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n-fold 2-nilpotent(solvable) ideal of a BCK-algebra

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Abstract

In this paper, first we introduce the notions of k-nilpotent (solvable) ideals and k-nilpotent BCK-algebras. Specially, we show that every commutative ideal is 1-nilpotent (solvable). Second, we state an equivalent condition to k-nilpotency (solvablity) ideals and BCK-algebra. Finally, we study n-fold 2-nilpotent (solvable) ideals and BCK-algebras as a generalization of n-fold commutative ideals and BCK-algebras, and we study the relation between these two concepts. Basically, we compare 2-nilpotent and solvable ideals (BCK-algebras).

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1 Introduction

In 1966, Y. Imai and K. Iseki [2, 4], defined an algebra of type (2, 0), also known as BCK-algebra, as a generalization of the notion of algebra sets with the subtraction set with only a fundamental, non-nullary operation and the notion of implication algebra [3, 5] on the other hand. Since then many scholars have studied in this area. It has been used in other branches of mathematics such as hyperstructures and fuzzy sets, too (see [6, 7]).
Nilpotency is a vital concept is used in structures such as groups and rings. Different types of commutators of BCI-algebras are defined. Najafi and et.all [9], introduced the notion of commutators in a BCI-algebra to study solvable BCI-algebras. Then, we defined nilpotent BCI-algebras by a new definition of commutators [8]. Now, we redefine the notions of commutators and introduce k-nilpotent BCK-algebras. In particular, with an example, we show that these two notions are different. In addition, we try to generalize the concept of commutative ideals of BCK-algebras to k-nilpotent(solvable) ideals of BCK-algebras and we get some main results on k-nilpotent BCK-algebras. Then, using ideals we characterize nilpotent BCK-algebras. We extend some results of

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n-fold commutative ideals to n-fold 2-nilpotent(solvable) ideals. Finally, we show that every n-fold 2-nilpotent ideal is solvable, but the converse is not valid.

2 Preliminary

Definition 2.1. *An algebra $(X, *, 0)$ of type (2, 0) is called a BCI-algebra, if for any $x, y, z \in X$, the following conditions hold.*
$(I1)$ $((x * y) * (x * z)) * (z * y) = 0,$
$(I2)$ $(x * (x * y)) * y = 0,$
$(I3)$ $x * x = 0,$
$(I4)$ $x * y = y * x = 0$ *implies* $x = y$.

Adding the condition $0 * x = 0$, make X a BCK-algebra.
For a BCK-algebra X, the order $\leq$ is defined as follows:

$$x \leq y \Leftrightarrow x * y = 0.$$

Theorem 2.2. [10] *Suppose that X is a BCK-algebra and x, y, z are arbitrary elements of X. Then we have the following statements.*
(i) $(x * y) * z = (x * z) * y,$
(ii) $x * y \leq x,$
(iii) $x \leq y$ *implies that* $x * z \leq y * z$ *and* $z * y \leq z * x,$
(iv) $x * 0 = x.$

Definition 2.3. *A non-empty subset I of BCK-algebra X is called*
(i) *an ideal (we write $I \lhd X$) if $0 \in I$ and for any $x, y \in X$ if $x * y \in I$ and $y \in I$, then $x \in I$.*
(ii) *a subalgebra of X if $x * y \in I$, whenever $x, y \in I$.*

A BCK-algebra X is said to be commutative if it satisfies $x * (x * y) = y * (y * x)$ for any $x, y \in X$.

Definition 2.4. *Let S be a subset of a BCK-algebra X. We call the least ideal of X containing S, the generated ideal of X by S, denoted by $< S >$.*

Note. From now on, let $(X, *, 0)$ be a BCK-algebra unless we notify.

Definition 2.5. [8] *Let $[x, y] = (y * (y * x)) * (x * (x * y))$, for any $x, y \in X$, $V_1(X) = [X, X] = \langle \{[x, y]$ for any $x, y \in X\} \rangle$ and for any $k \in \mathbb{N}$,*

$$V_k(X) = [V_{k-1}(X), V_{k-1}(X)].$$

The BCK-algebra X is called k-solvable if $V_k(X) = \{0\}$. We use `kSBCK` *for the set of all k-solvable BCK algebras.*

Definition 2.6. [8] *Let $Z_0(X) = \{0\}$, $Z_1(X) = \langle \{x \in X : [x, y] = 0,$ for any $y \in X\} \rangle$ and for any $k \in \mathbb{N}$,*

$$Z_k(X) = \langle \{x \in X : [[[x, y_1], y_2], ..., y_k] = 0, \text{ for any } y_1, y_2, ..., y_k \in X\} \rangle.$$

The BCK-algebra X is called nilpotent of class k if $Z_k(X) = X$.

Definition 2.7. [10] *Let $I \subseteq X$, $x, y \in X$ and $z \in I$. Then I is called a commutative ideal of X if $0 \in I$ and $(xy)z \in I$ implies $x(y(yx)) \in I$.*

3 k-nilpotent BCK-algebras

In this section, we redefine a nilpotent BCK-algebra to introduce nilpotent ideals. In addition, we introduce k-nilpotent BCK-algebras. Then we state an equivalent condition to k-nilpotency of a BCK-algebra. Although most of the results on nilpotent BCK-algebras are valid with this new definition, with an example we show that these are not the same.

