Journal of Algebraic Hyperstructures and Logical Algebras

Volume 3, Number 3, (2022), pp. 65-84

On graded absorbing hyperideals in graded Krasner hyperrings

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Abstract

Let G be a group (monoid) with identity e and R be a commutative Krasner hyperring. In this paper, we introduce the concepts of graded absorbing hyperideals of a graded Krasner hyperring such as, graded 2-absorbing hyperideals, graded n-absorbing hyperideals and graded 2-absorbing subhypermodules. Some basic properties of these structures and characterizations of these graded absorbing hyperideals and homogeneous components are proved.

Article Information

Corresponding Author:
P. Ghiasvand;
Received: July 2022;
Revised: August 2022;
Accepted: August 2022;
Paper type: Original.

Keywords:

Graded Krasner hyperring, graded hyperideal, graded n-absorbing hyperideal, graded 2-absorbing subhypermodule.

1 Introduction

Algebraic hyperstructures represent a natural generalization of classical algebraic structures. Hyperstructure theory was born in 1934, when Marty, a French mathematician, at the 8th Congress of Scandinavian Mathematicians gave the definition of hypergroup and illustrated some of their applications, with utility in the study of groups, algebraic functions and rational fractions. The first example of hypergroups, which motivated the introduction of these new algebraic structures, was the quotient of a group by any, not necessary normal, subgroup. More exactly, if the subgroup is not normal, then the quotient is not a group, but it is always a hypergroup with respect to a certain hyperoperation. The notion of hyperrings was introduced by M. Krasner [18]. Prime, primary, and maximal subhypermodules of a hypermodule were discussed by M. M. Zahedi and R. Ameri in [23]. Also, R. Ameri et al introduced Krasner (m, n)-hyperrings in [1] and in [2] studied

https://doi.org/10.52547/HATEF.JAHLA.3.3.5

prime and primary subhypermodules of (m, n)-hypermodules. The principal notions of algebraic hyperstructure theory can be found in [6, 7, 9, 10, 21].

Furthermore, the study of graded rings arises naturally out of the study of affine schemes and allows them to formalize and unify arguments by induction [22]. However, this is not just an algebraic trick. The concept of grading in algebra, in particular graded modules is essential in the study of homological aspect of rings. Much of the modern development of the commutative algebra emphasizes graded rings. Graded rings play a central role in algebraic geometry and commutative algebra. Gradings appear in many circumstances, both in elementary and advanced level. In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied (see [5, 11, 12, 19] and [20]). Theory of graded hyperrings and greded hypermodules can be considered as an extension theory of hyperrings and hypermodules. In addition, graded hyperrings and graded hypermodules are extensions of graded rings and graded modules [13, 14, 15, 16].

In 2007, Badawi [4] introduced the concept of 2-absorbing ideals of commutative rings with identity, which is a generalization of prime ideals, and investigated some properties of them. He defined a 2-absorbing ideal P of a commutative ring R with identity to be a proper ideal of R and if whenever $a, b, c \in R$ with $abc \in P$, then $ab \in P$ or $bc \in P$ or $ac \in P$. In 2011, Darani and Soheilnia [8] introduced the concept of 2-absorbing submodules of modules over commutative rings. A proper submodule P of a module M over a commutative ring R with identity is said to be a 2-absorbing submodule of M if whenever $a, b \in R$ and $m \in M$ with $abm \in P$, then $abM \subseteq P$ or $am \in P$ or $bm \in P$. One can see that 2-absorbing submodules are generalization of prime submodules. Moreover, it is obvious that 2-absorbing ideals are special cases of 2-absorbing submodules.

Recently, this notion is generalized to the hypercase by introducing the 2-absorbing hyperideals in a multiplicative hyperring [3]. The notion of the 2-absorbing hyperideals on Krasner hyperrings is introduced by Kamali Ardekani and B. Davvaz [17]. In this paper we introduce the notions of graded n-absorbing hyperideals, graded 2-absorbing hyperideals and graded 2-absorbing subhypermoudules in a graded Krasner hyperring and some properties are proved.

In the next section, we recall some preliminary definitions and results. In the third section, we study the concept of a graded 2-absorbing hyperideal of a graded Krasner hyperring R and we will investigate some properties of such graded hyperideals. Some examples of graded 2-absorbing hyperideals are given. Moreover, we investigate the behavior of this structure under homogeneous components, graded hyperring homomorphisms, Cartesian product. In continuing, we introduce and study graded n-absorbing hyperideals of a graded krasner hyperring $(R, +, \cdot)$. For example, we proved that a graded n-absorbing hyperideal of a graded Krasner hyperring R for $n \geq 2$ is not necessarily a graded prime hyperideal of R. Also, we gave a sufficient condition for a graded n-absorbing hyperideal to be graded primary hyperideal. Finally, we introduce and study graded 2-absorbing subhypermodules of a graded Krasner hyperring $(R, +, \cdot)$. Also, we prove some basic properties of graded 2-absorbing subhypermodules.

Throughout this work, all Krasner hyperrings are commutative Krasner hyperrings with identity and all hypermodules are unitary hypermodules.

2 Basic definitions and results

In this section we give some definitions and results of hyperstructures which we need to develop our paper. We refer to [9, 13] for these basic properties and information on hyperstructures.

Definition 2.1. [9] (a) *Let H be a non-empty set and $P^*(H)$ denotes the set of all non-empty subsets of H. If $+ : H \times H \longrightarrow P^*(H)$ is a map such that the following conditions hold, then we say that $(H, +)$ is a canonical hypergroup.*

(i) *for every $x, y, z \in H$, $x + (y + z) = (x + y) + z$;*

(ii) *for every $x, y \in H$, $x + y = y + x$;*

(iii) *there exists $0 \in H$ such that $0 + x = \{x\}$ for every $x \in H$;*

(iv) *for every $x \in H$ there exists a unique element $x' \in R$ such that $0 \in x + x'$, it is denoted by $-x$;*

(v) *for every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.*

(b) *Let $A \subset H$. Then A is called a subhypergroup of H if $0 \in H$ and $(A, +)$ is itself a hypergroup.*

Definition 2.2. [9] *A Krasner hyperring is an algebraic hyperstructure $(R, +, \cdot)$ which satisfies the following axioms:*

(1) *$(R, +)$ is a canonical hypergroup;*

(2) *$(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$;*

(3) *the operation "$\cdot$" is distributive over the hyperoperation "$+$", which means that for all x, y, z of R we have:*

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ and } (x + y) \cdot z = x \cdot z + y \cdot z.$$

A Krasner hyperring $(R, +, \cdot)$ is called *commutative with identity* $1 \in R$; if we have

(i) $xy = yx$ for all $x, y \in R$,

(ii) $1x = x1$ for all $x \in R$.

Definition 2.3. [9] (a) *Let $(R, +, \cdot)$ be a Krasner hyperring and $S \subset R$. Then S is said to be a subhyperring of R if $(S, +, \cdot)$ is itself a hyperring.*

(b) *A subhyperring I of a Krasner hyperring R is a left (right) hyperideal of R if $rx \in I (xr \in I)$ for all $r \in R$, $x \in I$. I is called a hyperideal if I is both a left and a right hyperideal.*

Definition 2.4. [23] (a) *Let $(M, +)$ be a canonical hypergroup and $(R, +, \cdot)$ be a Krasner hyperring with identity. M is a left hypermodule over a hyperring R if there exists a map*

$$\cdot : R \times M \longrightarrow M; \quad (a, m) \mapsto a \cdot m$$

such that for all $r_1, r_2 \in R$ and $m_1, m_2, m \in M$, the following are satisfied:

(1) *$r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_2 \cdot m_2$;*

(2) *$(r_1 + r_2) \cdot m = (r_1 \cdot m) + (r_2 \cdot m)$;*

(3) *$(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$;*

(4) *$1m = m$ and $0m = 0$.*

(b) A non-empty subset N of an R-hypermodule M is called a subhypermodule if N is an R-hypermodule with the operations of M.

Definition 2.5. [13] *Let G be a group (monoid) with identity e. A Krasner hyperring (R, G) is called a G-graded Krasner hyperring, if there exists a family $\{R_g\}_{g\in G}$ of canonical subhypergroups of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. For simplicity, we will denote the graded Krasner hyperring (R, G) by R. An element of a graded Krasner hyperring R is called homogeneous if it belongs to $\bigcup_{g\in G} R_g$ and this set of homogeneous elements is denoted by $h(R)$. If $x \in R_g$ for some $g \in G$, then we say that x is of degree g, and it is denoted by $\deg x$.*

If $x \in R$, then there exist unique elements $x_g \in h(R)$ such that $x \in \sum_{g\in G} x_g$. In fact, every Krasner hyperring is trivially a G-graded Krasner hyperring by letting $R_e = R$ and $R_g = 0$ for all $g \neq e$.

