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Near Krasner hyperrings on nearness approximation space

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Abstract

Krasner hyperrings are a generalization of rings. Indeed, in a Krasner hyperring the addition is a hyperoperation, while the multiplication is an ordinary operation. On the other hand, a generalization of rough set theory is the near set theory. Now, in this paper we are interested in combining these concepts. We study and investigate the notion of near Krasner hyperrings on a nearness approximation space. Also, we define near subhyperring, near hyperideal, near homomorphism and prove some results and present several examples in this respect.

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1 Introduction

For the first time, the idea of rough sets was presented by Pawlak [19, 20], in 1982. The theory of rough sets is an extension of set theory. The basis of this theory is an equivalence relation on the universal set. Pawlak defined the lower and upper approximations of a subset by using an equivalence relation. Many mathematicians extended and utilized the rough theory in algebraic structure. For sample, the notion of rough subring and ideal investigated by Davvaz [6]. This topic was studied and analyzed by several researchers. We refer the readers to [2, 5, 13]. A generalization of rough set theory is the near set theory. This topic was studied by Peter in 2007 [21]. Peters described an indiscernibility relation by utilizing the property of the objects to find the nearness of objects. In 2012, Inan and Ozturk investigated the notion of near groups [9, 8]. In 2013, Ozturk introduced nearness group of weak cosets [18]. In 2015, Inan and Ozturk investigated the nearness

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semigroups [10]. In 2019, Ozturk and Inan introduced nearness rings [16]. For the work done on near sets, we point out to [1, 17].

The hyperstructures theory is presented by Marty in 1934 [14]. In this theory, the composition of two members is a non-empty set. In the field of hypergroups, the first book was written by Corsini in 1993 [3]. In [4], Corsini and Leoreanu showed that the theory of hyperstructures has many applications in: geometry, hypergraghs, binary relations, codes, median algebras. In [12], Krasner defined the idea of the hyperfields and hyperrings. A hyperfield and a hyperring are a generalization of a field and a ring. The first kind of hyperrings was presented by Krasner where addition is a hyperopration but product is a binary operation. In 1982, the second kind of hyperrings was defined by Rota [24]. These hyperrings is called a multiplicative hyperring. The last kind hyperring was created by Mittas where both are hyperoperation. Theses hyperrings is called general hyperrings [15]. A monograph on hyperring theory is written by Davvaz and Leorenu-Fotea [7]. Jun studied algebraic geometry over hyperrings [11]. The theory of hyperstructures has been reviewed in [7, 25, 26, 27]

In this article, we first present and study the idea of near Krasner hyperring theory, which extends the notion of a near ring. Then, we define near (prime) hyperideals and prove some theorems and lemmas about them and present some examples. Also, we show that the intersection of two near prime hyperideals of R is not a near prime hyperideal. Finally, in the last part, we introduce the concept of near homomorphism and analyze several characterizations of them.

2 Preliminaries

In the section, we introduce the basic definitions and properties of near sets. For more results, we refer to [18, 22, 23, 19].

Symbol	Interpretation
R	Set of real numbers
$\mathcal{O}$	Set of perceptual objects
M	$M \subseteq \mathcal{O}$, Sets of sample objects
m	$m \in \mathcal{O}$, Sample object
$\mathcal{F}$	A set of functions representing object features
B	$B \subseteq \mathcal{F}$
θ	$\theta : \mathcal{O} \longrightarrow R^n$, Object description
n	n is a description length,
θ_i	$\theta_i \in B$, where $\theta_i : \mathcal{O} \longrightarrow R$
$\theta(x)$	$\theta(x) = \big(\theta_1(x), ..., \theta_n(x)\big), description$

Table 1: Description symbols

We denote $NAS = (\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, v_{N_r})$ and is named nearness approximation space where $\mathcal{O}$ is a set of percieved objects, $\mathcal{F}$ is a set of prob functions,

$$\sim_{B_r} = \{(m, n) \in \mathcal{O} \times \mathcal{O} : \theta_i \in B, \theta_i(m) = \theta_i(n)\},$$

is indiscernibility relation with $B_r \subseteq B \subseteq \mathcal{F}$, $N_r(B)$ is a collection of partition. The subscript r denotes the cardinally of the restricted subset B_r, where we consider $\binom{|B|}{r}$, i.e., $|B|$ functions $i \in \mathcal{F}$ taken r at a time to define the relation $\sim_{B_r}$. Also, $N_r(B)_*M = \{m \in \mathcal{O} : [m]_{B_r} \subseteq M\}$ is called lower approximation and $N_r(B)^*M = \{m \in \mathcal{O} : [m]_{B_r} \cap M \neq \emptyset\}$ is said to be upper approximation of M. We suppose that $\wp(\mathcal{O})$ is power set of $\mathcal{O}$, the function v_{N_r} is defined by $v_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \to [0, 1]$.

Proposition 2.1. *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and $M, N \subseteq \mathcal{O}$. Then the approximations have the following properties:*

(1) $N_r(B)_*M \subseteq M \subseteq N_r(B)^*M$,

(2) $N_r(B)^*(M \cup N) = N_r(B)^*M \cup N_r(B)^*N$,

(3) $N_r(B)_*(M \cap N) = N_r(B)_*M \cap N_r(B)_*N$,

(4) $M \subseteq N$ *implies* $N_r(B)_*M \subseteq N_r(B)_*N$,

(5) $M \subseteq N$ *implies* $N_r(B)^*M \subseteq N_r(B)^*N$,

(6) $N_r(B)^*(M \cap N) \subseteq N_r(B)^*M \cap N_r(B)^*N$,

(7) $N_r(B)^*M \cup N_r(B)^*N \subseteq N_r(B)^*(M \cup N)$.

Suppose that $* : H \times H \to \wp^*(H)$ is a hyperoperation.

Remark 2.2. *Suppose that M and N are non-empty subsets of H and $x \in H$. We define*

$$M * N = \bigcup_{\substack{m \in M \\ n \in N}} m * n, \quad M * x = M * \{x\}.$$

The pair $(H, *)$ is said to be a semihypergroup if for every a, b, c in H, we have $(a*b)*c = a*(b*c)$. Also, $(R, +, \cdot)$ is called a Krasner hyperring [12] if for any a, b, c in R:

(i) $(R, +)$ is a canonical hypergroup,
  1. $a + (b + c) = (a + b) + c$,
  2. $a + b = b + a$,
  3. there exists $0 \in R$ such that $0 + a = \{a\}$,
  4. there exists a unique element denoted by $-a \in R$ such that $0 \in a + (-a)$,
  5. $c \in a + b$ implies $b \in -a + c$ and $a \in c - b$.

(ii) $(R, \cdot)$ is a semigroup having 0 as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$.

(iii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

A non-empty subset M of R is said to be a subhyperring of R if $x - y \subseteq M$ and $x \cdot y \in M$ for every $x, y \in M$. A subhyperring N of R is normal if and only if $x + N + x \subseteq N$ for any $x \in R$.

The non-empty subset I is called a left (right) hyperideal of R if for every a, b in I, we have $a - b \subseteq I$ and $r.a \in I$, for any $r \in R$. If N is a normal hyperideal of a Krasner hyperring R, then we define the relation of $x \equiv y(modN) \Leftrightarrow (x - y) \cap N \neq \emptyset$. This relation is denotes by xN^*y. Let $(R, +, \otimes)$ and $(R', \uplus, \diamond)$ be two Krasner hyperrings. A mapping $\phi : R \to R'$ is said to be a homomorphism if for any $a, b \in R$, we have $\phi(a + b) = \phi(a) \uplus \phi(b)$, $\phi(a \otimes b) = \phi(a) \diamond \phi(b)$ and $\phi(0) = 0$.

3 Near Krasner hyperring and near subhyperring

In this part, we present the idea of a near hyperrings on nearness approximation spaces and give some examples. Moreover, we study and analyze some of its features.

Definition 3.1. *Suppose that $R \subseteq \mathcal{O}$. Then $(R, \oplus, \otimes)$ is named a near Krasner hyperring on NAS if the following are satisfied for every $m, n, p \in R$:*

(1) $m \oplus (n \oplus p) = (m \oplus n) \oplus p$ *hold in* $N_r(B)^*R$,

(2) $m \oplus n = n \oplus m$,

(3) *There exists* $0 \in N_r(B)^*R$ *such that* $0 \oplus m = \{m\}$,

(4) *There exists a unique element* $n \in R$ *such that* $0 \in m \oplus n$,
(We shall write $n = -m$ and we named it the near opposite of m.)

(5) $p \in m \oplus n$ *implies* $n \in -m \oplus p$ *and* $m \in p - n$,

(6) $(R, \otimes)$ *is a near semigroup having zero as a near bilaterally absorbing element, i.e.,* $m \otimes 0 = 0 \otimes m = 0$,

(7) $m \otimes (n \oplus p) = (m \otimes n) \oplus (m \otimes p)$ *and*
$(m \oplus n) \otimes p = (m \otimes p) \oplus (n \otimes p)$ *keep properties in* $N_r(B)^*R$.

R is said to be commutative if $m \otimes n = n \otimes m$. Also, R is said be to a near Krasner hyperring with identity if $1_R \in N_r(B)^*R$ we get $1_R \otimes m = m \otimes 1_R = m$.

Remark 3.2. *Throughout this article, we assume $(R, \oplus, \otimes)$ and $R', \uplus, \odot)$ are two near Krasner hyperrings on NAS.*

Example 3.3. *Suppose that $\mathcal{O} = \{0, 1, m, n, p\}$ with a hyperoperation "$\oplus$" and an operation "$\otimes$" defined as follows:*

$\oplus$	0	1	m	n	p
0	0	1	m	n	p
1	1	$\{0, m\}$	$\{1, n\}$	m	n
m	m	$\{1, n\}$	$\{0, m\}$	1	m
n	n	m	1	0	p
p	0	1	m	n	p

$\otimes$	0	1	m	n	p
0	0	0	0	0	0
1	0	1	m	n	0
m	0	m	m	n	0
n	0	n	n	m	0
p	0	0	0	0	1

Then $(\mathcal{O}, \oplus, \otimes)$ is not a hyperring, because $(\mathcal{O}, \oplus)$ is not associative, for instance

$$(m \oplus n) \oplus p = 1 \oplus p = n \neq m = m \oplus p = m \oplus (n \oplus p).$$

Suppose that $B = \{\Theta_1, \Theta_2, \Theta_3\}$ is a subset of $\mathcal{F}$, where Θ_i's are functions. Assume that $\Theta_1 : \mathcal{O} \to \{\rho_1, \rho_2, \rho_3\}$, $\Theta_2 : \mathcal{O} \to \{\rho_1, \rho_2\}$ and $\Theta_3 : \mathcal{O} \to \{\rho_1, \rho_2, \rho_3\}$ are given in the following table:

	0	1	m	n	p
Θ_1	ρ_1	ρ_2	ρ_2	ρ_3	ρ_3
Θ_2	ρ_1	ρ_1	ρ_1	ρ_2	ρ_2
Θ_3	ρ_2	ρ_1	ρ_3	ρ_3	ρ_1