Note. From now on, let $x_1, x_2, ..., x_k$ be arbitrary elements of BCK-algebra X and $n, k \in \mathbb{N}$, unless we notify. Also, for any $x, y \in X$, we use xy and $\mathbf{0}$ instead of $x * y$ and zero ideal of X, respectively.

We consider $A_1 = [x_1] = x_1$ and we define the commutator of x_2 and x_1, by $A_2 = [x_2, x_1] = (x_1(x_1x_2))(x_2(x_2x_1))$ and inductively for any $x_1, ..., x_k, y_1, ..., y_k \in X$, we have

$$A_k = [x_k, [x_{k-1}, ..., [x_3, [x_2, x_1]]...], C_k = [y_k, [y_{k-1}, ..., [y_3, [y_2, y_1]]...].$$

Definition 3.1. *Let $S_0(X) = \{0\}$, $S_1(X) = \{x \in X : [y, x] = 0,$ for any $y \in X\}$ and for any $k \in \mathbb{N}$,*

$$S_k(X) = \{x \in X : [y_k, , ..., [y_2, [y_1, x]]...] = 0, \text{ for any } y_1, y_2, ..., y_k \in X\}.$$

The BCK-algebra X is called k-nilpotent if $S_k(X) = X$. We use `kNBCK` *for the set of all k-nilpotent BCK algebras.*

Remark 3.2. *If $X \in$* `kNBCK`*, then $X \in$* `kSBCK`*.*

By the following two examples, we state a difference between $kNBCK$ and the definition of nilpotency in [8]. Also, we see that the converse of Remark 3.2, is not valid.

Example 3.3. *Let $X = [0, 1]$ and operation "$*$" be given by:*

$$x * y = \begin{cases} 0, & x \leq y \\ x, & \text{otherwise} \end{cases}$$

Then $(X, *, 0)$ is a BCK-algebra. If $x, y \in X$ such that $x \leq y$, then $[x, y] = 0$ and so $[y, x] = (x(xy))(y(yx)) = x(y(yx)) = x$. From $x \leq y$, we get $[y, ..., [y, [y, x]]...] = x \neq 0$. Therefore, $X \notin$ `kNBCK` *for some $k \in \mathbb{N}$. On the other hand if $x \leq y$, then $[y, x] = x$ and so $[[y, x], x] = [x, x] = 0$ and $[[y, x], y] = [x, y] = 0$. Consequently, X is nilpotent by Definition 2.6.*

Theorem 3.4. [1] *Every finite BCK-algebra is solvable.*

Example 3.5. *Assume $(X, *, 0)$, where $X = \{0, 1, 2, ..., n\}(n \in \mathbb{N})$ and the operation $*$ is as Example 3.3. Then by Theorem 3.4, X is solvable. Similar to Example 3.3, X is not k-nilpotent. Therefore, every solvable BCK-algebra is not k-nilpotent while the converse is holds by Remark 3.2.*

Theorem 3.6. *X is a commutative BCK-algebra if and only if $X \in 1NBCK(X \in 1SBCK.)$.*

Proof. X is a commutative BCK-algebra if and only if for any $x, y \in X$, $x(xy) = y(yx)$ if and only if $[x, y] = 0$ if and only if $S_1(X) = X$ if and only if $X \in 1NBCK$. □

Theorem 3.7. *$X \in$* `kNBCK` *if and only if for any $y_1, y_2, ..., y_k \in X$, $[y_k, , ..., [y_2, [y_1, x]]...] = 0$.*

Proof. By Definition 3.1, $X \in$ `kNBCK` if and only if $S_k(X) = X$ if and only if for any $y_1, y_2, ..., y_k \in X$, $[y_k, , ..., [y_2, [y_1, x]]...] = 0$. □

Example 3.8. *Let $X = \{0, 1, 2\}$. Define the operation " $*$ " on X as follows. Then $X \in$ 1NBCK.*

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Theorem 3.9. $[X, S_k(X)] \subseteq S_{k-1}(X)$.

Proof. Consider $x \in S_k(X)$. Then by Theorem 3.7, for any $y_1, y_2, ..., y_k \in X$, we have $[y_k, , ..., [y_2, [y_1, x]]...] = 0$, i.e $[y_1, x] \in S_{k-1}(X)$. Therefore, $[X, S_k(X)] \subseteq S_{k-1}(X)$. □

Theorem 3.10. *If $X \in kNBCK$, then $X \in (k+1)NBCK$.*

Proof. Assume $x_1, ..., x_k$ are arbitrary elements of X. By $X \in kNBCK$, we get

$$[[x_k, ..., [x_3, [x_2, x_1]]...] = 0.$$

Then

$$[x_{k+1}, [x_k, ..., [x_3, [x_2, x_1]]...]] = [x_{k+1}, A_k] = [x_{k+1}, 0] = 0.$$

Therefore, $X \in (k+1)NBCK$. □

It is interesting that kNBCKs have almost the same properties as nilpotent BCK-algebras of class k that were introduced in Definition 2.6. In what follows, we state some of them. Since the proofs are similar to the ones in [8], we omit the proofs.