Lemma 2.6. *If $R = \bigoplus_{g\in G} R_g$ is a graded Krasner hyperring, then R_e is a subhyperring of R where e is the identity element of monoid G.*

Example 2.7. *In Definition 2.5, let $G = (\mathbb{Z}_2, \cdot)$ be the monoid with identity $e = 1$ and $R = \{0, 1, 2, 3\}$. Consider the Krasner hyperring $(R, +, \cdot)$, where hyperoperation $+$ and operation $\cdot$ are defined on R as follows:*

+	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{1}	{0,1}	{3}	{2,3}
2	{2}	{3}	{0}	{1}
3	{3}	{2,3}	{1}	{0,1}

.	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	2
3	0	0	2	2

It is easy to see that $R_0 = \{0, 1\}$ and $R_1 = \{0, 2\}$ are subhypergroups of $(R, +)$. We have $0 \in 0 + 0$, $1 \in 1 + 0$, $2 \in 0 + 2$ and $3 \in 1 + 2$. Furthermore, we have $R_iR_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}_2$. Hence, $R = R_0 \oplus R_1$ and so R is a $\mathbb{Z}_2$-graded Krasner hyperring and $h(R) = \{0, 1, 2\}$.

Example 2.8. *In Definition 2.5, let $G = (\mathbb{Z}_4, +)$ be the cyclic group of order 4 and $R = \{0, a, b, c, d\}$. Consider the Krasner hyperring $(R, +, \cdot)$, where hyperoperation $+$ and operation $\cdot$ are defined on R as follows:*

+	0	a	b	c	d
0	{0}	{a}	{b}	{c}	{d}
a	{a}	{0}	{c}	{b, d}	{c}
b	{b}	{c}	{0}	{a}	{0}
c	{c}	{b, d}	{a}	{0}	{a}
d	{d}	{c}	{0}	{a}	{0}

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	b	c	d
b	0	b	0	b	0
c	0	c	b	a	d
d	0	d	0	d	0

Let $R_0 = \{0, a\}$, $R_2 = \{0, c\}$ and $R_1 = R_3 = \{0\}$. Then it is easy to verify that R_0, R_1, R_2 and R_3 are canonical hypergroups of $(R, +)$ and we can write $0 \in 0 + 0 + 0 + 0$, $a \in a + 0 + 0 + 0$, $b \in a+0+c+0$, $c \in 0+0+c+0$ and $d \in a+0+c+0$ uniquely. Furthermore, we have $R_iR_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}_4$. Hence, $R = R_0 \oplus R_1 \oplus R_2 \oplus R_3$ and so (R, G) is a graded Krasner hyperring and $h(R) = \{0, a, c\}$.

Definition 2.9. [13] *(a) Let $R = \bigoplus_{g\in G} R_g$ be a graded Krasner hyperring. A subhyperring S of R is called a graded subhyperring of R, if $S = \bigoplus_{g\in G}(S \cap R_g)$. Equivalently, S is graded if for every element $f \in S$, all the homogeneous components of f (as an element of R) are in S.*

(b) *Let I be a hyperideal of a graded Krasner hyperring R. Then I is a graded hyperideal, if $I = \bigoplus_{g \in G}(I \cap R_g)$. For any $a \in I$ and for some $r_g \in h(R)$ that $a \in \sum_{g \in G} r_g \subseteq I$, then $r_g \in I \cap R_g$ for all $g \in G$.*

Example 2.10. *Consider the graded Krasner hyperring of Example 2.7. Take $I = \{0, 2\}$. It is clear that I is a hyperideal. Since $I = \{0, 2\} = (\{0, 2\} \cap R_0) \oplus (\{0, 2\} \cap R_1)$ where $R_0 = \{0, 1\}$ and $R_1 = \{0, 2\}$, then I is a graded hyperideal of R.*

Let $R = \bigoplus_{g \in G} R_g$ be a graded Krasner hyperring and I a graded hyperideal of R. Then the quotient hyperring $(R/I, \oplus, \circ)$ where $(a+I) \circ (b+I) = ab+I$, for any $a, b \in R$ and $(a+I) \oplus (b+I) = \{t + I | t \in a + b\}$, for any $a, b \in R$ is also a graded Krasner hyperring with $R/I = \bigoplus_{g \in G}(R/I)_g$, where $(R/I)_g = (R_g + I)/I$.

Definition 2.11. [13] (a) *If $P \neq R$ is a graded hyperideal of a graded Krasner hyperring R, then P is called a graded prime hyperideal of R, if $a_g b_h \in P$, then $a_g \in P$ or $b_h \in P$ for $a_g, b_h \in h(R)$.*

(b) *A graded hyperring $R = \bigoplus_{g \in G} R_g$ is a graded hyperintegral domain, if $a_g b_h = 0$, for $a_g, b_h \in h(R)$, then $a_g = 0$ or $b_h = 0$.*

(c) *The graded hyperideal M of a graded Krasner hyperring R is said to be maximal, if for every graded hyperideal J of R; $M \subseteq J \subseteq R$, implies that $J = M$ or $J = R$.*

Definition 2.12. [13] *A non-empty subset S of $h(R)$ of a graded Krasner hyperring R is called multiplicative closed subset if $s_1 s_2 \in S$ for all $s_1, s_2 \in S$.*

Let G be a group and let R be a G-graded Krasner hyperring and $S \subseteq h(R)$ a multiplicative close subset of R. Then the hyperring of fractions $S^{-1}R$ is a graded Krasner hyperring which is called the graded Krasner hyperring of fractions. Indeed, $S^{-1}R = \bigoplus_{g \in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s | r \in R, s \in S; g = (degs)^{-1}(degr)\}$.

Definition 2.13. [13] *Let I be a graded hyperideal in a commutative graded Krasner hyperring R with identity. The graded radical of I (in abbreviation, $Grad(I)$) is the set of all $x \in R$ such that for each $g \in G$ there exists a positive integer n_g such that $x_g^{n_g} \in I$ where $x_g^{n_g} = x_g \cdots x_g (n_g$ times). Note that, if r is a homogeneous element of R, then $r \in Grad(I)$ iff $r^n \in I$ for some positive integer n.*

Definition 2.14. [13] (a) *Let $R = \bigoplus_{g \in G} R_g$ and $S = \bigoplus_{g \in G} S_g$ be graded Krasner hyperrings. A mapping ϕ from R into S is said to be a graded good homomorphism, if for all $a, b \in R$;*

(1) *$\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(0) = 0$,*

(2) *$\phi(ab) = \phi(a)\phi(b)$,*

(3) *for any $g \in G$; $\phi(R_g) \subseteq S_g$.*

(b) *A graded good homomorphism $\phi : R \to S$ is a graded isomorphism, if ϕ is one to one and onto and we write $R \cong S$.*

Definition 2.15. [13] *Let M be an R-hypermodule. Then M is said to be a G-graded R-hypermodule if there exists a family of canonical subhypergroups $\{M_g\}_{g \in G}$ of M such that*

(1) *$M = \bigoplus_{g \in G} M_g$,*

(2) *$R_g M_h \subseteq M_{gh}$ for all $g, h \in G$.*

The set of all homogeneous elements of M is denoted by $h(M)$, and so $h(M) = \bigcup_{g \in G} M_g$.

Definition 2.16. [13] (a) *A non-empty subset N of a graded R-hypermodule M is called a graded subhypermodule, if N is a graded R-hypermodule with the operations of M restricted to N.*

(b) *A proper graded subhypemodule N of a graded R-hypermodule M is said to be graded prime, if $r_g m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then $m_h \in N$ or $r_g \in (N : M)$.*

3 Graded 2-absorbing hyperideals

In this section, we study the concept of a graded 2-absorbing hyperideal of a graded Krasner hyperring R and we investigate some properties of such graded hyperideals.

Definition 3.1. *Let $R = \bigoplus_{g \in G} R_g$ be a graded Krasner hyperring. A proper graded hyperideal I is called a graded 2-absorbing hyperideal of R; if $a_g b_h c_k \in I$, then $a_g b_h \in I$ or $a_g c_k \in I$ or $b_h c_k \in I$ for all $a_g, b_h, c_k \in h(R)$.*

Example 3.2. *Let $G = (\mathbb{Z}_2, \cdot)$ be the monoid with identity $e = 1$ and $R = \{0, 1, 2, 3\}$. Consider the hyperring $(R, +, \cdot)$, where hyperoperation "$+$" and operation "$\cdot$" are defined on R as follows:*

$+$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,2\}$	$\{1,3\}$	$\{2\}$
2	$\{2\}$	$\{1,3\}$	$\{0,2\}$	$\{1\}$
3	$\{3\}$	$\{2\}$	$\{1\}$	$\{0\}$

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	0
3	0	3	0	3

Let $R_0 = \{0, 2\}$ and $R_1 = \{0, 3\}$. Then it is easy to verify that R_0 and R_1 are canonical hypergroups of $(R, +)$ and we can write $0 \in 0+0$, $1 \in 2+3$, $2 \in 2+0$ and $3 \in 0+3$ uniquely, hence $R = R_0 \bigoplus R_1$. Also, $R_i R_j \subseteq R_{ij}$ for any $i, j \in \mathbb{Z}_2$ and so R is a $\mathbb{Z}_2$-graded Krasner hyperring. Let $I = \{0, 2\}$. Then I is a graded 2-absorbing hyperideal of R.