Assume that $R = \{0, 1, m, n\} \subseteq \mathcal{O}$ *with a hyperoperation* "$\oplus$" *and an operation* "$\otimes$" *defined in the following tables:*

$\oplus$	0	1	m	n
0	0	1	m	n
1	1	$\{0, m\}$	$\{1, n\}$	m
m	m	$\{1, n\}$	$\{0, m\}$	1
n	n	m	1	0

$\otimes$	0	1	m	n
0	0	0	0	0
1	0	1	m	n
m	0	m	m	n
n	0	n	n	m

$$[0]_{\Theta_1} = \{\alpha \in \mathcal{O} \mid \Theta_1(0) = \Theta_1(\alpha) = \rho_1\} = \{0\},$$
$$[1]_{\Theta_1} = \{\alpha \in \mathcal{O} \mid \Theta_1(1) = \Theta_1(\alpha) = \rho_2\} = \{1, m\} = [m]_{\Theta_1},$$
$$[n]_{\Theta_1} = \{\alpha \in \mathcal{O} \mid \Theta_1(n) = \Theta_1(\alpha) = \rho_3\} = \{n, p\} = [p]_{\Theta_1}.$$

We obtain $\xi_{\Theta_1} = \{[0]_{\Theta_1}, [1]_{\Theta_1}, [n]_{\Theta_1}\}$.

$$[0]_{\Theta_2} = \{\alpha \in \mathcal{O} \mid \Theta_2(0) = \Theta_2(\alpha) = \rho_1\} = \{0, 1, m\} = [m]_{\Theta_2} = [1]_{\Theta_2},$$
$$[n]_{\Theta_2} = \{\alpha \in \mathcal{O} \mid \Theta_2(b) = \Theta_2(\alpha) = \rho_2\} = \{n, p\} = [p]_{\Theta_2}.$$

We get $\xi_{\Theta_2} = \{[0]_{\Theta_2}, [n]_{\Theta_2}\}$. *In the same way,* $\xi_3 = \{[0]_{\Theta_3}, [1]_{\Theta_3}, [m]_{\Theta_3}\}$.
Hence, for $r = 1$ *a partition of* $\mathcal{O}$ *is* $N_1 = \{\xi_{\Theta_1}, \xi_{\Theta_2}, \xi_{\Theta_3}\}$. *So, we obtain* $N_r(B)^* R = \{0, 1, m, n, p\}$. *Therefore, in the sense of Definition 3.1,* $(R, \oplus, \otimes)$ *is a near Krasner hyperring.*

We defined a collection of partitions $N_1(B)$, where $N_1(B) = \{\xi_{\mathcal{O},B_1} \mid B_1 \subseteq B\}$. Families of neighborhoods are constructed for any combination of functions in B using $\binom{|B|}{r}$, that means of, $|B|$ functions taken 1 at a time. We can give an example for $r = 2$.

Example 3.4. *In Example3.3, consider* $R = \{0, 1, m, n\}$ *with a hyperoperation* "$\oplus$" *and an operation* "$\otimes$" *by below tables:*

$\oplus$	0	1	m	n
0	0	1	m	n
1	1	$\{0, m\}$	$\{1, n\}$	m
m	m	$\{1, n\}$	$\{0, m\}$	1
n	n	m	1	0

$\otimes$	0	1	m	n
0	0	0	0	0
1	0	1	m	n
m	0	m	m	n
n	0	n	n	m

We define a set of functions $B = \{\phi_1, \phi_2, \phi_3, \phi_4\} \subseteq \mathcal{F}$. *The values of functions* $\phi_1 : \mathcal{O} \to \{\gamma_1, \gamma_2, \gamma_5\}$, $\phi_2 : \mathcal{O} \to \{\gamma_2, \gamma_3, \gamma_4, \gamma_5\}$, $\phi_3 : \mathcal{O} \to \{\gamma_1, \gamma_3, \gamma_4, \gamma_5\}$ *and* $\phi_4 : \mathcal{O} \to \{\gamma_3, \gamma_4, \gamma_5\}$ *are given in the below table:*

	0	1	m	n	p
ϕ_1	γ_1	γ_2	γ_2	γ_5	γ_1
ϕ_2	γ_4	γ_2	γ_3	γ_5	γ_4
ϕ_3	γ_1	γ_5	γ_3	γ_1	γ_4
ϕ_4	γ_5	γ_5	γ_3	γ_4	γ_5

$$[1]_{\{\phi_1,\phi_2\}} = \{\alpha \in \mathcal{O} \mid \phi_1(\alpha) = \phi_2(\alpha) = \phi_1(1) = \phi_2(1) = \gamma_2\} = \{1\},$$
$$[n]_{\{\phi_1,\phi_2\}} = \{\alpha \in \mathcal{O} \mid \phi_1(\alpha) = \phi_2(\alpha) = \phi_1(n) = \phi_2(n) = \gamma_5\} = \{n\}.$$

We obtain $\xi_{\{\phi_1,\phi_2\}} = \{[1]_{\{\phi_1,\phi_2\}}, [n]_{\{\phi_1,\phi_2\}}\}$.

$$[0]_{\{\phi_1,\phi_3\}} = \{\alpha \in \mathcal{O} \mid \phi_1(\alpha) = \phi_3(\alpha) = \phi_1(0) = \phi_3(0) = \gamma_1\} = \{0, p\} = [p]_{\{\phi_1,\phi_3\}}.$$

We get $\xi_{\{\phi_1,\phi_3\}} = \{[0]_{\{\phi_1,\phi_3\}}\}$.

$$[m]_{\{\phi_2,\phi_3\}} = \{\alpha \in \mathcal{O} \mid \phi_2(\alpha) = \phi_3(\alpha) = \phi_2(m) = \phi_3(m) = \gamma_3\} = \{m\},$$
$$[p]_{\{\phi_2,\phi_3\}} = \{\alpha \in \mathcal{O} \mid \phi_2(\alpha) = \phi_3(\alpha) = \phi_2(p) = \phi_3(p) = \gamma_4\} = \{p\}.$$

So, we have $\xi_{\{\phi_2,\phi_3\}} = \{[m]_{\{\phi_2,\phi_3\}}, [p]_{\{\phi_2,\phi_3\}}\}$.

$$[m]_{\{\phi_2,\phi_4\}} = \{\alpha \in \mathcal{O} \mid \phi_2(\alpha) = \phi_4(\alpha) = \phi_2(m) = \phi_4(m) = \gamma_3\} = \{m\}.$$

We get $\xi_{\{\phi_2,\phi_4\}} = \{[m]_{\{\phi_2,\phi_4\}}\}$.

$$[1]_{\{\phi_3,\phi_4\}} = \{\alpha \in \mathcal{O} \mid \phi_3(\alpha) = \phi_4(\alpha) = \phi_3(1) = \phi_4(1) = \gamma_5\} = \{1\},$$
$$[m]_{\{\phi_3,\phi_4\}} = \{\alpha \in \mathcal{O} \mid \phi_3(\alpha) = \phi_4(\alpha) = \phi_3(m) = \phi_4(m) = \gamma_3\} = \{m\}.$$

We obtain $\xi_{\{\phi_3,\phi_4\}} = \{[1]_{\{\phi_3,\phi_4\}}, [m]_{\{\phi_3,\phi_4\}}\}$.
Thus, for $r = 2$, *a set of partitions of* $\mathcal{O}$ *is* $N_2(B) = \{\xi_{\{\phi_1,\phi_3\}}, \xi_{\{\phi_2,\phi_3\}}, \xi_{\{\phi_2,\phi_4\}}, \xi_{\{\phi_3,\phi_4\}}\}$.

So, we write $N_2(B)^*(R) = \{0, 1, m, n, p\}$. *By Definition 3.1,* $(R, \oplus, \otimes)$ *is a near Krasner hyperring on NAS.*

Corollary 3.5. *Every Krasner hyperrings is a near Krasner hyperring on NAS.*

Proof. It is straightforward. □

Definition 3.6. [8] *Let* $NAS = (\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, v_{N_r})$ *be a nearness approximation space and let* $\cdot$ *be a binary operation on* $\mathcal{O}$. *A subset* G *of perceptual objects* $\mathcal{O}$ *is called a near group if the following properties are satisfied:*

(1) $\forall x, y \in G$, $x \cdot y \in N_r(B)^*G$;

(2) $\forall x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *property holds in* $x \cdot y \in N_r(B)^*G$;

(3) $\exists e \in N_r(B)^*G$ *such that for all* $x \in G$, $x \cdot e = e \cdot x = x$, e *is called the near identity element;*

(4) $\forall x \in G$, $\exists y \in G$ *such that* $x \cdot y = y \cdot x = e$, y *is called the near inverse element of* x.

Definition 3.7.

(1) *Assume that* $(R, +, \cdot)$ *is a near Krasner hyperring. Then* R *is said to be a near hyperfield if* $(R \setminus \{0\}, \cdot)$ *is a commutative near group.*

(2) *If* R *is a commutative near Krasner hyperring with unit element and* $mn = 0$ *implies* $m = 0$ *or* $n = 0$ *for all* $m, n \in R$, *then* R *is a near hyperdomine.*

Example 3.8. *Let* $\mathcal{O} = \{0, 1, a, b\}$ *be a set with a hyperoperation* "$\boxplus$" *and an operation* "$\boxtimes$" *by the following tables:*

$\boxplus$	0	1	a	b
0	0	1	a	b
1	1	$\{0,1\}$	1	$\{1,b\}$
a	a	1	$\{0,a\}$	0
b	b	$\{1,b\}$	0	$\mathcal{O}$

$\boxtimes$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	1	a
b	0	b	a	1

Then $(\mathcal{O}, \boxplus, \boxtimes)$ *is not a hyperring, because* $(a \boxplus b) \boxplus 1 = 0 \boxplus 1 = 1 \neq 0 = a \boxplus (b \boxplus 1)$. *Suppose that* $B = \{\gamma_1, \gamma_2\} \subseteq \mathcal{F}$ *where* $\gamma_1 : \mathcal{O} \to \{1, 2\}$ *and* $\gamma_2 : \mathcal{O} \to \{1, 2\}$ *are given in below Tables.*

	0	1	a	b
γ_1	1	2	1	2
γ_2	2	1	1	2

we obtain

$$[0]_{\gamma_1} = \{0, a\} = [a]_{\gamma_1},\ [1]_{\gamma_1} = \{1, b\} = [b]_{\gamma_1},$$
$$[0]_{\gamma_2} = \{0, b\} = [b]_{\gamma_2},\ [1]_{\gamma_2} = \{1, a\} = [a]_{\gamma_2}.$$

Now, we assume that $R = \{0, 1, a\} \subseteq \mathcal{O}$ *with a hyperoperation* "$\boxplus$" *and an operation* "$\boxtimes$" *by the below tables:*

$\boxplus$	0	1	a
0	0	1	a
1	1	$\{0,1\}$	1
a	a	1	$\{0,a\}$

$\boxtimes$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	1

Therefore, $N_r(B)^*(R) = \{0, 1, a, b\}$.*So, by Definition 3.1,* $(R, \boxplus, \boxtimes)$ *is a near Krasner hyperring with identity element* 1 *and it is commutative. Because for all* $m, n \in R$ *we have* $m \boxtimes n = n \boxtimes m$. *Also,* $(R, \boxplus, \boxtimes)$ *is a near hyperdomain. Hence* $1 \boxtimes 0 = a \boxtimes 0 = b \boxtimes 0 = 0$.
On the other hand, $(R \setminus \{0\}, \boxtimes)$ *is a commutative near group.*

(1) *for every* $a, b, c \in R \setminus \{0\}$, *we have* $(a \boxtimes b) \boxtimes c = a \boxtimes (b \boxtimes c)$,

(2) *for every* $a \boxtimes b = b \boxtimes a$,

(3) *there exists* $1 \in N_r(B)^*(R \setminus \{0\})$, *for all* $x \in R \setminus \{0\}$, *we have* $x \boxtimes 1 = x$,

(4) *for all* $x \in R \setminus \{0\}$, *there exists* $y \in R \setminus \{0\}$, *we have* $x \boxtimes y = 1$.

