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Interval-valued grey (hyper)group

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Abstract

In this research, we apply the notations of the kernel and relative measure of an interval-valued grey to introduce *grey groups* (*groups are based on interval-valued grey*) and *grey hypergroups* (*hypergroups are based on interval-valued grey*). The primary method used in this research is based on linear inequalities related to elements of grey (hyper)groups and (hyper)groups. It found a relation between grey hypergroups and grey groups via the fundamental relation and proves that the identity element of any given group plays a main role in the grey groups and show that its measure is greater than or equal to its degree of greyness and less than or equal to its kernel, respectively. We show that any given grey group is a generalization of a group and analyze that interval-valued grey groups are different from the interval-valued fuzzy group.

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1 Introduction

The hyper structures theory as an extension of classic structures was firstly introduced by F. Marty in 1934 [17]. In an algebraic hyper system, the output from the hyperoperation on elements is a set and so any algebraic system is an algebraic hyper system. Marty extended the concept of groups to hypergroups and other researchers presented the algebraic hyper structures concepts such as hyperring, hypermodule, hyperfield, hypergraph, polygroup, multiring, etc. in a similar way [18]. Algebraic hyper structures have been applied in several branches of sciences such as artificial intelligence and (hyper) complex networks [6]. Recently some researchers have investigated on fields of classic logical (hyper) structures, fuzzy (neutro) logical (hyper) structures, on subhypergroups of cyclic hypergroups [2], on neutro-d-subalgebras [8], on (2-closed) regular hypergroups [9] and residuated lattices derived from filters(ideals) in double Boolean algebras [11]. As an essential approach to the study of uncertainty, *grey system theory* provides applied models for systems with

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both known and unknown data. Today, the theory has found many critical applications [7]. The introduction of grey systems was inspired by the *vagueness* of real-world data. Although fuzzy sets or rough sets have also been applied in the study of ambiguity or incompleteness of information, grey systems and fuzzy sets provide entirely different representations, and there is a fundamental difference between grey numbers and fuzzy subsets. Recently, some researchers have studied grey systems and their subsystems known as *interval-valued grey* . Several types of interval-valued grey exist; for instance, according to Lin et al. [15, 12], a grey number is a number that belongs to a known range, but its exact value is unknown. Moreover, a grey number according to Yang et al. [21], is a number for which the upper bound and lower bound are specified, and whose exact position is unknown within the specified bounds. In some other studies, interval-valued grey such as black and white numbers have been defined. Recall that a *black number* is a number for which has neither an upper limit nor a lower limit or upper limit and lower limit of the number are grey numbers. Also, a number whose upper limit is equal to its lower limit is a *white number*. Further details on grey numbers and their applications in branches of other sciences can be found in uncertainty representation and information measurement of grey numbers such as [3], grey clustering model based on kernel and information field [16], a new approach to the degree of greyness [19], a study of the grey relational model of interval numbers for panel data [23] and a new grey comprehensive relational model based on weighted mean distance and induced intensity and its application [22]. To extend grey systems, some researchers studied grey (logical) algebraic systems and introduced some operations on grey algebraic systems and interval-valued grey. Recently Hamidi et.al applied the concept of interval-valued grey and introduced *grey vertices*, *grey edges*, and *grey graphs* (*graphs are based on interval-valued grey*). They proposed a method that can be applied for grey numbers in an extension of graphs and applied it for grey numbers in the real world and via grey graphs [10].

Regarding these points, we try to extend the group structures and hypergroup structures to construct complex group structures and complex hypergroup structures. We apply the concept of grey numbers in this regard and find some relations between two systems. This problem makes motivation for using the interval-valued grey for generalizing group structures and hypergroup structures and we introduce the concepts of grey groups and grey hypergroups. Linear inequalities have the main tools in this study and we investigate the properties of grey groups and grey hypergroups based on solving linear inequalities. Indeed, grey groups are an extension of groups and grey hypergroups are an extension of hypergroups. We apply the notation of fundamental relation to the grey hypergroup and find a connection between grey hypergroup and grey group.

2 Preliminaries

In this section, we recall some definitions and results, which we need in what follows.

Definition 2.1. [5] *Let $H \neq \emptyset$ be a set and $P^*(H) = \{S \mid \emptyset \neq S \subseteq H\}$. Every map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$, is a hyperoperation, a hyperstructure $(H, \circ)$ is a hypergroupoid and for all $\emptyset \neq A, B \subseteq H, A \circ B = \bigcup\limits_{a \in A, b \in B} (a \circ b)$. Recall that a hypergroupoid $(H, \circ)$ is a semihypergroup if for all $x, y, z \in H, (x \circ y) \circ z = x \circ (y \circ z)$ and a semihypergroup $(H, \circ)$ is a hypergroup if for all $x \in H, x \circ H = H \circ x = H$(reproduction axiom).*

Definition 2.2. [4] *Let U be an universal set, $\Omega \subseteq \mathbb{R}$ and $a \in U$. A grey number is a number with clear upper and lower boundaries but which has an unknown position within the boundaries. There are several types of grey numbers. A grey number that only takes a finite number or a countable* *number of potential values is known as discrete and if grey number can potentially take any value within an interval, then it is known as continuous. An interval grey number is a grey number which is expressed mathematically as* $(a^{\pm} \in [a^-, a^+], a^- < a^+)$, *where* $[a^-, a^+] = \{t \mid a^- \leq t \leq a^+\}$, t *is an information,* a^- *and* a^+ *are the lower and upper limits of the information. If* $a^{\pm} \in [a^-, +\infty)$, *is called lower-limit grey number, if* $a^{\pm} \in (-\infty, a^+]$, *is called upper-limit grey number, if* $a^{\pm} \in (-\infty, +\infty)$ *is called a black number (it is a number that neither the exact value nor the range is known) and if* $a^- = a^+$, *it is called a white number (it is an exact value).*

Definition 2.3. [13, 14] *Let* $\Omega = [c, d]$ *be the universe and* $a^{\pm} \in \Omega$ *be an interval grey number. Then*

(i) *the kernel of an interval grey number* $a^{\pm}$ *is defined as* $Ker(a^{\pm}) = \dfrac{a^- + a^+}{2}$.