Let $(X, *, 0)$ and $(Y, \cdot, 0')$ be two BCK-algebras. A mapping f from $(X, *, 0)$ to $(Y, \cdot, 0')$ is called a *homomorphism* of BCK-algebras if for any $x, y \in X$, $f(x * y) = f(x) \cdot f(y)$. Also, $X \times Y$ with the operation $\bullet$ is a BCK-algebra where

$$(x_1, y_1) \bullet (x_2, y_2) = (x_1 * x_2, y_1 \cdot y_2),$$

for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ (see [10]).

Theorem 3.11. *Let $f : X \to Y$ be an isomorphism of BCK-algebras. Then $X \in$ kNBCK if and only if $Y \in$ kNBCK.*

Proof. Since f is an isomorphism for any $y_i \in Y$ there exist $x_i \in X$ such that $f(x_i) = y_i$ $(1 \leq i \leq k)$. Then,

$$[y_k, ..., [y_3, [y_2, y_1]]...] = [f(x_k), ..., [f(x_3), [f(x_2), f(x_1)]]...] = f[x_k, ..., x_3, [x_2, x_1]]...].$$

If $X \in$ kNBCK, then $0 = f(0) = f[x_k, ..., x_3, [x_2, x_1]]...] = [y_k, ..., y_3, [y_2, y_1]]...]$. Therefore, $Y \in$ kNBCK. Similarly, we have the converse. □

Corollary 3.12. . *If $X \in kNBCK$, then any subalgebra of X is k-nilpotent. Also if $I \trianglelefteq X$, then $X/I \in kNBCK$.*

Lemma 3.13. *$X/S_1(X) \in nNBCK$ if and only if $X \in (n+1)NBCK$.*

Theorem 3.14. *Let $I \trianglelefteq X$ and $n, m \in N$. If $I \in mNBCK$ and $X/I \in nNBCK$, then $X \in (n+m)NBCK$.*

Lemma 3.15. *Let $X \in nNBCK$ and M be a non-trivial ideal of X. Then $M \bigcap S(X) \neq 0$.*

Proof. First note that if $x \in X$ and $m \in M$, then $[x, m] \in M$, because

$$[x, m] = (m(mx))(x(xm)) \leq m(mx) \leq m.$$

Now, the proof is similar to [8, Theorem 4.11]. □

Theorem 3.16. *Let $X \in nNBCK$. If M is a minimal ideal of X, then $M \leq S(X)$.*

Proof. The proof is similar to [8, Theorem 4.11]. □

Theorem 3.17. *Every BCK-algebra of order less than 5, is k-nilpotent for some $k \in \mathbb{N}$.*

Theorem 3.18. *If $X, Y \in kNBCK$, then $X \bigcap Y, X \times Y \in kNBCK$.*

Proof. It is straightforward. □

4 k-nilpotent(solvable) ideals

In this section, first we extend the notion of commutative ideals and define k-nilpotent(solvable) ideals and investigate some main theorems. Then, using K-nilpotent(solvable) BCK-algebras we obtain a relation between k-nilpotency(solvableity) of a BCK-algebra and ideals.

Definition 4.1. *Assume $B \lhd X$. Then B is called*
(i) a k-nilpotent ideal of X (we write $B \blacktriangleleft_k X$) if $A_k z \in B$ implies $A_k(z(zA_k)) \in B$ for any $z \in X$.
(ii) a k-solvable ideal of X (we write $B \triangle_k X$) if $A_k C_k \in B$ implies $A_k(C_k(C_k A_k)) \in B$ for any $C_k \in X$.

Note. If we replace z with C_k in Definition 4.1(i), we can see that every k-nilpotent ideal is k-solvable. Therefore, we state and prove some results on k-nilpotent ideals. Then in a similar way, by replacing z with C_k, you can get the results on k-solvable ideals. This caused us to omit the proof when B is a k-solvable ideal. Although, the results are similar with these two definitions, we see they are not the same.

Theorem 4.2. *$B \blacktriangleleft_1 X (B \triangle_1 X)$ if and only if B is a commutative ideal of X.*

Proof. Since for any $x_1 \in X$, we have $A_1 = [x_1] = x_1$. Then we get the result by definitions. □

Example 4.3. *Assume $Y = X \bigcup \{1\}$ is the Iséki's extension of X (see [10]). Then X is a commutative ideal of Y. By Theorem 4.2, $X \blacktriangleleft_1 Y$.*

Theorem 4.4. *Let $B \trianglelefteq X$, $X \in \mathtt{kNBCK}$ and $z(zA_k) \in B$. Then $B \blacktriangleleft_k X$.*

Proof. By $X \in \mathtt{kNBCK}$ for any $x_1, ..., x_k, z \in X$ we have $0 = [z, [x_k, ..., x_1]] = [z, A_k] = (A_k(A_k z))(z(zA_k))$. Then $B \trianglelefteq X$ and $z(zA_k) \in B$ implies $A_k(A_k z) \in B$ (*). Therefore, if $A_k z \in B$, then by $B \trianglelefteq X$ and (*), we obtain $A_k \in B$. Consequently, $A_k(z(zA_k)) \leq A_k \in B$ and so $A_k(z(zA_k)) \in B$. Therefore, $B \blacktriangleleft_k X$. □

