Journal of Algebraic Hyperstructures and Logical Algebras
Volume 3, Number 4, (2022), pp. 51-64

Primary decomposition of A-ideals in MV-semimodules

S. Saidi Goraghani[1]

[1]*Department of Mathematics, Farhangian University, Tehran, Iran*

SiminSaidi@yahoo.com

Abstract

In [16], by using an MV-semiring and an MV-algebra, we introduced the new definition of MV-*semimodule* and studied some of their basic properties. In this paper, we study and present definitions of primary ideals of MV-semirings, decomposition of ideals in MV-semirings, primary A-ideals of MV-*semimodules*, and decomposition of A-ideals in MV-*semimodules*. Then we present some conditions that an A-ideal can have a reduced primary decomposition.

Article Information

Corresponding Author:
S. Saidi Goraghani;
Received: September 2022;
Revised: October 2022;
Accepted: October 2022;
Paper type: Original.

Keywords:

MV-semiring, MV-algebra, MV-semimodule, MV-module, primary ideal, primary decomposition

1 Introduction

In 1935, Vandiver introduced the concept of semirings. Since then the semirings have been studied by many authors. Nowadays, the theory of idempotent semirings has many applications in other fields, such as discrete mathematics, computer science and languages, linguistic problems, optimization difficulties, discrete event systems, computational difficulties, and so on. We know the theory arising from the substitution of the fields of real and complex numbers with idempotent semirings or semifields. It is often referred to as idempotent or topical mathematics.
Chang [3, 4] defined MV-algebras as algebras corresponding to the Łukasiewicz infinite valued propositional calculus. MV-algebras have equivalent presentation such as CN-algebras, Wajsberg algebras, bounded commutative BCK-algebras and so on. It is proved that MV-algebras are naturally related to the Murray-von Neumann order of projections in operator algebras on Hilbert spaces

https://doi.org/10.52547/HATEF.JAHLA.3.4.4

and that they have an interesting role to play as invariant of approximately finite-dimensional C^*-algebras. Also Ulam's searching games with lies is naturally related to MV-algebras. MV-algebras admit a natural order structure (natural lattice reduction). Chang established this fact that nontrivial MV-algebras are subdirect products of MV-chains. That is, MV-algebras are totally ordered and so some essential properties can be derived from this fact. He introduced the notion of prime ideal in MV-algebras to prove this important result. We know that there is a categorical equivalence between MV-algebras and lu-groups. It leads some researches to define a product operation on MV-algebras to obtain structures corresponding to l-rings. A *product MV-algebra* (PMV-algebra) is an MV-algebra with an addition operation (an associative binary operation "."). PMV-algebras have been widely studied and researched recently. Then, their equivalence with a certain class of l-rings with strong unit was proved. The introduction of modules over such algebras seemed natural for generalizing the divisible MV-algebras and the MV-algebras obtained from Riesz spaces and to prove natural equivalence theorems. Hence, Di Nola [6] introduced the notion of MV-modules as an action of a PMV-algebra over an MV-algebra. By presentation of definition of MV-modules, some researches were encouraged for working on MV-modules and related structures (see, for instance, [2, 15]). Specially, some researches obtained some results on ideals and decomposition of ideals in MV-algebras and MV-modules (see, for instance, [9, 12, 13, 14]). In recent years, Di Nola et al. studied the notions of MV-semiring and semimodules and investigated related results [7, 8, 11]. MV-semirings are a special class of idempotent semirings strictly connected to MV-algebras, the algebraic semantics of ukasiewicz propositional logic. In particular, in [8], Di Nola and Russo showed that the two aforementioned categories are isomorphic. This fact allows us to import results and techniques from semiring and ring theory into the study of MV-algebras. It is well known that an effective way to study rings is to study the way in which a ring R acts on its modules. There is a connection between an special category of additively idempotent semirings and MV-algebras. This connection was first observed in [7] and eventually enforced in [1]. On the other hand, every MV-algebra has two semirings reducts isomorphic to each other. Also, the category of MV-semirings, defined in [1], is isomorphic to the one of MV-algebras. Since MV-semirings are a branch of semirings, it seems that MV-modules can be defined over MV-semirings. There are many researchers who are interested in modules structures. In [16], we presented a new definition for MV-modules. We used MV-semirings instead of PMV-algebras. In fact, we defined MV-modules over MV-semirings. During the last years, MV-modules have been defined over PMV-algebras. Since MV-semirings are a special class of semirings, we hope that new definition of MV-modules helps us to explain MV-modules better than the last. For example, in [14], we introduced the concept of primary decomposition of A-ideals in MV-modules that it was not easy.

In this paper, we study and present definitions of primary ideals of MV-semirings, primary A-ideals of MV-semimodules, and primary (reduced primary) decomposition of A-ideals in MV-semimodules. Then we obtain some results about primary decomposition of A-ideals in MV-semimodules. For example, we present that every proper A-ideals in a Noetherian A-semimodule has a reduced primary decomposition.

2 Preliminaries

In this section, we review the material that we will use in the next sections.

Definition 2.1. [8, 10] *A semiring is an algebraic structure* $(S, \dotplus, ., 0, 1)$ *of type* $(2, 2, 0, 0)$ *such that;*
(i) $(S, \dotplus, 0)$ *is a commutative monoid,*

(ii) $(S, ., 1)$ *is a monoid,*
(iii) "$.$" *distributes over* "$\dot{+}$" *from either side.*

A semiring S *is called commutative if* $x.y = y.x$*, and it is called idempotent if it satisfies the equation* $x \dot{+} x = x$*, for every* $x, y \in S$*. A left ideal of semiring* S *is a nonempty subset* I *of* S *that satisfying the following conditions:*
(1) *if* $a, b \in I$*, then* $a \dot{+} b \in I$*,*
(2) *if* $a \in I$ *and* $r \in S$*, then* $r.a \in I$*.*
The proper ideal P *of* A *is called a prime ideal of* A *if* $a.b \in P$ *implies* $a \in P$ *or* $b \in P$*, for any* $a, b \in A$*. The proper ideal* I *of* S *is a maximal ideal of* S *if and only if no proper ideal of* S *strictly contains* I*. For* $J \subseteq S$*, the generated ideal of* J *is* $\prec J \succ = \bigcap_{J \subseteq I} I$ *where* I *shows any ideal of* S*.*
An MV-semiring is a commutative and additive idempotent semiring $(A, \dot{+}, ., 0, 1)$ *such that there exists a map* $' : A \to A$ *that satisfying the following conditions:*
(i) $a.b = 0$ *if and only if* $b \leq a'$ *(where* $\leq$ *is naturally defined by means of* $\dot{+}$*),*
(ii) $a + b = (a'.(a'.b)')'$*, for every* $a, b \in A$*.*