Example 3.3. *Let $R = \{0, a, b, c, d, f\}$ and $G = (\mathbb{Z}_2, \cdot)$. Consider the hyperring $(R, +, \cdot)$, where hyperoperation "$+$" and operation "$\cdot$" are defined on R as follows:*

$+$	0	a	b	c	d	f
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{f\}$
a	$\{a\}$	$\{0\}$	$\{a,b\}$	$\{d\}$	$\{a,d\}$	$\{c\}$
b	$\{b\}$	$\{a,b\}$	$\{0\}$	$\{f\}$	$\{b,d\}$	$\{a\}$
c	$\{c\}$	$\{d\}$	$\{f\}$	$\{0\}$	$\{f\}$	$\{c\}$
d	$\{d\}$	$\{a,d\}$	$\{b,d\}$	$\{f\}$	$\{0\}$	$\{d,f\}$
f	$\{f\}$	$\{c\}$	$\{a\}$	$\{c\}$	$\{d,f\}$	$\{0\}$

$\cdot$	0	a	b	c	d	f
0	0	0	0	0	0	0
a	0	a	b	0	a	b
b	0	b	b	a	b	a
c	0	0	a	c	c	c
d	0	a	b	c	d	f
f	0	b	a	c	f	f

It is easy to see that $R_0 = \{0, a, b\}$ and $R_1 = \{0, c\}$ are subhypergroups of $(R, +)$. We have R is a $\mathbb{Z}_2$-graded Krasner hyperring. Let $I = \{0, a, b\}$. Then I is a graded 2-absorbing hyperideal of R.

Proposition 3.4. *If P is a graded prime hyperideal of a graded Krasner hyperring $R = \bigoplus R_g$, then P is a graded 2-absorbing hyperideal of R.*

Proof. Assume that P is a graded prime hyperideal of a graded Krasner hyperring R. Let $a_g, b_h, c_k \in h(R)$ such that $a_g b_h c_k \in P$. Since P is graded prime, we have $a_g b_h \in P$ or $c_k \in P$. If $a_g b_h \in P$ then we done. If $c_k \in P$ then $a_g c_k \in P$ and it follows that P is a graded 2-absorbing hyperideal. □

Proposition 3.4, guarantees that every graded prime hyperideal of a graded Krasner hyperring R is a graded 2-absorbing hyperideal. The converse does not necessarily hold in general.

Example 3.5. *Let $R = \{0, a, b, c, d\}$ and $G = (\mathbb{Z}_2, \cdot)$ be the monoid. Consider the hyperring $(R, +, \cdot)$, where hyperoperation "$+$" and operation "$\cdot$" are defined on R as follows:*

$+$	0	a	b	c	d
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
a	$\{a\}$	$\{0\}$	$\{c,d\}$	$\{b,d\}$	$\{c\}$
b	$\{b\}$	$\{c,d\}$	$\{0\}$	$\{a\}$	$\{c\}$
c	$\{c\}$	$\{b,d\}$	$\{a\}$	$\{0\}$	$\{a\}$
d	$\{d\}$	$\{c\}$	$\{c\}$	$\{a\}$	$\{0\}$

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	d
b	0	a	0	b	0
c	0	a	b	c	d
d	0	d	0	d	0

It is easy to see that $R_0 = \{0, a\}$ and $R_1 = \{0, b\}$ are subhypergroups of $(R, +)$. We have $0 \in 0 + 0$, $a \in a + 0$, $b \in 0 + b$, $c \in a + b$ and $d \in a + b$. Hence, $R = R_0 \bigoplus R_1$ also, $R_i R_j \subseteq R_{ij}$ for any $i, j \in \mathbb{Z}_2$ and so R is a $\mathbb{Z}_2$-graded Krasner hyperring. Let $I = \{0, d\}$. Then I is a graded 2-absorbing hyperideal of R, because, $abd \in I \to ad = d \in I$, $acd \in I \to ad = d \in I$ and $bcd \in I \to bd = 0 \in I$. But it is not a graded prime hyperideal, because, $b^2 \in I$ and $b \notin I$.

Proposition 3.6. *The intersection of each pair of graded prime hyperideals of a graded Krasner hyperring R is a graded 2-absorbing hyperideal of R.*

Proof. Let P and K be two graded prime hyperideals. If $P = K$; then $P \cap K$ is a graded prime hyperideal of R so that $P \cap K$ is a graded 2-absorbing hyperideal of R. Assume that P and K are distinct. Since P and K are proper graded hyperideals of R, it follows that $P \cap K$ is a proper graded hyperideal. Next, let $a_g, b_h, c_k \in h(R)$ such that $a_g b_h c_k \in P \cap K$ but $a_g c_k \notin P \cap K$ and $a_g b_h \notin P \cap K$. Then, we can conclude that:

(i) $a_g c_k \notin P$ or $a_g c_k \notin K$, and
(ii) $a_g b_h \notin P$ or $a_g b_h \notin K$.

These two conditions give four cases:

(1) $a_g c_k \notin P$ and $a_g b_h \notin P$;

(2) $a_g c_k \notin P$ and $a_g b_h \notin K$;

(3) $a_g c_k \notin K$ and $a_g b_h \notin P$;

(4) $a_g c_k \notin K$ and $a_g b_h \notin K$.

First, we consider Case (1). Since $a_g b_h c_k \in P \cap K \subseteq P$ and $a_g b_h \notin P$, we get $c_k \in P$, hence $a_g c_k \in P$, a contradiction. Similarly, Case (4) is not possible.
Now, Case (2) is considered. $a_g b_h c_k \in K$ and $a_g b_h \notin K$ implies $c_k \in K$ and so $b_h c_k \in K$. $a_g(b_h c_k) \in P$ implies $a_g \in P$ or $b_h c_k \in P$ because P is a graded prime hyperideal. Since $a_g c_k \notin P$, $a_g \in P$ is not possible. Hence $b_h c_k \in P \cap K$. The proof of Case (3) is similar to that of Case (2). □

Proposition 3.7. *Let P, W be two graded hyperideals of R and $P \subseteq W$. If P is a graded 2-absorbing hyperideal of R, then P is a graded 2-absorbing hyperideal of W.*

Proof. Note that we can write $W = \bigoplus W_g$, where $W_g = R_g \cap W$. Let $a_g b_h c_k \in P$, where $a_g, b_h, c_k \in h(W)$. Since $a_g, b_h, c_k \in h(W) \subseteq h(R)$ and P is a graded 2-absorbing hyperideal of R, either $a_g c_k \in P$ or $b_h c_k \in P$ or $a_g b_h \in P$. Hence P is a graded 2-absorbing hyperideal of W. □

Lemma 3.8. *Let P be a graded 2-absorbing hyperideal of a graded hyperring R. If $a_h, b_{h'} \in h(R)$ and $K = \oplus_{g\in G} K_g$ is a graded hyperideal of R such that $a_h b_{h'} K_g \subseteq P$, for some $g \in G$, then $a_h b_{h'} \in P$ or $a_h K_g \subseteq P$ or $b_{h'} K_g \subseteq P$.*

Proof. Let $a_h b_{h'} K_g \subseteq P$ and $a_h b_{h'} \notin P$, $a_h K_g \nsubseteq P$ and $b_{h'} K_g \nsubseteq P$. Then there exist k_g, k'_g in K_g such that $a_h k_g \notin P$ and $b_{h'} k'_g \notin P$. Since $a_h b_{h'} k_g \in a_h b_{h'} K_g \subseteq P$ and $a_h b_{h'} \notin P$, $a_h k_g \notin P$, we get $b_{h'} k_g \in P$. Also, since $a_h b_{h'} k'_g \in a_h b_{h'} K_g \subseteq P$ and $a_h b_{h'} \notin P$, $b_{h'} k'_g \notin P$, we get $a_h k'_g \in P$. Now, since $a_h b_{h'}(k_g + k'_g) \subseteq a_h b_{h'} K_g \subseteq P$, so $a_h b_{h'} t_g \subseteq P$ for any $t_g \in k_g + k'_g$ and as $a_h b_{h'} \notin P$ we get $a_h t_g \in P$ or $b_{h'} t_g \in P$. Thus $a_h(k_g + k'_g) \subseteq P$ or $b_{h'}(k_g + k'_g) \subseteq P$. If $a_h(k_g + k'_g) \subseteq P$, i.e., $(a_h k_g + a_h k'_g) \subseteq P$, then $a_h k_g \in P$ which is contradiction. If $b_{h'} k_g + b_{h'} k'_g \subseteq P$, then $b_{h'} k'_g \in P$ which is a contradiction. Thus either $a_h b_{h'} \in P$ or $a_h K_g \subseteq P$ or $b_{h'} K_g \subseteq P$. □