(ii) *the degree of greyness of* $a^{\pm}$ *is defined as* $g^{\circ}(a^{\pm}) = \dfrac{\mu(a^{\pm})}{\mu(\Omega)}$, *where* $\mu(a^{\pm}) = a^+ - a^-$ *and* $\mu(\Omega) = d - c$.

Definition 2.4. [20] *Let* $(G, *)$ *be a group and the map* $\mu : G \to [0, 1]$. *Then* μ *is called a fuzzy subgroup of* G, *if for all* $x, y \in G$,
$(i) : \mu(x * y) \geq \mu(x) \wedge \mu(y)$
$(ii) : \mu(x^{-1}) \geq \mu(x)$.

Definition 2.5. [1] *Let* $D(I)$ *be the set of all closed subintervals of the unit interval* $[0, 1]$ *and* $(G, *)$ *be a group. A mapping* $A : G \to D(I)$ *is called an interval-valued fuzzy group on* G, *denoted by* $A = [A^L, A^U]$, *if* A^L *and* A^U *are fuzzy subgroups of* G *such that* $A^L(x) \leq A^U(x)$, *for all* $x \in G$.

3 Interval-valued Grey Group

In this section, we introduce the concept of group based on interval grey numbers and investigate some basic properties in this complex structures.

From now on, for all $x, y \in [0, 1]$, we consider $Min(x, y) = \min\{x, y\}, Max(x, y) = \max\{x, y\}$ and $Pr(x, y) = xy$.

Definition 3.1. *Let* $\Omega \subseteq \mathbb{R}$, $(G, *)$ *be a group,* $G^{\pm} = \{x^{\pm} = (x, [x^-, x^+]) \mid x \in G, [x^-, x^+] \subseteq \Omega\}$ *and the maps* $\mu^-, \mu^+ : G^{\pm} \to \mathbb{R}$ *such that* $\mu^- \leq \mu^+$. *An algebraic structure* $(G, \Omega, *, \mu^{\pm})$ *or simplify* $\mu^{\pm} = [\mu^-, \mu^+]$ *is called a grey group based on group* G, *if for all* $x, y \in G, \mu^{\pm}((x * y)^{\pm}) = [\mu^-((x * y)^{\pm}), \mu^+((x * y)^{\pm})]$, *where* $\mu^-((x * y)^{\pm}) = Sup\{g^{\circ}(a^{\pm}) \wedge g^{\circ}(b^{\pm}) \mid a * b = x * y\}$ *and* $\mu^+((x * y)^{\pm}) = Inf\{|Ker(a^{\pm})| \vee |Ker(b^{\pm})| \mid a * b = x * y\}$.

From now on, when we say that $(G, \Omega, *, \mu^{\pm})$ is a grey group, it means that $(G, *)$ is a group and $(G, \Omega, *, \mu^{\pm})$ is a grey group based on group G and for any given $x \in G$, for simplify we will show $x^{\pm} = (x, [x^-, x^+])$ by $x^{\pm}$.

Theorem 3.2. *Let* $(G, \Omega, *, \mu^{\pm})$ *be a grey group and* $x, y \in G$. *Then*

(i) $\mu^-((x * y)^{\pm}) \geq g^{\circ}(x^{\pm}) \wedge g^{\circ}(y^{\pm})$,

(ii) $\mu^-((x)^{\pm}) \geq g^{\circ}(x^{\pm}) \wedge g^{\circ}(e^{\pm})$,

(iii) $\mu^+((x * y)^{\pm}) \leq |Ker(x^{\pm})| \vee |Ker(y^{\pm})|$,

(iv) $\mu^+((x)^\pm) \leq |Ker(x^\pm)| \vee |Ker(e^\pm)|$,

(v) $g^\circ(x^\pm) \wedge g^\circ(y^\pm) \leq |Ker(x^\pm)| \vee |Ker(e^\pm)|$.

Proof. Immediate by definition. $\square$

Example 3.3. *(i) Let $\Omega = [0, 10]$ and consider the Klein four-group $(V_4, *)$. Then $(V_4, \Omega, *, \mu^\pm)$ is a grey group in Table 1.*

Table 1: Interval-valued Grey Group $(V_4, \Omega, *, \mu^\pm)$

$\mu^\pm$	$(e, [1, 2])$	$(a, [3, 5])$	$(b, [2, 6])$	$(c, [1, 9])$
$(e, [1, 2])$	$(e, [\frac{8}{10}, \frac{3}{2}])$	$(a, [\frac{4}{10}, 4])$	$(b, [\frac{2}{10}, 4])$	$(c, [\frac{2}{10}, 4])$
$(a, [3, 5])$	$(a, [\frac{4}{10}, 4])$	$(e, [\frac{8}{10}, \frac{3}{2}])$	$(c, [\frac{2}{10}, 4])$	$(b, [\frac{2}{10}, 4])$
$(b, [2, 6])$	$(b, [\frac{2}{10}, 4])$	$(c, [\frac{2}{10}, 4])$	$(e, [\frac{8}{10}, \frac{3}{2}])$	$(a, [\frac{4}{10}, 4])$
$(c, [1, 9])$	$(c, [\frac{2}{10}, 4])$	$(b, [\frac{2}{10}, 4])$	$(a, [\frac{4}{10}, 4])$	$(e, [\frac{8}{10}, \frac{3}{2}])$

(ii) Let $\Omega = [-5, 5]$ and consider the Klein four-group $(V_4, *)$. If

$$G^\pm = \{(e, [-2, -1]), (a, [-2, 0]), (b, [-2, 1]), (c, [-2, 2])\},$$

simple computations show that $\mu^-((a * a)^\pm) = 0.4 > 0 = \mu^+((a * a)^\pm)$. Thus $(V_4, \Omega, *, \mu^\pm)$ is not a grey group.

