x \leq z'$, *then* $x' \to (z' \to y) = z' \to (y' \to x)$,
$(vii)$ $(x \wedge y')' \to (y' \to ((y' \to x)' \wedge z)) = ((y' \to x)' \wedge z)' \to (y' \to x)$.

**Definition 3.72.** [45] *We call $(A, \to, 0, 1)$ an $LE - L$-*algebra*, if it is an $L$-algebra with 0 and satisfies one of the equivalent conditions of Proposition 3.71*

**Theorem 3.73.** [45] *A lattice-ordered effect algebra can be regarded as an $LE - L$-algebra. Conversely, an $LE - L$-algebra can be converted into a lattice-ordered effect algebra.*

**Theorem 3.74.** [45] $\mathcal{E}(\mathcal{L}(X)) = X$ *and* $\mathcal{L}(\mathcal{E}(A)) = A$, *where $(X, \oplus, 0, 1)$ is a lattice-ordered effect algebra, $(A, \to, 1)$ is an $LE - L$-algebra.*

**Definition 3.75.** [30] *Recall that an* orthomodular lattice (OML) *is a structure $(A, \leq, ', 0, 1)$, where $(A, \leq, 0, 1)$ is a bounded lattice, $'$ is a unary operation on $A$ such that the following conditions are satisfied, for all $x, y \in A$:*
$(O1)$ $x'' = x$,
$(O2)$ *if* $x \leq y$, *then* $y' \leq x'$,
$(O3)$ $x \vee x' = 1$,
$(O4)$ $x \wedge x' = 0$,
$(O5)$ *if* $x \leq y$, *then* $y = x \vee (x' \wedge y)$.

*If $A$ satisfies all the above properties except possibly $(O5)$, then $A$ is called an ortholattice $(OL)$.*

**Definition 3.76.** [44] *An orthomodular lattice (OML) is an* ortholattice *satisfying* $(O5)$.

**Definition 3.77.** [44] *We call $(A, \to, 0, 1)$ an $OM - L$-*algebra*, if it is a bounded $L$-algebra and satisfies the following condition:*

$$x' \leq y \text{ implies } y \to x = x.$$

**Proposition 3.78.** [44] *Every $OM - L$-algebra is an orthomodular lattice, and vice versa.*

**Theorem 3.79.** [45] $(A, \odot, ', 1)$ *is an $MV$-algebra if and only if $(A, \to, 0, 1)$ is a bounded $L$-algebra and satisfies the following condition:*

$$x \to y = y' \to x'.$$

**Theorem 3.80.** [45] $(A, \to, 0, 1)$ *is an $OM - L$-algebra if and only if $A$ is an $LE - L$-algebra and satisfies the following condition:*

$$x \leq x' \text{ if and only if } x = 0.$$

**Corollary 3.81.** [45] *A lattice-ordered effect algebra is an $OML$ if and only if*

$$x \leq x' \text{ if and only if } x = 0.$$

**Theorem 3.82.** [44] *Every orthomodular lattice* $(A, \leq,', 0, 1)$ *gives rise to an* $L$*-algebra* $(A, \rightarrow, 1)$.

**Theorem 3.83.** [44] *Let* $(A, \rightarrow, 0, 1)$ *be an* $OM - L$*-algebra. Then,* $(A, \leq,', 0, 1)$ *is an* $OML$*. This lattice is denoted by* $\mathcal{L}(A)$ *and call it the associated orthomodular lattice of* $A$.

**Proposition 3.84.** [44] $\mathcal{A}(\mathcal{L}(X)) = X$ *and* $\mathcal{L}(\mathcal{A}(A)) = A$*, where* $X$ *is an* $OM - L$*-algebra, and* $A$ *is an* $OML$.

**Proposition 3.85.** [44] *An* $OM - L$*-algebra can be regarded as an* $OML$*-algebra. Conversely, an* $OML$*-algebra can be converted into an* $OM - L$*-algebra.*

**Theorem 3.86.** [44] *Let* $(A, \rightarrow, 0, 1)$ *be a bounded* $L$*-algebra. Then,* $(A, \rightarrow, 0, 1)$ *is an* $OM - L$*-algebra if and only if* $A$ *satisfies* $(O1)$, $(O2)$ *and*

$$x' \leq y \text{ if and only if } x \rightarrow y = y,$$

*for all* $x, y \in A$.

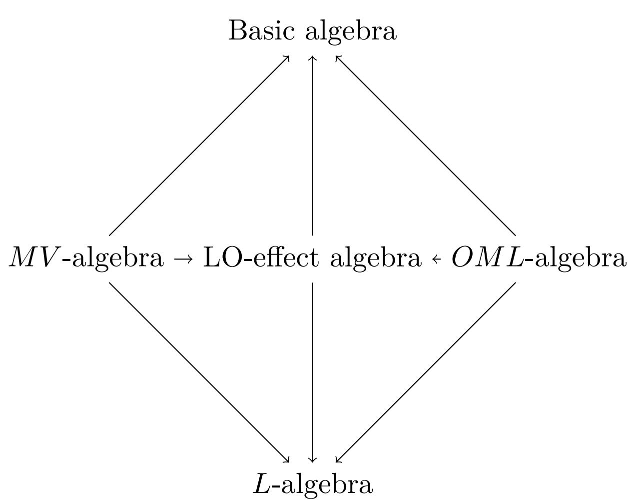
An $OM - L$-algebra $(A, \rightarrow, 0, 1)$ is called an *$OM - KL$-algebra*, if it satisfies $(K)$.

**Theorem 3.87.** [44] $(A, \rightarrow, 0, 1)$ *is an* $OM - KL$*-algebra if and only if* $A$ *is a Boolean algebra.*

**Corollary 3.88.** [44] *Let* $(A, \rightarrow, 0, 1)$ *be an* $OM - L$*-algebra. Then,* $A$ *is a Boolean algebra if and only if*

$$(x \rightarrow y) \vee (x \rightarrow y') = 1,$$

*for all* $x, y \in A$.




## 4 Conclusion

The aim of this paper was to study the relations between $L$-algebras and other logical algebras such as residuated lattice, Hilbert and $MTL$-algebras, etc. In this way, we can see under what conditions $L$-algebras are equivalent to these logical algebras. So, many results that are proved for these logical algebras, hold for $L$-algebras.

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