Theorem 3.9. *Let P be a graded 2-absorbing hyperideal of a graded hyperring $R = \oplus_{g\in G} R_g$. Then if I, J and K are graded hyperideals of R and $g \in G$ such that $IJK_g \subseteq P$, then $IK_g \subseteq P$ or $JK_g \subseteq P$ or $IJ \subseteq P$.*

Proof. Suppose $IJK_g \subseteq P$ and $IJ \nsubseteq P$. We show that $IK_g \subseteq P$ or $JK_g \subseteq P$. Suppose $IK_g \nsubseteq P$ and $JK_g \nsubseteq P$. Then there exist $a_{g_1} \in h(R) \cap I$ and $a_{g_2} \in h(R) \cap J$ such that $a_{g_1} K_g \nsubseteq P$ and $a_{g_2} K_g \nsubseteq P$. But $a_{g_1} a_{g_2} K_g \subseteq IJK_g \subseteq P$. Since P is a graded 2-absorbing hyperideal it follows from Lemma 3.8 that $a_{g_1} a_{g_2} \in P$. Since $IJ \nsubseteq P$, there exist $b_{h_1} \in h(R) \cap I$ and $b_{h_2} \in h(R) \cap J$ such that $b_{h_1} b_{h_2} \notin P$. Now, since P is graded 2-absorbing and $b_{h_1} b_{h_2} K_g \subseteq IJK_g \subseteq P$ and also $b_{h_1} b_{h_2} \notin P$ it follows from Lemma 3.8 that $b_{h_1} K_g \subseteq P$ or $b_{h_2} K_g \subseteq P$. We have the following cases:

Case (1): $b_{h_1} K_g \subseteq P$ and $b_{h_2} K_g \nsubseteq P$. Since $a_{g_1} b_{h_2} K_g \subseteq IJK_g \subseteq P$ and $a_{g_1} K_g \nsubseteq P$ and $b_{h_2} K_g \nsubseteq P$ it follows from Lemma 3.8 that $a_{g_1} b_{h_2} \in P$. Since $b_{h_1} K_g \subseteq P$ and $a_{g_1} K_g \nsubseteq P$, we conclude $(a_{g_1} + b_{h_1}) K_g \nsubseteq P$. On the other hand since $(a_{g_1} + b_{h_1}) b_{h_2} K_g \subseteq P$ and neither $(a_{g_1} + b_{h_1}) K_g \subseteq P$ nor $b_{h_2} K_g \subseteq P$, we get that $(a_{g_1} + b_{h_1}) b_{h_2} \subseteq P$ by Lemma 3.8. Since $(a_{g_1} + b_{h_1}) b_{h_2} = (a_{g_1} b_{h_2} + b_{h_1} b_{h_2}) \subseteq P$ and $(a_{g_1} + b_{h_1}) b_{h_2} \subseteq P$, we get $b_{h_1} b_{h_2} \in P$ which is a contradiction.

Case (2): $b_{h_2} K \subseteq P$ and $b_{h_1} K_g \nsubseteq P$. By a similar argument to Case (1) we get a contradiction.

Case (3): $b_{h_1} K_g \subseteq P$ and $b_{h_2} K_g \subseteq P$. $b_{h_2} K_g \subseteq P$ and $a_{g_2} K_g \nsubseteq P$ gives $(a_{g_2} + b_{h_2}) K_g \nsubseteq P$. But $a_{g_1}(a_{g_2} + b_{h_2}) K_g \subseteq P$ and neither $a_{g_1} K_g \subseteq P$ nor $(a_{g_2} + b_{h_2}) K_g \subseteq P$, hence $a_{g_1}(a_{g_2} + b_{h_2}) \subseteq P$ by Lemma 3.8. Since $a_{g_1} a_{g_2} \in P$ and $(a_{g_1} a_{g_2} + a_{g_1} b_{h_2}) \subseteq P$, we have $a_{g_1} b_{h_2} \in P$. Since $(a_{g_1} + b_{h_1}) a_{g_2} K_g \subseteq P$ and neither $a_{g_2} K_g \subseteq P$ nor $(a_{g_1} + b_{h_1}) K_g \subseteq P$, we conclude $(a_{g_1} + b_{h_1}) a_{g_2} \subseteq P$ by Lemma 3.8. But $(a_{g_1} + b_{h_1}) a_{g_2} = a_{g_1} a_{g_2} + b_{h_1} a_{g_2}$, so $(a_{g_1} a_{g_2} + b_{h_1} a_{g_2}) \subseteq P$ and since $a_{g_1} a_{g_2} \in P$, we get $b_{h_1} a_{g_2} \in P$. Now, since $(a_{g_1} + b_{h_1})(a_{g_2} + b_{h_2}) K_g \subseteq P$ and neither $(a_{g_1} + b_{h_1}) K_g \subseteq P$ nor $(a_{g_2} + b_{h_2}) K_g \subseteq P$, we have $(a_{g_1} + b_{h_1})(a_{g_2} + b_{h_2}) = (a_{g_1} a_{g_2} + a_{g_1} b_{h_2} + b_{h_1} a_{g_2} + b_{h_1} b_{h_2}) \subseteq P$ by Lemma 3.8. But $a_{g_1} a_{g_2},\ a_{g_1} b_{h_2},\ b_{h_1} a_{g_2} \in P$, so $b_{h_1} b_{h_2} \in P$ which is a contradiction. Consequently $IK_g \subseteq P$ or $JK_g \subseteq P$. □

Proposition 3.10. *Let P and K be graded hyperideals of a graded hyperring R with $K \nsubseteq P$. If P is a graded 2-absorbing hyperideal of R, then $K \cap P$ is a graded 2-absorbing hyperideal of K.*

Proof. Since P, K are graded hyperideals and $K \nsubseteq P$, it follows that $K \cap P$ is a proper graded hyperideal of K. Assume that P is a graded 2-absorbing hyperideal of R. Let $a_g, b_h, c_k \in h(K)$ be such that $a_g b_h c_k \in K \cap P$. Since K is a graded hyperideal of R, we obtain that $a_g b_h \in K$ or $a_g c_k \in K$ or $b_h c_k \in K$. Moreover, since $a_g b_h c_k \in K \cap P \subseteq P$ and P is a graded 2-absorbing hyperideal of R, it follows that $a_g b_h \in P$ or $a_g c_k \in P$ or $b_h c_k \in P$. Thus $a_g b_h \in K \cap P$ or $a_g c_k \in K \cap P$ or $b_h c_k \in K \cap P$. Therefore, $K \cap P$ is a graded 2-absorbing hyperideal of K. □

Remark 3.11. *It is well known that if P and K are graded hyperideals of any graded hyperring R with $K \nsubseteq P$ and P is a graded prime hyperideal of R, then $K \cap P$ is a graded prime hyperideal and hence a graded 2-absorbing hyperideal of K.*

Lemma 3.12. *Let $R = \bigoplus R_g$ be a graded hyperring and I be a graded 2-absorbing hyperideal of R. Then $Grad(I)$ is a graded 2-absorbing hyperideal of R.*

Proof. Suppose that $a_g b_h c_k \in Grad(I)$ where $a_g, b_h, c_k \in h(R)$. So, $(a_g b_h c_k)^n \in I$ for some $n \in \mathbb{N}$. Thus, $a_g^n b_h^n c_k^n \in I$ and this implies that $(a_g b_h)^n = a_g^n b_h^n \in I$ or $(a_g c_k)^n = a_g^n c_k^n \in I$ or $(b_h c_k)^n = b_h^n c_k^n \in I$. Therefore, at least one of $a_g b_h$, $a_g c_k$ and $b_h c_k$ belongs to $Grad(I)$. □

Theorem 3.13. *Let R and S be commutative graded hyperrings with unit element and $\varphi : R \to S$ be a graded good homomorphism. Then the following hold:*

(i) If J is a graded 2-absorbing hyperideal of S, then $\varphi^{-1}(J)$ is a graded 2-absorbing hyperideal of R;

(ii) If φ is an epimorphism and I is a graded 2-absorbing hyperideal of R containing $Ker\varphi$, then $\varphi(I)$ is a graded 2-absorbing hyperideal of S.