Remark 3.4. *In Example 3.3 (i), we see that $\mu^\pm((a * b)^\pm) = \mu^\pm((a * c)^\pm)$, while $(b, [b^-, b^+]) \neq (c, [c^-, c^+])$, then the cancellation laws are not necessarily true.*

Theorem 3.5. *Let $(G, *)$ be a group.*

(i) If $G^\pm = \{x^\pm \mid x^\pm$ is a white number $(x^- = x^+)\}$, then $(G, \Omega, *, \mu^\pm)$ is a grey group.

(ii) If $\mu(\Omega) = 2$ and $G^\pm = \{x^\pm = (x, [0, x^+]) \mid x \in G\}$, then $(G, \Omega, *, \mu^\pm)$ is a grey group.

(iii) If $\mu(\Omega) = 2$ and $G^\pm = \{x^\pm = (x, [x^-, 0]) \mid x \in G\}$, then $(G, \Omega, *, \mu^\pm)$ is a grey group.

(iv) If $G^\pm = \{x^\pm = (x, [-x^-, x^-]) \mid x \in G, x^- < 0\}$, then $(G, \Omega, *, \mu^\pm)$ is not a grey group.

Proof. (i) Let $x, y \in G$. Since $x^\pm$ and $y^\pm$ are white numbers, we get that $\mu^-((x * y)^\pm) = 0$ and $\mu^+((x * y)^\pm) = Inf\{|a^+| \vee |b^+| \mid a * b = x * y\}$. Hence $\mu^-((x * y)^\pm) \leq \mu^+((x * y)^\pm)$ and so $(G, \Omega, *, \mu^\pm)$ is a grey group.

(ii) Let $x, y \in G$. Since $x^\pm = (x, [0, x^+])$ and $y^\pm = (y, [0, y^+])$, we get that $\mu^-((x * y)^\pm) = Sup\{\frac{a^+}{2} \wedge \frac{b^+}{2} \mid a * b = x * y\}$ and $\mu^+((x * y)^\pm) = Inf\{|\frac{a^+}{2}| \vee |\frac{b^+}{2}| \mid a * b = x * y\}$. Hence $\mu^-((x * y)^\pm) \leq \mu^+((x * y)^\pm)$ and so $(G, \Omega, *, \mu^\pm)$ is a grey group.

(iii) It is similar to item (ii).

(iv) Let $x, y \in G$. Since $x^\pm = (x, [x^-, x^+])$ and $y^\pm = (y, [y^-, y^+])$, we get that $\mu^-((x * y)^\pm) = Sup\{\frac{a^-}{\mu(\Omega)} \wedge \frac{b^-}{\mu(\Omega)} \mid a * b = x * y\}$ and $\mu^+((x * y)^\pm) = 0$. Hence $\mu^-((x * y)^\pm) > \mu^+((x * y)^\pm)$ and so $(G, \Omega, *, \mu^\pm)$ is not a grey group. $\square$

Let $(G, \Omega, *, \mu^\pm)$ be a grey group and $x, y \in G$. If $\mu^-((x * y)^\pm) = \mu^+((x * y)^\pm)$, then $\mu^\pm((x * y)^\pm) = [\mu^-((x * y)^\pm), \mu^-((x * y)^\pm)]$ or $\mu^\pm((x * y)^\pm)$ is a white number.

Table 2: Interval-valued Grey Group $(\mathbb{Z}_3, \Omega, +, \mu^{\pm})$

$\mu^{\pm}$	$(\overline{0}, [0,1])$	$(\overline{1}, [0,1])$	$(\overline{2}, [0,1])$
$(\overline{0}, [0,1])$	$(\overline{0}, [\frac{1}{2}, \frac{1}{2}])$	$(\overline{1}, [\frac{1}{2}, \frac{1}{2}])$	$(\overline{2}, [\frac{1}{2}, \frac{1}{2}])$
$(\overline{1}, [0,1])$	$(\overline{1}, [\frac{1}{2}, \frac{1}{2}])$	$(\overline{2}, [\frac{1}{2}, \frac{1}{2}])$	$(\overline{0}, [\frac{1}{2}, \frac{1}{2}])$
$(\overline{2}, [0,1])$	$(\overline{2}, [\frac{1}{2}, \frac{1}{2}])$	$(\overline{0}, [\frac{1}{2}, \frac{1}{2}])$	$(\overline{1}, [\frac{1}{2}, \frac{1}{2}])$

Corollary 3.6. *Any grey group is a generalization of a group.*

Example 3.7. *Let $\Omega = [0, 2]$ and consider the finite group $(\mathbb{Z}_3, +)$. Then $(\mathbb{Z}_3, \Omega, +, \mu^{\pm})$ is a grey group in Table 2.*

Proposition 3.8. *Let $(G, \Omega, *, \mu^{\pm})$ be a grey group and $x, y \in G$. Then*

(i) $\mu^-(e^{\pm}) \geq g^\circ(e^{\pm})$,

(ii) $\mu^+(e^{\pm}) \leq |Ker(e^{\pm})|$,

(iii) $g^\circ(e^{\pm}) \leq |Ker(e^{\pm})|$.

Proof. $(i), (ii)$ Let $x \in G$. Since $x = x * e$, we get that $\mu^-(e^{\pm}) \geq g^\circ(e^{\pm}) \wedge g^\circ(e^{\pm}) = g^\circ(e^{\pm})$ and $\mu^+(e^{\pm}) \leq |Ker(e^{\pm})| \vee |Ker(e^{\pm})| = |Ker(e^{\pm})|$.