Definition 2.2. [5] *An MV-algebra is an algebraic structure* $M = (M, \oplus, ', 0)$ *of type* $(2, 1, 0)$ *satisfying the following equations*
$(MV1)$ $(M, \oplus, 0)$ *is an Abelian monoid,*
$(MV2)$ $(a')' = a$*,*
$(MV3)$ $0' \oplus a = 0'$*,*
$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$*, for every* $a, b \in M$*.*
If we define the constant $1 = 0'$ *and operations* $\odot$ *and* $\ominus$ *by* $a \odot b = (a' \oplus b')'$ *and* $a \ominus b = a \odot b'$*, then*
$(MV5)$ $a \oplus b = (a' \odot b')'$*,*
$(MV6)$ $a \oplus 1 = 1$*,*
$(MV7)$ $(a \ominus b) \oplus b = (b \ominus a) \oplus a$*,*
$(MV8)$ $a \oplus a' = 1$*,*
for every $a, b \in M$*. Clearly,* $(M, \odot, 1)$ *is an Abelian monoid. Now, if we define auxiliary operations* $\vee$ *and* $\wedge$ *on* M *by* $a \vee b = (a \odot b') \oplus b$ *and* $a \wedge b = a \odot (a' \oplus b)$*, for every* $a, b \in M$*, then* $(M, \vee, \wedge, 0)$ *is a bounded distributive lattice. Let* $\emptyset \neq S \subseteq M$*. We say that* S *is* $\wedge$*-closed, if* $a \wedge b \in S$*, for all* $a, b \in S$*. In an MV-algebra* M*, the following conditions are equivalent:*
(i) $a' \oplus b = 1$*,*
(ii) $a \odot b' = 0$*,*
(iii) $b = a \oplus (b \ominus a)$*,*
(iv) $\exists c \in M$ *such that* $a \oplus c = b$*, for every* $a, b, c \in M$*.*
For any two elements a, b *of the MV-algebra* M*,* $a \leq b$ *if and only if* a, b *satisfy the above equivalent conditions* $(i) - (iv)$*. An ideal of MV-algebra* M *is a subset* I *of* M*, satisfying the following conditions:*
$(I1)$*:* $0 \in I$*,*
$(I2)$*:* $x \leq y$ *and* $y \in I$ *imply* $x \in I$*,*
$(I3)$*:* $x \oplus y \in I$*, for every* $x, y \in I$*.*
We denote that $\mathcal{I}(M)$ *is the set of all ideals of* M*. A proper ideal* I *of* M *is a prime ideal of* M *if and only if* $x \ominus y \in I$ *or* $y \ominus x \in I$ *(or* $x \wedge y \in I$ *implies* $x \in I$ *or* $y \in I$*), for every* $x, y \in M$*. Let* $I \in \mathcal{I}(M)$*. Then the intersection of all prime ideals of* M*, including* I*, is called radical of* I *and it is denoted by* $rad_M(I)$ *or briefly* $rad(I)$*. If there is not any prime ideal of* M *including* I*, then we let* $rad(I) = M$ *[16]. A partial addition on MV-algebra* M *is defined as follows:* $x + y$ *is defined if and only if* $x \leq y'$ *and in this case,* $x + y = x \oplus y$*, for every* $x, y \in M$*. Moreover, if*

$x+z \leq y+z$, then $x \leq y$, for every $x, y, z \in M$.

Proposition 2.3. [5] *The following equations hold in every MV-algebra:*
(i) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
(ii) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,
(iii) *if* $x \leq y$, *then* $y' \leq x'$ *and* $x \odot z \leq y \odot z$, *for every* $x, y, z \in M$.

Lemma 2.4. [14] *In every MV-algebra M, the following conditions are equivalent:*
(i) $x = x \ominus (y \ominus x)$,
(ii) $x \ominus y = (x \ominus y) \ominus y$,
(iii) $(x \ominus z) \ominus (y \ominus z) = (x \ominus y) \ominus z$,
(iv) $x \wedge x' = 0$,
(v) $x \vee x' = 1$,
(vi) $x = x \ominus x'$(*or* $x = x \oplus x$),
(vii) $x' = x' \ominus x$,
$(viii)$ $y' \wedge x = x \ominus y$,
(ix) $y \wedge x = x \ominus y'$
(x) $x \wedge (y \ominus z) = (x \wedge y) \ominus z$, *for every* $x, y, z \in M$.

Proposition 2.5. [8] (i) *Let* $A = (A, \oplus, ', 0)$ *be an MV-algebra. Then* $(A, \vee, \odot, 0, 1)$ *is a semiring.*
(ii) *Let* $A = (A, \dot{+}, ., 0, 1)$ *be an MV-semiring. Then* $(A, \oplus, ', 0)$ *is an MV-algebra, where* $a \oplus b = (a'.b')'$, *for all* $a, b \in A$.

Proposition 2.6. [5] *Every proper ideal of an MV-algebra is an intersection of prime ideals.*

Lemma 2.7. [14] *Let* $I \in \mathcal{I}(M)$, $S \subseteq M$ *be* $\wedge$*-closed and* $S \cap I = \emptyset$. *Then there exists a maximal ideal* P *of* M *such that* $P \supseteq I$ *and* $P \cap S = \emptyset$. *Furthermore,* P *is a prime ideal of* M.

Theorem 2.8. [14] *Let M be an MV-algebra and $I_1, \cdots, I_n$ be ideals of M. Then*

$$rad(\bigcap_{k=1}^{n} I_k) = \bigcap_{k=1}^{n} rad(I_k).$$

Definition 2.9. [14] *Let Q be a proper ideal of MV-algebra M. Then Q is called a primary ideal of M if $a \wedge b \in Q$, then there exists $c \in M \setminus P$ such that $c \wedge b \in Q$ or $a \wedge c \in Q$, for every prime ideal P of M that contains Q and $a, b \in M$. In an MV-algebra, every prime ideal is primary ideal.*

Definition 2.10. [16] *Let $A = (A, \dot{+}, ., 0, 1)$ be an MV-semiring, $M = (M, \oplus, ', 0)$ be an MV-algebra, and the operation $\phi : A \times M \longrightarrow M$ be defined by $\phi(a, m) = am$, which satisfies the following axioms, for every $a, b \in A$ and $x, y \in M$:*
$(SMV1)$ *if $x + y$ is defined in M, then $ax + ay$ is defined in M and $a(x + y) = ax + ay$;*
$(SMV2)$ $(a \dot{+} b)x = ax \oplus bx$;
$(SMV3)$ $(a.b)x = a(bx)$. *Then M is called a (left) MV-semimodule over A or briefly an A-semimodule. We say that M is a unitary A-semimodule if A has a unity 1_A for the product, that is*
$(SMV4)$ $1_A x = x$, *for every* $x \in M$.