Example 4.5. *Let $X = \{0, 1, 2\}$. Define the operation "$*$" on X as Example 3.8. Then X is a BCK-algebra. Put $B = \{0, 1\}$. Clearly $B \trianglelefteq X$. For any $x, y \in X$ we have $A = [x, y] = 0$. It implies that for any $z \in X$ if $Az \in B$, then $A(z(zA)) = 0(z(zA) = 0 \in B$, i.e. $B \blacktriangleleft_2 X$.*

Theorem 4.6. *If $B \trianglelefteq X$ and $X \in 1NBCK$, then $B \blacktriangleleft_2 X$.*

Proof. By $X \in 1NBCK$ for any $x, y, z \in X$ we have $A = [x, y] = 0$ and so $A(z(zA)) = 0 \in B$. Thus, $B \blacktriangleleft_2 X$. □

Example 4.7. *Consider X as Example 3.3. If $x \leq y$, then $[y, x] = x$ and so $[y, [y, x]] = [y, x] = x$.*
(i) Take $x = 0.6, y = z = 0.7$ and B is the interval $[0, 0.5]$. Clearly, $B \lhd X$ and $A = [0.7, 0.6] = 0.6$. Then $Az = 0.6 * 0.7 = 0 \in B$ but $A(z(zA)) = 0.6 \notin B$. Therefore, $B \not\blacktriangleleft_2 X$.
(ii) Clearly, $X \in BCK^*$.

In what follows, we see that for an ideal B of X, there is not any k such that $B \blacktriangleleft_k X$.

Example 4.8. *Let X, operation " $*$ " and B be as Example 4.7. Then for $x \leq y$, we get $A_k = [y, ..., [y, [y, x]]...] = x$. Now, put $x = 0.6, y = z = 0.7$. Then $A_k z = 0.6 * 0.7 = 0 \in B$ and $A_k * (z * (z * A_k)) = 0.6 \notin B$.*

Theorem 4.9. *If $B \blacktriangleleft_k X$, then $B \blacktriangleleft_{k+1} X$.*

Proof. Let $B \blacktriangleleft_k X$, $C = [x_2, x_1]$ and $A_{k+1} z \in B$. Then

$$A_{k+1} = [x_{k+1}, [x_k, ...x_3, [x_2, x_1]]...] = [x_{k+1}, [x_k, ...x_3, x']...] = A'_k.$$

Since $B \blacktriangleleft_k X$ we get $A'_k(z(zA'_k)) \in B$ for any $z \in X$ and so $A_{k+1}(z(zA_{k+1})) = A'_k(z(zA'_k)) \in B$, i.e $B \blacktriangleleft_{k+1} X$. □

Theorem 4.10. *Let $f : X \to Y$ be an epimorphism of BCK-algebras and $J, B_1, B_2 \blacktriangleleft_k X$, $C_1 \blacktriangleleft_k Y$ and $I \lhd Y$ with $J = f^{-1}(I)$. Then*
(i) $J \blacktriangleleft_k X$ if and only if $I \blacktriangleleft_k Y$.
(ii) $B_1 \bigcap B_2 \blacktriangleleft_k X$.
(iii) $H = B_1 \times C_1 \blacktriangleleft_k X \times Y$.

Proof. (i) Let $J \blacktriangleleft_k X$ and $A_k z \in I$, where $z, y_1, y_2, ..., y_k \in Y$, $A_k = [y_k, ..., [y_2, y_1]...]$. Then $f^{-1}(A_k) f^{-1}(z) = f^{-1}(A_k z) \in f^{-1}(I) = J$ and so by $J \blacktriangleleft_k X$ we have

$$f^{-1}(A_k)(f^{-1}(z)(f^{-1}(z)f^{-1}(A_k))) = f^{-1}(A_k(z(zA_k))) \in J = f^{-1}(I).$$

Then $A_k(z(zA_k)) \in I$. Therefore, $I \blacktriangleleft_k Y$. The converse of the theorem is proved similarly.
(ii) It is straightforward.
(iii) Let $(A_k, A'_k)(z_1, z_2) = (A_k z_1, A'_k z_2) \in H$ where $z_1, x_1, x_2, ..., x_k \in X$ and $z_2, y_1, y_2, ..., y_k \in Y$, $A_k = [x_1, x_2, ..., x_k]$, $A'_k = [y_1, y_2, ..., y_k]$. Then by $B_1 \blacktriangleleft_k X$, $C_1 \blacktriangleleft_k Y$ we have

$$(A_k, A'_k) \bullet ((z_1, z_2) \bullet ((z_1, z_2) \bullet (A_k, A'_k))) = (A_k(z_1(z_1 A_k)), A'_k(z_2(z_2 A'_k))) \in H.$$