(iii) Since $\mu^-(e^{\pm}) \leq \mu^+(e^{\pm})$, using items (i) and (ii), we get that $g^\circ(e^{\pm}) \leq \mu^-(e^{\pm}) \leq \mu^+(e^{\pm}) \leq |Ker(e^{\pm})|$. □

Example 3.9. *Let $(G, *, e)$ be a group and $\Omega = [-1, 1]$. If $e^{\pm} = (e, [-0.5, 0.75])$, then $(G, \Omega, *, e, \mu^{\pm})$ can't be a grey group.*

Theorem 3.10. *Let $(G, \Omega, *, \mu^{\pm})$ be a grey group and $x, y \in G$. Then*

(i) if $\mu(\Omega) \leq 1$ or $\big(\mu(\Omega) > 1$ and $e^{\pm}$ is a white number$\big)$, then $\mu^-(e^{\pm}) = g^\circ(e^{\pm})$,

(ii) if $e^- \leq \min\{3e^+, \dfrac{e^+}{3}\}$, then $\mu^+(e^{\pm}) = |Ker(e^{\pm})|$.

Proof. (i) If $\mu(\Omega) \leq 1$, then $\dfrac{(e^+ - e^-)}{\mu(\Omega)} \geq (e^+ - e^-)$ and so $\mu^-(e^{\pm}) \leq g^\circ(e^{\pm})$. If $\big(\mu(\Omega) > 1$ and $e^{\pm}$ is a white number$\big)$, then $\mu^-(e^{\pm}) = g^\circ(e^{\pm}) = 0$. Thus in any cases $\mu^-(e^{\pm}) \leq g^\circ(e^{\pm})$ and by Proposition 3.8, we get that $\mu^-(e^{\pm}) = g^\circ(e^{\pm})$.

(ii) Since $e^- \leq \min\{3e^+, \dfrac{e^+}{3}\}$, we get that $e^- \leq 3e^+$ and $e^- \leq \dfrac{e^+}{3}$. It follows that $e^+ + e^- \geq 2e^- - 2e^+$ and $e^+ + e^- \leq 2e^+ - 2e^-$, hence $\dfrac{|e^+ + e^-|}{2} \leq e^+ - e^-$. Applying Proposition 3.8, we get that $\mu^+(e^{\pm}) = |Ker(e^{\pm})|$. □

Theorem 3.11. *Let $(G, \Omega, *, \mu^{\pm})$ be a grey group and $x, y \in G$. Then*

(i) $\mu^-(e^{\pm}) \geq g^\circ(x^{\pm}) \wedge g^\circ(x^{-1\pm})$,

(ii) if $x^2 = e$, then $\mu^-(e^{\pm}) \geq g^\circ(x^{\pm})$,

(iii) $\mu^+(e^{\pm}) \leq |Ker(x^{\pm})| \vee |Ker(x^{-1\pm})$,

(*iv*) *if* $x^2 = e$, *then* $\mu^+(e^\pm) \leq |Ker(x^\pm)|$.

Proof. $(i), (ii)$ Let $x \in G$. Since $e = x * x^{-1}$, we get that $\mu^-(e^\pm) \geq g^\circ(x^{-1^\pm}) \wedge g^\circ(x^\pm)$. If $x^2 = e$, then $x = x^{-1}$ and so by item (i), $\mu^-(e^\pm) \geq g^\circ(x^\pm)$.

$(iii), (iv)$ In a similar way to item (i), can see that $\mu^+(e^\pm) \leq |Ker(x^\pm)| \vee |Ker(x^{-1^\pm})$. In addition $x^2 = e$, implies that $x = x^{-1}$ and so $\mu^+(e^\pm) = |Ker(x^\pm)| \vee |Ker(x^{-1^\pm})$. □

Corollary 3.12. *Let* $(G, \Omega, *, \mu^\pm)$ *be a grey group,* $x \in G$ *and* $x^2 = e$. *Then*

(*ii*) *if* $\mu(\Omega) \leq 1$ *or* $\big(\mu(\Omega) > 1$ *and* $e^\pm$ *is a white number*$\big)$, *then* $\mu^-(e^\pm) \geq g^\circ(x^\pm)$,

(*iv*) *if* $3e^- \geq e^+$ *or* $3e^+ \leq e^-$, *then* $\mu^+(e^\pm) \leq |Ker(x^\pm)|$,

Proposition 3.13. *Let* $2 \leq n \in \mathbb{N}, (G, \Omega, *, \mu^\pm)$ *be a grey group and* $x \in G$. *Then*

(*i*) $\mu^-((x^n)^\pm) \geq g^\circ(x^\pm)$,

(*ii*) $\mu^+((x^n)^\pm) \leq |Ker(x^\pm)|$,

(*iii*) $g^\circ(x^\pm) \leq |Ker(x^\pm)|$,

(*iv*) *if* $\mu(x^\pm) \leq \mu(e^\pm)$, *then* $\mu^-(x^\pm) \geq g^\circ(x^\pm)$.

Proof. (i) Let $n \in \mathbb{N}$ and $x \in G$. Then

$$\mu^-((x^n)^\pm) = \mu^-(\underbrace{(x * x * \ldots * x)^\pm}_{n-times}) \geq \underbrace{\mu^-((x)^\pm) \wedge \mu^-((x)^\pm) \wedge \ldots \wedge \mu^-((x)^\pm)}_{n-times} = \mu^-((x)^\pm).$$

(ii) It is similar to (i).

(iii) Let $x \in G$. Then by items $(i), (ii)$, we get that $g^\circ(x^\pm) \leq \mu^-((x^n)^\pm) \leq \mu^+((x^n)^\pm) \leq |Ker(x^\pm)|$. Thus $g^\circ(x^\pm) \leq |Ker(x^\pm)|$.

(iv) Let $x \in G$. Since $\mu(x^\pm) \leq \mu(e^\pm)$, we get

$$\mu^-(x^\pm) = \mu^-((x * e)^\pm) = Sup\{g^\circ(a^\pm) \wedge g^\circ(b^\pm) \mid a * b = x\} \geq g^\circ(x^\pm) \wedge g^\circ(e^\pm) = g^\circ(x^\pm).$$

□

Theorem 3.14. *Let* $n \in \mathbb{N}, (G, \Omega, *, \mu^\pm)$ *be a grey group,* $x \in G$ *and* $x^n = e$. *Then* $Max(g^\circ(x^\pm), g^\circ(e^\pm)) \leq Min(|Ker(x^\pm)|, |Ker(e^\pm)|)$.