Theorem 2.11. [16] *Let $A = (A, \dot{+}, ., 0, 1)$ be an MV-semiring such that $x.x = x$, for every $x \in A$, and P be an ideal of A. Then*
(i) *P is an ideal of A as an MV-algebra;*
(ii) *If $x.y', y \in P$, then $x \in P$, for any $x, y \in A$;*
(iii) *If P is a prime ideal of A as an MV-semiring, then P is a prime ideal of A as an MV-algebra;*
(iv) *P is a prime ideal of A if and only if $x.y' \in P$ or $y.x' \in P$, for every $x, y \in A$.*

3 Primary decomposition of A-ideals in MV-semimodules

In this section, we define the notions of primary and P-primary ideals (A-ideals) of an MV-semiring (MV-semimodule). Then we present definition of primary (reduced primary) decomposition of A-ideals in MV-semimodules and obtain some results on them. As a fundamental result, we introduce an MV-semimodule that all its proper A-ideals have reduced primary decomposition.

Note. From now on, in this paper, we let A be an MV-semiring and M be an MV-algebra. We set that $\mathcal{PI}(X)$ is the set of all prime ideals of X, and $\mathcal{PI}_J(X)$ is the set of all prime ideals of X that contain $J \in \mathcal{I}(X)$, where $X = M$ or $X = A$.

Proposition 3.1. *Let M be an A-semimodule and N be an A-ideal of M. Then*

$$Q_N = \{x \in A : xM \subseteq N\},$$

is an ideal of A.

Proof. Let $x, y \in Q_N$. Then $xm,\ ym \in N$ and so $xm \oplus ym \in N$, for every $m \in M$. By $(SMV2)$, $(x \dotplus y)m = xm \oplus ym \in N$, for every $m \in M$. Hence, $x \dotplus y \in Q_N$. Now, let $a \in A$ and $x \in Q_N$. Then $xm \in N$ and so by $(SMV3)$, $(a.x)m = a(xm) \in N$, for every $m \in M$. It means that $a.x \in Q_N$ and so Q_N is an ideal of A. □

Definition 3.2. (i) *The proper ideal Q of A is called a primary ideal of A if $a.b \in Q$ implies $c.b \in Q$ or $c.a \in Q$, where $c \in A \setminus P$, for every $P \in \mathcal{PI}_Q(M)$.*
(ii) *Let M be an A-semimodule. Then an ideal N of M is called an A-ideal of M if*
$(I4)$*: $ax \in N$, for every $a \in A$ and $x \in N$.*
A proper A-ideal N of M is called a prime A-ideal of M, if $am \in N$ implies $m \in N$ or $a \in Q_N = \{x \in A : xM \subseteq N\}$, for any $a \in A$ and $m \in M$. A proper A-ideal N of M is called a primary A-ideal of M, if for any $x \in A$ and $m \in M$, $xm \in N$ implies $m \in N$ or $\exists c \in A \setminus P$ such that $(c.x)M \subseteq N$, for every $P \in \mathcal{PI}_{Q_N}(A)$.

Example 3.3. (i) *Let $A = \{0, 1, 2, 3\}$ and the operations "$\dotplus$" and "." on A be defined as follows:*

$\dotplus$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Consider the map $' : A \longrightarrow A$ such that $0' = 3,\ 1' = 2,\ 2' = 1$ and $3' = 0$. Then it is easy to show that $(A, \dotplus, ., 0, 3)$ is an MV-semiring and $(A, \oplus, ', 0)$ is an MV-algebra, where $\oplus = \dotplus$. Now, let the operation $\phi : A \times A \longrightarrow A$ be defined by $\phi(a, b) = a.b = ab$, for every $a, b \in A$. Then A is an A-semimodule. It is easy to show that $I = \{0, 1\}$ and $J = \{0, 2\}$ are primary ideals of A, and $\{0\}$ is not a primary ideal of A. Also, I, J are primary A-ideals of M. Note that $\{0\}$ is not a primary A-ideal of M.
(ii) *Let $A = \{0, 1\}$, $a \dotplus b = \min\{1, a + b\}$, and the map $' : A \longrightarrow A$ be defined by $a' = 1 - a$, for every $a, b \in A$, where $+, -, .$ are ordinary operations in $\mathbb{R}$. Then it is routine to show that $(A, \dotplus, ., 0, 1)$ is an MV-semiring. Also, let $M = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and operations "$\dotplus$" and "$'$" be defined on M*

similar to A. Then it is routine to show that $(M, \dot{+}, ', 0)$ *is an MV-algebra. Now, let operation* $\phi : A \times M \longrightarrow M$ *be defined by* $\phi(a, b) = a.b = ab$, *for every* $a \in A$ *and* $b \in M$. *Then it is easy to show that M is an MV-semimodule over A. Obviously,* $I = \{0\}$ *is a primary ideal of A and I is a primary A-ideal of M.*

Proposition 3.4. *Let M be a unitary A-semimodule and N be a prime A-ideal of M. Then N is a primary A-ideal of M.*

Proof. Let $xm \in N$ and $m \notin N$, for $x \in A$ and $m \in M$. Then we consider $c = 1 \in A \setminus P$ and so $(c.x)M = xM \subseteq N$, for every $P \in \mathcal{PI}_{Q_N}(A)$. Hence N is a primary A-ideal of M. □

Theorem 3.5. *Let M be a unitary A-semimodule and N be a primary A-ideal of M. Then* Q_N *is a primary ideal of A.*

Proof. If $Q_N = A$, then $1 \in Q_N$ and so $M = N$, which is a contradiction. Let $a.b \in Q_N$ and $a \notin Q_N$, for $a, b \in A$. Then we have $(a.b)M \subseteq N$ and so $b(am) = (b.a)m = (a.b)m \in N$, for every $m \in M$. Since $a \notin Q_N$, there exists $m' \in M$ such that $am' \notin N$. Moreover, since $b(am') \in N$ and $am' \notin N$, there exists $c \in A \setminus P$ such that $(c.b)M \subseteq N$, for every $P \in \mathcal{PI}_{Q_N}(A)$. It means that $c.b \in Q_N$. Therefore, Q_N is a primary ideal of A. □

Proposition 3.6. *Let* $I \in \mathcal{I}(A)$, $S \subseteq A$ *be* $\cdot$*-closed and* $S \cap I = \emptyset$. *Then there exists a maximal ideal P of A such that* $P \supseteq I$ *and* $P \cap S = \emptyset$. *Furthermore, P is a prime ideal of A.*

Proof. By *Zorn's* Lemma, there exists an ideal P of A such that $P \supseteq I$ and $P \cap S = \emptyset$. Let $x.y \in P$, $x \notin P$ and $y \notin P$, for $x, y \in A$. Then we have $P \subsetneq \prec P \cup \{x\} \succ = P_1$ and $P \subsetneq \prec P \cup \{y\} \succ = P_2$. Hence by maximality of P, $P_1 \cap S \neq \emptyset$ and $P_2 \cap S \neq \emptyset$. Consider $s_i \in P_i \cap S$, for $i = 1, 2$. Since $s_1.s_2 \leq s_i \in P_i$, we have $s_1.s_2 \in P_1 \cap P_2 = P$. On the other hand, $s_1.s_2 \in S$, which is a contradiction. Therefore, P is a prime ideal of A. □