Consequently, $H \blacktriangleleft_k X \times Y$. □

Theorem 4.11. *$X \in \mathtt{kNBCK}$ ($X \in \mathtt{kSBCK}$) if and only if $A_k z = A_k(z(zA_k))$ ($A_k C_k = A_k(C_k(C_k A_k))$).*

Proof. ($\Rightarrow$) If $X \in \mathtt{kNBCK}$, then $0 = [z, [x_k, ..., [x_2, x_1]...]] = [z, A_k] = (A_k(A_k z))(z(zA_k))$ and so $A_k(A_k z) \leq z(zA_k)$. It follows by Theorem 2.2, $A_k(z(zA_k)) \leq A_k(A_k(A_k z)) = A_k z$. On the other hand $z(zA_k) \leq z$, implies $A_k z \leq A_k(z(zA_k))$. Consequently, $A_k z = A_k(z(zA_k))$.
($\Leftarrow$) By Theorem 2.2 and hypotheses, we obtain

$$[z, A_k] = (A_k(A_k z))(z(zA_k)) = (A_k(A_k(z(zA_k))))(z(zA_k)) = (A_k(z(zA_k)))(A_k(z(zA_k))) = 0.$$

Therefore, $X \in \mathtt{kNBCK}$. □

Definition 4.12. *X is called a BCK-algebra with condition $(*)$ if $A_k(A_kz) = [s_k, ..., [s_2, s_1]...]$ for some $s_1, s_2, ..., s_k \in X$. We use $\mathtt{BCK^{k*}}$ for the set of all BCK-algebras with condition $(*)$.*

Proposition 4.13. *Let $X \in BCK^{k*}$ and $I \blacktriangleleft_k X$. Then $X/I \in BCK^{k*}$.*

Proof. Since $X \in BCK^{k*}$ we have $A_k(A_kz) = [x_k, ..., [x_2, x_1]...]$ for some $x_1, x_2, ..., x_k \in X$ and so $I_{A_k}(I_{A_k}I_z) = I_{A_k(A_kz)} = I_{[x_k,...,[x_2,x_1]...]}$, i.e $X/I \in BCK^{k*}$. □

Theorem 4.14. *Suppose that $X \in \mathtt{BCK^{k*}}$ and $I, B \lhd X$ and $I \subseteq B$. If $I \blacktriangleleft_k X (I \triangle_k X)$, then $B \blacktriangleleft_k X (B \triangle_k X)$.*

Proof. Assume $u = A_kz \in B$. Then by Theorem 2.2,

$$(A_ku)z = (A_kz)u = (A_kz)(A_kz) = 0 \in I.$$

Now by $X \in \mathtt{BCK^{k*}}$ since $I \blacktriangleleft_k X$, we have

$$(A_ku)(z(z(A_ku)) \in I \subseteq B.$$

It follows by $B \unlhd X$ that $A_k(z(z(A_ku))) \in B$. Since $A_ku \leq A_k$ we have $z(z(A_ku)) \leq z(zA_k)$. Now, using Theorem 2.2, we have $A_k(z(zA_k)) \leq A_k(z(z(A_kz)))$. Therefore, $A_k(z(zA_k)) \in B$, i.e $B \blacktriangleleft_k X$. □

Corollary 4.15. *Assume $X \in \mathtt{BCK^{k*}}$. Then $\mathbf{0} \blacktriangleleft_k X (\mathbf{0} \triangle_k X)$ if and only if all ideals of X are k-nilpotent(solvable).*

Theorem 4.16. *If $X \in \mathtt{kNBCK}$, then $\mathbf{0} \blacktriangleleft_k X (\mathbf{0} \triangle_k X)$.*

Proof. Assume $X \in \mathtt{kNBCK}$ and $A_kz \in \mathbf{0}$. Then by assumption we have

$$0 = [z, A_k] = (A_k(A_kz))(z(zA_k)) = A_k(z(zA_k)).$$

Therefore, $\mathbf{0} \blacktriangleleft_k X$. □

Theorem 4.17. *Let $X \in \mathtt{BCK^{k*}}$ and $X \in \mathtt{kNBCK}(X \in \mathtt{kSBCK})$. Then all ideals of X are k-nilpotent(solvable).*

Proof. It is clear by Corollary 4.15 and Theorem 4.16. □

Now, we show that there is a 2-solvable ideal that is not a 2-nilpotent ideal.

Example 4.18. *Consider $(X, *, 0)$ as Example 3.5, $A = [5, 4] = 4$, $z = 5$ and $B = \{0, 1, 2\}$. Now, $Az = 0 \in B$ but $A(z(zA)) = 4(5(5(4))) = 4 \notin B$. Therefore, $B \not\blacktriangleleft_2 X$. Clearly, $X \in BCK^{2*}$. According to Theorem 3.4, $X \in \mathtt{2SBCK}$ and so Theorem 4.17, implies $B \triangle_2 X$*

Theorem 4.19. *Let $X \in \mathtt{BCK^{k*}}$ and $z \in X$. Then the following statements are equivalent.*
(i) $A_k \leq z$ implies $A_k \leq z(zA_k)$
(ii) $A_kz = A_k(z(zA_k))$.