Proof. Let $x \in G$ and $n \in \mathbb{N}$. Since $x^n = e$, using Proposition 3.13, $\mu^-(e^\pm) \geq g^\circ(x^\pm)$ and $\mu^+(e^\pm) \leq |Ker(x^\pm)|$. Thus by Proposition 3.8, $\mu^-(e^\pm) \geq g^\circ(e^\pm)$ and $\mu^+(e^\pm) \leq |Ker(e^\pm)|$, imply that $Max(g^\circ(x^\pm), g^\circ(e^\pm)) \leq Min(|Ker(x^\pm)|, |Ker(e^\pm)|)$. □

Definition 3.15. Let Ω be a universe and $[x^-, x^+] \subseteq \Omega$. Then, $x^\pm$ is *called* 1-*polar* if $Pr(x^-, x^+) \geq 0$ and $x^\pm$ is called 2-*polar* if $Pr(x^-, x^+) < 0$. Figures 1 and 2 show (*non-*)*negative-polar* and (*non-*)*positive-polar*, respectively and Figure 3 is denoted by 2-*polar*.

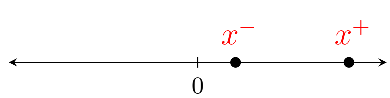

Figure 1: Position of the grey number $x^\pm$ on the real line ((non-)negative-polar).

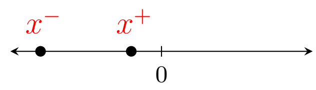

Figure 2: Position of the grey number $x^\pm$ on the real line ((non-)positive-polar).

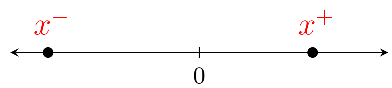

Figure 3: Position of the grey number $x^\pm$ on the real line (2-polar).

Theorem 3.16. *Let* $(G, \Omega, *, \mu^\pm)$ *be a grey group,* $x \in G$ *and* $\alpha = \dfrac{2+\mu(\Omega)}{2-\mu(\Omega)}$.

- (i) *If* $\mu(\Omega) > 2$, *then* $(x^+ \geq 0$ *and* $[x^-, x^+] \subseteq [\frac{1}{\alpha}x^+, x^+])$ *or* $(x^- \leq 0$ *and* $[x^-, x^+] \subseteq [x^-, \frac{1}{\alpha}x^-])$.
- (ii) *If* $\mu(\Omega) = 2$, *then* $x^\pm$ *is either a (non-)negative-polar or a (non-)positive-polar.*
- (iii) *If* $\mu(\Omega) < 2$, *then* $(x^+ \geq 0$ *and* $[x^-, x^+] \subseteq [\frac{1}{\alpha}x^+, x^+])$ *or* $(x^- \leq 0$ *and* $[x^-, x^+] \subseteq [x^-, \frac{1}{\alpha}x^-])$.

Proof. Let $x \in G$. By Proposition 3.13, we get that $g^\circ(x^\pm) \leq |Ker(x^\pm)|$. It follows that $\dfrac{x^+ - x^-}{\mu(\Omega)} \leq \dfrac{|x^+ + x^-|}{2}$ and so $x^+(2-\mu(\Omega)) \leq x^-(2+\mu(\Omega))$ or $x^+(2+\mu(\Omega)) \leq (2-\mu(\Omega))x^-$.

(i) If $\mu(\Omega) < 2$, then $x^- \leq x^+$ implies that $(x^+ \geq 0$ and $[x^-, x^+] \subseteq [\frac{1}{\alpha}x^+, x^+])$ or $(x^- \leq 0$ and $[x^-, x^+] \subseteq [x^-, \frac{1}{\alpha}x^-])$.

(ii) If $\mu(\Omega) = 2$, then $x^- \geq 0$ and so $x^\pm$ is a (non-)negative-polar or $x^+ \leq 0$ that it implies $x^\pm$ is a (non-)positive-polar.

(iii) If $\mu(\Omega) < 2$, then $\alpha > 0$ and so $(x^+ \geq 0$ and $[x^-, x^+] \subseteq [\frac{1}{\alpha}x^+, x^+])$ or $(x^- \leq 0$ and $[x^-, x^+] \subseteq [x^-, \frac{1}{\alpha}x^-])$. □

Theorem 3.17. *Let* $(G, \Omega, *, \mu^\pm)$ *be a grey group and* $x, y \in G$. *If* $\mu^-((x*y)^\pm) = \mu^+((x*y)^\pm)$, *then one of the following statements is true.*

- (i) $g^\circ(y^\pm) \leq |Ker(x^\pm)|$.
- (ii) $g^\circ(x^\pm) \leq |Ker(y^\pm)|$.

Proof. Let $x, y \in G$. Since

$$Sup\{g^\circ(a^\pm) \wedge g^\circ(b^\pm) \mid a*b = x*y\} = Inf\{|Ker(a^\pm)| \vee |Ker(b^\pm) \mid a*b = x*y\},$$

for all $a, b \in G$ such that $a*b = x*y$, we get that $g^\circ(a^\pm) \wedge g^\circ(b^\pm) \leq |Ker(a^\pm)| \vee |Ker(b^\pm)$, because of $g^\circ(a^\pm) \wedge g^\circ(b^\pm) = g^\circ(a^\pm)$ or $g^\circ(a^\pm) \wedge g^\circ(b^\pm) = g^\circ(b^\pm)$, and $|Ker(a^\pm)| \vee |Ker(b^\pm) = |Ker(a^\pm)|$ or $|Ker(a^\pm)| \vee |Ker(b^\pm) = |Ker(b^\pm)|$. Thus, we have $g^\circ(a^\pm) \leq |Ker(a^\pm)|$ or $g^\circ(b^\pm) \leq |Ker(a^\pm)|$ or $g^\circ(a^\pm) \leq |Ker(b^\pm)|$ or $g^\circ(b^\pm) \leq |Ker(b^\pm)|$.