So, R *is a near hyperfield.*

Suppose that $M \oplus N = \Big\{x \mid x \in m+n, m \in M \text{ and } n \in N\Big\}$ and

$$M \otimes N = \Big\{ \sum_{\text{finit}} m_i \otimes n_i \mid m_i \in M,\ n_i \in N \Big\},$$

where M, N are two subsets of R.

Theorem 3.9. *Suppose that M, N are two subsets of R. Then,*

(1) *If $M, N \subseteq R$, then $(N_r(B)^*M) \oplus (N_r(B)^*N) \subseteq N_r(B)^*(M \oplus N)$,*

(2) *If $M, N \subseteq R$, then $N_r(B)^*(M) \otimes N_r(B)^*(N) \subseteq N_r(B)^*(M \otimes N)$.*

Proof. (1) Assume that $x \in (N_r(B)^*M) \oplus (N_r(B)^*N)$. We have $x \in m \oplus n$; $m \in (N_r(B)^*M)$, $n \in (N_r(B)^*N)$. Since $m \in N_r(B)^*M$, it follows that $[m]_{B_r} \cap M \neq \emptyset$. Therefore, $y \in [m]_{B_r} \cap M$. Then $y \in [m]_{B_r}$ and $y \in M$. Likewise, $n \in N_r(B)^*N$, then $[n]_{B_r} \cap N \neq \emptyset$, there exists $z \in [n]_{B_r} \cap N$, so $z \in [n]_{B_r}$ and $z \in N$. Since $w \in y \oplus z \subseteq [m]_{B_r} \oplus [n]_{B_r} \subseteq [m \oplus n]_{B_r}$, it follows that $w \in [m \oplus n]_{B_r}$ and $w \in M \oplus N$. Thus, $w \in [m \oplus n]_{B_r} \cap (M \oplus N)$. Therefore, $[m \oplus n]_{B_r} \cap (M \oplus N) \neq \emptyset$, and so $x \in a \oplus b \subseteq N_r(B)^*(M \oplus N)$.

(2) The proof of (2) is similar to (1). □

Suppose that $\mathcal{O}_1, \mathcal{O}_2$ are two sets and " $\sim'_{B_r}$ ", " $\sim''_{B_r}$ " are two indispensability relations on $\mathcal{O}_1$, $\mathcal{O}_2$ respectively. Then, we define relation " $\sim_{B_r}$ " on $\mathcal{O}_1 \times \mathcal{O}_2$, for every $(a, b), (c, d) \in \mathcal{O}_1 \times \mathcal{O}_2$:

$$(a, b) \sim_{B_r} (c, d) \Leftrightarrow a \sim'_{B_r} c \text{ and } b \sim''_{B_r} d.$$

It is easy to see that the relation " $\sim_{B_r}$ " is an equivalence relation on $\mathcal{O}_1 \times \mathcal{O}_2$. Now, we define equivalence classes on elements $\mathcal{O}_1 \times \mathcal{O}_2$ as follows:

$$[(a, b)]_{\sim_{B_r}} = \{(c, d) \in \mathcal{O}_1 \times \mathcal{O}_2 \mid (a, b) \sim_{B_r} (c, d)\}.$$

If $X_1 \subseteq \mathcal{O}_1$ and $X_2 \subseteq \mathcal{O}_2$, then $N_r(B)^*(X_1 \times X_2) = \bigcup_{[(a,b)]_{B_r} \cap (X_1 \times X_2) \neq \emptyset} [(a, b)]_{B_r}$.

Suppose that R_1, R_2 are two near Krasner hyperrings. So, $R_1 \times R_2$ is not a near Krasner hyperring. See the following example:

Example 3.10. *In Example 3.8, we show $R = \{0, 1, a\}$ is a near Krasner hyperring with function $B = \{\gamma_1, \gamma_2\}$. Also, in Example 3.3, $R = \{0, 1, m, n\}$ is a near Krasner hyperring with function $B = \{\phi_1, \phi_2, \phi_3, \phi_4\}$. But, $R_1 \times R_2$ is not a near Krasner hyperring because $|B|_{\mathcal{O}_1} = 2$ and $|B|_{\mathcal{O}_2} = 4$, thus $|B|_{\mathcal{O}_1} \neq |B|_{\mathcal{O}_2}$.*

Theorem 3.11. *Assume that R, R' are two near Krasner hyperrings. If*

1. *The number of functions in $\mathcal{O}_1$ and $\mathcal{O}_2$ should be equal, that is $|B|_{\mathcal{O}_1} = |B|_{\mathcal{O}_2}$,*
2. *$N_r(B)^*R \times N_r(B)^*R' = N_r(B)^*(R \times R')$.*

Then $(R \times R', +, .)$ is a near Krasner hyperring.

Proof. First, we define hyperoperation "+", operation "·" between elements $R \times R'$. Let $m, m', \in R$ and $n, n' \in R'$:

$$(m, n) + (m', n') = \{(x, y) \mid x \in m \oplus m', y \in n \uplus n'\},$$
$$(m, n).(m', n') = (x, y), \quad x = m \otimes m', y = n \odot n'.$$

Let $m, m', m_1, m_2, m_3 \in R$ and $n, n', n_1, n_2, n_3 \in R'$.

(i) We show $(R \times R', +)$ is a near canonical hyperring.

(1) Associative property:

$$\begin{aligned}(m,n) + \big((m_1,n_1) + (m_2,n_2)\big) &= (m,n) + \{(m',n') \mid m' \in m_1 \oplus m_2, n' \in n_1 \uplus n_2\}\\ &= \{(m'',n'') \mid m'' \in m \oplus (m_1 \oplus m_2), n \in n \uplus (n_1 \uplus n_2)\}\\ &= \{(m'',n'') \mid m'' \in (m \oplus m_1) \oplus m_2, n'' \in (n \uplus n_1) \uplus n_2\}\\ &= \{(p,q) \mid p \in m \oplus m_1, q \in n \uplus n_1\} + (m_2,n_2)\\ &= \big((m,n) + (m_1,n_1)\big) + (m_2,n_2),\end{aligned}$$

(2) Commutative property:

$$\begin{aligned}(m_1,n_1) + (m_2,n_2) &= \{(m,n) \mid m \in m_1 \oplus m_2, n \in n_1 \uplus n_2\}\\ &= \{(m,n) \mid m \in m_2 \oplus m_1, n \in n_2 \uplus n_1\}\\ &= (m_2,n_2) + (m_1,n_1).\end{aligned}$$

(3) There exists $0_R \in N_r(B)^*(R)$ such that $0_R \oplus m = \{m\}$ and similar to for R', we have $0_{R'} \uplus n = \{n\}$. So, $(0_R, 0_{R'}) + (m_1, m_2) = \{(m_1, m_2)\}$.

(4) For any $m \in R$, there exists a unique element $-m$ such that $0_R \in m - m$. Similarly, for R', $0_{R'} \in n - n$. Therefore, $(0_R, 0_{R'}) \in (m,n) + (-m,-n)$.

(5) We prove that if $(m_1,n_1) \in (m_2,n_2) + (m_3,n_3)$, then $(m_2,n_2) \in (m_1,n_1) + (-m_3,-n_3)$ and $(m_3,n_3) \in (m_1,n_1) + (-m_2,-n_2)$.

$$\begin{aligned}m_1 \in m_2 \oplus m_3 \;&\Rightarrow\; m_2 \in m_1 - m_3 \;\; \textit{and} \;\; m_3 \in m_1 - m_2,\\ n_1 \in n_2 \uplus n_3 \;&\Rightarrow\; n_2 \in n_1 - n_3 \;\; \textit{and} \;\; n_3 \in n_1 - n_2.\end{aligned}$$

From $(1), (2)$, we get $(m_2,n_2) \in (m_1,n_1) + (-m_3,-n_3)$ and $(m_3,n_3) \in (m_1,n_1) + (-m_2,-n_2)$.

(ii) $(R \times R', \cdot)$ is a near semigroup. Whereas R, R' be two near semigroups, we obtain

$$(m_1,n_1).(m_2,n_2) = (m_1 \otimes m_2, n_1 \odot n_2) = (m_2 \otimes m_1, n_2 \odot n_1) = (m_2,n_2).(m_1,n_1).$$

(iii) Distributive property.

$$\begin{aligned}(m_1,n_1).\big[(m_2,n_2) + (m_3,n_3)\big] &= (m_1,n_1).\{(m,n) \mid m \in m_2 \oplus m_3, n \in n_2 \uplus n_3\}\\ &= \{(m',n') \mid m' \in m \otimes m_1, n' \in n \odot n_1, m \in m_2 \oplus m_3, n \in n_2 \uplus n_3\}\\ &= \{(m',n') \mid m' \in m_1 \otimes (m_2 \oplus m_3), n' \in n_1 \odot (n_2 \uplus n_3)\}\\ &= \{(m',n') \mid m' \in m_1 \otimes m_2 \oplus m_1 \otimes m_3, y' \in n_1 \odot n_2 \uplus n_1 \odot n_3\}\\ &= (m_1 \otimes m_2, n_1 \odot n_2) + (m_1 \otimes m_3, n_1 \odot n_3)\\ &= (m_1,n_1).(m_2,n_2) + (m_1,n_1).(m_3,n_3).\end{aligned}$$

□

Example 3.12. *In Example 3.8, we show* $(R = \{0,1,a\}, \boxplus, \boxtimes)$ *is a near Krasner hyperring by below tables.*