Proof. $(i \Rightarrow ii)$ Since $A_k(A_kz) \leq z$ by (i), we have $A_k(A_kz) \leq z(z(A_k(A_kz)))$. Then by Theorem 2.2,

$$A_k(z(z(A_k(A_kz)))) \leq A_k(A_k(A_kz)). \quad (I)$$

Also, since $A_k(A_kz) \leq A_k$, by Theorem 2.2, we have $zA_k \leq z(A_k(A_kz))$. It follows by Theorem 2.2, $A_k(z(zA_k)) \leq A_k(z(z(A_k(A_kz))))$. Then by (I), we obtain $A_k(z(zA_k)) \leq A_kz$, (II). On the other hand by $z(zA_k) \leq z$ and Theorem 2.2, we get $A_kz \leq A_k(z(zA_k))$. It follows by (II), that $A_kz = A_k(z(zA_k))$.
$(ii \Rightarrow i)$ Assume $A_k \leq z$. Then by (ii), we get $0 = A_kz = A_k(z(zA_k))$ and so $A_k \leq z(zA_k)$. □

Let $I \lhd X$ and $x, y \in X$. Define the congruence relation $\backsimeq$ on X as follows

$$x \backsimeq y \Leftrightarrow x * y,\ y * x \in I.$$

Take $I_x = [x]$ and $X/I = \{I_x; x \in X\}$. Then $(X/I, *)$ is a BCK-algebra, where $I_x * I_y = I_{x*y}$ (see [10]).

Theorem 4.20. *Let $X \in \mathtt{BCK}^{\mathtt{k}*}$ and $I \blacktriangleleft_k X (I \triangle_k X)$. Then $X/I \in \mathtt{kNBCK}(X/I \in \mathtt{kSBCK})$.*

Proof. We get the result from Corollary 4.15 and Theorem 4.17. □

Corollary 4.21. *Let $X \in \mathtt{BCK}^{\mathtt{k}*}$ and $I \blacktriangleleft_k X$. Then for any $z \in X$, $z(zA_k) \in I$ imply $A_k(A_k z) \in I$.*

Proof. By Theorem 4.20 and $I \blacktriangleleft_k X$, we have $X/I \in \mathtt{kNBCK}$ and so for any $z, x_1, ..., x_k \in X$,

$$I_0 = [I_z, [I_{x_k}, ..., I_{x_1}]] = [I_z, I_{[x_k,...,x_1]}] = [I_z, I_{A_k}] = (I_{A_k}(I_{A_k}I_z))(I_z(I_z I_{A_k}))\quad (*).$$

On the other hand $0 * z(zA_k) = 0 \in I$ if $z(zA_k) \in I$, then $z(zA_k) \simeq 0$. It follows that $I_{z(zA_k)} = I_0$ and so $I_z(I_z I_{A_k}) = I_0$. Consequently, by (*), $(I_{A_k}(I_{A_k}I_z)) = I_0$, i.e $A_k(A_k z) \in I$. □

Theorem 4.22. *Assume $X \in BCK^{k*}$, $f : X \to Y$ is an epimorphism. Then $Kern(f) \blacktriangleleft_k X$ if and only if $Y \in \mathtt{kNBCK}$.*

Proof. (⇒) Since $Kern(f) \blacktriangleleft_k X$ by Theorem 4.20, we get that $X \in \mathtt{kNBCK}$. Therefore, $X/kern(f) \in \mathtt{kNBCK}$. From $X/Kern(f) \approxeq Y$ we obtain $Y \in \mathtt{kNBCK}$.
(⇐) From Theorem 4.16 and $Y \in \mathtt{kNBCK}$ we obtain $\mathbf{0} \blacktriangleleft_k Y$. Consider $Az \in Kern(f)$. Then $f(A)f(z) = f(Az) = 0 \in \mathbf{0} \blacktriangleleft_k Y$ implies that $f(A)(f(z)(f(z)f(A))) = f(A(z(zA)) \in \mathbf{0}$. Therefore, $f(A(z(zA)) = 0$, i.e $A(z(zA)) \in Kern(f)$. Consequently, $Kern(f) \blacktriangleleft_k X$. □

Theorem 4.23. *Let $X \in \mathtt{BCK}^{2*}$ and $I \blacktriangleleft_2 X$. Then for any $x, y, z \in X$, $[z, [y, x]] \in I$.*

Proof. Since $I \blacktriangleleft_2 X$ by Theorem 4.20, we get $X/I \in \mathtt{2NBCK}$. Then for any $x, y, z \in X$, we have $[I_z, [I_y, I_x]] = I_0$ and so $I_{[z,[y,x]]} = I_0$. It implies $[z, [y, x]] \in I$, as we need. □

Theorem 4.24. *$X/I \in \mathtt{2NBCK}$ if and only if $[z, A] \in I$, where $A = [y, x]$ and x, y are arbitrary elements of X.*

Proof. (⇒) It is clear by the proof of Theorem 4.23.
(⇐) Assume for any $z \in X$, $[z, A] \in I$. Then $[z, A] * 0 = [z, A], 0 * [z, A] = 0 \in I$ and so $[z, A] \simeq 0$. Therefore, $I_{[z,A]} = I_0$, i.e. $I_0 = I_{[z,A]} = [I_z, I_A]$. Consequently, $X/I \in \mathtt{2NBCK}$. □

Clearly, if $X \in \mathtt{2NBCK}$ then $X/I \in \mathtt{2NBCK}$. In the following we obtain the converse.

