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On ringoids

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Abstract

In this paper, we introduce the notion of a ringoid, and we obtain left distributive ringoids over a field which are not rings. We introduce several different types of ringoids, and also we discuss on (r, s)-ringoids. Moreover, we discuss geometric observations of the parallelism of vectors in several ringoids.

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1 Introduction

The theory of groupoids [3, 4] has been introduced by some researchers. It has been combined with the theory of general algebraic structures [7, 9, 10]. One of the methods for the generalization of axioms is to employ special functions, i.e., by using of proper mappings, we may generalize axioms in mathematical structures. The notion of a linear groupoid has been applied to the Fibonacci sequences in groupoids. Using the notion of a flexibility, the linear groupoid was used to the study of the Fibonacci sequence in groupoids [5]. The notion of BCK-algebras was formulated by K. Iséki. The motivation of this notion is based on both set theory and propositional calculus (see [6, 8, 11]). Neggers and Kim introduced the notion of d-algebras which is a useful generalization of BCK-algebras, and they investigated several relations between d-algebras and BCK-algebras

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[12]. Allen et al. [1] developed a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type. Allen et al. [2] introduced the notion of deformation in d/BCK-algebras. Using such deformations they constructed d-algebras from BCK-algebras in such a manner as to maintain control over properties of the deformed BCK-algebras via the nature of the deformation employed, and observed that certain BCK-algebras cannot be deformed at all, leading to the notion of a rigid d-algebra, and consequently of a rigid BCK-algebra as well.

In this paper, we introduce new generalization of rings as ringoids, and discuss several properties of left distributive ringoids over a field which are not rings. Moreover, we will study several different types of ringoids. One of them is a notion of an (r, s)-ringoid, and present proper examples for (d-algebra, left zero, left distributive) ringoids and each (r, s)-ringoid. We will investigate geometric interpretations of the parallelism of vectors in several ringoids. It will be discussed along with several notions in linear algebras for further investigation.

2 Preliminaries

A *d-algebra* [12] is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A groupoid $(X, *)$ is said to be a *right zero semigroup* if $x * y = y$ for any $x, y \in X$, and a groupoid $(X, *)$ is said to be a *left zero semigroup* if $x * y = x$ for any $x, y \in X$.
Let $\mathbf{R}$ be the set of all real numbers. We define a binary operation "$*$" on $\mathbf{R}$ by

$$x * y := A + Bx + Cy,$$

for all $x, y \in \mathbf{R}$, where $A, B, C \in \mathbf{R}$. We call such a groupoid $(\mathbf{R}, *)$ is a *linear groupoid* [7, 9] over reals.

3 Ringoids

An algebra $(X, *, +, 0)$ of type $(2, 2, 0)$ is said to be a *ringoid* if it satisfies the following conditions:

(I) $(X, +, 0)$ is an abelian group,

(II) $(X, *)$ is a groupoid.

Example 3.1. *(i) Let $(\mathbf{R}, +, \cdot, 0, 1)$ be the field of real numbers. Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := x \cdot (x - y)$ for all $x, y \in \mathbf{R}$. Then $(\mathbf{R}, *, 0)$ is a d-algebra, but it is not an abelian group, since it does not contain the zero element. Hence $(\mathbf{R}, *, +, 0)$ is a ringoid, but it is neither a ring nor a recognized type of generalization of a ring such as semi-ring, near-ring, etc.*
(ii) Consider the abelian group $(\mathbf{R}, \cdot, 1)$. Define a binary operation "$*$" on $\mathbf{R}$ as follows:

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases}$$

Table 1: Groupoid $(\mathbf{Z_4},\, *)$

$*$	0	1	2	3
0	3	1	2	3
1	0	3	2	1
2	0	1	3	2
3	0	1	2	1

Then $(\mathbf{R},\, *,\, \cdot, 1\,)$ *is a ringoid.*
(iii) *Consider the abelian group* $(\mathbf{Z_4},\, +,\, 0\,)$*. Define a binary operation "$*$" on* $\mathbf{Z_4}$ *with the following Table 1. Then* $(\mathbf{Z_4},\, *,\, +,\, 0\,)$ *is a ringoid.*
(iv) *Consider the interval* $[0,1]$ *of real numbers. Define binary operations "$+$" and "$*$" on it as follows: for all* $x, y \in [0,1]$,

$$x + y = \begin{cases} x + y & \textit{if } x + y < 1, \\ x + y - 1 & \textit{if } x + y \geq 1, \end{cases}$$

and

$$x * y = \begin{cases} 1 & \textit{if } x = y = 0, \\ y^x & \textit{otherwise.} \end{cases}$$

Then $([0,1],\, *,\, +,\, 0\,)$ *is a ringoid.*

Clearly, every ring is a ringoid, but the converse need not be true in general. (See Theorem 4.4)

Given a ringoid $(X,\, *,\, +,\, 0\,)$, we consider its cartesian product X^n consisting of vectors $\overrightarrow{x} = (x_1, \cdots, x_n)$. Define two binary operations "$\oplus$" and "$\otimes$" on X^n by

$$\overrightarrow{x} \oplus \overrightarrow{y} := (x_1 + y_1, \cdots, x_n + y_n),$$

and

$$\overrightarrow{x} \otimes \overrightarrow{y} := (x_1 * y_1, \cdots, x_n * y_n),$$

for all $\overrightarrow{x} = (x_1, \cdots, x_n), \overrightarrow{y} = (y_1, \cdots, y_n) \in X^n$. We call two binary operations "$\oplus$" and "$\otimes$" a *natural vector addition* and *natural induced product* respectively.

The ringoid structure permits us to define a *natural scaler product*:

$$\overrightarrow{x} \star \overrightarrow{y} := x_1 * y_1 + \cdots + x_n * y_n,$$

and a *natural projection*:

$$\pi(\overrightarrow{x}) := x_1 + \cdots + x_n,$$

so that we obtain a general formula for all ringoids, viz., $\pi(\overrightarrow{x} \otimes \overrightarrow{y}) = \overrightarrow{x} \star \overrightarrow{y}$.