$\boxplus$	0	1	a
0	0	1	a
1	1	$\{0,1\}$	1
a	a	1	$\{0,a\}$

$\boxtimes$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	1

Now, we have

$$R\times R=\{(0,0),(0,1),(0,a),(1,0),(1,1),(1,a),(a,0),(a,1),(a,a)\}.$$

Also $N_r(B)^*(R)=\{0,1,a,b\}$. *We give*

$$\begin{aligned}N_r(B)^*(R)\times N_r(B)^*(R)=\{&(0,0),(0,1),(0,a),(0,b),(1,0),(1,1),(1,a),(1,b),\\&(a,0),(a,1),(a,a),(a,b),(b,0),(b,1),(b,a),(b,b)\}.\end{aligned}$$

$$\begin{aligned}[(0,0)]_{\gamma_1}&=\{(x,y)\in\mathcal{O}\times\mathcal{O}\mid (0,0)\sim_{B_r}(x,y)\}\\&=\{(0,0),(0,a),(a,0),(a,a)\},\\ [(0,1)]_{\gamma_1}&=\{(0,1),(0,b),(a,1),(a,b)\},\\ [(1,0)]_{\gamma_1}&=\{(1,0),(1,a),(b,0),(b,a)\},\\ [(1,1)]_{\gamma_1}&=\{(1,1),(1,b),(b,1),(b,b)\}.\end{aligned}$$

So, we obtain $\xi_{\gamma_1}=\{[(0,0)]_{\gamma_1},[(0,1)]_{\gamma_1},[(1,0)]_{\gamma_1},[(1,1)]_{\gamma_1}\}$.

$$\begin{aligned}[(0,0)]_{\gamma_2}&=\{(0,0),(0,b),(b,0),(b,b)\}\\ [(0,1)]_{\gamma_2}&=\{(0,1),(0,a),(b,1),(b,a)\}\\ [(1,0)]_{\gamma_2}&=\{(1,0),(1,b),(a,0),(a,b)\}\\ [(1,1)]_{\gamma_2}&=\{(1,1),(1,a),(a,1),(0,a)\}.\end{aligned}$$

Thus, we have $\xi_{\gamma_2}=\{[(0,0)]_{\gamma_2},[(0,1)]_{\gamma_2},[(1,0)]_{\gamma_2},[(1,1)]_{\gamma_2}\}$.
Therefore, we obtain

$$\begin{aligned}N_1(B)^*(R\times R)=\{&(0,0)(0,1),(1,0),(0,a),(a,0),(1,a),(a,1),(a,a),\\&(a,b),(b,a),(0,b),(b,0),(1,b),(b,1),(b,b)\}.\end{aligned}$$

Hence, $N_r(B)^*(R\times R)=N_r(B)^*(R)\times N_r(B)^*(R)$, *so* $R\times R$ *is a near Krasner hyperring.*

Theorem 3.13. (1) *If* $|B|_{\mathcal{O}_1}=|B|_{\mathcal{O}_2}$ *and* $N_r(B)^*R\times N_r(B)^*R'=N_r(B)^*(R\times R')$, *then* $R\times R'$ *is a commutative near Krasner hyperring where* R,R' *are two commutative near Krasner hyperrings.*

(2) *If* $|B|_{\mathcal{O}_1}=|B|_{\mathcal{O}_2}$ *and* $N_r(B)^*R\times N_r(B)^*R'=N_r(B)^*(R\times R')$, *then* $R\times R'$ *is a near Krasner hyperring with near unite element where* R,R' *are two near Krasner hyperrings with near unite elements.*

Proof. (1) We assume $m_1,m_2\in R$ and $n_1,n_2\in R'$. Since $m_1\otimes m_2=m_2\otimes m_1$ and $n_1\odot n_2=n_2\odot n_1$, we have

$$(m_1,n_1)(m_2,n_2)=(m_1\otimes m_2,n_1\odot n_2)=(m_2\otimes m_1,n_2\odot n_1)=(m_2,n_2)(m_1,n_1).$$

Therefore, $R\times R'$ is a commutative near Krasner hyperring.
(2) Whereas R,R' are two near Krasner hyperrings on NAS with near unite elements respectively $1_{R_1},1_{R'}$, we have $1_R\otimes m_1=m_1$ and $1_{R'}\odot m_2=m_2$ for every $m_1\in R,m_2\in R'$. Thus,

$$(1_R, 1_{R'})(m_1, m_2) = (1_{R_1} \otimes m_1, 1_{R'} \odot m_2) = (m_1, m_2).$$

So, $R \times R'$ is a near Krasner hyperring with near unite element $(1_R, 1_{R'})$. □

In the following, we present an example of Theorem 3.13.

Example 3.14.

(1) *In Example 3.12, we prove $(R\times R, +, .)$ is a near Krasner hyperring. On the other hand, $(R, \boxtimes)$ is a commutative, that means of $x, y \in R$, $x \boxtimes y = y \boxtimes x$. Now, for every $(a, b), (c, d) \in R \times R$, we obtain $(a, b).(c, d) = (c, d).(a, b)$. For example,*

$$(0, 1).(0, a) = (0, a) = (0, a).(0, 1),$$
$$(1, 1).(a, 0) = (a, 0) = (a, 0).(1, 1).$$

Therefore, $R \times R$ is a commutative near Krasner hyperring.

(2) *R is a near Krasner hyperring with near unite element 1. Thus, for every $x \in R$, we get $1 \otimes x = x \otimes 1 = x$. So, for any $(x, y) \in R \times R$ and $(1, 1) \in N_r(B)^*(R \times R)$, we obtain $(x, y).(1, 1) = (x, y)$. Thus $(1, 1)$ is a near unite element $R \times R$.*

Corollary 3.15.

(1) *If $N_r(B)^*(\prod_{i\in\Gamma} R_i) = \prod_{i\in\Gamma} N_r(B)^* R_i$, then $\prod_{i\in\Gamma} R_i$ is a near Krasner hyperring, where R_i are near Krasner hyperrings.*

(2) *If R_i are commutative near Krasner hyperrings on NAS, then $\prod_{i\in\Gamma} R_i$ is a commutative near Krasner hyperring.*

(3) *If R_i are near Krasner hyperrings with identity on NAS, then $\prod_{i\in I} R_i$ is a near Krasner hyperring with identity.*

Proof. It is straightforward. □

Definition 3.16. *Suppose $M \subseteq R$. Then M is said be to a near subhyperring of R. If $m_1 - m_2 \subseteq N_r(B)^*M$ and $m_1 \otimes m_2 \in N_r(B)^*M$ for every $m_1, m_2 \in M$.*

Example 3.17. *In Example 3.3, let $M = \{0, 1, n\} \subseteq R$ and consider the following tables:*

$\oplus$	0	1	n
0	0	1	n
1	1	$\{0, m\}$	m
n	n	m	0

$\otimes$	0	1	n
0	0	0	0
1	0	1	n
n	0	n	1

We obtain $N_1(B)^*M = \{0, 1, m, n, p\}$. So, by Definition 3.16, M is a near subhyperring of R.

Also, we suppose $M' = \{n\} \subseteq R$ and $B = \{\Theta_1, \Theta_2, \Theta_3\}$. In this case, we show M' is not a near subhyperring of R. Consider the below tables .

	0	1	m	n	p
Θ_1	ρ_1	ρ_2	ρ_2	ρ_3	ρ_3
Θ_2	λ_1	λ_1	λ_1	λ_2	λ_2
Θ_3	γ_2	γ_1	γ_3	γ_3	γ_1

We obtain $N_1(B)^*M' = \bigcup_{[x]_{\Theta_i} \cap M' \neq \emptyset} [x]_{\Theta_i} = \{n, p, m\}$. *By Definition 3.16, we have*

$$n \oplus n = 0 \notin \{n, p, m\} = N_1(B)^*M'.$$

Thus, M' *is not a near subhyperring of* R.

Lemma 3.18. *If* M *is a subhyperring of* R*, then,* M *is a near subhyperring on NAS.*

Proof. By the definition of subhyperring, for every $a, b \in M$ we have $a - b \subseteq M$ and $a \otimes b \in M$. On the other hand, $M \subseteq N_r(B)^*(M)$. So, $a - b \subseteq N_r(B)^*M$ and $a \otimes b \in N_r(B)^*(M)$. □

In the following example we see that the converse of Lemma 3.18 is not true. In fact, we show that a near subhyperring is not a subhyperring.

Example 3.19. *In Example 3.17,* $M = \{0, 1, n\}$ *is a near subhyperring of* R*. But,* M *is not a subhyperring, because* $1 \oplus 1 = \{0, m\} \nsubseteq M$.

Theorem 3.20. *The intersection of two near subhyperring is a near subhyperring if* $(N_r(B)^*M_1) \cap (N_r(B)^*M_2) = N_r(B)^*(M_1 \cap M_2)$ *where* M_1, M_2 *are near subhyperrings of* R.

Proof. Suppose $m_1, m_2 \in M_1 \cap M_2$. Then $m_1, m_2 \in M_1$ and $m_1, m_2 \in M_2$. Since M_1, M_2 are two near subhyperrings, we have $m_1 - m_2 \subseteq N_r(B)^*M_1$ and $m_1 - m_2 \subseteq N_R(B)^*M_2$. By assumption, $N_r(B)^*M_1 \cap N_r(B)^*M_2 = N_r(B)^*(M_1 \cap M_2)$, $m_1 - m_2 \subseteq N_r(B)^*(M_1 \cap M_2)$.
Whereas M_1, M_2 are two near subhyperring, we get $m_1 \otimes m_2 \in N_r(B)^*M_1$ and $m_1 \otimes m_2 \in N_r(B)^*M_2$. Then $m_1 \otimes m_2 \in (N_r(B)^*M_1) \cap (N_r(B)^*M_2) = N_r(B)^*(M_1 \cap M_2)$. Therefore, $M_1 \cap M_2$ is a near subhyperring of R. □

Example 3.21. *Suppose that* $\mathcal{O} = \{0, 1, h, k, s, t, z, w\}$ *is a set*

$+$	0	1	h	k	s	t	z	w
0	0	1	h	k	s	t	z	w
1	1	1	$\{0,1,h,k\}$	1	t	t	$\{s,t,z,w\}$	t
h	h	$\{0,1,h,k\}$	h	h	z	$\{s,t,z,w\}$	z	z
k	k	1	h	0	w	t	z	s
s	s	t	z	w	$\{0,s\}$	$\{1,t\}$	$\{h,z\}$	$\{k,w\}$
t	t	t	$\{s,t,z,w\}$	t	$\{1,t\}$	$\{1,t\}$	$\mathcal{O}$	$\{1,t\}$
z	z	$\{s,t,z,w\}$	z	z	$\{h,z\}$	$\mathcal{O}$	$\{h,z\}$	$\{h,z\}$
w	w	t	$\{z,w\}$	s	$\{k,w\}$	$\{1,t\}$	$\{h,z\}$	$\{0,s\}$

$.$	0	1	h	k	s	t	z	w
0	0	s	h	k	s	t	z	w
1	0	s	h	k	0	1	h	k
h	0	h	s	k	0	h	t	k
k	0	0	0	0	0	0	0	0
s	0	0	0	0	s	s	s	s
t	0	1	h	k	s	t	z	w
z	0	h	t	k	s	z	t	w
w	0	0	0	0	s	s	s	s

$(\mathcal{O}, +, .)$ *is not a Krasner hyperring. Because*

$$(w+h)+w=\{0,h,z,s\}\neq\{h,z\}=w+(h+w)$$

Assume that $B=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$ *is a subset of* $\mathcal{F}$*, where* γ_i*'s are functions. We define* $\gamma_1:\mathcal{O}\to\{a,b,c\}$*,* $\gamma_2:\mathcal{O}\to\{a,c,d\}$*,* $\gamma_3:\mathcal{O}\to\{a,b,d\}$ *and* $\gamma_4:\mathcal{O}\to\{a,b,c,d\}$ *are given in the following tables:*