Theorem 4.25. *Let $X \in \mathtt{BCK}^{2*}$, $I \blacktriangleleft_2 X$ and I be a k-nilpotent subalgebra of X. Then $X \in (\mathtt{k}+2)\mathtt{NBCK}$.*

Proof. By Theorem 4.23, for any $x, y, z \in X$, $[z, [y, x]] \in I$. Since I is a kNBCK for any $x_k, ..., x_2 \in X$, we have $[x_k, ..., [x_2, [z, [y, x]]]...] = 0$, i.e $X \in (\mathtt{k}+2)\mathtt{NBCK}$. □

5 n-fold 2-nilpotent(solvable) ideals

In this section, we generalize the notion of n-fold commutative ideals(BCK-algebra) to n-fold k-nilpotent(solvable) ideals of BCK-algebra. Specially, we study the case $k = 2$.

Definition 5.1. *Let $A = [x, y]$, $C = [s, t]$ and $x, y, s, t \in X$. Then X is called*
(i) n-fold 2-nilpotent if there exists a fixed integer $n \geq 0$ such that $Az = A(z(zA^n))$.
(ii) n-fold 2-solvable if there exists a fixed integer $n \geq 0$ such that $AC = A(C(CA^n))$,
We use `nF2NBCK` *and* `nF2SBCK` *for the set of all n-fold 2-nilpotent and solvable BCK-algebras, respectively.*

Proposition 5.2. *$X \in$ `1F2NBCK` if and only if $X \in$ `2NBCK` and $X \in 1F1NBCK$ if and only if $X \in$ `1NBCK` if and only if X is commutative.*

Proof. It follows by Theorems 4.11 and 3.6. □

Example 5.3. *Let $X = \{0, 1, ..., n\}$ $(n \geq 4)$. Define the operation "$*$" on X as follows. Then by Theorem 3.6 and Proposition 5.2, $X \in$ `2F1NBCK` but $X \notin$ `1F1NBCK`.*

$$x * y = \begin{cases} 0 & x \leq y \\ x & y=0 \\ n-y & x=0 \\ 1 & 0 < y < x < n. \end{cases}$$

Theorem 5.4. *Every* `nF2NBCK` *is* $(\mathtt{n}+\mathtt{1})$`F2NBCK`*.*

Proof. Let $X \in$ `nF2NBCK`. Then $Az = A(z(zA^n))$. Clearly, $0 \leq zA^{n+1} \leq zA^n$. Thus,

$$z = z0 \geq z(zA^{n+1}) \geq z(zA^n)$$

and so $Az \leq A(z(zA^{n+1})) = A(z(zA^n)) = Az$. Therefore, $Az = A(z(zA^{n+1}))$, i.e $X \in (\mathtt{n}+\mathtt{1})$`F2NBCK`. □

Definition 5.5. *Assume $B \lhd X$, $z \in X$. Then B is called a*
(i) n-fold 2-nilpotent ideal of X (we write $B \blacktriangleleft_{nf} X$) if $Az \in B$ implies $A(z(zA^n)) \in B$.
(ii) n-fold 2-solvable ideal of X (we write $B \triangle_{nf} X$) if $AC \in B$ implies $A(C(CA^n)) \in B$.

Theorem 5.6. *If $B \blacktriangleleft_{nf} X (B \triangle_{nf} X)$, then $B \blacktriangleleft_{(n+1)f} X (B \triangle_{(n+1)f} X)$.*

Proof. Assume $Az \in B$. Then $A(z(zA^n)) \in B$. Also, by $zA^{n+1} \leq zA^n$, we get

$$A(z(zA^{n+1})) \leq A(z(zA^n)) \in B.$$

Consequently, $A(z(zA^{n+1})) \in B$, i.e $B \blacktriangleleft_{(n+1)f} X$. □

Obviously, the notions of 2-nilpotent ideals and 1-fold 2-nilpotent ideals are the same.

Theorem 5.7. *If I is a commutative ideal of X, then $I \blacktriangleleft_{nf} X$.*

Proof. Using Proposition 5.2 and Theorem 5.6, we get the result. □

Theorem 5.8. *Let $f : X \to Y$ be an epimorphism of BCK-algebras and $J, B_1, B_2 \blacktriangleleft_{nf} X$, $C_1 \blacktriangleleft_{nf} Y$ and $I \lhd_{nf} Y$ with $J = f^{-1}(I)$. Then the following statements hold.*
(i) $J \blacktriangleleft_{nf} X$ if and only if $I \blacktriangleleft_{nf} Y$,
(ii) $B_1 \bigcap B_2 \blacktriangleleft_{nf} X$,
(iii) $K = B_1 \times C_1 \blacktriangleleft_{nf} X \times Y$.