Note that $(X^n,\, \otimes,\, \oplus,\, \overrightarrow{0}\,)$ becomes again a ringoid, and also we have

$$\begin{aligned} \pi(\overrightarrow{x} \oplus \overrightarrow{y}) &= \pi(x_1 + y_1, \cdots, x_n + y_n) \\ &= (x_1 + y_1) + \cdots + (x_n + y_n) \\ &= (x_1 + \cdots + x_n) + (y_1 + \cdots + y_n) \\ &= \pi(\overrightarrow{x}) + \pi(\overrightarrow{y}), \end{aligned}$$

so that $\pi : (X^n,\, \oplus,\, \overrightarrow{0}\,) \to (X,\, +,\, 0\,)$ is in fact a group homomorphism of abelian groups and $Ker\pi := \{\overrightarrow{x} \mid \pi(\overrightarrow{x}) = 0\}$ is a subgroup of an abelian group $(X^n,\, \oplus,\, \overrightarrow{0}\,)$.

Since $\pi(\overrightarrow{x} \otimes \overrightarrow{y}) \neq \pi(\overrightarrow{x}) * \pi(\overrightarrow{y})$ in general, it fails to be a homomorphism of ringoids.

Example 3.2. *Consider the ringoid* $(\mathbf{R}, *, +, 0)$ *in Example 3.1(i). Since* $x * y = x \cdot (x - y)$ *for all* $x, y \in \mathbf{R}$*, we obtain* $(1, 2) \otimes (2, 7) = (-1, -10)$*. Thus*

$$\pi((1,2) \otimes (2,7)) = \pi(-1,-10) = -11 \neq \pi(1,2) * \pi(2,7) = 3 * 9 = -18.$$

The extent to which π fails to be a homomorphism of ringoids can be measured by looking at the subgroup of $(X, +, 0)$ generated by the expressions:

$$h : (X^n, \otimes, \oplus, \overrightarrow{0}) \times (X^n, \otimes, \oplus, \overrightarrow{0}) \longrightarrow (X, *, +, 0)$$

$$h(\overrightarrow{x}, \overrightarrow{y}) := \pi(\overrightarrow{x} \otimes \overrightarrow{y}) - \pi(\overrightarrow{x}) * \pi(\overrightarrow{y}).$$

Remark 3.3. *Two binary operations* $\otimes$ *and* $\star$ *are playing differently their roles of the commutativity in general.*

Example 3.4. *Consider the ringoid* $(\mathbf{R}, *, +, 0)$ *in Example 3.1(i). Since* $x * y = x \cdot (x - y)$ *for all* $x, y \in \mathbf{R}$*, we obtain* $(5,2) \otimes (\sqrt{29}, 0) = (25 - 5\sqrt{29}, 4) \neq (29 - 5\sqrt{29}, 0) = (\sqrt{29}, 0) \otimes (5, 2)$ *and* $(5,2) \star (\sqrt{29}, 0) = 29 - 5\sqrt{29} = (\sqrt{29}, 0) \star (5, 2)$.

A ringoid $(X, *, +, 0)$ is said to be a *left zero ringoid* if $(X, *)$ is a left zero semigroup.

Example 3.5. *Let* $(\mathbf{Q}, +, \cdot, 0, 1)$ *be the field of rational numbers. Define two binary operations "$*$" and "$\oplus$" on* $\mathbf{Q}$ *by* $x * y = x$ *and* $x \oplus y = x \cdot y$ *(resp.* $x \oplus y = x + y$*) for all* $x, y \in \mathbf{Q}$*. Then* $(\mathbf{Q}, *, \oplus, 1)$ *(resp.* $(\mathbf{Q}, *, \oplus, 0)$*) is a left zero ringoid.*

Proposition 3.6. *If* $(X, *, +, 0)$ *is a left zero ringoid, then* $(X^n, \otimes, \oplus, \overrightarrow{0})$ *is a left zero ringoid and the natural projection* π *is an epimorphism of ringoids.*

Proof. Since $(X, *, +, 0)$ is a left zero ringoid, $x * y = x$, for all $x, y \in X$. It follows that

$$\overrightarrow{x} \otimes \overrightarrow{y} = (x_1 * y_1, \cdots, x_n * y_n) = (x_1, \cdots, x_n) = \overrightarrow{x},$$

for all $\overrightarrow{x} = (x_1, \cdots, x_n), \overrightarrow{y} = (y_1, \cdots, y_n) \in X^n$. Hence, $(X^n, \otimes, \oplus, \overrightarrow{0})$ is a left zero ringoid. Moreover, we have

$$\begin{aligned} \pi(\overrightarrow{x} \otimes \overrightarrow{y}) &= \overrightarrow{x} \star \overrightarrow{y} \\ &= x_1 + \cdots + x_n \\ &= \pi(\overrightarrow{x}) \\ &= \pi(\overrightarrow{x}) * \pi(\overrightarrow{y}), \end{aligned}$$

proving the proposition. □

A ringoid $(X, *, +, 0)$ is said to be a *d-algebra ringoid* if $(X, *, 0)$ is a d-algebra.

Example 3.7. ([1]) *Consider the real number* $\mathbf{R}$*, and suppose that* $(\mathbf{R}, *, e)$ *has the multiplication*

$$x * y = (x - y)(x - e) + e.$$

Then $(\mathbf{R}, *, +, e)$ *is a d-algebra ringoid.*

Theorem 3.8. *If* $(X, *, +, e)$ *is a d-algebra ringoid, then*

(i) $(X^n, \otimes, \vec{e}\,)$ *is a d-algebra,*

(ii) $\vec{x} \star \vec{x} = \vec{e} \star \vec{x} = ne$,

(iii) *if* $e = 0$*, then* $h(\vec{x}, \vec{x}) = 0 = h(\vec{e}, \vec{x})$*, for all* $\vec{x} \in X^n$.