Lemma 3.7. *Let* $a.a = a$, *for every* $a \in A$ *and* $Q \in \mathcal{I}(A)$. *Then*

$$rad(Q) = \{a \in A \mid \forall P \in \mathcal{PI}_Q(A), \exists c \in A \setminus P \ \ \textit{such that } c.a \in Q\}.$$

Proof. Let

$$T = \{a \in A \mid \forall P \in \mathcal{PI}_Q(A), \exists c \in A \setminus P \ \ \textit{such that } c.a \in Q\}.$$

If $a \in T$, then for every $P \in \mathcal{PI}_Q(A)$, there is $c \in A \setminus P$ such that $c.a \in Q$ and so $c.a \in P$. Since $c \notin P$, we have $a \in P$, for all $P \in \mathcal{PI}_Q(A)$ and so $a \in rad(Q)$. Hence $T \subseteq rad(Q)$.
Let $a \in rad(Q)$. If $a \notin T$, then there exists $P_1 \in \mathcal{PI}_Q(A)$ such that $c.a \notin Q$, for every $c \in A \setminus P_1$. Consider

$$S = \{c.a + x \mid x \in Q \textit{ and } c \in A \setminus P_1\}.$$

First, we show that S is $\cdot$ -closed. Let $c_1.a + x_1,\ c_2.a + x_2 \in S$, where $c_1, c_2 \in A \setminus P_1$ and $x_1, x_2 \in Q$. We have

$$\begin{aligned} (c_1.a + x_1).(c_2.a + x_2) &= (c_1.c_2).a.a + (c_1.a).x_2 + +x_1.(c_2.a) + x_1.x_2 \\ &= (c_1.c_2).a + (c_1.a).x_2 + +x_1.(c_2.a) + x_1.x_2 \in S. \end{aligned}$$

Since $c_1.c_2 \in A \setminus P_1$ and $(c_1.a).x_2 + x_1.(c_2.a) + x_1.x_2 \in Q$. Hence S is $\cdot$-closed. Now, we prove that $S \cap Q = \emptyset$. Let $S \cap Q \neq \emptyset$. Then there is $\alpha \in S \cap Q$ such that $\alpha = c'.a + x$, where $c' \in A \setminus P_1$ and $x \in Q$. It results that $c'.a \in Q$ which is a contradiction. So by Proposition 3.6, there exists a prime ideal P of A such that $P \supseteq Q$ and $P \cap S = \emptyset$. On the other hand, we have $a = 1.a + 0 \in P \cap S$ and so $P \cap S \neq \emptyset$ which is a contradiction. It means that $a \in T$ and so $rad(Q) \subseteq T$. Therefore, $rad(Q) = T$. □

Theorem 3.8. *Let Q be an ideal of A, where $a.a = a$, for every $a \in A$. Then Q is a primary ideal of A if and only if $a.b \in Q$ implies $a \in rad(Q)$ or $b \in rad(Q)$, for any $a, b \in A$.*

Proof. ($\Rightarrow$) Let Q be a primary ideal of A and $a.b \in Q$, for $a, b \in A$. If $a \in Q$, then $a \in rad(Q)$. Let $a \notin Q$. Then there exists $c \in A \setminus P$ such that $c.b \in Q$ or $a.c \in Q$, for every $P \in \mathcal{PI}_Q(A)$. If $c.b \in Q$, then $c.b \in P$, for every $P \in \mathcal{PI}_Q(A)$. Since $c \notin P$, we have $b \in P$, for every $P \in \mathcal{PI}_Q(A)$. It results that $b \in \bigcap_{Q \subseteq P} P = rad(Q)$. Similarly, if $a.c \in Q$, then $a \in rad(Q)$.
($\Leftarrow$) Suppose $a.b \in Q$. Then $a \in rad(Q)$ or $b \in rad(Q)$, for $a, b \in A$ and so by Lemma 3.7, there exists $c \in A \setminus P$ such that $c.b \in Q$ or $a.c \in Q$, for every $P \in \mathcal{PI}_Q(A)$. It means that Q is a primary ideal of A. □

Lemma 3.9. *Let M be additive idempotent and $I \in \mathcal{I}(M)$. Then*

$$rad(I) = \{x \in M \mid \forall P \in \mathcal{PI}_I(M), \exists c \in M \setminus P \textit{ such that } c \wedge x \in I\}.$$

Proof. Let

$$T = \{x \in M \mid \forall P \in \mathcal{PI}_I(M), \exists c \in M \setminus P \textit{ such that } c \wedge x \in I\},$$

and $x \in rad(I)$. Then $x \in P$, for every $P \in \mathcal{PI}_I(M)$. If $x \in I$, then by considering $c = 1$, we have $x \in T$. Now, let $x \notin I$. If $x \notin T$, then there exists $P_1 \in \mathcal{PI}_I(M)$ such that $c \wedge x \notin I$, for every $c \in M \setminus P_1$. Let

$$S = \{(c \wedge x) \ominus y \mid y \in I \textit{ and } c \in M \setminus P_1\}.$$

First, we show that S is $\wedge$-closed. Let $(c_1 \wedge x) \ominus y_1, (c_2 \wedge x) \ominus y_2 \in S$, where $c_1, c_2 \in M \setminus P_1$ and $y_1, y_2 \in I$. By Lemma 2.4(ix) and (x),

$$\begin{aligned}
((c_1 \wedge x) \ominus y_1) \wedge ((c_2 \wedge x) \ominus y_2) &= ((c_1 \wedge x) \ominus y_1) \wedge (c_2 \wedge x)) \ominus y_2 \\
&= ((c_2 \wedge x) \wedge ((c_1 \wedge x) \ominus y_1)) \ominus y_2 \\
&= (((c_2 \wedge x) \wedge (c_1 \wedge x)) \ominus y_1) \ominus y_2 \\
&= y_2' \wedge (((c_1 \wedge c_2) \wedge x) \ominus y_1) \\
&= (y_2' \wedge ((c_1 \wedge c_2) \wedge x)) \ominus y_1 \\
&= ((y_2' \wedge c_1 \wedge c_2) \wedge x) \ominus y_1.
\end{aligned}$$