Therefore, for $a = x$ and $b = y$, we get that $\big(g^\circ(x^\pm) \leq g^\circ(y^\pm)$ and $|Ker(x^\pm)| \leq |Ker(y^\pm)|\big)$ or $\big(g^\circ(x^\pm) \leq g^\circ(y^\pm)$ and $|Ker(y^\pm)| \leq |Ker(x^\pm)|\big)$ or $\big(g^\circ(y^\pm) \leq g^\circ(x^\pm)$ and $|Ker(x^\pm)| \leq |Ker(y^\pm)|\big)$ or $\big(g^\circ(y^\pm) \leq g^\circ(x^\pm)$ and $|Ker(y^\pm)| \leq |Ker(x^\pm)|\big)$. Now, by Proposition 3.13, since for all $x, y \in G, g^\circ(x^\pm) \leq |Ker(x^\pm)|$ and $g^\circ(y^\pm) \leq |Ker(y^\pm)|$, we get that $g^\circ(x^\pm) \leq |Ker(y^\pm)|$ or $g^\circ(y^\pm) \leq |Ker(x^\pm)|$. □

Theorem 3.18. *Let $(G, \Omega, *, \mu^{\pm})$ be a grey group $x, y \in G$ and $\mu(\Omega) = 2$. If $\mu^-((x*y)^{\pm}) = \mu^+((x*y)^{\pm})$, then one of the following statements is true:*

(i) $(y^+ - y^- \leq x^+ - x^-, |y^+ + y^-| \leq |x^+ + x^-|$ *and* $(x^+ + x^- + y^- \geq y^+$ *or* $x^+ + x^- + y^+ \leq y^-)$,

(ii) $(x^+ - x^- \leq y^+ - y^-, |x^+ + x^-| \leq |y^+ + y^-|$ *and* $(y^+ + y^- + x^- \geq x^+$ *or* $y^+ + y^- + x^+ \leq x^-)$,

Proof. It is clear by Theorem 3.17. □

Example 3.19. (i) *Consider the finite group $(\mathbb{Z}_4, +)$ and $\Omega = [0,1]$. Then $(\mathbb{Z}_4, +, [A^L, A^U])$ is an interval-valued fuzzy group in Table 3. Since $A^L(\overline{1}+\overline{2}) = 0.4 \neq 0.1 = Sup\{g^\circ(\overline{0}) \wedge g^\circ(\overline{3}), g^\circ(\overline{1}) \wedge g^\circ(\overline{2})\}$, so it is not a grey group.*

Table 3: Interval-Valued Fuzzy group $(\mathbb{Z}_4, +, [A^L, A^U])$

$[A^L, A^U]$	$(\overline{0}, (0.7, 0.8))$	$(\overline{1}, (0.4, 0.5))$	$(\overline{2}, (0.45, 0.6))$	$(\overline{3}, (0.4, 0.5))$
$(\overline{0}, (0.7, 0.8))$	$(\overline{0}, (0.7, 0.8))$	$(\overline{1}, (0.4, 0.5))$	$(\overline{2}, (0.45, 0.6))$	$(\overline{3}, (0.4, 0.5))$
$(\overline{1}, (0.4, 0.5))$	$(\overline{1}, (0.4, 0.5))$	$\{(\overline{2}, (0.45, 0.6))$	$(\overline{3}, (0.4, 0.5))\}$	$(\overline{0}, (0.7, 0.8))$
$(\overline{2}, (0.45, 0.6))$	$(\overline{2}, (0.45, 0.6))$	$(\overline{3}, (0.4, 0.5))$	$(\overline{0}, (0.7, 0.8))$	$(\overline{1}, (0.4, 0.5))$
$(\overline{3}, (0.4, 0.5))$	$(\overline{3}, (0.4, 0.5))$	$(\overline{0}, (0.7, 0.8))$	$(\overline{1}, (0.4, 0.5))$	$(\overline{2}, (0.45, 0.6))$

(ii) *Consider the finite group $(\mathbb{Z}_4, +)$ and $\Omega = [0,1]$. Then $(\mathbb{Z}_4, +, \mu^{\pm})$ is grey group in Table 4. Since $\mu^-(\overline{3}+\overline{3}) = 0.3 < 0.4 = Min(\mu^-(\overline{3}), \mu^-(\overline{3}))$, so it is not an interval-valued fuzzy group.*

Table 4: Interval-valued Grey Group $(\mathbb{Z}_4, +, \mu^{\pm})$

$\mu^{\pm}$	$(\overline{0}, [0.8, 0.9])$	$(\overline{1}, [0.7, 1])$	$(\overline{2}, [0.3, 0.8])$	$(\overline{3}, [0.4, 1])$
$(\overline{0}, [0.8, 0.9])$	$(\overline{0}, [0.5, 0.55])$	$(\overline{1}, [0.5, 0.7])$	$(\overline{2}, [0.6, 0.7])$	$(\overline{3}, [0.3, 0.85])$
$(\overline{1}, [0.7, 1])$	$(\overline{1}, [0.5, 0.7])$	$\{(\overline{2}, [0.6, 0.7])$	$(\overline{3}, [0.3, 0.85])\}$	$(\overline{0}, [0.5, 0.55])$
$(\overline{2}, [0.3, 0.8])$	$(\overline{2}, [0.6, 0.7])$	$(\overline{3}, [0.3, 0.85])$	$(\overline{0}, [0.5, 0.55])$	$(\overline{1}, [0.5, 0.7])$
$(\overline{3}, [0.4, 1])$	$(\overline{3}, [0.3, 0.85])$	$(\overline{0}, [0.5, 0.55])$	$(\overline{1}, [0.5, 0.7])$	$(\overline{2}, [0.6, 0.7])$

Remark 3.20. *Example 3.19, shows that the group based on grey numbers is different to group based on interval-valued fuzzy subsets.*

4 Interval-valued grey hypergroups

In this section, we introduce the concept of hypergroup based on interval-valued grey and investigate some basic properties in this complex hyper structures.