Proof. (i) Let $a_g b_h c_k \in \varphi^{-1}(J)$ for $a_g, b_h, c_k \in h(R)$. Thus $\varphi(a_g)\varphi(b_h)\varphi(c_k) = \varphi(a_g b_h c_k) \in J$. Since J is a graded 2-absorbing hyperideal of S, so $\varphi(a_g b_h) = \varphi(a_g)\varphi(b_h) \in J$ or $\varphi(b_h c_k) = \varphi(b_h)\varphi(c_k) \in J$ or $\varphi(a_g c_k) = \varphi(a_g)\varphi(c_k) \in J$. Therefore, $a_g b_h \in \varphi^{-1}(J)$ or $b_h c_k \in \varphi^{-1}(J)$ or $a_g c_k \in \varphi^{-1}(J)$, as needed.

(ii) Let $a'_g b'_h c'_k \in \varphi(I)$ for some $a'_g, b'_h, c'_k \in h(S)$. Since φ is an epimorphism, $a'_g = \varphi(a_g)$, $b'_h = \varphi(b_h)$ and $c'_k = \varphi(c_k)$ for some $a_g, b_h, c_k \in h(R)$. Thus $\varphi(a_g b_h c_k) = \varphi(a_g)\varphi(b_h)\varphi(c_k) \in \varphi(I)$, so $(a_g b_h c_k - x) \cap Ker(\varphi) \neq \emptyset$ for some $x \in I$. Consider $t \in (a_g b_h c_k - x) \cap Ker(\varphi)$. Hence $a_g b_h c_k \in t + x \subseteq Ker(\varphi) + I \subseteq I$. This implies that $a_g b_h \in I$ or $b_h c_k \in I$ or $a_g c_k \in I$, so $a'_g b'_h = \varphi(a_g b_h) \in \varphi(I)$ or $b'_h c'_k = \varphi(b_h c_k) \in \varphi(I)$ or $a'_g c'_k = \varphi(a_g c_k) \in \varphi(I)$, as required. □

The following corollary is deduced directly from Theorem 3.13.

Corollary 3.14. *Let I and J be distinct proper graded hyperideals of R. If $J \subseteq I$ and I is a graded 2-absorbing hyperideal of R, then I/J is a graded 2-absorbing hyperideal of R/J.*

Let R_1 and R_2, be two G-graded Krasner hyperrings where $R_1 = \bigoplus_{g\in G}(R_1)_g$ and $R_2 = \bigoplus_{g\in G}(R_2)_g$. Then $(R_1 \times R_2, +, \cdot)$ is a Krasner hyperring with operation $\cdot$ and the hyperoperation $+$ are defined respectively as $(x, y) \cdot (z, t) = (x \cdot z, y \cdot t)$ and $(x, y) + (z, t) = \{(a, b) \in R \,|\, a \in x + z, b \in y + t\}$ for all $(x, y), (z, t) \in R_1 \times R_2$. Also, $(R_1 \times R_2, +, \cdot)$ becomes a G-graded hyperring with homogeneous elements $h(R_1 \times R_2) = \bigcup_{g\in G}(R_1 \times R_2)_g$, where $(R_1 \times R_2)_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Note that each graded hyperideal of $R_1 \times R_2$ is the Cartesian product of graded hyperideals of R_1 and R_2.

Example 3.15. *Consider the $\mathbb{Z}_2$-graded hyperring of Example 3.5 with R_1 that $(R_1)_0 = \{0, a\}$, $(R_1)_1 = \{0, b\}$ and the $\mathbb{Z}_2$-graded hyperring of Example 3.2 with R_2 that $(R_2)_0 = \{0, 2\}$, $(R_2)_1 = \{0, 3\}$. Then $R = R_1 \times R_2$ is a $\mathbb{Z}_2$-graded Krasner hyperring with*

$$(R_1 \times R_2)_0 = \{(0, 0), (0, 2), (a, 0), (a, 2)\},$$

$$(R_1 \times R_2)_1 = \{(0,0),(0,3),(b,0),(b,3)\},$$

and

$$h(R_1 \times R_2) = \{(0,0),(0,2),(a,0),(a,2),(0,3),(b,0),(b,3)\}.$$

We observe that $(R_1 \times R_2)_0 = (R_1)_0 \times (R_2)_0$ *and* $(R_1 \times R_2)_1 = (R_1)_1 \times (R_2)_1$.

Theorem 3.16. *Let* R_1, R_2 *be* G*-graded Krasner hyperrings and* $R = R_1 \times R_2$. *If* I_1 *(*I_2*, respectively) is a graded* 2*-absorbing hyperideal of* R_1 *(*R_2*, respectively), then* $I_1 \times R_2$ *(*$R_1 \times I_2$*, respectively) is a graded 2-absorbing hyperideal of* R.

Proof. Let I_1 be a graded 2-absorbing hyperideal of G-graded Krasner hyperring R_1 and also $(a_g, a'_g)(b_h, b'_h)(c_k, c'_k) \in I_1 \times R_2$ where $(a_g, a'_g)(b_h, b'_h)(c_k, c'_k) \in h(R_1 \times R_2)$. Thus $a_g b_h c_k \in I_1$, so $a_g b_h \in I_1$ or $a_g c_k \in I_1$ or $b_h c_k \in I_1$ since I_1 is a graded 2-absorbing hyperideal of R_1. Therefore, $(a_g, a'_g)(b_h, b'_h) \in I_1 \times R_2$ or $(a_g, a'_g)(c_k, c'_k) \in I_1 \times R_2$ or $(b_h, b'_h)(c_k, c'_k) \in I_1 \times R_2$. Hence $I_1 \times R_2$ is a graded 2-absorbing hyperideal of R. Similarly, if I_2 is a graded 2-absorbing hyperideal of R_2, then $R_1 \times I_2$ is a graded 2-absorbing hyperideal of R. □

Theorem 3.17. *Let* I *be a graded hyperring of a graded hyperring* R *and* $S \subseteq h(R)$ *be a multiplicatively closed subset of* R. *Then the following statements hold:*

(*i*) *If* I *is a graded* 2*-absorbing hyperideal of* R *and* $S \cap I = \emptyset$*, then* $S^{-1}I$ *is a graded 2-absorbing hyperideal of* $S^{-1}R$.

(*ii*) *If* $S^{-1}I$ *is a graded 2-absorbing hyperideal of* $S^{-1}R$ *and* $S \cap Z_R(R/I) = \emptyset$*, where* $Z_R(R/I) = \{r + I \in R/I \mid$ *there exists* $s \in R \setminus I$ *such that* $rs \in I\}$*, then* I *is a graded 2-absorbing hyperideal of* R.

Proof. (*i*) Let I be a graded 2-absorbing hyperideal of R. Since $S \cap I = \emptyset$, then $S^{-1}I \neq S^{-1}R$. Assume that $(a/s)(b/t)(c/k) \in S^{-1}I$ where $a/s, b/t, c/u \in h(S^{-1}R)$. Then there exists $s' \in S$ such that $(s'a)bc \in I$. Hence $(s'a)b \in I$ or $(s'a)c \in I$ or $bc \in I$ because I is a graded 2-absorbing hyperideal of R. Therefore, $(a/s)(b/t) = (s'ab)/(s'st) \in S^{-1}I$ or $(a/s)(c/u) = (s'ac)/(s'su) \in S^{-1}I$ or $(b/t)(c/u) = (bc)/(tu) \in S^{-1}I$. Thus $S^{-1}I$ is a graded 2-absorbing hyperideal of $S^{-1}R$.

(*ii*) Let $abc \in I$ where $a, b, c \in h(R)$. We have $(abc)/1 = (a/1)(b/1)(c/1) \in S^{-1}I$. Thus $(a/1)(b/1) \in S^{-1}I$ or $(a/1)(c/1) \in S^{-1}I$ or $(b/1)(c/1) \in S^{-1}I$ since $S^{-1}I$ is a graded 2-absorbing hyperideal of $S^{-1}R$. Hence $ab \in I$ or $ac \in I$ or $bc \in I$ since $S \cap Z_R(R/I) = \emptyset$. Consequently, I is a graded 2-absorbing hyperideal of R. □

4 Graded n-absorbing hyperideals

In this section, we introduce and study the concept of graded n-absorbing hyperideals of a graded Krasner hyperring and investigate the basic properties of this notion in commutative graded Krasner hyperrings.

Definition 4.1. *A proper graded hyperideal* I *of a graded Krasner hyperring* R *is called a graded* n*-absorbing hyperideal of* R *if whenever*

$$a_{g_1} a_{g_2} \cdots a_{g_{n+1}} \in I, \quad a_{g_1}, a_{g_2}, \ldots, a_{g_{n+1}} \in h(R),$$

then there are n *of the* $a_{g_i}^{,}s$ *whose product is in* I.