Proof. (i) Given $\vec{x} \in X^n$, we have

$$\vec{x} \otimes \vec{x} = (x_1, \cdots, x_n) \otimes (x_1, \cdots, x_n) = (x_1 * x_1, \cdots, x_n * x_n) = (e, \cdots, e) = \vec{e},$$

and

$$\vec{e} \otimes \vec{x} = (e, \cdots, e) \otimes (x_1, \cdots, x_n) = (e * x_1, \cdots, e * x_n) = (e, \cdots, e) = \vec{e}\ .$$

Assume $\vec{x} \otimes \vec{y} = \vec{y} \otimes \vec{x} = \vec{e}$. Then $(x_1 * y_1, \cdots, x_n * y_n) = (e, \cdots, e)$ and $(y_1 * x_1, \cdots, y_n * x_n) = (e, \cdots, e)$, and so $x_i * y_i = y_i * x_i = e$ for all $i = 1, \cdots, n$. Since $(X, *, e)$ is a d-algebra, we obtain $x_i = y_i$ for all $i = 1, \cdots, n$, i.e., $\vec{x} = \vec{y}$. Hence $(X^n, \otimes, \vec{e}\,)$ is a d-algebra.

(ii) Given $\vec{x} \in X^n$, we have

$$\vec{x} \star \vec{x} = x_1 * x_1 + \cdots + x_n * x_n = e + \cdots + e = ne,$$

and

$$\vec{x} \star \vec{x} = \pi(\vec{x} \otimes \vec{x}) = \pi(\vec{e}) = ne.$$

(iii) Given $\vec{x}, \vec{y} \in X^n$, we have

$$\begin{aligned} h(\vec{x}, \vec{y}) &= \pi(\vec{x} \otimes \vec{y}) - \pi(\vec{x}) * \pi(\vec{y}) \\ &= \pi((x_1 * y_1, \cdots, x_n * y_n)) - (x_1 + \cdots + x_n) * (y_1 + \cdots + y_n) \\ &= \sum_{i=1}^{n} x_i * y_i - \sum_{i=1}^{n} x_i * \sum_{i=1}^{n} y_i\,. \end{aligned}$$

It follows that

$$h(\vec{x}, \vec{x}) = \pi(\vec{x} \otimes \vec{x}) - \pi(\vec{x}) * \pi(\vec{x}) = \pi(\vec{e}) - e = ne - e = (n-1)e.$$

Similarly, we obtain

$$h(\vec{e}, \vec{x}) = \sum_{i=1}^{n} e * x_i - \sum_{i=1}^{n} e * \sum_{i=1}^{n} x_i = ne - ne * \sum_{i=1}^{n} x_i.$$

If we let $e := 0$, then we obtain $h(\vec{x}, \vec{x}) = h(\vec{0}, \vec{x}) = 0$. This proves the theorem. □

A groupoid $(X, *)$ is said to be a *leftoid* over a function $\varphi : X \to X$ if $x * y := \varphi(x)$ for all $x, y \in X$.

A ringoid $(X, *, +, 0)$ is said to be a *φ-leftoid ringoid* if $(X, *)$ is a leftoid over φ. A mapping $f : X^n \to X^n$ is said to be *additive* if $f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)$ for all $x_1, \cdots, x_n \in X$ where $n \geq 1$.

Theorem 3.9. *Let* $(X, *, +, 0)$ *be a φ-leftoid ringoid. If φ is additive, then* $h(\vec{x}, \vec{y}) = 0$ *for all* $\vec{x}, \vec{y} \in X^n$.

Proof. Given $\overrightarrow{x}= (x_1, \cdots, x_n), \overrightarrow{y}= (y_1, \cdots, y_n) \in X^n$, we have

$$\overrightarrow{x} \otimes \overrightarrow{y}= (x_1 * y_1, \cdots, x_n * y_n) = (\varphi(x_1), \cdots, \varphi(x_n)).$$

It follows from φ is additive that

$$\begin{aligned}
\pi(\overrightarrow{x} \otimes \overrightarrow{y}) &= \pi((\varphi(x_1), \cdots, \varphi(x_n))) \\
&= \varphi(x_1) + \cdots + \varphi(x_n) \\
&= \varphi(x_1 + \cdots + x_n) \\
&= (x_1 + \cdots + x_n) * (y_1 + \cdots + y_n) \\
&= \pi(\overrightarrow{x}) * \pi(\overrightarrow{y}).
\end{aligned}$$

This shows that $h(\overrightarrow{x}, \overrightarrow{y}) = \pi(\overrightarrow{x} \otimes \overrightarrow{y}) - \pi(\overrightarrow{x}) * \pi(\overrightarrow{y}) = 0$, proving the theorem. □

4 Distributive ringoids

A ringoid $(X, *, +, 0)$ is said to be

- *left distributive* if $x * (y + z) = (x * y) + (x * z)$,
- *right distributive* if $(x + y) * z = (x * z) + (y * z)$

for all $x, y, z \in X$. A ringoid $(X, *, +, 0)$ is said to be a *distributive ringoid* if it is both left distributive and right distributive.