	0	1	h	k	s	t	z	w
γ_1	a	b	c	a	a	c	c	b
γ_2	a	c	a	c	d	a	d	d
γ_3	b	d	b	d	b	d	a	a
γ_4	c	d	a	b	c	a	d	d

Assume that $R=\{0,1,h,k,s\}$ *is a non-empty subset of* $\mathcal{O}$ *with a hyperoperation* "$+$" *and an operation* "$\cdot$" *defined in the following tables:*

$+$	0	1	h	k	s
0	0	1	h	k	s
1	1	1	$\{0,1,h,k\}$	1	t
h	h	$\{0,1,h,k\}$	h	h	z
k	k	1	h	0	w
s	s	t	z	w	$\{0,s\}$

$.$	0	1	h	k	s
0	0	s	h	k	s
1	0	s	h	k	0
h	0	h	s	k	0
k	0	0	0	0	0
s	0	0	0	0	s

$$\begin{aligned}
[0]_{\gamma_1} &= \{y\in\mathcal{O}\mid \gamma_1(0)=\gamma_1(y)=a\}=\{0,k,s\}=[k]_{\gamma_1}=[s]_{\gamma_1},\\
[1]_{\gamma_1} &= \{y\in\mathcal{O}\mid \gamma_1(1)=\gamma_1(y)=b\}=\{1,w\}=[w]_{\gamma_1},\\
[h]_{\gamma_1} &= \{y\in\mathcal{O}\mid \gamma_1(h)=\gamma_1(y)=c\}=\{h,t,z\}=[t]_{\gamma_1}=[z]_{\gamma_1}.
\end{aligned}$$

We obtain $\xi_{\gamma_1}=\{[0]_{\gamma_1},[1]_{\gamma_1},[h]_{\gamma_1}\}$.

$$\begin{aligned}
[0]_{\gamma_2} &= \{y\in\mathcal{O}\mid \gamma_2(0)=\gamma_2(y)=a\}=\{0,h,t\}=[h]_{\gamma_2}=[t]_{\gamma_2},\\
[1]_{\gamma_2} &= \{y\in\mathcal{O}\mid \gamma_2(1)=\gamma_2(y)=c\}=\{1,k\}=[k]_{\gamma_2},\\
[s]_{\gamma_2} &= \{y\in\mathcal{O}\mid \gamma_2(s)=\gamma_2(y)=d\}=\{s,z,w\}=[z]_{\gamma_2}=[w]_{\gamma_2}.
\end{aligned}$$

We get $\xi_{\gamma_2}=\{[0]_{\gamma_2},[1]_{\gamma_2},[s]_{\gamma_2}\}$.

$$\begin{aligned}
[0]_{\gamma_3} &= \{y\in\mathcal{O}\mid \gamma_3(0)=\gamma_3(y)=b\}=\{0,h,s\}=[h]_{\gamma_3}=[s]_{\gamma_3},\\
[1]_{\gamma_3} &= \{y\in\mathcal{O}\mid \gamma_3(1)=\gamma_3(y)=d\}=\{1,k,t\}=[k]_{\gamma_3}=[t]_{\gamma_3},\\
[z]_{\gamma_3} &= \{y\in\mathcal{O}\mid \gamma_3(s)=\gamma_3(y)=a\}=\{z,w\}=[w]_{\gamma_3}.
\end{aligned}$$

We get $\xi_{\gamma_3} = \{[0]_{\gamma_3}, [1]_{\gamma_3}, [z]_{\gamma_3}\}$.

$$\begin{aligned}
[0]_{\gamma_4} &= \{y \in \mathcal{O} \mid \gamma_4(0) = \gamma_4(y) = c\} = \{0, s\} = [s]_{\gamma_4},\\
[1]_{\gamma_4} &= \{y \in \mathcal{O} \mid \gamma_4(1) = \gamma_4(y) = d\} = \{1, z, w\} = [z]_{\gamma_4} = [w]_{\gamma_4},\\
[h]_{\gamma_4} &= \{y \in \mathcal{O} \mid \gamma_4(s) = \gamma_4(y) = a\} = \{h, t\} = [t]_{\gamma_4}\\
[k]_{\gamma_4} &= \{y \in \mathcal{O} \mid \gamma_4(s) = \gamma_4(y) = b\} = \{k\}.
\end{aligned}$$

We have $\xi_{\gamma_4} = \{[0]_{\gamma_4}, [1]_{\gamma_4}, [h]_{\gamma_4}, [k]_{\gamma_4}\}$. *Then, for* $r = 1$ *a partition of* $\mathcal{O}$ *is* $N_1 = \{\xi_{\gamma_1}, \xi_{\gamma_2}, \xi_{\gamma_3}, \xi_{\gamma_4}\}$. *Therefore, we get* $N_r(B)^*R = \{0, 1, h, k, s, t, z, w\} = \mathcal{O}$. *Thus, by Definition 3.1,* $(R, +, \cdot)$ *is a near krasner hyperring.*

Consider that $R' = \{0, h, s\}$ *and* $R'' = \{0, 1, s\}$ *are two subsets of* R. *It is easy to see* R' *and* R'' *are two near subhyperrings of* R. *We obtain*

$$\begin{aligned}
&N_r(B)^*(R' \cap R'') = N_r(B)^*(\{0, s\}) = \{0, h, k, s, t, z, w\},\\
&N_r(B)^*(R') = \{0, k, s, h, z, t, w\},\\
&N_r(B)^*(R'') = \mathcal{O}.
\end{aligned}$$

Also, $N_r(B)^*(R' \cap R'') = N_r(B)^*(R') \cap N_r(B)^*(R'')$. *Thus* $R' \cap R''$ *is a near subhyperring of* R.

Corollary 3.22. *The intersection of a family of near subhyperrings is a near subhyperring if* $N_r(B)^*\big(\bigcap_{i \in \Gamma} M_i\big) = \bigcap_{i \in \Gamma} \big(N_r(B)^* M_i\big)$, *where* M_i *are near subhyperrings.*

Proof. It is straightforward. □

Theorem 3.23. $M_1 \times M_2$ *is a near subhyperring of* $R \times R'$ *if* $|B|_{\mathcal{O}_1} = |B|_{\mathcal{O}_2}$ *and*

$$N_r(B)^*(M_1 \times M_2) = N_r(B)^*(M_1) \times N_r(B)^*(M_2),$$

where M_1, M_2 *are two near subhyperrings of* R, R', *respectively.*

Proof. Let $(m_1, n_1), (m_2, n_2) \in M_1 \times M_2$. Then, by Definition 3.16, we get

$$\begin{aligned}
(m_1, n_1) - (m_2, n_2) &= \{(m_1 - m_2, n_1 - n_2) \mid m_1, m_2 \in M_1, n_1, n_2 \in M_2\}\\
&\subseteq N_r(B)^* M_1 \times N_r(B)^* M_2 = N_r(B)^*(M_1 \times M_2).\\
(m_1, n_1).(m_2, n_2) &= (m_1.m_2, n_1.n_2) \in N_r(B)^*(M_1) \times N_r(B)^*(M_2)\\
&= N_r(B)^*(M_1 \times M_2).
\end{aligned}$$

Therefore, $M_1 \times M_2$ is a near subhyperring of $R \times R'$. □

Example 3.24. *In Example 3.17, let* $M = \{0, 1\} \subseteq R$. *Then, we obtain* $N_r(B)^*M = \{0, 1, a, b\}$. *So, by Definition 3.16,* M *is a near subhyperring. Thus,* $M \times M = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

$$\begin{aligned}
N_r(B)^*(M) \times N_r(B)^*(M) &= \{0, 1, a, b\} \times \{0, 1, a, b\}\\
&= \{(0, 0), (0, 1), (0, a), (0, b), (1, 0), (1, 1), (1, a), (1, b),\\
&\quad (a, 0), (a, 1), (a, a), (a, b), (b, 0), (b, 1), (b, a), (b, b)\}.
\end{aligned}$$

On the other hand we see in Example 3.12, $\xi_{\gamma_1} = \{[(0, 0)]_{\gamma_1}, [(0, 1)]_{\gamma_1}, [(1, 0)]_{\gamma_1}, [(1, 1)]_{\gamma_1}\}$ *and* $\xi_{\gamma_2} = \{[(0, 0)]_{\gamma_2}, [(0, 1)]_{\gamma_2}, [(1, 0)]_{\gamma_2}, [(1, 1)]_{\gamma_2}\}$. *Therefore,*

$$\begin{aligned}
N_r(B)^*(M \times M) = \{&(0, 0)(0, 1), (1, 0), (0, a), (a, 0), (1, a), (a, 1), (a, a),\\
&(a, b), (b, a), (0, b), (b, 0), (1, b), (b, 1), (b, b)\}.
\end{aligned}$$

Thus, $N_R(B)^*(M \times M) = N_r(B)^*M \times N_r(B)^*M$. *So,* $M \times M$ *is a near subhyperring of* $R \times R$.

Corollary 3.25. $\prod_{i\in\Gamma} M_i$ *is a near subhyperring* $\prod_{i\in\Gamma} R_i$ *if* $N_r(B)^*\big(\prod_{i\in\Gamma} M_i\big) = \prod_{i\in\Gamma}\big(N_r(B)^* M_i\big)$*, where that* M_i *is a near subhyperring of* R_i*.*

Proof. It is straightforward. □

Let A be a non-empty subset of near Krasner hyperring R. Then, we defined

$$r \otimes A = \Big\{ \sum_{i=1}^{n} r \otimes a_i \mid a_i \in A \Big\},$$

for $r \in R$.

Lemma 3.26. *For each* $a \in R$*,* $a \otimes R$ *is a right near subhyperring of* R *.*

Proof. Suppose $x \in \sum_{finit} a \otimes x_i$ and $y \in \sum_{finit} a \otimes y_i$ for x_i, y_i in R. Thus,

$$x - y \subseteq \sum_{finit} a \otimes x_i - \sum_{finit} a \otimes y_i = a \otimes \sum_{finit} z_i \subseteq a \otimes (N_r(B)^* R),$$

for all $z_i \in R$. There exists $w \in N_r(B)^* R$ such that $x - y = a \otimes w$ for any $a \in R$. Then $[w]_{B_r} \cap R \neq \emptyset$, and so $p \in [w]_{B_r}$ and $p \in R$. We get $w \sim_{B_r} p$ and $p \in R$. Therefore, $a \otimes w \sim_{B_r} a \otimes p$. Hence, $a \otimes p \in [a \otimes w]_{B_r}$ and $a \otimes p \in a \otimes R$, then $[a \otimes w]_{B_r} \cap a \otimes R \neq \emptyset$. We obtain $x - y = a \otimes w \in N_r(B)^*(a \otimes R)$. Now, we assume $x \in a \otimes R$ and $y \in R$. Consider

$$x \otimes y \in \Big(\sum_{finit} a \otimes r\Big) \otimes y = \sum_{finit} a \otimes (r \otimes y) = a \otimes \sum_{finit} r \otimes y \subseteq a \otimes N_r(B)^*(R),$$

then there exists $z \in N_r(B)^* R$ such that $x \otimes y = a \otimes z$ for every $y \in R$. Therefore, $[z]_{B_r} \cap R \neq \emptyset$, hence $c \in [z]_{B_r}, c \in R$ and $c \sim_{B_r} z$. We have $a \otimes c \sim_{B_r} a \otimes z, c \in R$, $a \otimes z \in [a \otimes c]_{B_r}$ and $a \otimes z \in a \otimes R$. Therefore, $[a \otimes z]_{B_r} \cap (a \otimes R) \neq \emptyset$, so we have $x \otimes y = a \otimes z \in N_r(B)^*(a \otimes R)$. We get $(a \otimes R) \otimes R \subseteq N_r(B)^*(a \otimes R)$. □

4 Near hyperideals and near prime hyperideals

In the following, we introduce the idea of near (prime) hyperideal and investigate some results. This section presents definitions and theorems for left near hyperideals. These definitions and theorems are also true for near hyperideals. Also, it is true for right near hyper ideals.