Definition 4.1. *Let $\Omega \subseteq \mathbb{R}$, $(H, \circ)$ be a hypergroup, $H^{\pm} = \{x^{\pm} = (x, [x^-, x^+]) \mid x \in H, [x^-, x^+] \subseteq \Omega\}$ and the maps $\mu^-, \mu^+ : H^{\pm} \to \mathbb{R}$ such that $\mu^- \leq \mu^+$. An algebraic hyper structure $(H, \Omega, *, \mu^{\pm})$ or simplify $\mu^{\pm} = [\mu^-, \mu^+]$ is called a grey hypergroup based on hypergroup H, if for all $x, y \in H$, $\mu^{\pm}((x \circ y)^{\pm}) = [\mu^-((x \circ y)^{\pm}), \mu^+((x \circ y)^{\pm})]$, where $\mu^-((x \circ y)^{\pm}) = Sup\{g^\circ(a^{\pm}) \wedge g^\circ(b^{\pm}) \mid a \circ b = x \circ y\}$ and $\mu^+((x \circ y)^{\pm}) = Inf\{|Ker(a^{\pm})| \vee |Ker(b^{\pm})| \mid a \circ b = x \circ y\}$.*

From now on, when we say that $(H, \Omega, \circ, \mu^{\pm})$ is a grey hypergroup, it means that $(H, \circ)$ is a hypergroup and $(H, \Omega, \circ, \mu^{\pm})$ is a grey hypergroup based on hypergroup H.

Example 4.2. *Let* $\Omega = [0, 10]$.

(i) Consider the hypergroup $(H, \circ)$ *as shown in Table 5. Then* $(H, \Omega, \circ, \mu^{\pm})$ *is a grey hypergroup as shown in Table 6.*

Table 5: Hypergroup $(H, \circ)$

$\circ$	a	b	c
a	a	b	c
b	b	$\{a, c\}$	b
c	c	b	a

Table 6: Interval-valued Grey Hypergroup $(H, \circ, \mu^{\pm})$

$\mu^{\pm}$	$(a, [1, 3])$	$(b, [3, 7])$	$(c, [3, 9])$
$(a, [1, 3])$	$(a, [\frac{6}{10}, 2])$	$(b, [\frac{4}{10}, 5])$	$(c, [\frac{2}{10}, 6])$
$(b, [3, 7])$	$(b, [\frac{4}{10}, 5])$	$\{(a, [\frac{6}{10}, 2]), (c, [\frac{2}{10}, 6])\}$	$(b, [\frac{4}{10}, 5])$
$(c, [3, 9])$	$(c, [\frac{2}{10}, 6])$	$(b, [\frac{4}{10}, 5])$	$(a, [\frac{6}{10}, 2])$

(ii) Consider the hypergroup $(\mathbb{N}, \circ)$, *by* $x \circ y = \{x, y\}$. *Define* $(n, [n^-, n^+]) = \begin{cases} (n, [0, 5]) & n \textit{ is an odd,} \\ (n, [0, 8]) & n \textit{ is an even} \end{cases}$.

Then for all $m, n \in \mathbb{N}$, *computations show that* $\mu^{\pm}((m \circ n)^{\pm}) = \begin{cases} (\{m, n\}, [\frac{5}{10}, \frac{5}{2}]) & m, n \textit{ are odd,} \\ (\{m, n\}, [\frac{8}{10}, 4]) & m, n \textit{ are even,} \\ (\{m, n\}, [\frac{5}{10}, 4]) & o.w \end{cases}$.

Theorem 4.3. *Let* $(H, \circ, \mu^{\pm})$ *be a grey hypergroup and* $x, y, z \in H$. *Then*

(i) $\mu^-(x \circ x) \geq g^\circ(x^{\pm})$,

(ii) $\mu^+(x \circ x) \leq |Ker(x^{\pm})|$,

(iii) $g^\circ(x^{\pm}) \leq |Ker(x^{\pm})|$.

Proof. Proofs are clear by Definition 4.1. $\square$

Corollary 4.4. *Any grey hypergroup is a generalization of a hypergroup.*

For an equivalence relation ρ on a hypergroup (H, ϑ), a hyperoperation θ on H/ρ is defined by $\theta(\rho(a), \rho(b)) = \{\rho(c) \mid c \in \vartheta(\rho(a), \rho(b))\}$. It was shown that $(H/\rho, \theta)$ is a hypergroup if and only if ρ is a regular and it is a group if and if only ρ is a strongly regular [5]. One can see that $(H/\beta^*, \theta)$ is a group (*fundamental group*), where for the set of all finite hyperproduct of elements H, $\mathcal{U}(H)$, $a \; \beta \; b \Longleftrightarrow \exists \; u \in \mathcal{U}(H)$ such that $\{a, b\} \subseteq u$ and β^* is the *transitive closure* of β. Let $(H, \circ, \mu^{\pm})$ be a grey hypergroup.

Define $H/\beta^* = \{(\beta^*(x))^{\pm} \mid x \in H\}$(will denote it's grey group by $\mu^{\pm}/\beta^*$) and for all $x \in H, g^\circ((\beta^*(x))^{\pm}) = \bigwedge\limits_{y \in \beta^*(x)} g^\circ(y^{\pm})$ and $|Ker((\beta^*(x))^{\pm})| = \bigvee\limits_{y \in \beta^*(x)} |Ker(y^{\pm})|$. So have the following results.

Theorem 4.5. *Let* $(H, \circ, \mu^{\pm})$ *be a grey hypergroup and* $x \in H$. *Then*

(i) $g^{\circ}((\beta^*(x))^{\pm}) \leq g^{\circ}(x^{\pm})$,

(ii) $|Ker((\beta^*(x))^{\pm})| \geq |Ker(x^{\pm})|$,

(iii) $g^{\circ}((\beta^*(x))^{\pm}) \leq |Ker((\beta^*(x))^{\pm})|$,

(iv) *if* $x \in y \circ z$, *then* $g^{\circ}(x^{\pm}) \geq g^{\circ}((\beta^*(y \circ z))^{\pm})$.