Example 4.1. (i) *Every ring is a distributive ringoid.*
(ii) *Consider the abelian group* $(\mathbf{Z_2}, +, 0)$*. Define a binary operation "$*$" on* $\mathbf{Z_2}$ *with the following Table 2. Then* $(\mathbf{Z_2}, *, +, 0)$ *is a left distributive ringoid.*

Table 2: Groupoid $(\mathbf{Z_2}, *)$

$*$	0	1
0	0	1
1	0	0

(iii) *Consider the ringoid* $(\mathbf{Z_4}, *, +, 0)$ *in Example 3.1*(iii)*. It is not a left distributive, since*

$$2 * (1 + 3) = 2 * 0 = 0 \neq 2 * 1 + 2 * 3 = 1 + 2 = 3.$$

(iv) *The ringoid* $([0, 1], *, +, 0)$ *in Example 3.1*(iv) *is not a left distributive, since*

$$0.1 * (0.2 + 0.4) = 0.1 * 0.6 = 0.6^{0.2} \neq 0.1 * 0.2 + 0.1 * 0.4 = 0.2^{0.1} + 0.4^{0.1}.$$

(v) *The d-algebra ringoid in Example 3.7 is not a left distributive, since*

$$\begin{aligned}
2 * (5 + 3) &= 2 * 8 \\
&= (2 - 8)(2 - e) + e \\
&= -12 + 7e \\
&\neq 2 * 5 + 2 * 3 \\
&= (2 - 5)(2 - e) + e + (2 - 3)(2 - e) + e \\
&= -8 + 6e.
\end{aligned}$$

In this section, we assume that $(K, \cdot, +, 0, 1)$ is a field, and that $A, B, C, \alpha, \beta, \gamma \in K$. Define two binary operations "$*$" and "$\oplus$" on K by

$$\begin{aligned} x * y &:= A + Bx + Cy, & (1)\\ x \oplus y &:= \alpha + \beta x + \gamma y, & (2)\end{aligned}$$

for all $x, y \in K$. In this section, we discuss left and right distributive ringoids related to linear groupoids over a field K.

Lemma 4.2. *Let $(K, *, \oplus)$ be an algebra defined as (1) and (2). If it satisfies the left distributive law and $\beta + \gamma \neq 1$, then $x * y = \frac{\alpha(1-C)}{1-(\beta+\gamma)} + Cy$ and $x \oplus y = \alpha + \beta x + \gamma y$ for all $x, y \in K$.*

Proof. Given $x, y, z \in K$, we have

$$\begin{aligned} x * (y \oplus z) &= A + Bx + C(y \oplus z)\\ &= A + Bx + C(\alpha + \beta y + \gamma z)\\ &= (A + C\alpha) + Bx + C\beta y + C\gamma z, & (3)\end{aligned}$$

and

$$\begin{aligned} (x * y) \oplus (x * z) &= \alpha + \beta(x * y) + \gamma(x * z)\\ &= \alpha + \beta(A + Bx + Cy) + \gamma(A + Bx + Cz)\\ &= \alpha + A(\beta + \gamma) + B(\beta + \gamma)x + C\beta y + C\gamma z. & (4)\end{aligned}$$

Since the left distributive law holds, by (3) and (4), we obtain

$$\begin{aligned} A + C\alpha &= \alpha + (\beta + \gamma)A, & (5)\\ B &= B(\beta + \gamma). & (6)\end{aligned}$$

It follows that $A(1 - \beta - \gamma) = \alpha(1 - C)$ and $B(1 - \beta - \gamma) = 0$.
Since $\beta + \gamma \neq 1$, we obtain $B = 0$ and $A = \frac{\alpha(1-C)}{1-(\beta+\gamma)}$. Hence we obtain

$$x * y = \frac{\alpha(1 - C)}{1 - (\beta + \gamma)} + Cy, \tag{7}$$

$$x \oplus y = \alpha + \beta x + \gamma y, \tag{8}$$

proving the lemma. □

Example 4.3. *Let $(\mathbf{Q}, +, \cdot, 0, 1)$ be the field of rational numbers. Define two binary operations "$*_1$" and "$\oplus_1$" on $\mathbf{Q}$ by $x *_1 y = -\frac{5}{2} - y$ and $x \oplus_1 y = 5 + 2x + 3y$ for all $x, y \in \mathbf{Q}$. Then $(\mathbf{Q}, *_1, \oplus_1)$ is a left distributive. Now, if we define $x *_2 y = 1 + x - y$ and $x \oplus_2 y = 5 + 2x + 3y$ for all $x, y \in \mathbf{Q}$. Then $(\mathbf{Q}, *_2, \oplus_2)$ is not a left distributive, since*

$$2 *_2 (4 \oplus_2 1) = 2 *_2 16 = -13 \neq (2 *_2 4) \oplus_2 (2 *_2 1) = (-1) \oplus_2 2 = 9.$$

Theorem 4.4. *Let $(K, *, \oplus)$ be an algebra with (1) and (2). Then $(K, *, \oplus, -\alpha)$ is a left distributive ringoid if and only if*

$$\begin{aligned} x * y &= (C - 1)\alpha + Cy, & (9)\\ x \oplus y &= \alpha + x + y, & (10)\end{aligned}$$

for all $x, y \in K$.

Proof. Assume $(K, \oplus, \xi)$ be an abelian group. Then $x \oplus \xi = \xi \oplus x = x$ for all $x \in K$. It follows that

$$\alpha + \beta x + \gamma\xi = \alpha + \beta\xi + \gamma x = x. \tag{11}$$

for all $x \in K$. Hence $\beta x + \gamma\xi = \beta\xi + \gamma x$, and hence $\beta(x - \xi) = \gamma(x - \xi)$, i.e., $(\beta - \gamma)(x - \xi) = 0$ for all $x \in K$. This proves that $\beta = \gamma$. If we let $\beta = \gamma$ in (11), then

$$\alpha + \beta(x + \xi) = x. \tag{12}$$

for all $x \in K$. If we put $x := -\xi$ in (12), then we obtain $\alpha = -\xi$. Hence $x \oplus y = -\xi + \beta(x + y)$ for all $x, y \in K$. Now, for any $x \in K$, we have

$$\begin{aligned} x &= x \oplus \xi \\ &= -\xi + \beta(x + \xi) \\ &= \beta x + \xi(\beta - 1). \end{aligned}$$

It follows that $\beta = \gamma = 1$. Hence $x \oplus y = \alpha + x + y$ for all $x, y \in K$. Since $\beta + \gamma \neq 1$, we obtain $x * y = \frac{\alpha(1-C)}{1-(\beta+\gamma)} + Cy = -\alpha(1 - C) + Cy = (C - 1)\alpha + Cy$.