Proposition 4.2. *Let $P_1, \ldots, P_n$ be graded prime hyperideals of a graded Krasner hyperring R. Then $P_1 \cap \cdots \cap P_n$ is a graded n-absorbing hyperideal of R.*

Proof. It is straightforward. □

The following example shows that a graded n-absorbing hyperideal of a graded Krasner hyperring R for $n \geq 2$ is not necessarily a graded prime hyperideal of R.

Example 4.3. (a) *Let $R = \mathbb{Z}[i]$ be the Gaussian integers ring and $G = (\mathbb{Z}_2, +)$ be the cyclic group of order 2. Consider the Krasner hyperring $(R, +, \cdot)$ with the hyperoperation "$+$" and operation "$\cdot$" as follows: $(a+bi)+(c+di) = \{t_1+t_2 i \mid t_1 \in a+c, t_2 \in b+d\}$ and $(a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i$. Let $R_0 = i\mathbb{Z}$ and $R_1 = \mathbb{Z}$, R be the $\mathbb{Z}_2$-graded Krasner hyperring and $I = \{2a + 2bi \ : \ a, b \in \mathbb{Z}\}$ is a graded 2-absorbing hyperideal of R.*

(b) *In the graded Krasner hyperring $R = (\mathbb{Z}[i], +, \cdot)$, the graded hyperideal $J = \langle 6 \rangle \oplus \langle 0 \rangle$ of R is a graded n-absorbing hyperideal for $n \geq 2$, but it is not a graded prime hyperideal. Because $(2, 0) \cdot (3i, 0) = 6i \in J$, but $(2, 0) \notin J$ and $(3i, 0) \notin J$.*

Theorem 4.4. *Let I be a graded n-absorbing hyperideal of a graded Krasner hyperring R. Then $Grad(I)$ is a graded n-absorbing hyperideal of R and for all $x_g \in h(R)$ such that $x_g \in Grad(I)$, then $x_g^n \in I$.*

Proof. Let $a_{g_1} a_{g_2} \cdots a_{g_{n+1}} \in I$ for some $a_{g_1}, a_{g_2}, \ldots, a_{g_{n+1}} \in h(R)$. Therefore $(a_{g_1} a_{g_2} \cdots a_{g_{n+1}})^n = a_{g_1}^n a_{g_2}^n \cdots a_{g_{n+1}}^n \in I$. Since I is a graded n-absorbing hyperideal of R, we may assume that $a_{g_1}^n a_{g_2}^n \cdots a_{g_n}^n \in I$. Hence $(a_{g_1} a_{g_2} \cdots a_{g_n})^n \subseteq I$, so $a_{g_1} a_{g_2} \cdots a_{g_n} \in Grad(I)$. Thus $Grad(I)$ is a graded n-absorbing hyperideal of R. Now, let $x_g \in Grad(I)$. Then $x_g^t \in I$ for some $t \in \mathbb{N}$. If $t \leq n$, we are done. Let $t \geq n$. By using the graded n-absorbing property on products $x_g^n x_g^{t-n}$, we conclude that $x_g^n \in I$. □

Let I be a proper graded hyperideal of graded Krasner hyperring R. It is clear that a graded n-absorbing hyperideal is also a graded m-absorbing hyperideal for all $m \geq n$. If I is a graded n-absorbing hyperideal of R, we define $Gab(I) = min\{n \,|I \textit{ is a graded } n\textit{-absorbing hyperideal of } R\}$, otherwise, set $Gab(I) = \infty$. We define $Gab(R) = 0$. Hence $Gab(I) = 1$ if and only if I is a graded prime hyperideal of R.

Example 4.5. *In Example 3.2, the graded hyperideal $I = \{0, b\}$, $Gab(I) = 1$ and in Example 3.5, the graded hyperideal $I = \{0, d\}$, $Gab(I) = 2$.*

Proposition 4.6. *Let $I \subseteq P$ be graded hyperideals of a graded Krasner hyperring R such that P is a graded prime hyperideal. Then the following statements are equivalent:*

- (i) *P is a minimal graded prime hyperideal of R.*
- (ii) *For any $a_g \in P \cap h(R)$, there exist $b_{g'} \in h(R) \setminus P$ and a non-negative integer n such that $b_{g'} a_g^n \in I$.*

Proof. $(i) \Rightarrow (ii)$ Let P be a minimal graded prime hyperideal of I and $P_i^{,}s$ be other minimal graded prime hyperideals of I. Then $Grad(I) = P \cap (\bigcap_{P_i \in Min_{gr}(I)} P_i)$. Suppose that $a_g \in P$ but $a_g \notin Grad(I)$. We may assume that $a_g \in P \cap (\bigcap_{i=1}^t P_i)$ such that $a_g \notin \bigcup_{i \geq t+1} P_i$. Let $x_h \in \bigcap_{i \geq t+1} P_i \setminus P$. Thus $x_h a_g \in P \cap (\bigcap_{i=1}^t P_i) \cap (\bigcap_{i \geq t+1} P_i)$ and so $x_h a_g \in Grad(I)$. Hence $(x_h a_g)^n = x_h^n a_g^n \in I$. Let $b_{h^n} \in x_h^n$ and set $h^n = g'$. Therefore $b_{g'} a_g^n \in I$.

$(ii) \Rightarrow (i)$ Assume P is not a minimal graded prime hyperideal I of R. Hence $I \subseteq Q \subset P$ for some graded prime hyperideal Q of R. Let $a_g \in (P \cap h(R)) \setminus Q$. Thus there exists $b_h \in h(R) \setminus P$ and $n \in \mathbb{N}$ such that $b_h a_g^n \in I \subseteq Q$. This is a contradiction since $a_g, b_h \notin Q$. □

Theorem 4.7. *Let $P_1, \dots, P_n$ be graded prime hyperideals of a graded Krasner hyperring R that are pairwise coprime. Then $I = P_1 \cdots P_n$ is a graded n-absorbing hyperideal of R. Moreover, $Gab(I) = n$.*

Proof. We have $I = P_1 \cdots P_n = P_1 \cap \cdots \cap P_n$ since $P_1, \dots, P_n$ are pairwise coprime. Thus I is a graded n-absorbing hyperideal of R by Proposition 4.2. Since $P_1, \dots, P_n$ are incomparable, then for each $1 \le i \le n$, there is $a_{g_i} \in (P_i \cap h(R)) \setminus \bigcup_{j \neq i} P_j$. Hence $a_{g_1} a_{g_2} \cdots a_{g_n} \in P_1 \cap \cdots \cap P_n$, but no proper subproduct of the a_{g_i}'s is in $P_1 \cap \cdots \cap P_n$. Hence $Gab(P_1 \cdots P_n) = Gab(P_1 \cap \cdots \cap P_n) \ge n$. On the other hand, we have $Gab(P_1 \cdots P_n) = Gab(P_1 \cap \cdots \cap P_n) \le n$. Therefore, $Gab(I) = Gab(P_1 \cdots P_n) = n$. □

Corollary 4.8. *Let $M_1, M_2, \dots, M_n$ be distinct graded maximal hyperideals of a graded Krasner hyperring R. Then $I = M_1 \cdots M_n$ is a graded n-absorbing hyperideal of R.*

Proof. Apply Theorem 4.7. □

Lemma 4.9. *Let M be a graded maximal hyperideal of a graded Krasner hyperring R and n be a positive integer. Then M^n is a graded n-absorbing hyperideal of R such that $Gab(M) \le n$. Moreover, if $M^{n+1} \subset M^n$, then $Gab(M^n) = n$.*

Proof. Let $a_{g_1} a_{g_2} \cdots a_{g_{n+1}} \in I$ where $a_{g_1}, \dots, a_{g_{n+1}} \in h(R)$. If we have $a_{g_1}, a_{g_2}, \dots, a_{g_{n+1}} \in M$, then we are done. We may assume that $a_{g_{n+1}} \notin M$. Thus $\langle M, a_{g_{n+1}} \rangle = R$, hence there exist $m \in M$ and $b \in R$ such that $1 \in m + a_{g_{n+1}} b$. Therefore,

$$a_{g_1} a_{g_2} \cdots a_{g_n} = a_{g_1} \cdots a_{g_n} 1 \in (a_{g_1} \cdots a_{g_n}) m + (a_{g_1} \cdots a_{g_{n+1}}) b \subset M^n.$$

Hence M^n is a graded n-absorbing hyperideal of R. Now, let $M^{n+1} \subset M^n$. Then there are $a_{g_1}, \dots, a_{g_{n+1}} \in M \cap h(R)$ such that $a_{g_1} \cdots a_{g_n} \in M^n \setminus M^{n+1}$. Thus all products of $n-1$ of a_{g_i}'s are not in M^n, since otherwise $a_{g_1} \cdots a_{g_n} \in M^{n+1}$ which is a contradiction. Therefore M^n is not a graded $(n-1)$-absorbing hyperideal of R. Thus $Gab(M^n) = n$ since M^n is a graded n-absorbing hyperideal of R. □