Now, we show that $y_2' \wedge c_1 \wedge c_2 \in M \setminus P_1$. Let $y_2' \wedge c_1 \wedge c_2 \in P_1$. Since $c_1 \wedge c_2 \notin P_1$, we have $y_2' \in P_1$ and so $1 \in P_1$. We get $x \in P_1$ and so $P_1 = M$, which is a contradiction. Hence

$$\begin{aligned}
y_2' \wedge c_1 \wedge c_2 \in M \setminus P_1 &\implies ((y_2' \wedge c_1 \wedge c_2) \wedge x) \ominus y_1 \in S \\
&\implies ((c_1 \wedge x) \ominus y_1) \wedge ((c_2 \wedge x) \ominus y_2) \in S,
\end{aligned}$$

and so S is $\wedge$-closed. Now, we prove that $S \cap I = \emptyset$. If $S \cap I \neq \emptyset$, then there exist $c_3 \in M \setminus P_1$ and $y' \in I$ such that $(c_3 \wedge x) \ominus y' \in I$. It results that $c_3 \wedge x \in I$. But, by definition of S, $c_3 \wedge x \notin I$, for every $c \in M \setminus P_1$, which is a contradiction. Then $S \cap I = \emptyset$ and so by Lemma 2.7, there exists $P_2 \in \mathcal{PI}_I(M)$ such that $P_2 \cap S = \emptyset$. Since $(c \wedge x) \ominus x = 0 \in P$ and $x \in P$, we have $c \wedge x \in P$, for every $c \in M \setminus P$ and for every $P \in \mathcal{PI}_I(M)$. Then $c \wedge x \in P_2$. On the other hand, $c \wedge x = (c \wedge x) \ominus 0 \in S$. Hence, $c \wedge x \in P_2 \cap S$, which is a contradiction. It implies that $x \in T$. Therefore, $rad(I) \subseteq T$.
Now, let $x \in T$. Hence, for every $P \in \mathcal{PI}_I(M)$ there exists $c \in M \setminus P$ such that $c \wedge x \in I \subseteq P$. Since $c \notin P$, we get $x \in P$, for every $P \in \mathcal{PI}_I(M)$. It means that $x \in rad(I)$ and so $T \subseteq rad(I)$. Therefore, $T = rad(I)$. □

Theorem 3.10. (i) *The radical of every primary ideal of* A *is a prime ideal of* A, *where* $a.a = a$, *for every* $a \in A$.
(ii) *If* M *is additive idempotent, then the radical of every primary ideal of* M *is a prime ideal of* M.

Proof. (i) Let Q be a primary ideal of A. If $rad(Q) = A$, then $1 \in rad(Q)$ and so for every $P \in PI_Q(A)$ there exists $c \in A \setminus Q$ such that $c.1 \in Q \subseteq P$. Hence $c \in P$ that is a contradiction. Now, let $a.b \in rad(Q)$. Then by Lemma 3.7, for every $P \in PI_Q(A)$, there is $c' \in A \setminus P$ such that $(c'.a).b = c'.(a.b) \in Q$. If $a \notin rad(Q)$, then there is $P_1 \in PI_Q(A)$ such that for every $c \in A \setminus P_1$, we have $c.a \notin Q$ and so $c'.a \notin Q$. Consider $c'_1 \in A \setminus P_1$ such that $c'_1.(a.b) \in Q$. Since $((c'_1.a).b) \in Q$, $c'_1.a \notin Q$ and Q is a primary ideal of A, there exists $c'' \in A \setminus P$ such that $c''.b \in Q$, for every $P \in PI_Q(A)$. It results that $b \in rad(Q)$ and so $rad(Q)$ is a prime ideal of A.
(ii) Let Q be a primary ideal of M. If $rad(Q) = M$, then $1 \in rad(Q)$. Hence, by Lemma 3.9, for every $P \in \mathcal{PI}_Q(M)$, there exists $c \in M \setminus P$ such that $c \wedge 1 = c \in Q \subseteq P$ and so $c \in P$, which is a contradiction. Now, let $a \wedge b \in rad(Q)$, for $a, b \in M$. Then there exists $c_1 \in M \setminus P$ such that $(c_1 \wedge a) \wedge b = c_1 \wedge (a \wedge b) \in Q$, for every $P \in \mathcal{PI}_Q(M)$. If $a \notin rad(Q)$, then by Lemma 3.9, there is $P \in \mathcal{PI}_Q(M)$ such that $c_1 \wedge a \notin Q$, for every $c_1 \in M \setminus P$. Since Q is a primary ideal of M and $(c_1 \wedge a) \wedge b \in Q$, there is $c_2 \in M \setminus P$ such that $c_2 \wedge b \in Q$, for every $P \in \mathcal{PI}_Q(M)$ and so $b \in rad(Q)$. Therefore, $rad(Q)$ is a prime ideal of M. □

Note. If M is a unitary A-semimodule, and N is a primary A-ideal of M, where $a.a = a$, for every $a \in A$, then by Theorems 3.5 and 3.10(i), $rad(Q_N)$ is a prime ideal of A and by Lemma 3.7,

$$rad(Q_N) = \{x \in A \mid \forall P \in \mathcal{PI}_{Q_N}(A), \exists c \in A \setminus P \textit{ such that } (c.x)M \subseteq N\}.$$

Lemma 3.11. *Let* $A = (A, \dotplus, ., 0, 1)$ *be an* MV*-semiring. Then*
(i) $a \le c$ *if and only if there is* $b \in A$ *such that* $a \dotplus b = c$;
(ii) $a \le b$ *implies* $a.c \le b.c$;
(iii) $ac' \le (ac)'$, *where* $a, b, c \in A$.

Proof. (i) Let $a, c \in A$ and there is $b \in A$ such that $a \dotplus b = c$. Then we have $(a'.(a'.b)')' = c$ and so $a'.(a'.b)' = c'$. Hence $c'.a = a'.(a'.b)'.a = a.a'.(a'.b)' = 0$ and so $a \le c$. The conversely is proved, similarly.
(ii) Let $a \le b$. Then by (i), there exists $t \in A$ such that $a \dotplus t = b$ and so $(a \dotplus t).c = a.c \dotplus t.c = b.c$. It results that $a.c \le b.c$.
(iii) Since $a \le 1$, by (i), we have $a.c' \le c'$. Also, since $(a.c).c' = 0$, we have $a.c \le c$ and so $c' \le (a.c)'$. Hence $a.c' \le (a.c)'$. □

Proposition 3.12. *Let* $A = (A, \dotplus, ., 0, 1)$ *be an* MV*-semiring,* $M = (M, \oplus, ', 0)$ *be an* MV*-algebra, and* M *be an* A*-semimodule. Then*
(i) $a \le b$ *implies* $ax \le bx$,
(ii) $ax \vee bx \le (a \dotplus b)x$,
(iii) $x \le y$ *implies* $ax \le ay$,
(iv) $ax' \le (ax)'$,
(v) $(ax) \odot (ay)' \le a(x \odot y')$, *for every* $x, y \in M$ *and every* $a, b, c \in A$.