Conversely, since the conditions (9) and (10) are special cases of (7) and (8) respectively, the algebra $(K, *, \oplus)$ satisfies the left distributive law and $(K, \oplus, -\alpha)$ is an abelian group. Hence $(K, *, \oplus, -\alpha)$ is a left distributive ringoid. □

Remark 4.5. *The left distributive ringoid $(K, *, \oplus, -\alpha)$ discussed in Theorem 4.4 need not be a ring in general. It is enough to show that $(K, *)$ is not a semigroup. Given $x, y, z \in K$, we have $(x * y) * z = (C - 1)\alpha + Cz$ and $x * (y * z) = (C - 1)\alpha + C(y * z) = (C - 1)\alpha + C[(C - 1)\alpha + Cz] = (C + 1)(C - 1)\alpha + C^2 z$. If $C \neq 1$ or $\alpha \neq 0$, then $(K, *)$ is not a semigroup.*

Example 4.6. *Consider the left distributive algebra $(\mathbf{Q}, *_2, \oplus_2)$ in Example 4.3. Then $(\mathbf{Q}, *_2, \oplus_2, 0)$ is not a ringoid, since $2 \oplus_2 5 = 24 \neq 5 \oplus_2 2 = 21$ and so $(\mathbf{Q}, \oplus_2, 0)$ is not an abelian group. Now, if we define $x *_3 y = 2 + 3y$ and $x \oplus_3 y = 1 + x + y$, then by using Lemma 4.1, $(\mathbf{Q}, *_3, \oplus_3)$ is a left distributive algebra and by Theorem 4.3, $(\mathbf{Q}, *_3, \oplus_3, -1)$ is a left distributive ringoid, but not a ring, since*

$$(3 *_3 5) *_3 2 = 17 *_3 2 = 8 \neq 3 *_3 (5 *_3 2) = 3 *_3 8 = 26,$$

so $(\mathbf{Q}, *_3)$ is not a semigroup.

In Lemma 4.2 and Theorem 4.4, we have discussed the case of $\beta + \gamma \neq 1$. From now on, we discuss the case of $\beta + \gamma = 1$.

Theorem 4.7. *Let $(K, *, \oplus)$ be an algebra defined as (1) and (2). If $\beta + \gamma = 1$, then there is no left distributive ringoid over K.*

Proof. If we assume $\beta + \gamma = 1$, then, by (5) and (6), we obtain $\alpha(1 - C) = 0$ and B is arbitrary. Hence we have two cases: (i) $\alpha = 0$; (ii) $C = 1$. Consider $\alpha = 0$. Then $x \oplus y = \beta x + \gamma y = \beta x + (1 - \beta)y$. Assume that $(K, \oplus, \xi)$ is an abelian group with zero ξ. Then $x \oplus \xi = x$ for all $x \in K$, and hence $\beta x + (1 - \beta)\xi = x$, i.e., $(1 - \beta)(x - \xi) = 0$ for all $x \in K$. It follows that $\beta = 1$, which shows that $x \oplus y = x$. This means $(K, \oplus)$ is a left zero semigroup which is not a group, a contradiction.

Consider $C = 1$. Then $x \oplus y = \alpha + \beta x + (1-\beta)y$ and $x * y = A + Bx + y$ for all $x, y \in K$. Assume $(K, \oplus, \xi)$ is an abelian group with zero ξ. Then $x \oplus \xi = \xi \oplus x = x$ for all $x \in K$. It follows that

$$\alpha + \beta x + (1-\beta)\xi = \alpha + \beta\xi + (1-\beta)x, \tag{13}$$

and hence $(2\beta - 1)(x - \xi) = 0$ for all $x \in K$. We obtain $\beta = \frac{1}{2}$, and apply it to (13). Then

$$\alpha + \frac{1}{2}x + \frac{1}{2}\xi = x, \tag{14}$$

for all $x \in K$. If we let $x := \xi$, then $\alpha + \xi = \xi$, i.e., $\alpha = 0$. This $x \oplus y = \frac{1}{2}(x+y)$ for all $x, y \in K$. It follows that $x = x \oplus \xi = \frac{1}{2}(x + \xi)$, i.e., $2x = x + \xi$ for all $x \in K$. Hence $x = \xi$ for all $x \in K$, which proves that $|K| = 1$, a contradiction. This proves the theorem. □

Example 4.8. *Let $(\mathbf{R}, +, \cdot, 0, 1)$ be a field of real numbers. Define a binary operations "$*$" and "$\oplus$" on $\mathbf{R}$ by $x * y := 2 + 3y$ and $x \oplus y = 1 + \frac{3}{4}x + \frac{1}{4}y$ for all $x, y \in \mathbf{R}$. Then $(\mathbf{R}, *, \oplus)$ is a left distributive algebra by Lemma 4.2, but $(\mathbf{R}, \oplus)$ is not an abelian group, since*

$$2 \oplus 3 = \frac{13}{4} \neq 3 \oplus 2 = \frac{15}{4}.$$

Thus $(\mathbf{R}, *, \oplus, 0)$ is not a left distributive ringoid over $\mathbf{R}$.

By applying Theorem 4.4, we see that there are many left distributive ringoids which are linear groupoids. Moreover, we see that those left distributive ringoids are not rings in general.

5 (r, s)-ringoids

Given an abelian group $(X, +, 0)$ and integers $r, s \in \mathbf{Z}$, the expression $rx + sy$ is well-defined for all $x, y \in X$, since abelian groups are naturally $\mathbf{Z}$-modules. Thus, if we define $x * y := rx + sy$ for fixed $r, s \in \mathbf{Z}$ and for all $x, y \in X$, then $(X, *)$ is a groupoid and hence $(X, *, +, 0)$ is a ringoid. We call it an *(r, s)-ringoid.* Thus the $(1, 1)$-ringoid has $x * y = x + y$, while the $(1, -1)$-ringoid has $x * y = x - y$ for all $x, y \in X$.