Proof. (i) Let $J \blacktriangleleft_{nf} X$ and $Az \in I$ where $z, y_1, y_2 \in Y$, $A = [y_1, y_2]$. Then

$$f^{-1}(A)f^{-1}(z) = f^{-1}(Az) \in f^{-1}(I) = J$$

and so by $J \blacktriangleleft_{nf} X$ we have $f^{-1}(A(z(zA^n))) \in J = f^{-1}(I)$, i.e. $A(z(zA^n)) \in I$. Therefore, $I \blacktriangleleft_{nf} Y$.
(ii) and (iii) are similar to Theorem 4.10. □

Theorem 5.9. *Consider $X \in \mathtt{BCK}^{2*}$ and $I, B \lhd X$ and $I \subseteq B$. If $I \blacktriangleleft_{nf} X (I \triangle_{nf} X)$, then $B \blacktriangleleft_{nf} X (B \triangle_{nf} X)$.*

Proof. Assume $Az \in B$ and $u = A(Az)$. Then $uz = 0 \in I$. Since $I \blacktriangleleft_{nf} X$ and $X \in \mathtt{BCK}^{2*}$ we conclude that $u(z(zu^n)) \in I$, i.e $(A(Az))(z(zu^n)) \in I \subseteq B$. Then $(A(z(zu^n)))(Az) \in B$. It follows by $B \lhd X$ and $Az \in B$ that $A(z(zu^n)) \in B$, $(*)$. In other word, by $u \leq A$ we obtain $zA^n \leq zu^n$ and so $A(z(zA^n)) \leq A(z(zu^n))$. Hence by (*), $A(z(zA^n)) \in B$, i.e $B \blacktriangleleft_{nf} X$. □

Corollary 5.10. *Assume $X \in \mathtt{BCK}^{2*}$. Then $\mathbf{0} \blacktriangleleft_{nf} X (\mathbf{0} \triangle_{nf} X)$ if and only if all ideals of X are n-fold 2-nilpotent(solvable).*

Similarly, we have the following.

Theorem 5.11. *If $X \in \mathtt{nf2NBCK}(X \in \mathtt{nf2SBCK})$, then $\mathbf{0} \blacktriangleleft_{nf} X (\mathbf{0} \triangle_{nf} X.)$.*

Corollary 5.12. *Let $X \in \mathtt{BCK}^{2*}$ and $X \in \mathtt{nf2NBCK}(X \in \mathtt{nf2SBCK})$. Then all ideals of X are n-fold 2-nilpotent(solvable).*

Theorem 5.13. *let $f : X \to Y$ be an epimorphism. Then $Ker(f) \blacktriangleleft_{nf} X$.*

Proof. By Theorem 5.11, if $Y \in nf2NBCK$, then $0 \blacktriangleleft_{nf} Y$. Then by Theorem 4.10(i), $Kerf = f^{-1}(0) \blacktriangleleft_{nf} X$. □

Theorem 5.14. *Let $X \in \mathtt{BCK}^{2*}$ and $z \in X$. Then the following are equivalent.*
(i) $A \leq z$ implies $A \leq z(zA^n)$
(ii) $Az = A(z(zA^n))$.

Proof. $(i \Rightarrow ii)$ Since $A(Az) \leq z$ by Theorem 2.2, we have $zA^n \leq z(A(Az))^n$ and so

$$A(z(zA^n)) \leq A(z(z(A(Az))^n)) \qquad (*)$$

Then $A(Az) \leq z$ implies that $A(Az) \leq z(z(A(Az))^n)$. Therefore,

$$A(z(z(A(Az))^n)) \leq A(A(Az)) = Az.$$

It follows by (*) that $A(z(zA^n)) \leq Az$. In addition, by $z(zA^n) \leq z$ we have $Az \leq A(z(zA^n))$. Consequently $Az = A(z(zA^n))$.
$(ii \Rightarrow i)$ It is similar to Theorem 4.19. □

Now, we give a characterization of n-fold 2-nilpotent BCK-algebras. In addition, we obtain a relation between n-fold 2-nilpotency of a BCK-algebra and all ideals of it.

Theorem 5.15. *Let $X \in \mathtt{BCK}^{2*}$. Then $X \in \mathtt{nF2NBCK}(X \in \mathtt{nF2SBCK})$ if and only if all ideals of X are n-fold 2-nilpotent(solvable).*

Theorem 5.16. *Let $X \in \mathtt{BCK}^{2*}$ and $I \blacktriangleleft_{nf} X (I \triangle_{nf} X)$. Then $X/I \in \mathtt{nF2NBCK}(X/I \in \mathtt{nF2SBCK})$.*

6 Conclusions

First the notion of k-nilpotent BCK-algebra was introduced. Also, an equivalent condition to k-nilpotency of a BCK-algebra was obtained. Then k-nilpotent ideal of a BCK-algebra as a generalization of commutative ideals was defined and investigated. In addition, most of the theorems on commutative ideals were obtained on k-nilpotent ideals. Finally, n-fold 2-nilpotent ideals were studied. Continuing this method, we can define k-Engels and solvable ideals of BCI-algebras, too. This reduce or exchange some problems of BCI-algebras.

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