Proof. (i) Let $a, b \in A$, $x \in M$ and $a \le b$. Then by Lemma 3.11(i), there exists $c \in A$ such that $a \dotplus c = b$ and so $ax \oplus cx = (a \dotplus c)x = bx$. It results that $ax \le bx$.
(ii) By Proposition 2.5(ii), $(A, \oplus, ', 0)$ is an MV-algebra, where $a \oplus b = (a'.b')'$, for all $a, b \in A$.

Then we have $a \vee b = a \oplus (b' \oplus a)'$. Hence $a \leq a \oplus (b' \oplus a)'$ and $b \leq a \oplus (b' \oplus a)'$. So, $ax \leq (a \oplus (b' \oplus a)')x$ and $bx \leq (a \oplus (b' \oplus a)')x$. It results that $ax \vee bx \leq (a \vee b)x$. Now, we have

$$a \vee b = a \oplus (b' \oplus a)' = (a'.(b.a')')' = a \dotplus b,$$

and so $ax \vee bx \leq (a \dotplus b)x$.
(iii) Since $x \leq y$, there exists $c \in M$ such that $x \oplus c = y$ and so by $(SAM1)$, $ax \oplus ac = ay$. It means that $ax \leq ay$.
(iv) Since $x + x$ and $ax + ax$ are defined, we have

$$ax' + ax = a(x + x') = a1 \leq 1 = (ax)' + ax$$

and so $ax \leq (ax)'$.
(v) By (iii), we have $ax \vee ay \leq a(x \vee y)$. Then

$$((ax) \odot (ay)') \oplus ay = ax \vee ay \leq a(x \vee y) = a((x \odot y') \oplus y).$$

Now, since $x \odot y' \leq y'$, by (iii) and (iv), we have $a(x \odot y') \leq ay' \leq (ay)'$. Then $a(x \odot y') + ay$ is defined and so

$$a(x \odot y' \oplus y) = a(x \odot y' + y) = a(x \odot y') + ay.$$

It results that

$$((ax) \odot (ay)') + ay \leq a(x \odot y') + ay \text{ and so } (ax) \odot (ay)' \leq a(x \odot y').$$

□

Definition 3.13. *Let M be an A-semimodule and N be a proper A-ideal of M. Then N is called a P-primary A-ideal of M, if N is a primary A-ideal of M and $rad(Q_N) = P$.*

Example 3.14. *Consider A is the MV-semiring as Example 3.3(i). We have*

$$Q_I = \{x \in A : xA \subseteq A\} = \{0, 1\} = P,$$

and $rad(Q_I) = P$. Then I is a P-primary A-ideal of A.

Lemma 3.15. *Let M be an A-semimodule and $N_1, \cdots, N_k$ be P'-primary A-ideal of M. Then $\bigcap_{i=1}^{k} N_i$ is a P'-primary A-ideal of M.*

Proof. It is clear that $\bigcap_{i=1}^{k} N_i \neq M$. Let $xm \in \bigcap_{i=1}^{k} N_i$ and $m \notin \bigcap_{i=1}^{k} N_i$, for $x \in A$ and $m \in M$. Then $xm \in N_i$, for every $1 \leq i \leq k$ and there exists $1 \leq j \leq k$ such that $m \notin N_j$. Since $xm \in N_j$ and $m \notin N_j$, there exists $c_j \in A \setminus P$ such that $(c_j.x)M \subseteq N_j$, for every $P \in \mathcal{PI}_{Q_{N_j}}(A)$. It results that $x \in rad(Q_{N_j})$. Since N_i and N_j are P'-primary, we have $x \in rad(Q_{N_j}) = P' = rad(Q_{N_i})$, for every $1 \leq i \leq k$. Hence, there exists $c_i \in A \setminus P$ such that $(c_i.x)M \subseteq N_i$, for every $P \in \mathcal{PI}_{Q_{N_i}}(A)$. Let $c = c_1.c_2 \cdots .c_k$. Since $c.c_i' = 0$, we have $c \leq c_i$, for every $1 \leq i \leq k$. By Lemma 3.11(ii), we have $c.x \leq c_i.x$ and by Proposition 3.12(i), we get $(c.x)m \leq (c_i.x)m \in N_i$. Then $(c.x)m \in N_i$, for every $1 \leq i \leq k$. It results that $(c.x)m \in \bigcap_{i=1}^{k} N_i$, for every $m \in M$. Hence $\bigcap_{i=1}^{k} N_i$ is a primary A-ideal of M. Now, we show that $rad(Q_{\bigcap_{i=1}^{k} N_i}) = P'$. For every $1 \leq i \leq k$,

$$x \in Q_{\bigcap_{i=1}^{k} N_i} \Leftrightarrow xM \subseteq \bigcap_{i=1}^{k} N_i \Leftrightarrow xM \subseteq N_i \Leftrightarrow x \in Q_{N_i} \Leftrightarrow x \in \bigcap_{i=1}^{k} Q_{N_i}.$$

Then $Q_{\bigcap_{i=1}^{k} N_i} = \bigcap_{i=1}^{k} Q_{N_i}$ and so by Theorem 2.8,

$$rad(Q_{\bigcap_{i=1}^{k} N_i}) = rad(\bigcap_{i=1}^{k} Q_{N_i}) = \bigcap_{i=1}^{k} rad(Q_{N_i}) = \bigcap_{i=1}^{k} P' = P'.$$

Therefore, $\bigcap_{i=1}^{k} N_i$ is a P'-primary A-ideal of M. □

Remark 3.16. *Let $M = (M, \oplus, ', 0)$ be an MV-algebra. If a, b are two idempotent elements of M, then $a \oplus b$ and $a \odot b$ as well; moreover, we have $a \oplus b = a \vee b$, $a \odot b = a \wedge b$, $a \vee a' = 1$ and $a \wedge a' = 0$ (see [8]).*

Theorem 3.17. *Every MV-semiring A is a unitary MV-semimodule on itself, where $x.x = x$, for every $x \in A$.*

Proof. Let $A = (A, \dot{+}, ., 0, 1)$ be an MV-semiring. Then by Proposition 2.5(ii), $(A, \oplus, ', 0)$ is an MV-algebra, where $a \oplus b = (a'.b')'$, for every $a, b \in A$. It is routine to see that $a \dot{+} b = a \vee b$ and $a.b = a \wedge b$, for every $a, b \in A$. Now, if the operation $\phi : A \times A \longrightarrow A$ is defined by $\phi(a, b) = ab = a \odot b$, for every $a, b \in A$, then A is an A-semimodule:
$(SMV1)$ If $b + c$ is defined in A, then $b \leq c'$ and so by Lemma 3.11(ii) and (iii), $ab \leq ac' \leq (ac)'$. It means that $ab + ac$ is defined in A. Now, by Remark 3.16 and Proposition 2.3(i), for every $a, b \in A$, we have

$$a(b + c) = a(b \oplus c) = a \odot (b \vee c) = (a \odot b) \vee (a \odot c) = (a \odot b) \oplus (a \odot c) = ab \oplus ac.$$

$(SMV2)$ For every $a, b, c \in A$,

$$(a \dot{+} b) \odot c = (a \dot{+} b).c = a.c \dot{+} b.c = a.c \vee b.c = (a \odot c) \oplus (b \odot c) = ac \oplus bc.$$

$(SMV3)$ and $(SMV4)$ are clear. □

Definition 3.18. *Let M be an A-semimodule, N be a proper A-ideal of M and there exist proper A-ideals $A_1, A_2, \cdots, A_n$ of M such that A_i is a P_i-primary of M, for every $1 \leq i \leq n$ and $N = A_1 \cap A_2 \cap \cdots \cap A_n$. Then we say $A_1 \cap A_2 \cap \cdots \cap A_n$ is a primary decomposition of N and so N has a primary decomposition. Furthermore, this decomposition is reduced if*
(i) $A_i \not\supseteq \bigcap_{i \neq j} A_j$,
(ii) $rad(Q_{A_i}) \neq rad(Q_{A_j})$, for every $1 \leq i, j \leq n$.