We construct an (m, n)-ringoid which is not a ring as follows:

Example 5.1. *Let $(X, +, 0)$ be an abelian group. Assume $26x = 0$ for all $x \in X$. Define a binary operation "$*$" on X by $x * y := 13x + 14y$ for all $x, y \in X$. Then $(X, *, +, 0)$ is clearly $(13, 14)$-ringoid. We claim that $(X, *)$ is a semigroup. For all $x, y, z \in X$, we have $(x * y) * z = 13(13x + 14y) + 14x = 169x + 182y + 14x = 13x + 14z = 13x + 14(13y + 14z) = x * (y * z)$. Given $x, y, z \in X$, we have $x * (y + z) = 13x + 14y + 14z$, while $x * y + x * z = 26x + 14y + 14z = 14y + 14z$, proving that $x * (y + z) \neq x * y + x * z$. Thus $(13, 14)$-ringoid $(X, *, +, 0)$ is not a left distributive.*

Theorem 5.2. *Let $(X, *, +, 0)$ be an (m, n)-ringoid. Then $(X, *)$ is a semigroup if and only if $(m^2 - m)x = 0 = (n - n^2)z$ for all $x, z \in X$.*

Proof. Given $x, y, z \in X$, we consider the expressions $(x * y) * z$ and $x * (y * z)$:

$$\begin{aligned} (x * y) * z &= m(x * y) + nz \\ &= m(mx + ny) + nz \\ &= m^2 x + mny + nz, \end{aligned} \tag{15}$$

and

$$\begin{aligned} x*(y*z) &= mx+n(y*z) \\ &= mx+n(my+nz) \\ &= mx+mny+n^2z. \end{aligned} \tag{16}$$

By (15) and (16), we obtain

$$x*(y*z)-x*(y*z) \;=\; (m^2-m)x+(n-n^2)z, \tag{17}$$

for all $x, y, z \in X$. Assume $(X, *)$ is a semigroup. Then

$$(m^2-m)x+(n-n^2)z \;=\; 0, \tag{18}$$

for all $x, z \in X$. If we let $x := 0$ and $z := 0$ in (18), then we obtain $(m^2-m)x = 0$ and $(n-n^2)z = 0$ for all $x, z \in X$. Conversely, if we assume $(m^2-m)x = 0$ and $(n-n^2)z = 0$ for all $x, z \in X$, then, by (17) and (18), we obtain $x*(y*z)-x*(y*z) = 0$, proving that $(X, *)$ is a semigroup. □

Example 5.3. *Let* $(\mathbf{Z}, +, 0)$ *be the abelian group of integers. Consider* $(1, -1)$*-ringoid* $(\mathbf{Z}, *, +, 0)$ *where* $x*y = x-y$ *for all* $x, y \in \mathbf{Z}$*. Then* $(\mathbf{Z}, *)$ *is not a semigroup, since*

$$(x*y)*z = (x-y)-z \neq x*(y*z) = x-(y-z).$$

Moreover, we have $-5*5 = -5-5 = -2\cdot 5 = (-1-(-1)^2)\cdot 5 = -10 \neq 0$.

6 Geometric interpretations

Consider a field $(X, +, \cdot, 0, 1)$ and its Cartesian product X^n consisting of vectors $\vec{x}= (x_1, \cdots, x_n)$ with its natural vector addition $\oplus$. The natural scalar product $\vec{x} \star \vec{y}$ is defined by the ordinary inner product of $\vec{x}$ and $\vec{y}$. The square of the cosine of the angle θ between $\vec{x}$ and $\vec{y}$ in X^n can be represented as

$$\cos^2\theta = \frac{(\vec{x}\otimes\vec{y})\star(\vec{y}\otimes\vec{x})}{(\vec{x}\otimes\vec{x})\star(\vec{y}\otimes\vec{y})}.$$

Since $\sqrt{\vec{x}\otimes\vec{x}}$ may not be defined in the field X, we avoid difficulties by doing it this way. In particular, if this ratio is 1, then we consider the cosine to be $+1$ or -1 and thus we have reason to consider $\vec{x}$ and $\vec{y}$ to be parallel. Given $\vec{x}, \vec{y}\in X^n$, we define two functions as follows:

$$S(\vec{x},\vec{y}) = (\vec{x}\otimes\vec{y})\star(\vec{y}\otimes\vec{x}) + (\vec{y}\otimes\vec{x})\star(\vec{x}\otimes\vec{y}),$$

and

$$T(\vec{x},\vec{y}) = (\vec{x}\otimes\vec{x})\star(\vec{y}\otimes\vec{y}) + (\vec{y}\otimes\vec{y})\star(\vec{x}\otimes\vec{x}).$$

Two vectors $\vec{x}, \vec{y}\in X^n$ are said to be *parallel* if $S(\vec{x},\vec{y}) = T(\vec{x},\vec{y})$. Observe that $S(\vec{x},\vec{x}) = T(\vec{x},\vec{x})$ and that $S(\vec{x},\vec{y}) = T(\vec{x},\vec{y})$ implies $S(\vec{y},\vec{x}) = T(\vec{y},\vec{x})$. A transitivity relationship may or may not exist.

Proposition 6.1. *If* $(X, *, +, e)$ *is a d-algebra ringoid, then* $T(\vec{x},\vec{y}) = 2ne$ *for all* $\vec{x}, \vec{y}\in X^n$.