Proof. $(i), (ii)$ Let $x \in H$. Since $x^{\pm} \in (\beta^*(x))^{\pm}$, we get that $g^{\circ}(x^{\pm}) \geq g^{\circ}((\beta^*(x))^{\pm})$ and $|Ker((\beta^*(x))^{\pm})| \geq |Ker(x^{\pm})|$.

(iii) Let $x \in H$. Using Proposition 3.13, $g^{\circ}(x^{\pm}) \leq |Ker(x^{\pm})|$, so by items $(i), (ii)$, we get that $g^{\circ}((\beta^*(x))^{\pm}) \leq g^{\circ}(x^{\pm}) \leq |Ker(x^{\pm})| \leq |Ker((\beta^*(x))^{\pm})|$.

(iv) Let $x, y, z \in H$. Since $x \in y \circ z$, we get that $x^{\pm} \in (\beta^*(x))^{\pm} \in \beta^*((y \circ z)^{\pm})$, and so by item (i), $g^{\circ}(x^{\pm}) \geq g^{\circ}((\beta^*(x))^{\pm}) \geq g^{\circ}((\beta^*(y \circ z))^{\pm})$. □

Definition 4.6. *Let* $(H, \circ, \mu^{\pm})$ *be a grey hypergroup and* $\mu^{\pm}/\beta^* = [\mu^-/\beta^*, \mu^+/\beta^*]$*(will denote by* $\nu^{\pm} = \mu^{\pm}/\beta^*, \nu^- = \mu^-/\beta^*, \nu^+ = \mu^+/\beta^*$ *for simplify), where* $\nu^-, \nu^+ : H/\beta^* \to \mathbb{R}$. *For all* $x \in H$, *define* $\nu^-(\beta^*(x))) = n - \dfrac{m}{2}\mu(\Omega)$ *and* $\nu^+(\beta^*(x))) = n + \dfrac{m}{2}\mu(\Omega)$, *where*

$$m = \bigvee_{\beta^*(a \circ b) = \beta^*(x \circ e)} (g^{\circ}((\beta^*(a))^{\pm} \wedge g^{\circ}((\beta^*(b))^{\pm}),$$
$$n = \bigwedge_{\beta^*(a \circ b) = \beta^*(x \circ e)} (|Ker((\beta^*(a))^{\pm})| \vee |Ker((\beta^*(b))^{\pm}))|,$$

and $\beta^*(e)$ *is the identity element in the group* H/β^*.

Theorem 4.7. *Let* $(H, \circ, \mu^{\pm})$ *be a grey hypergroup. Then* $(H/\beta^*, \theta, \nu^{\pm})$ *is a grey group.*

Proof. Let $x, y \in H$. Clearly H/β^* is a group. Since $(H, \circ, \mu^{\pm})$ is a grey hypergroup, for all $t \in \beta^*(x \circ y)$

$$\begin{aligned}
\nu^-((\theta(\beta^*(x), \beta^*(y)))^{\pm}) &= \nu^-((\beta^*(x \circ y))^{\pm}) = \frac{\nu^+(\beta^*(t)) - \nu^-(\beta^*(t))}{\mu(\Omega)} \\
&= \frac{n + \frac{m}{2}\mu(\Omega) - n + \frac{m}{2}\mu(\Omega)}{\mu(\Omega)} = m \\
&= \bigvee_{\beta^*(a \circ b) = \beta^*(x \circ e)} (g^{\circ}((\beta^*(a))^{\pm} \wedge g^{\circ}((\beta^*(b))^{\pm}).
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\nu^+((\theta(\beta^*(x), \beta^*(y)))^{\pm}) &= \nu^+((\beta^*(x \circ y))^{\pm}) = \frac{|\nu^+(\beta^*(t)) + \nu^-(\beta^*(t))|}{2} \\
&= \frac{n + \frac{m}{2}\mu(\Omega) + n - \frac{m}{2}\mu(\Omega)}{\mu(\Omega)} = n \\
&= \bigwedge_{\beta^*(a \circ b) = \beta^*(x \circ e)} (|Ker((\beta^*(a))^{\pm})| \vee |Ker((\beta^*(b))^{\pm}))|.
\end{aligned}$$

It follows by Definitions 3.1 and 4.6, that $(H/\beta^*, \theta, \mu^{\pm}/\beta^*)$ is a grey group. □

Example 4.8. *Consider the grey hypergroup* $(H, \circ, \mu^{\pm})$ *in Example 4.2, Table 6. Then* $(\mu^{\pm}/\beta^*, \theta)$ *is a grey group as depicted in Table 7.*

Table 7: Interval-valued Grey Group $(H/\beta^*, \theta, \mu^{\pm}/\beta^*)$

$\mu^{\pm}/\beta^*$	$(\beta^*(a), [3,7])$	$(\beta^*(b), [5,7])$
$(\beta^*(a), [3,7])$	$(\beta^*(a), [\frac{4}{10}, 5])$	$(\beta^*(b), [\frac{2}{10}, 6])$
$(\beta^*(b), [5,7])$	$(\beta^*(b), [\frac{2}{10}, 6])$	$(\beta^*(a), [\frac{4}{10}, 5])$

Conclusion

In this paper, we apply the concept of interval grey numbers and construct the structures of interval-valued grey groups and interval-valued grey hypergroups. Also, it is shown that:

(i) any interval-valued grey group is a generalization of a group,

(ii) any interval-valued grey group hypergroup is a generalization of a hypergroup,

(ii) if the measure of universe set is equal to two, then obtain the (non-)negative-polars or (non-)positive-polars.

(iii) the fundamental relation makes the main tool in the connection between grey groups and grey hypergroups,

(iv) the quotient of any grey hypergroup on fundamental relation is a grey group, under some conditions.

We hope that these results are helpful for further studies in the theory of groups and hypergroups. In our future studies, we hope to obtain more results regarding rings, hyperrings, modules, and hypermodules based on grey numbers as a generalization of rings, hyperrings, modules, and hypermodules and obtain some results in this regard.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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