Example 3.19. *In Example 3.3(i), $\{0, 2\} \cap \{0, 1\}$ is a primary decomposition of $\{0\}$. This decomposition is reduced, too.*

Remark 3.20. *Let $x.x = x$, for every $x \in A$. By Theorem 3.17, we consider A as A-semimodule, where $xy = x.y$, for every $x, y \in A$. In this case, every prime ideal of A is a prime A-ideal of A. Then by Theorem 2.11(iii), every prime ideal of A as MV-semiring is a prime A-ideal of A as MV-algebra. Hence, by Proposition 2.6, every proper A-ideal of A has a primary decomposition.*

Theorem 3.21. *Let M be an A-semimodule and N be an A-ideal of M that has a primary decomposition. Then N has a reduced primary decomposition.*

Proof. Let $N = A_1 \cap \cdots \cap A_n$, where A_i is a primary ideal of M, for every $1 \leq i \leq n$. If $A_j \supseteq \bigcap_{i=1}^{n} A_i$, where $i \neq j$, then we set $N = A_1 \cap \cdots \cap A_{j-1} \cap A_{j+1} \cap \cdots \cap A_n$, for every $1 \leq j \leq n$ and so by renumbering,

$$N = \bigcap_{i=1}^{k} A'_i, \ where\ k \leq n\ and\ A'_j \not\supseteq \bigcap_{i=1}^{k} A'_i, \ for\ every\ 1 \leq j \leq k.$$

Set $rad(Q_{A'_i}) = P_i$, for some $1 \leq i \leq k$, and let $T = \{P_1, \cdots, P_m\}$, where $P_i \neq P_j$ and $m \leq k$, for every $1 \leq i, j \leq m$. Now, we resume

$$N = (A'_{i_1} \cap \cdots \cap A'_{i_t}) \cap (A'_{j_1} \cap \cdots \cap A'_{j_l}) \cap \cdots \cap (A'_{s_1} \cap \cdots \cap A'_{s_w}).$$

On the other hand, by Lemma 3.15, we have

$$rad(Q_{\bigcap_{h=1}^{t} A'_{i_h}}) = \bigcap_{h=1}^{t} rad(Q_{A'_{i_h}}) = \bigcap_{h=1}^{t} p_1 = p_1,$$

$$\vdots$$

$$rad(Q_{\bigcap_{h=1}^{w} A'_{s_h}}) = \bigcap_{h=1}^{w} rad(Q_{A'_{s_h}}) = \bigcap_{h=1}^{w} p_m = p_m.$$

Therefore, I has a reduced primary decomposition. □

Definition 3.22. *Let M be an A-semimodule. Then*
(i) M is called Noetherian if M satisfies the ascending chain condition (ACC): that is any chain $N_1 \subseteq N_2 \subseteq \cdots$ of A-ideals of M is stationary.
(ii) We say M satisfies the maximum condition, if every non-empty family of A-ideals of M has a maximum element.

Theorem 3.23. *Let M be an A-semimodule. Then M is Noetherian if and only if M has maximum condition.*

Proof. The proof is routine. □

Theorem 3.24. *Let M be a Noetherian A-semimodule. Then every proper A-ideal of M has a reduced primary decomposition.*

Proof. Let

$$T = \{N \subsetneq M \mid N \text{ is an } A\text{-ideal of M such that N has no any reduced primary decomposition}\}.$$

We show that $T = \emptyset$. Let $T \neq \emptyset$. Since M is Noetherian, by Theorem 3.23, T has a maximum element G. It is clear that G is not a primary A-ideal of M. So there exists $x \in A$ and $m \in M$ such that $xm \in G$, $m \notin G$ and for every $c \in A \setminus P$, $(c.x)M \nsubseteq G$, where $P \in \mathcal{PI}_{Q_G}(A)$. We give an index $i \geq 1$ to every $c \in A \setminus P$. Let

$$B_i = \{m \in M \mid (c_1.c_2 \cdots .c_i.x)m \in G\},$$

for every $i \geq 1$ and $m \in B_i$. Then

$$\begin{aligned}(c_1.c_2.\cdots.c_i.c_{i+1}.x)m &\leq (c_1.\cdots.c_i.x)m \in G\\ &\Longrightarrow (c_1.c_2.\cdots.c_i.c_{i+1}.x)m \in G.\end{aligned}$$

Hence, $m \in B_{i+1}$ and so $B_i \subseteq B_{i+1}$, for every $i \geq 1$. Since M is Noetherian, there exists $k \in \mathbb{N}$ such that $B_k = B_n$, for every $n \geq k$. We show that B_k is an A-ideal of M. Let $m_1, m_2 \in B_k$. Then

$$(c_1.\cdots.c_k.x)m_1 \in G \textit{ and } (c_1.\cdots.c_k.x)m_2 \in G.$$

Since G is an ideal of M, we have

$$\begin{aligned}(c_1.\cdots.c_k.x).(m_1 \oplus m_2) &= (c_1.\cdots.c_k.x)m_1 \oplus (c_1.\cdots.c_k.x)m_2 \in G\\ &\Longrightarrow m_1 \oplus m_2 \in B_K.\end{aligned}$$