Proof. Since $(X, *, e)$ is a d-algebra, we have

$$\vec{x} \otimes \vec{x} = (x_1, \cdots, x_n) \otimes (x_1, \cdots, x_n) = (x_1 * x_1, \cdots, x_n * x_n) = (e, \cdots, e),$$

for all $\vec{x} \in X^n$. It follows that

$$\begin{aligned} T(\vec{x}, \vec{y}) &= (\vec{x} \otimes \vec{x}) \star (\vec{y} \otimes \vec{y}) + (\vec{y} \otimes \vec{y}) \star (\vec{x} \otimes \vec{x}) \\ &= (e, \cdots, e) \star (e, \cdots, e) + (e, \cdots, e) \star (e, \cdots, e) \\ &= ne + ne = 2ne. \end{aligned}$$

□

Example 6.2. *Consider the d-algebra ringoid $(\mathbf{R}, *, +, e)$ in Example 3.7. Let $n := 2$ and $\vec{x} = (a, b) \in \mathbf{R^2}$. Using* (I), *we get*

$$\vec{x} \otimes \vec{x} = (a, b) \otimes (a, b) = (a * a, b * b) = (e, e).$$

It follows that

$$\begin{aligned} T(\vec{x}, \vec{y}) &= (\vec{x} \otimes \vec{x}) \star (\vec{y} \otimes \vec{y}) + (\vec{y} \otimes \vec{y}) \star (\vec{x} \otimes \vec{x}) \\ &= (e, e) \star (e, e) + (e, e) \star (e, e) \\ &= e * e + e * e + e * e + e * e \\ &= e + e + e + e = 4e. \end{aligned}$$

Corollary 6.3. *Let $(X, *, +, 0)$ be a d-algebra ringoid. If $S(\vec{x}, \vec{y}) = 0$ for some $\vec{x}, \vec{y} \in X^n$, then $\vec{x}$ and $\vec{y}$ are parallel.*

Proof. If we let $e := 0$ in Proposition 6.1, then $T(\vec{x}, \vec{y}) = 0$. By assumption we obtain $S(\vec{x}, \vec{y}) = 0 = T(\vec{x}, \vec{y})$. Hence $\vec{x}$ are $\vec{y}$ are parallel. □

Proposition 6.4. *Let $(X, *, +, 0)$ be a left-zero ringoid. Then $\vec{x}$ and $\vec{y}$ are parallel for all $\vec{x}, \vec{y} \in X^n$.*

Proof. Given $\vec{x}, \vec{y} \in X^n$, since $(X, *, +, 0)$ is a left-zero ringoid, we obtain $\vec{x} \otimes \vec{y} = (x_1 * y_1, \cdots, x_n * y_n) = (x_1, \cdots, x_n) = \vec{x}$. It follows that

$$\begin{aligned} S(\vec{x}, \vec{y}) &= (\vec{x} \otimes \vec{y}) \star (\vec{y} \otimes \vec{x}) + (\vec{y} \otimes \vec{x}) \star (\vec{x} \otimes \vec{y}) \\ &= \vec{x} \star \vec{y} + \vec{y} \star \vec{x} \\ &= x_1 * y_1 + \cdots + x_n * y_n + y_1 * x_1 + \cdots + y_n * x_n \\ &= x_1 + \cdots + x_n + y_1 + \cdots + y_n \\ &= \pi(\vec{x}) + \pi(\vec{y}) \\ &= \pi(\vec{x} \oplus \vec{y}). \end{aligned}$$

Similarly, we compute

$$T(\vec{x}, \vec{y}) = (\vec{x} \otimes \vec{x}) \star (\vec{y} \otimes \vec{y}) + (\vec{y} \otimes \vec{y}) \star (\vec{x} \otimes \vec{x}) = \vec{x} \star \vec{y} + \vec{y} \star \vec{x} = \pi(\vec{x} \oplus \vec{y}),$$

which shows that $S(\vec{x}, \vec{y}) = T(\vec{x}, \vec{y})$. Hence $\vec{x}$ and $\vec{y}$ are parallel. □

Theorem 6.5. *Let $(X, *, +, 0)$ be an (r,s)-ringoid where $r, s \in \mathbf{Z}$. Then $\overrightarrow{x}$ and $\overrightarrow{y}$ are parallel for all $\overrightarrow{x}, \overrightarrow{y} \in X^n$.*

Proof. Let $(X, *, +, 0)$ be an (r,s)-ringoid and let $\overrightarrow{x}, \overrightarrow{y} \in X^n$. Then

$$\overrightarrow{x} \otimes \overrightarrow{y} = (x_1 * y_1, \cdots, x_n * y_n) = (rx_1 + sy_1, \cdots, rx_n + sy_n).$$

It follows that

$$\begin{aligned}
(\overrightarrow{x} \otimes \overrightarrow{y}) \star (\overrightarrow{y} \otimes \overrightarrow{x}) &= (x_1 * y_1, \cdots, x_n * y_n) \star (y_1 * x_1, \cdots, y_n * x_n) \\
&= (rx_1 + sy_1, \cdots, rx_n + sy_n) \star (ry_1 + sx_1, \cdots, ry_n + sx_n) \\
&= (rx_1 + sy_1) * (ry_1 + sx_1) + \cdots + (rx_n + sy_n) * (ry_n + sx_n) \\
&= r(rx_1 + sy_1) + s(ry_1 + sx_1) + \cdots + r(rx_n + sy_n) + s(ry_n + sx_n) \\
&= r^2 x_1 + 2rsy_1 + s^2 x_1 + \cdots + r^2 x_n + 2rsy_n + s^2 x_n \\
&= r^2(x_1 + \cdots + x_n) + 2rs(y_1 + \cdots + y_n) + s^2(x_1 + \cdots + x_n) \\
&= r^2 \pi(\overrightarrow{x}) + 2rs\pi(\overrightarrow{y}) + s^2 \pi(\overrightarrow{x}).
\end{aligned}$$