Definition 4.1. *Suppose that* R *is a near Krrasner hyperring on NAS and* $M \subseteq R$*. Then,* M *is said to be a left (right) near hyperideal of* R *if* $\alpha - \beta \subseteq N_r(B)^* M$*, and*

$$r' \otimes \alpha \in N_r(B)^* M \ \ (\alpha - \beta \subseteq N_r(B)^* M, \alpha \otimes r' \in N_r(B)^* M),$$

for every $\alpha, \beta \in M$ *and* $r' \in R$*.*

If M *is both a left and a right near hyperideal, then* M *is called a near hyperideal on* R*.*

Example 4.2. *In Example 3.3, suppose that* $I = \{m, n\} \subseteq R$ *and* $B = \{\varphi_1, \varphi_2, \varphi_3\}$*. Consider the following tables:*

$\oplus$	m	n
m	$\{0,m\}$	1
n	1	0

$\otimes$	m	n
m	m	n
n	n	m

So, we obtain $N_1(B)^*(I) = \bigcup\limits_{[x]_{\varphi_i}\cap I\neq\emptyset} [x]_\varphi = \{0,1,m,n,p\}$. *In the sense of Definition 4.1, we get*

$$m - m = \{0, m\}, \quad n - n = 0, \quad m - n = n - m = 1.$$

They are all subsets of $N_1(B)^*(I)$. *For every* $r \in R, x \in I$, *we get* $r \otimes x \in N_1(B)^*(I)$. *Therefore,* I *is a near hyperideal of* R.

Lemma 4.3. *Every near hyperideal is a near subhyperring of* R.

Remark 4.4. *The converse of Lemma 4.3 is not correct, see the following example.*

Example 4.5. *In Example 3.21, we show* $R' = \{0, h, s\}$ *is a near subhyperring of* R *but* R' *is not a near hyperideal of* R. *Because, we suppose* $x = h \in R'$, *we have* $h - h = h + (-h) = h + 1 = \{0, 1, h, k\} \nsubseteq N_r(B)^*(R') = \{0, k, s, h, t, z, w\}$.

Theorem 4.6. *Suppose that* I_1, I_2 *are two near hyperideals of* R.

(1) *If* $N_r(B)^*(I_1 \cap I_2) = N_r(B)^*(I_1) \cap N_r(B)^*(I_2)$, *then* $I_1 \cap I_2$ *is a near hyperideal.*

(2) *Union of two near hyperideals is a near hyperideal if*

$$N_r(B)^*(I_1 \cup I_2) = N_r(B)^*(I_1) \cup N_r(B)^*(I_2).$$

Proof.

(1) Assume $\alpha, \beta \in I_1 \cap I_2$. Then $\alpha, \beta \in I_1$ and $\alpha, \beta \in I_2$. By Definition 4.1, we have $\alpha - \beta \subseteq N_r(B)^*(I_1)$ and $\alpha - \beta \subseteq N_R(B)^*(I_2)$. Thus, $\alpha - \beta \subseteq N_r(B)^*(I_1) \cap N_r(B)^*(I_2)$. By hypothesis, $\alpha - \beta \subseteq N_r(B)^*(I_1 \cap I_2)$. Also, we suppose $r \in R$ and $\alpha \in I_1 \cap I_2$. Therefore, $r \otimes \alpha \in N_r(B)^*(I_1)$ and $r \otimes \alpha \in N_r(B)^*(I_2)$. By hypothesis, we have $r \otimes \alpha \in N_r(B)^*(I_1 \cap I_2)$. We prove $I_1 \cap I_2$ is a near hyperideal of R.

(2) The proof is straightforward. □

Example 4.7. 1. *In Example 4.2, we see* $I = \{m, n\}$ *is a near hyperideal of* R *and* $N_r(B)^*(I) = \{0, 1, m, n, p\}$. *Also,* $I' = \{0, m\}$ *is a near hyperideal of* R. *We give* $N_r(B)^*(I') = \{0, 1, m\}$. *Then* $N_r(B)^*(I) \cap N_r(B)^*(I') = \{0, 1, m\}$, *on the other hand* $N_r(B)^*(I \cap I') = N_r(B)^*(\{m\}) = \{0, 1, m\}$. *So,* $I \cap I'$ *is a near hyperideal of* R.
2. *We obtain* $N_r(B)^*(I \cup I') = N_r(B)^*(\{0, m, n\}) = \{0, 1, m, n, p\}$. *On the other hand,* $N_r(B)^*(I) \cup N_r(B)^*(I') = \{0, 1, m, n, p\}$. *Therefore,* $I \cup I'$ *is a near hyperideal of* R.

Corollary 4.8. *Suppose that* $\{I_i\}_{i\in\Lambda}$ *is a non-empty family of near hyperideals of* R. *Then*

(1) *Intersection of near hyperideals is a near hyperideal if* $N_r(B)^*\big(\bigcap\limits_{i\in\Lambda} I_i\big) = \bigcap\limits_{i\in\Lambda} \big(N_r(B)^* I_i\big)$.

(2) *Union of near hyperideals is a near hyperideal of* R.

Proof. It is straightforward. □

Definition 4.9. *Assume that N is a non-empty subset R. Then N is said to be a near left (right, two sided) hyperideal of R if $N_r(B)^*N$ is a left (right, two side) hyperideal of R.*

Example 4.10. *In Example 3.3, we show that $(R, \oplus, \otimes)$ is a near Krasner hyperring, where $R = \{0, 1, m, n\}$. Let $N = \{0, m, n\}$ be a non-empty subset of R. We obtain $N_r(B)^*(N) = \{0, 1, m, n, p\}$. Therefore, $N_r(B)^*N$ is a hyperideal of R. Because, for every $x, y \in N_r(B)^*(N)$, we have $x - y \subseteq N_r(B)^*N$ and for every $r \in R$, we have $r \otimes x \in N_r(B)^*N$. For example*

$$0 - 0 = 0,\ 0 - m = m,\ 0 - n = n,$$
$$m - 0 = m,\ m - m = \{0, m\},\ m - n = 1.$$

Also, $1 \otimes m = m \in N_r(B)^*N$, $m \otimes n = n \in N_r(B)^*N$, etc. It is easily prove that N is a hyperideal of R.

Proposition 4.11. *Suppose that M and N are a near subhyperring and a near hyperideal of R, respectively. Then*

(1) *$M \oplus N$ is a near subhyperring of R.*

(2) *$M \cap N$ is a near hyperiseal of M.*

Proof.

1. Suppose $m, n \in M \oplus N$. Then there exist $m', m'' \in M$, and $n', n'' \in N$ such that $m \in m' \oplus n'$ and $n \in m'' \oplus n''$. Therefore,

$$m - n \subseteq (m' + n') - (m'' \oplus n'') = (m' - m'') \oplus (n' - n'') \subseteq N_r(B)^*M \oplus N_r(B)^*N \subseteq N_r(B)^*(M \oplus N).$$

Now, we have

$$\begin{aligned} m \otimes n \in (m' \oplus n') \otimes (m'' \oplus n'') & \\ &\subseteq m'm'' \oplus m'n'' \oplus m''n' \oplus n'n'' \\ &= m'm'' \oplus (m'n'' \oplus m''n' \oplus n'n'') \\ &\subseteq N_r(B)^*M \oplus N_r(B)^*N \\ &\subseteq N_r(B)^*(M \oplus N), \end{aligned}$$

since N is a near hyperideal of R. Consequently $M \oplus N$ is a near subhyperring of R.

2. Let $m, n \in M \cap N$ implies $m, n \in M$ and $m, n \in N$. Since M is a near subhyperring and N is a near hyperideal, we get $m - n \subseteq N_r(B)^*M$ and $m - n \subseteq N_r(B)^*N$, which implies $m - n \subseteq N_r(B)^*M \cap N_r(B)^*N = N_r(B)^*(M \cap N)$.

Let $m' \in M$. Then $m' \otimes n \in N_r(B)^*M$. Also, $m' \otimes n \in N_r(B)^*N$, since N is a near hyperideal and M is a near subhyperring. Hence, $M \cap N$ is a near hyperideal of M. □

Definition 4.12. *A near hyperideal P is called a near prime hyperideal of R if $a \otimes b \in N_r(B)^*P$. Then $a \in P$ or $b \in P$, for each near hyperideal a, b of R.*

Example 4.13. *Suppose that $\mathcal{O} = \{0, a, b, c, d, e\}$ with a hyperoperation "$+$" and an operation "$\bullet$" defined by the below tables.*

$+$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	$\{0,a\}$	$\{a,b,d\}$	$\{a,c\}$	$\{0,a,b,d\}$	$\{a,e\}$
b	b	$\{a,b\}$	$\{0,b\}$	$\{b,c\}$	$\{b,d\}$	$\{b,d,e\}$
c	c	$\{a,c\}$	$\{b,c\}$	$\{0,c\}$	$\{c,d\}$	$\{c,d,e\}$
d	d	$\{a,d\}$	$\{b,d\}$	$\{c,d\}$	$\{0,d\}$	$\{e,d\}$
e	e	$\{a,d\}$	$\{b,d,e\}$	$\{c,d,e\}$	$\{d,e\}$	$\{0,e\}$

$\bullet$	0	a	b	c	d	e
0	0	0	0	0	a	0
a	0	a	0	b	c	e
b	0	0	0	0	0	0
c	0	c	0	0	e	e
d	0	a	0	c	e	e
e	0	e	0	0	e	e

$(\mathcal{O},+,\bullet)$ is not a hyperring, because $(\mathcal{O},+)$ is not associative, for instance

$$(a+d)+b=\{0,b,d\}\neq\{0,a,b,d\}=a+(b+d).$$

Assume that $B=\{\varphi_1,\varphi_2\}$ is a subset of $\mathcal{F}$, where φ_i's are functions. Suppose that $\varphi_1:\mathcal{O}\to\{1,2,3\}$ and $\varphi_2:\mathcal{O}\to\{1,2\}$ are given in the following table:

	0	a	b	c	d	e
φ_1	1	2	3	3	2	1
φ_2	2	2	1	2	1	1

We obtain

$$[0]_{\varphi_1}=\{0,c\}=[c]_{\varphi_1},\ [a]_{\varphi_1}=\{a,d\},[b]_{\varphi_1}=[e]_{\varphi_1}=\{e,b\},$$
$$[0]_{\varphi_2}=\{0,a,\}=[a]_{\varphi_2},\ [c]_{\varphi_2}=[b]_{\varphi_2}=\{b,c\},\ [d]_{\varphi_2}=[e]_{\varphi_2}.$$

Suppose that $R=\{0,a,b,c,e\}$ is a non-empty subset of $\mathcal{O}$ with a hyperoperation "$+$" and an operation "$\bullet$" defined in the following tables:

$+$	0	a	b	c	e
0	0	$\{0,a\}$	b	c	e
a	0	a	0	b	e
b	b	$\{a,b\}$	$\{0,b\}$	$\{b,c\}$	$\{b,d,e\}$
c	c	$\{a,c\}$	$\{b,c\}$	$\{0,c\}$	$\{c,d,e\}$
e	e	$\{a,d\}$	$\{b,d,e\}$	$\{c,e\}$	$\{0,e\}$

$\bullet$	0	a	b	c	e
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	0
c	0	0	0	e	e
e	0	0	0	0	e

It is see that $(R, +, \bullet)$ is not a hyperring. Because $b + e = \{b, d, e\} \nsubseteq R$. We have $N_r(B)^*R = \{0, a, b, c, d, e\}$. By Definition 3.1, $(R, +, \bullet)$ is a near Krasner hyperring on NAS. Let $P = \{0, c\} \subseteq R$. We get $N_r(B)^*P = \{0, a, c\}$. By Definition 4.12, P is a near prime hyperideal of R.

Remark 4.14. *Every prime hyperideal is not a near prime hyperideal of R. See Example 4.15.*

Example 4.15. *In Example 3.21, $(R, +, \cdot)$ is a near krasner hyperring where $R = \{0, 1, s, h, k\}$. Now, Assume that $P = \{0, s, k, h\}$. Then P is a prime hyperideal of R, but P is not a near prime hyperideal. We obtain $N_r(B)^*(P) = \{0, 1, s, h, k, t, w, z\}$. On the other hand, $z.t = z \in N_r(B)^*(P)$, but $z \notin P$ and $t \notin P$.*

.	0	h	k	s
0	0	h	k	s
h	0	s	k	0
k	0	0	0	0
s	0	0	0	s

In Example 4.16, we show that if P_1 and P_2 are two near prime hyperideals of R, then $P_1 \cap P_2$ is not a near prime hyperideal of R.

Example 4.16. *In Example 4.13, we see that $P = \{0, c\}$ is a near prime hyperideal of R. Suppose $P' = \{b, c\}$. Then $N_r(B)^*P' = \{b, c, e\}$. It is easily seen that P' is a near prime hyperideal. We obtain $N_r(B)^*P \cap N_r(B)^*P' = \{0, a, c\} \cap \{b, c, e\} = \{c\}$. On the other hand, $N_r(B)^*(P \cap P') = N_r(B)^*(B)\{c\} = \{0, a, c\}$.*

Theorem 4.17. *Suppose that P_1 and P_2 are two near prime hyperideals of R and $N_r(B)^*(P_1 \cap P_2) = N_r(B)^*(P_1) \cap N_r(B)^*(P_2)$. Then $P_1 \cap P_2$ is a near prime hyperideal of R if $P_1 \cap P_2$ is a near prime hyperideal of $P_1 \cup P_2$.*

Proof. By Theorem 4.6, we show $P_1 \cap P_2$ is a near hyperideal of R. Assume $a \otimes b \in N_r(B)^*(P_1 \cap P_2)$. Because $N_r(B)^*(P_1 \cap P_2) = N_r(B)^*(P_1) \cap N_r(B)^*(P_2)$, we have $a \otimes b \in N_r(B)^*(P_1) \cap N_r(B)^*(P_2)$. Thus, $a \otimes b \in N_r(B)^*(P_1)$ and $a \otimes b \in N_r(B)^*(P_2)$. Hence, P_1 and P_2 are two near prime hyperideals of R, we get $a \in P_1$ or $b \in P_1$. Also, $a \in P_2$ or $b \in P_2$. Consequently, $a, b \in P_1 \cup P_2$. As, $P_1 \cap P_2$ is a near prime hyperideal of $P_1 \cap P_2$, we obtaim $a \in P_1 \cap P_2$ or $b \in P_1 \cap P_2$. □

Theorem 4.18. *If P_1 and P_2 are two near prime hyperideals of R. Then $P_1 \cup P_2$ is a near prime hyperideal of R if $N_r(B)^*(P_1 \cup P_2) = N_r(B)^*(P_1) \cup N_r(B)^*(P_2)$.*

Proof. We suppose that $a \otimes b \subseteq N_r(B)^*(P_1 \cup P_2)$ for all $a, b \in R$. Because $N_r(B)^*(P_1 \cup P_2) = N_r(B)^*P_1 \cup N_r(B)^*P_2$, so $a \otimes b \subseteq N_r(B)^*P_1$ or $a \otimes b \subseteq N_r(B)^*P_2$. Hence, P_1 is a near prime hyperideal, we have $a \in P_1$ or $b \in P_1$. Also, P_2 is a near prime hyperideal, we obtain $a \in P_2$ or $b \in P_2$. Therefore, $a \in P_1 \cup P_2$ or $b \in P_1 \cup P_2$. Consequently, $P_1 \cup P_2$ is a near prime hyperideal of R. □

Example 4.19. *In Example 4.16, we show $P = \{0, c\}$ and P' are two near prime hyperideals. Also, we obtained $N_r(B)^*(P) = \{0, a, c\}$ and $N_r(B)^*(P') = \{b, c, e\}$. On the other hand, $N_r(B)^*(P) \cup N_r(B)^*(P') = \{0, a, b, c, e\}$ and $N_r(B)^*(P \cup P') = N_r(B)^*(\{0, b, c\}) = \{0, a, b, c, e\}$. Hence, $N_r(B)^*(P) \cup N_r(B)^*(P') = \{0, a, b, c, e\} = N_r(B)^*(P \cup P')$. Therefore, $P \cup P'$ is a near prime hyperideal.*

Corollary 4.20. *Assume that $\{P_i \mid i \in \Lambda\}$ is a near prime hyperideal of R. Then*

(1) *An intersection of near prime hyperideals is a near prime hyperideal, and*

$$N_r(B)^*(\bigcap_{i\in I} P_i) = \bigcap_{i\in I} N_r(B)^* P_i.$$

Then, $\bigcap_{i\in I} P_i$ *is a near prime hyperideal of* R*, if* $\bigcap_{i\in I} P_i$ *is a near prime hyperideal of* $\bigcup_{i\in I} P_i$.

(2) *A union of near prime hyperideals is also a near prime hyperideal if* $N_r(B)^*(\bigcup_{i\in I} P_i) = \bigcap_{i\in I} N_r(B)^* P_i$.

Proof. It is straightforward. $\square$

Theorem 4.21. *If* $X \oplus X' = \bigcup_{x\in X, x'\in X'} (x \oplus x')$*, then* $X \oplus X'$ *is a near hyperideal of* R *where* X *and* X' *are two near hyperideals.*

Proof. Let $x, x' \in X \oplus X'$. Then there exist $x_1, x_2 \in X$ and $x'_1, x'_2 \in X'$, such that $x \in x_1 \oplus x'_1$ and $y \in x_2 \oplus x'_2$. So, $x - y \subseteq (x_1 \oplus x'_1) - (x_2 \oplus x'_2)$, and by using associativity and commutativity of $(R, \oplus, \otimes)$. We have $x \oplus y \subseteq (x_1 - x_2) \oplus (x'_1 - x'_2)$. Since X, X' are near hyperideals of R, we get $x_1 - x_2 \subseteq N_r(B)^* X$ and $x'_1 - x'_2 \subseteq N_r(B)^* X'$. This implies $(x_1 - x_2) \oplus (x'_1 - x'_2) \subseteq N_r(B)^* A \oplus N_r(B)^* B$. By Theorem 3.9, $(x_1 - x_2) \oplus (x'_1 - x'_2) \subseteq N_r(B)^*(X \oplus X')$. Hence, $x - y \subseteq N_r(B)^*(X \oplus X')$.
Let $r \in R$ and $x \in X \oplus X'$. Then there exist $x_1 \in X$ and $x'_1 \in X'$ such that $x \in x_1 \oplus x'_1$. Consider $r \otimes x \in r \otimes (x_1 \oplus x'_1) = r \otimes x \oplus r \otimes x'_1$ by distributivity of R. Whereas X, X' are two near hyperideals of R, for each $x \in X, x_1 \in X'_1$ and $r \in R$, we obtain $r \otimes x \in N_r(B)^* X$ and $r \otimes x'_1 \in N_r(B)^* X'$ implies $(r \otimes x) \oplus (r \otimes x'_1) \subseteq N_r(B)^* X \oplus N_r(B)^* X'$. By Theorem 3.9(i), $r \otimes x \in N_r(B)^*(X \oplus X')$. $\square$

Theorem 4.22. *Suppose that* $p, q \in R$*. If* P *is a near right prime hyperideal of* R *such that* $N_r(B)^*(N_r(B)^* P) = N_r(B)^* P$*, then* $p \otimes R \otimes q \subseteq N_r(B)^* P$ *implies* $p \in P$ *or* $q \in P$*.*

Proof. Let $p \otimes R \otimes q \subseteq N_r(B)^* P$. We have $(p \otimes R \otimes q) \otimes R \subseteq (N_r(B)^* P) \otimes R \subseteq N_r(B)^* P$ in the sense of Theorem 3.9 (ii). Therefore, by Lemma 3.26, $p \otimes R$ and $q \otimes R$ are right near hyperideals of R and hence P is a right near prime hyperideal of R, $p \otimes R \subseteq P$ or $q \otimes R \subseteq P$. There exists $e \in N_r(B)^* R$ such that $r = e \otimes r$ for all $r \in R$. Therefore, either $p \in P$ or $q \in P$. $\square$

5 Near homomorphisme of a near Krasner hyperring

In this segment, we present the idea of near homomorphism and investigate some of its near hyperring homomorphism theorems. Also, we defined kernel of near hyperring homomorphism.

Definition 5.1. *The mapping* Γ *is from* $N_r(B)^* R$ *into* $N_r(B)^* R'$*. Then* Γ *said to be a near hyperring homomorphism for every* $\alpha, \beta \in N_r(B)^* R$,

$$\Gamma(\alpha \oplus \beta) = \Gamma(\alpha) \uplus \Gamma(\beta), \;\; \Gamma(\alpha \otimes \beta) = \Gamma(\alpha) \odot \Gamma(\beta) \text{ and } \Gamma(0_R) = 0_{R'}.$$

A near hyperring homomorphism Γ *said to be a near isomorphism if* Γ *is one-one and onto and we write* $R \simeq_n R'$.