Now, let $m_1 \leq m_2 \in B_k$. Since

$$(c_1.\cdots.c_k.x)m_1 \leq (c_1.\cdots.c_k.x)m_2 \in G,$$

we have $(c_1.\cdots.c_k.x)m_1 \in G$ and so $m_1 \in B_k$. On the other hand,

$$\begin{aligned}(c_1.\cdots.c_k.x)(am) &= ((c_1.\cdots.c_k.x).a)m \leq (c_1.\cdots.c_k.x)m \in G\\ &\Longrightarrow (c_1.\cdots.c_k.x)(am) \in B_K \Longrightarrow am \in B_k,\end{aligned}$$

for every $a \in A$ and $m \in B_k$. Hence, B_k is an A-ideal of M. Let

$$D = \{(c_1.\cdots.c_k.x)m' \oplus g \mid m' \in M \textit{ and } g \in G\}.$$

We show that D is an A-ideal of M. Let $d_1, d_2 \in D$. It is easy to show that $d_1 \oplus d_2 \in D$. Let $d \in D$ and $a \in A$. So there exist $m' \in M$ and $g \in G$ such that

$$\begin{aligned}ad &= a((c_1.\cdots.c_k.x)m' \oplus g) = a((c_1.\cdots.c_k.x)m') \oplus ag\\ &= (a.(c_1.\cdots.c_k.x))m' \oplus ag = (a.c_1.\cdots.c_k.x)m' \oplus ag\\ &\leq (c_1.\cdots.c_k.x)m' \oplus ag \in D.\end{aligned}$$

Hence, $ad \in D$ and so D is an A-ideal of M. Now, we prove that $G = D \cap B_k$, $G \subsetneq D$ and $G \subsetneq B_k$. Let $g \in G$. Then $g = (c_1.\cdots.c_k.x)0 \oplus g \in D$. On the other hand, $(c_1.\cdots.c_k.x)g \in G$. Then $g \in B_k$ and so $G \subseteq D \cap B_k$. Let $m \in D \cap B_k$. Since $m \in B_k$, we have $(c_1.\cdots.c_k.x)m \in G$ and since $m \in D$, there exist $m' \in M$ and $g \in G$ such that $m = (c_1.\cdots.c_k.x)m' \oplus g$. Since

$$\begin{aligned}((c_1.\cdots.c_k.x).(c_1.\cdots.c_k.x))m' \oplus (c_1.\cdots.c_k.x)g &= (c_1.\cdots.c_k.x)((c_1.\cdots.c_k.x)m') \oplus (c_1.\cdots.c_k.x)g\\ &= (c_1.\cdots.c_k.x)((c_1.\cdots.c_k.x)m' \oplus g)\\ &= (c_1.\cdots.c_k.x)m \in G,\end{aligned}$$

We have

$$(c_1.\cdots.c_k.x).((c_1.\cdots.c_k.x))m' \oplus (c_1.\cdots.c_k.x).g \in G.$$

Now, since $(c_1.\cdots.c_k.x)g \in G$, we have

$$(c_1.\cdots.c_k.x)((c_1.\cdots.c_k.x)m') \in G \Longrightarrow m' \in B_{2K} = B_K.$$

Hence

$$(c_1.\cdots.c_k.x)m' \in G \Longrightarrow m = (c_1.\cdots.c_k.x)m' \oplus g \in G.$$

Hence, $D \cap B_k \subseteq G$. It is enough to show that $G \subsetneq D$ and $G \subsetneq B_k$. We have $(c.x)M \nsubseteq G$, for every $c \in A \setminus P$, where $P \in \mathcal{PI}_{Q_G}(A)$. Then there exists $t \in M$ such that $(c.x)t \notin G$. But if $c = c_1.\cdots.c_k$, then $(c.x)t = (c.x)t + 0 \in D$ and so $G \subsetneq D$. On the other hand, there exist $m \in M$ and $x \in A$ such that $xm \in G$ and $m \notin G$, but

$$(c_1.\cdots.c_k.x)m = ((c_1.\cdots.c_k).x)m = (c_1.\cdots.c_k)(xm) \in G.$$

It means that $m \in B_k$ and so $G \subsetneq B_k$. By the maximality of G, each of sets D and B_k has a primary decomposition. It results that G has a primary decomposition, which is a contradiction. Therefore, $T = \emptyset$.

□

4 Conclusion

In this paper, definitions of primary ideals of MV-semirings, primary A-ideals of MV-semimodules, and primary (reduced primary) decomposition of A-ideals in MV-semimodules were presented, and some results about primary decomposition of A-ideals were obtained. We intend to study MV-semimodules in specific cases, too. For examples, free MV-semimodules, projective(injective) MV-semimodules, and so on. We hope that we have taken an effective step in this regard.

5 Acknowledgements

The author would like to thank referee for some very helpful comments in improving several aspects of this paper.

References

[1] L.P. Belluce, A. Di Nola, *Commutative rings whose ideals form an MV-algebra*, Mathematical Logic Quarterly, 55(5) (2009), 468–486.

[2] R.A. Borzooei, S. Saidi Goraghani, *Free MV-modules*, Journal of Intelligent and Fuzzy Systems, 31(1) (2016), 151–161.

[3] C.C. Chang, *Algebraic analysis of many-valued logic*, Transactions of the American Mathematical Society, 88 (1958), 467–490.

[4] C.C. Chang, *A new proof of the completeness of the ukasiewicz axioms*, Transactions of the American Mathematical Society, 93 (1959), 74–80.

[5] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, *Algebric foundation of many-valued reasoning*, Kluwer Academic, Dordrecht, 2000.

[6] A. Di Nola, P. Flondor, I. Leustean, *MV-modules*, Journal of Algebra, 267 (2003), 21–40.

[7] A. Di Nola, B. Gerla, *Algebras of Łukasiewicz's logic and their semiring reducts*, Contemporary Mathematics, 377 (2005), 131–144.

[8] A. Di Nola, C. Russo, *Semiring and semimodules issues in MV-algebras*, Communications in Algebra, 41(3) (2013), 1017–1048.

[9] F. Forouzesh, E. Eslami, A. Borumand Saeid, *Radical of A-ideals in MV-modules*, Analele Stiintifice ale Universitatii Al l Cuza din Iasi-Mathematica, LXII (f. 1) (2016), 33–57.

[10] J. S. Golan, *Semirings and their applications*, Kluwer Academic Publishers, Dordrecht, 1999.

[11] L. Peter Belluce, A. Di Nola, A. Rita Ferraioli, *MV-semirings and their sheaf representations*, Order, 30 (2013), 165–179.

[12] S. Saidi Goraghani, R.A. Borzooei, *Most results on A-ideals in MV-moduls*, Journal of Algebraic Systems, 5(1) (2017), 1–13.

[13] S. Saidi Goraghani, R.A. Borzooei, *Results on prime ideals in PMV-algebras and MV-modules*, Italian Journal of Pure and Applied Mathematics, 37 (2017), 183–196.

[14] S. Saidi Goraghani, R.A. Borzooei, *Decomposition of A-ideals in MV-modules*, Annals of the University of Craiova-mathematics and Computer Science Series, 45(1) (2018), 66–77.

[15] S. Saidi Goraghani, R.A. Borzooei, *On injective MV-modules*, Bulletin of of the Section of Logic, 47(4) (2018), 283–298.

[16] S. Saidi Goraghani, R.A. Borzooei, S.S. Ann, *New definition of MV-semimoduls*, Journal of Intelligent and Fuzzy Systems, 41 (2021), 3715–3726.