By a similar argument, we obtain

$$\begin{aligned}
(\overrightarrow{y} \otimes \overrightarrow{x}) \star (\overrightarrow{x} \otimes \overrightarrow{y}) &= r^2(y_1 + \cdots + y_n) + 2rs(x_1 + \cdots + x_n) + s^2(y_1 + \cdots + y_n) \\
&= r^2 \pi(\overrightarrow{y}) + 2rs\pi(\overrightarrow{x}) + s^2 \pi(\overrightarrow{y}).
\end{aligned}$$

Using this results, we have

$$\begin{aligned}
S(\overrightarrow{x}, \overrightarrow{y}) &= (\overrightarrow{x} \otimes \overrightarrow{y}) \star (\overrightarrow{y} \otimes \overrightarrow{x}) + (\overrightarrow{y} \otimes \overrightarrow{x}) \star (\overrightarrow{x} \otimes \overrightarrow{y}) \\
&= r^2 \pi(\overrightarrow{x}) + 2rs\pi(\overrightarrow{y}) + s^2 \pi(\overrightarrow{x}) + r^2 \pi(\overrightarrow{y}) + 2rs\pi(\overrightarrow{x}) + s^2 \pi(\overrightarrow{y}) \\
&= (r^2 + 2rs + s^2)\pi(\overrightarrow{x}) + (r^2 + 2rs + s^2)\pi(\overrightarrow{y}) \\
&= (r+s)^2[\pi(\overrightarrow{x}) + \pi(\overrightarrow{y})] \\
&= (r+s)^2 \pi(\overrightarrow{x} \oplus \overrightarrow{y}).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
(\overrightarrow{x} \otimes \overrightarrow{x}) \star (\overrightarrow{y} \otimes \overrightarrow{y}) &= (x_1 * x_1, \cdots, x_n * x_n) \star (y_1 * y_1, \cdots, y_n * y_n) \\
&= (rx_1 + sx_1, \cdots, rx_n + sx_n) \star (ry_1 + sy_1, \cdots, ry_n + sy_n) \\
&= (rx_1 + sx_1) * (ry_1 + sy_1) + \cdots + (rx_n + sx_n) * (ry_n + sy_n) \\
&= r(rx_1 + sx_1) + s(ry_1 + sy_1) + \cdots + r(rx_n + sx_n) + s(ry_n + sy_n) \\
&= r^2 x_1 + rsx_1 + sry_1 + s^2 y_1 + \cdots + r^2 x_n + rsx_n + sry_n + s^2 y_n \\
&= r^2(x_1 + \cdots + x_n) + rs(x_1 + \cdots + x_n) + sr(y_1 + \cdots + y_n) + s^2(y_1 + \cdots + y_n) \\
&= r^2 \pi(\overrightarrow{x}) + rs[\pi(\overrightarrow{x}) + \pi(\overrightarrow{y})] + s^2 \pi(\overrightarrow{y}).
\end{aligned}$$

If we change $\overrightarrow{x}$ and $\overrightarrow{y}$ in the above equation, then

$$(\overrightarrow{y} \otimes \overrightarrow{y}) \star (\overrightarrow{x} \otimes \overrightarrow{x}) = r^2 \pi(\overrightarrow{y}) + rs[\pi(\overrightarrow{y}) + \pi(\overrightarrow{x})] + s^2 \pi(\overrightarrow{x}).$$

Using this results, we obtain

$$\begin{aligned}
T(\vec{x},\vec{y}) &= (\vec{x}\otimes\vec{x})\star(\vec{y}\otimes\vec{y})+(\vec{y}\otimes\vec{y})\star(\vec{x}\otimes\vec{x})\\
&= r^2\pi(\vec{x})+rs(\pi(\vec{x})+\pi(\vec{y}))+s^2\pi(\vec{y})+r^2\pi(\vec{y})+rs(\pi(\vec{y})+\pi(\vec{x}))+s^2\pi(\vec{x})\\
&= r^2[\pi(\vec{x})+\pi(\vec{y})]+2rs[\pi(\vec{y})+\pi(\vec{x})]+s^2[\pi(\vec{x})+\pi(\vec{y})]\\
&= (r^2+2rs+s^2)[\pi(\vec{x})+\pi(\vec{y})]\\
&= (r+s)^2\pi(\vec{x}\oplus\vec{y}).
\end{aligned}$$

This shows that $S(\vec{x},\vec{y})=T(\vec{x},\vec{y})$, and hence $\vec{x}$ and $\vec{y}$ are parallel. □

Corollary 6.6. *If* $(X,*,+,0)$ *is an* $(1,0)$*-ringoid, then* $S(\vec{x},\vec{y})=\pi(\vec{x}\oplus\vec{y})$ *for all* $\vec{x},\vec{y}\in X^n$*.*

Proof. Let $r:=1, s:=0$ in the proof of Theorem 6.5. □

7 Conclusions

In this paper, a new algebraic structure as a generalization of a ring has been introduced, and called it ringoid, and discussed several properties of d-algebra ringoids, left zero ringoids, and left distributive ringoids. It is shown that there are many left distributive ringoids over a field which are linear groupoids, but not rings in general. Beside, the notion of an (r,s)-ringoid is defined, and we investigated geometric interpretations of the parallelism of vectors in several ringoids.

In this fashion we will discuss a study of the right distributivity law in linear groupoids, and will combine with the previous obtained results on the left distributive ringoids which are linear groupoids. This future research will investigate the existence of distributive ringoids which are not rings. Moreover, we will investigate other cases of polynomial algebras $(K,*,\oplus,\xi)$ as follows: (i) $(K,*)$ is linear and $(K,\oplus)$ is quadratic; (ii) $(K,*)$ is quadratic and $(K,\oplus)$ is linear; (iii) $(K,*)$ and $(K,\oplus)$ are quadratic. Notice that $x*y:=xy, x\oplus y:=x+y$ is an example of a quadratic-linear distributive ringoid. Polynomially defined ringoids may have interesting properties.

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