Theorem 4.10. *Let $M_1, \dots, M_n$ be graded maximal hyperideals of a graded Krasner hyperring R. Then $I = M_1 \cdots M_n$ is a graded n-absorbing hyperideal of R. Moreover, $Gab(I) \le n$.*

Proof. Let $M_1, \dots, M_n$ be distinct graded maximal hyperideals of R and $n_1, \dots, n_k$ be positive integers such that $n = n_1 + \cdots + n_k$. We show that $I = M_1^{n_1} \cdots M_n^{n_k}$ is a graded n-absorbing hyperideal of R. By Lemma 4.9, for all $1 \le i \le k$, $M_i^{n_i}$ is a graded n_i-absorbing hyperideal of R. Hence $I = M_1^{n_1} \cdots M_n^{n_k} = M_1^{n_1} \cap \cdots \cap M_n^{n_k}$ is a graded n-absorbing hyperideal of R. □

Theorem 4.11. *Let P be a graded prime hyperideal of R and I be a graded P-primary hyperideal of R such that $P^n \subseteq I$ for some $n \in \mathbb{N}$. Then I is a graded n-absorbing hyperideal of R with $Gab(I) \le n$.*

Proof. Let $a_{g_1} a_{g_2} \cdots a_{g_{n+1}} \in I$ for $a_{g_1}, a_{g_2}, \dots, a_{g_{n+1}} \in h(R)$. Suppose that one of the a_{g_i}'s is not in P. Since I is a graded P-primary hyperideal of R, then we conclude that the other a_{g_i}'s is in P. Hence we may assume that $a_{g_i} \in P$ for any $1 \le i \le n$. Since $P^n \subseteq I$, we have $a_{g_1} \cdots a_{g_n} \in I$. Therefore I is a graded n-absorbing hyperideal of R. □

We next give a sufficient condition for a graded n-absorbing hyperideal to be graded primary.

Definition 4.12. *Let R be a graded Krasner hyperring. A proper graded prime hyperideal P of R is called divided graded prime, if for all $a_g \in h(R) - P$, $P \subset Ra_g$.*

Example 4.13. *Consider the $\mathbb{Z}$-graded Krasner hyperring $(\mathbb{Z}_5, +, \cdot)$. Then the graded prime hyperideal $P = (0)$ is divided graded prime, because for all $x \in h(\mathbb{Z}_5) - P = \mathbb{Z}_5 - \{0\}$, $P \subset Rx$.*

Theorem 4.14. *Let P be a divided graded prime hyperideal of a graded Krasner hyperring R, and let I be a graded n-absorbing hyperideal of R with $Grad(I) = P$. Then I is a graded P-primary hyperideal of R.*

Proof. Let $a_g b_h \in I$ for $a_g, b_h \in h(R)$ and $b_h \notin Grad(I) = P$. Then $a_g \in P$ since P is a graded prime hyperideal. Since P is a divided graded prime hyperideal of R, we have $P \subset Rb_h^{n-1}$ because $b_h^{n-1} \notin P$. Thus $a_g = c_k b_h^{n-1}$ for some $c_k \in h(R)$. As $a_g b_h = c_k b_h^n \in I$ and $b_h^n \notin I$ and I is a graded n-absorbing hyperideal, we have $a_g = b_h^{n-1} c_k \in I$. Therefore I is a graded n-absorbing hyperideal of R. □

Lemma 4.15. *Let I be a graded hyperideal of a graded Krasner hyperring R. Then for any $x_g \in h(R)$, $(I :_R x_g) = \{r \in R \mid rx_g \in I\}$ is a graded hyperideal of R.*

Proof. It is clear. □

Proposition 4.16. *Let I be a graded n-absorbing hyperideal of a graded Krasner hyperring R. Then $(I :_R x_g)$ is a graded n-absorbing hyperideal of R containing I for all $x_g \in h(R)\backslash I$. Moreover, $Gab((I :_R x_g)) \leq Gab(I)$ for all $x_g \in h(R)$.*

Proof. Let $x_{g_1} x_{g_2} \cdots x_{g_{n+1}} \in (I :_R x_g)$ for $x_{g_1}, x_{g_2}, \ldots, x_{g_{n+1}} \in h(R)$. Then $(x_g x_{g_1}) x_{g_2} \cdots x_{g_{n+1}} \in I$. Thus either the product of $x_g x_{g_1}$ with $n-1$ of the x_{g_i}'s for $2 \leq i \leq n+1$ is in I or $x_{g_2} \cdots x_{g_{n+1}} \in I$. Hence there exists a product of n of the x_{g_i}'s that is $(I :_R x_g)$. Therefore $(I :_R x_g)$ is a graded n-absorbing hyperideal of R. If $x_g \in I$, then $(I :_R x_g) = R$, so $Gab((I :_R x_g)) = 0 \leq Gab(I)$. □

Definition 4.17. *Let I be a proper graded hyperideal of a graded Krasner hyperring R and $g \in G$ such that $I_g \neq R_g$. Then I is said to be a g-n-absorbing hyperideal of R, if whenever $x_1, x_2, \ldots, x_{n+1} \in R_g$ such that $x_1 x_2 \cdots x_{n+1} \in I$, then there are n of the x_i's whose product is in I.*

Theorem 4.18. *Let I be a g-n-absorbing hyperideal of a graded Krasner hyperring R. Let $x \in R_g \backslash I$ such that $x^k \subseteq I$ for some $k \geq 2$. Then if $(I :_R x^{k-1})_g \neq R_g$, then $(I :_R x^{k-1})$ is a g-$(n-k+1)$-absorbing hyperideal of R.*

Proof. Suppose that I is a g-n-absorbing hyperideal of R. Since $2 \leq k \leq n$, then $n - t + 1 \geq 1$. It is clear that $I \subseteq (I :_R x^{k-1})$. Let $x_1 \cdots x_{n+1} \in (I :_R x^{k-1})$ for $x_1, \ldots, x_{n+1} \in R_g$. Then $x^{k-1} x_1 \cdots x_{n+1} \in I$ and so either $x^{k-2} x_1 \cdots x_{n-k+2} \in I$ or the product of x^{k-1} with some $n-k+1$ of the x_i's is in I. In the second case, we are done. Now, let the product of x^{k-1} with any $n-k+1$ of the x_i's is not in I. Thus $x^{k-2} x_1 \cdots x_{n-k+2} \in I$. We have $xx^{k-2} x_1 \cdots x_{n-k+1}(c_{n-k+2} + x) = x^{k-1} x_1 \cdots x_{n-k+1} c_{n-k+2} + x^k x_1 \cdots x_{n-k+1} c_{n-k+2} \subseteq I$. Since I is a g-n-absorbing hyperideal of R, so $x^{k-2} x_1 \cdots x_{n-k+1}(c_{n-k+2} + x) \subseteq I$. Hence $x^{k-1} x_1 \cdots x_{n-k+1} \in I$ because $x^{k-2} \circ x_1 \cdots x_{n-k+2} \in I$, which is a contradiction. Hence the product of x^{k-1} with some $n-k+1$ of the x_i's is in I and so $(I :_R x^{k-1})$ is a g-$(n-k+1)$-absorbing hyperideal of R. □

Theorem 4.19. *Let R and S be graded Krasner hyperrings and let $f : R \to S$ be a graded good homomorphism. Then the following statements hold:*

(*i*) *If J is a graded n-absorbing hyperideal of S, then $f^{-1}(J)$ is a graded n-absorbing hyperideal of R.*

(*ii*) *If f is onto and I is a graded n-absorbing hyperideal of R containing $Ker(f)$, then $f(I)$ is a graded n-absorbing hyperideal of S.*

Proof. The proof is similar that Theorem 3.13. □

Corollary 4.20. *Let I, J be graded hyperideals of a graded Krasner hyperring R such that $J \subseteq I$. If I is a graded n-absorbing hyperideal of R, then I/J is a graded n-absorbing hyperideal of R/J.*

Proof. Consider the function $f : R \to R/J$ defined by $f(r) = r + J$. It is clear that f is a good epimorphism. Since $Ker(f) = J \subseteq I$ and I is a graded n-absorbing hyperideal of R, then the proof follows from Theorem 4.19(ii). □

Corollary 4.21. *Let U be a graded subring of R. If I is a graded n-absorbing hyperideal of R such that $U \nsubseteq I$, then $I \cap U$ is a graded n-absorbing hyperideal of U.*

Proof. Define $i : U \to R$ by $i(x) = x$. It is clear that $i^{-1}(I) = I \cap U$. Hence $I \cap U$ is a graded n-absorbing hyperideal of U by Theorem 4.19(i). □

Theorem 4.22. *Let R and S be graded Krasner hyperrings. Then the following statements hold:*

(*i*) *I is a graded n-absorbing hyperideal of R if and only if $I \times S$ is a graded n-absorbing hyperideal of $R \times S$.*

(*ii*) *J is a graded n-absorbing hyperideal of S if and only if $R \times J$ is a graded n-absorbing hyperideal of $R \times S$.*

Proof. (i) ($\Rightarrow$) Let I be a graded n-absorbing hyperideal of R and $(a_{g_1}, b_{g_1}) \cdots (a_{g_{n+1}}, b_{g_{n+1}}) \in I \times S$ for $(a_{g_1}, b_{g_1}), \ldots, (a_{g_{n+1}}, b_{g_{n+1}}) \in h(R \times S)$. Then $a_{g_1} \cdots a_{g_{n+1}} \in I$. Since I is a graded n-absorbing hyperideal of R, there are n of a_{g_i}'s is in I. We may assume that $a_{g_1} \cdots a_{g_n} \in I$. This implies that $(a_{g_1}, b_{g_1}) \cdots (a_{g_n}, b_{g_n}) \in I \times S$. Therefore $I \times S$ is a graded n-absorbing hyperideal of $R \times S$.