Example 5.2. *In Example 3.8, we see $R = \{0, 1, a\}$ is a near krasner hyperring. We get $N_r(B)^*R = \{0, 1, a, b\}$. Let $R' = \{0, 1\}$ be a non-empty subset of $\mathcal{O}$. Then, $N_r(B)^*(R') = \{0, 1, a, b\}$. It is easily seen R' is a near Krasner hyperring.*

$\boxplus$	0	1	a
0	0	1	a
1	1	$\{0,1\}$	1
a	a	1	$\{0,a\}$

$\boxtimes$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	1

We define

$$\Gamma : N_r(B)^*R \to N_r(B)^*R'$$
$$\Gamma(0_R) = 0_{R'},\ \Gamma(1_R) = 1_{R'},\ \Gamma(a) = a,\ \Gamma(b) = b.$$

Then Γ is a near homomorphisme.

Lemma 5.3. *Suppose that $\Gamma : N_r(B)^*(R) \to N_r(B)^*(R')$ is a near homomorphism. Then $\Gamma(-\alpha) = -\Gamma(\alpha)$ for every α in R.*

Proof. Whoever α in R, we have $0_R \in \alpha - \alpha$. So, we get $\Gamma(0) \in \Gamma(\alpha) \uplus \Gamma(-\alpha)$ or $0_{R'} \in \Gamma(\alpha) \uplus \Gamma(-\alpha)$. We yields $\Gamma(-\alpha) \in -\Gamma(\alpha) \oplus 0$, therefore $\Gamma(-\alpha) = -\Gamma(\alpha)$ for every α in R. □

We write $\Gamma(X) = \{\Gamma(\alpha) : \alpha \in X\}$ where in X is a near subhyperring of R. Now, in following theorem, we show that $\Gamma(X)$ is a near subhyperring of R'.

Theorem 5.4. *Suppose that $\Gamma : N_r(B)^*R \to N_r(B)^*R'$ is a near homomorphism . Also, we assume that X is a near subhyperring of R, Then $\Gamma(X)$ is a near subhyperring of R', if $\Gamma(N_r(B)^*X) = N_r(B)^*\Gamma(X)$.*

Proof. We claim $\Gamma(X) \neq \emptyset$. Since $0 \in N_r(B)^*X$ and in the sense of Definition 5.1, $\Gamma(0_R) = 0_{R'}$. So, we have $0_{R'} = \Gamma(0_R) \in \Gamma(N_r(B)^*X) = N_r(B)^*\Gamma(X)$. This yields $\Gamma(X) \neq \emptyset$. Now, we assume $\Gamma(\alpha), \Gamma(\beta)$ in $\Gamma(X)$, where α, β in X. As X is a near subhyperring of R, we have $\alpha - \beta \subseteq N_r(B)^*X$. Wherefore, $\Gamma(\alpha) - \Gamma(\beta) = \Gamma(\alpha - \beta) \subseteq \Gamma(N_r(B)^*X) = N_r(B)^*\Gamma(X)$. Also, $\alpha \otimes \beta \in N_r(B)^*X$, hence $\Gamma(\alpha) \odot \Gamma(\beta) = \Gamma(\alpha \otimes \beta) \in \Gamma(N_r(B)^*X) = N_r(B)^*\Gamma(X)$. Consequently, in the sense of Definition 3.16, $\Gamma(X)$ is a near subhyperring of R'. □

Theorem 5.5. *Suppose that $\Gamma : N_r(B)^*R \to N_r(B)^*R'$ is a near homomorphism. Also, we assume that X is a near commutative subhyperring of a near Krasner hyperring R. Then $\Gamma(X)$ is a near commutative subhyperring of R' if $\Gamma(N_r(B)^*X) = N_r(B)^*\Gamma(X)$.*

Proof. In the sense of Theorem 5.4, $\Gamma(X)$ is a near subhyperring of R'. Whoever $\Gamma(\alpha), \Gamma(\beta)$ in $\Gamma(X)$, we have

$$\Gamma(\alpha) \odot \Gamma(\beta) = \Gamma(\alpha \otimes \beta) = \Gamma(\beta \otimes \alpha) = \Gamma(\beta) \odot \Gamma(\alpha).$$

So $\Gamma(X)$ is a commutative near subhyperring of R'. □

Theorem 5.6. *Assume that $\Gamma : N_r(B)^*R \to N_r(B)^*R'$ is a near homomorphism. Moreover, let K be a near hyperideal on R. Then $\Gamma(K)$ is a near hyperideal of R', if $\Gamma(N_r(B)^*K) = N_r(B)^*\Gamma(K)$.*

Proof. As K is a near hyperideal, we have $\alpha - \beta \subseteq N_r(B)^*K$ and $r \otimes \alpha \in N_r(B)^*K$ where α, β in K and r in R. Therefore, we get $\Gamma(\alpha) - \Gamma(\beta) = \Gamma(\alpha - \beta) \subseteq \Gamma(N_r(B)^*K) = N_r(B)^*\Gamma(K)$ and $\Gamma(r) \odot \Gamma(\alpha) = \Gamma(r \otimes \alpha) \subseteq \Gamma(N_r(B)^*K) = N_r(B)^*\Gamma(K)$ for every $\Gamma(\alpha), \Gamma(\beta)$ in $\Gamma(K)$. Then $\Gamma(K)$ is a near hyperideal of R'. □

Definition 5.7. *Suppose that* $\Gamma : N_r(B)^*R \to N_r(B)^*R'$ *is a near homomorphism. We denoted* $ker\Gamma = \{\alpha \in R \mid \Gamma(\alpha) = 0\}$.

Example 5.8. *In Example 5.2, we obtain* $Ker\Gamma = \{0\}$.

Theorem 5.9. *Suppose that* $\Gamma : N_r(B)^*R \to N_r(B)^*R'$ *is a near homomorphism where* R *and* R' *are two near Krasner hyperrings on NAS. Then* $Ker\Gamma \neq \emptyset$ *is a near hyperideal of* R.

Proof. By the definition of $Ker\Gamma$, we have $\Gamma(\alpha - \beta) = \Gamma(\alpha) - \Gamma(\beta) = 0 - 0 = 0 \in N_r(B)^*R'$ and $\alpha - \beta \subseteq N_r(B)^*(Ker\Gamma)$ for every α, β in $Ker\Gamma$ and r in R. Then $\Gamma(r \otimes \alpha) = \Gamma(r) \odot \Gamma(\alpha) = \Gamma(r) \odot 0 = 0 \in N_r(B)^*R'$ and $r \otimes \alpha \in N_r(B)^*(Ker\Gamma)$. Similarly, $\alpha \otimes r \in N_r(B)^*(Ker\Gamma)$. Hence, by Definition 4.1, $Ker\Gamma$ is a near hyperideal of R. □

6 Conclusion

We combined the notions of near sets and Krasner hyperrings to obtain a generalization of near rings. Some properties of this algebraic hyperstructure are drived and several examples are given. As a future work, we will focus on the other algebraic hyperstructures.

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References

[1] I.H. Bekmezci, *Gamma nearness semirings*, MSc. Thesie, Adiyaman University, Graduate School of Natural and Applied Sciences, Supervisor: M. A. Oztrk, 2019.

[2] R. Chinram, *Rough prime ideas and rough fuzzy prime ideals in gamma-semigroups*, Communications of the Korean Mathematical Society, 24(3) (2009), 341–351.

[3] P. Corsini, *Prolegomena of hypergroup theory*, Second Edition, Aviain Editor, 1993.

[4] P. Corsini, V. Leoreanu, *Applications of hyperstructure theory*, Kluwer Academic Publishers, 2003.

[5] B. Davvaz., *Rough sets in a fundamental ring*, Bulletin of the Iranian Mathematical Society, 24(2) (1998), 49–61.

[6] B. Davvaz, *Roughness in rings*, Information Sciences, 164 (2004), 147–163.

[7] B. Davvaz, V. Leoreanu-Fotea, *Hyperring theory and applications*, International Academic Press; Cambridge, MA,USA, 2007.

[8] E. Inan, M.A. Oztrk, *Near groups on nearness approximation spaces*, Hacettepe Journal of Mathematics and Statistics, 41(4) (2012), 545–558.

[9] E. Inan, M.A. Oztrk, *Erratum and notes for near groups on nearness approximation spaces*, Hacettepe Journal of Mathematics and Statistics, 43(2) (2014), 279–281.

[10] E. Inan, M.A. Oztrk, *Near semigroups on nearnes approximation spaces*, Annals of Fuzzy Mathematics and Informatics, 10(2) (2015), 287–297.

[11] J. Jun, *Algebraic geometry over hyperrings*, Advances in Mathematics, 323 (2018), 142–192.

[12] M. Krasner, *A class of hyperrings and hyperfields*, International Journal of Mathematics and Mathematical Sciences, 6(2) (1983), 307–311.

[13] N. Kuroki, *Rough ideals in semigroups*, Information Sciences, 100 (1997), 139–163.

[14] F. Marty, *Une generalization de la notion de group*, 8th 519 Congress Math Scandenaves, Stockholm, (1934), 45–49.

[15] J. Mittas, *Hypergroupes canoniques*, Math. Balk., 2 (1972), 165–179.

[16] M.A. Oztrk, E. Anan, *Nearness rings*, Annals of Fuzzy Mathematics and Informatics, 17(2) (2019), 115–131.

[17] M.A. Oztrk, Y. B. Jun, *Nobusawa gamma nearness semigroups*, New Mathematics and Natural Computation, 15(2) (2019), 373–394.

[18] M.A. Oztrk, M. Uckun, E. Inan, *Near groups of weak costes on nearness approximation spaces*, Fund Information, 133(4) (2014), 433–448.

[19] Z. Pawlak, *Classification of objects by means of attributes*, Institute for Computer Science, Polish Academy of Sciences, Tech. Rep. PAS 429, 1981.

[20] Z. Pawlak, *Rough sets*, International Journal of Computer Science and Information, 11(5) (1982), 341–356.

[21] J.F. Peters, *Near set, General theory about nearness of objects*, Applied Mathematical Sciences (Ruse), 1 (53-56) (2007), 2609–2629.

[22] J.F. Peters, *Classification of perceptual objects by means of features*, International Journal of Information Technology and Intelligent Computing, 2(2) (2008), 1–35.

[23] J.F. Peters, P. Wasilewski, *Foundations of near sets*, Information Sciences, 179 (2009), 3091–3109.

[24] R. Rota, *Sugli iperanelli moltiplicativi*, Rendiconti del Matematico, Series VII, 4(2) (1982), 711–724.

[25] V. Vahedi, M. Jafarpour, H. Aghabozorgi, I. Cristea, *Extension of elliptic curves on Krasner hyperfields*, Communications in Algebra, 47 (2019), 4806–4823.

[26] T. Vougiouklis, *The fundamental relation in hyperring, The general hyperfield*, In Proc. Fourth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1990); World Scientific: Singapore, (1991), 203–211.

[27] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press, Florida, 1994.