($\Leftarrow$) Assume that $I \times S$ is a graded n-absorbing hyperideal of $R \times S$. Let $a_{g_1} \cdots a_{g_{n+1}} \in I$ for $a_{g_1}, \ldots, a_{g_{n+1}} \in h(R)$. Then $(a_{g_1}, 0) \cdots (a_{g_{n+1}}, 0) \in I \times S$. Since $I \times S$ is a graded n-absorbing hyperideal of $R \times S$, then there are n of $(a_{g_i}, 0)$'s is in $I \times S$. We may assume that $(a_{g_1}, 0) \cdots (a_{g_n}, 0) \in I \times S$ and so $a_{g_1} \cdots a_{g_n} \in I$. Thus I is a graded n-absorbing hyperideal of R.

(ii) It is similar to that (i). □

5 Graded 2-absorbing subhypermodules

In this section, we introduce the concept of graded 2-absorbing subhypermodules of a graded Krasner R-hypermodule of M and we investigate some properties of such graded subhypermodules.

Definition 5.1. *Let R be a graded hyperring and N be a proper graded subhypermodule of a graded Krasner R-hypermodule M. Then N is a graded 2-absorbing subhypermodule of M if $a_g b_h m_k \in N$ implies $a_g b_h \in (N : M)$ or $a_g m_k \in N$ or $b_h m_k \in N$ for all $a_g, b_h \in h(R)$ and $m_k \in h(M)$.*

Example 5.2. *Let $(R, +, \cdot)$ be the graded Krasner hyperring in Example 3.2. Set $M = R$ and $\oplus = +$, then $(M, \oplus)$ is an R-hypermodule with the following operation:*

$$\forall (r, m) \in R \times M; \quad r \cdot m = rm.$$

We know that $M_0 = \{0, b\}$ and $M_1 = \{0, c\}$ are subhypergroups of $(M, \oplus)$ and $M = M_0 \bigoplus M_1$. Moreover, $R_0M_0 \subseteq M_0$, $R_0M_1 \subseteq M_0$, $R_1M_0 \subseteq M_0$, $R_1M_1 \subseteq M_1$. Hence M is a graded Krasner R-hypermodule. It is clear that $\{0\}$ is a graded 2-absorbing subhypermodule of M and so it is a graded 2-absorbing subhypermodule of M.

The interested reader can easily prove the following lemma.

Lemma 5.3. *Let M be a graded hypermodule over a graded Krasner hyperring R. Then the following hold:*

(*i*) *If N is a graded subhypermodule of M, I a graded hyperideal of R, $r \in h(R)$ and $x \in h(M)$, then Rx, IN and rN are graded subhypermodules of M.*

(*ii*) *If N and K are graded subhypermodules of M, then $N + K$ and $N \cap K$ are also graded subhypermodules of M and $(N : M)$ is a graded hyperideal of R.*

(*iii*) *Let $\{N_\lambda\}$ be a collection of graded subhypermodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded subhypermodues of M.*

Lemma 5.4. *Let $M = \bigoplus_{g\in G} M_g$ be a graded Krasner R-hypermodule and $N = \bigoplus_{g\in G} N_g$ be a proper graded subhypermodule of M. If N be a graded 2-absorbing subhypermodule of M, then N_g is a g-2-absorbing R_e-subhypermodule of M_g for all $g \in G$.*

Proof. Let $a, b \in R_e$ and $m \in M_g$ with $abm \in N_g$. Since N is a graded 2-absorbing subhypermodule of M and $N_g = N\cap M_g \subset N$, we get either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. If $ab \in (N :_R M)$, then $ab \in (N_g :_{R_e} M_g)$ as $(N : M) \subset ((N \cap M_g) :_{R_e} M_g) = (N_g :_{R_e} M_g)$. Suppose that $am \in N$. Since $am \in M_g$ and $am \in N$, we have $am \in N \cap M_g = N_g$. If $bm \in N$, then similarly we conclude $bm \in N_g$. Therefore N_g is a g-2-absorbing R_e-subhypermodule of M_g. □

Proposition 5.5. *If N is a graded prime subhypermodule of a graded Krasner R-hypermodule M, then N is a graded 2-absorbing subhypermodule of M.*

Proof. Assume that N is a graded prime subhypermodule of M and let $a_g b_h m_k \in N$ but $a_g m_k \notin N$ for some $a_g, b_h \in h(R)$ and $m_k \in h(M)$. Hence $a_g M \subseteq N$ since N is a graded prime subhypermodule of M. Therefore, $a_g b_h M \subseteq a_g M \subseteq N$, so N is a graded 2-absorbing subhypermodule of M. □

Lemma 5.6. *Let N be a proper graded subhypermodule of a graded Krasner R-hypermodule M. Let $g \in G$. If N_g is a g-2-absorbing R_e-subhypermodule of M_g, then $(N_g :_{R_e} M_g)$ is a 2-absorbing hyperideal of R_e.*

Proof. Let $a, b, c \in R_e$ with $abc \in (N_g :_{R_e} M_g)$ and suppose that $ac \notin (N_g :_{R_e} M_g)$ and $bc \notin (N_g :_{R_e} M_g)$. We show that $ab \in (N_g :_{R_e} M_g)$. Since $ac, bc \notin (N_g :_{R_e} M_g)$, there exist $m_1, m_2 \in M_g$ such that $acm_1 \notin N_g$ and $bcm_2 \notin N_g$. Now $abc(m_1+m_2) \in N_g$. So $ab \in (N_g :_{R_e} M_g)$ or $ac(m_1+m_2) \in N_g$ or $bc(m_1+m_2) \in N_g$. If $ac(m_1+m_2) \in N_g$, then $acm_1 \notin N_g$ since $acm_1 \notin N_g$. Similarly, $bcm_2 \notin N_g$. Since $abcm_2 \in N_g$ and $bcm_2 \notin N_g$ and $acm_2 \notin N_g$ we have $ab \in (N_g :_{R_e} M_g)$. Hence $(N_g :_{R_e} M_g)$ is a 2-absorbing hyperideal of R_e. □

Proposition 5.7. *The intersection of each pair of graded prime subhypermodules of a graded Krasner R-hypermodule M is a graded 2-absorbing subhypermodule of M.*

Proof. Let N and K be two graded prime subhypermodules of M. If $N = K$, then $N \cap K$ is a graded prime subhypermodule of M, so that $N \cap K$ is a graded 2-absorbing subhypermodule of M. Assume that N and K are distinct. Since N and K are proper subhypermodules of M, then $N \cap K$ is a proper subhypermodule of M. Now, let $a_g, b_{g'} \in h(R)$ and $m_h \in h(M)$ be such that $a_g b_{g'} m_h \in N \cap K$ but $a_g m_h \notin N \cap K$ and $a_g b_{g'} M \nsubseteq N \cap K$. Then we can conclude that

(a) $a_g m_h \notin N$ or $a_g m_h \notin K$, and
(b) $a_g b_{g'} M \nsubseteq N$ or $a_g b_{g'} M \nsubseteq K$.

These two conditions give four cases:

(1) $a_g m_h \notin N$ and $a_g b_{g'} M \nsubseteq N$;
(2) $a_g m_h \notin N$ and $a_g b_{g'} M \nsubseteq K$;
(3) $a_g m_h \notin K$ and $a_g b_{g'} M \nsubseteq N$;
(4) $a_g m_h \notin K$ and $a_g b_{g'} M \nsubseteq K$.

We consider Case (1). Since $a_g b_{g'} m_h \in N$ and N is graded prime, we have $a_g M \subseteq N$ or $b_{g'} m_h \in N$. If $a_g M \subseteq N$, then $a_g b_{g'} M \subseteq a_g M \subseteq N$ which is not possible. So suppose that $b_{g'} m_h \in N$. Therefore $b_{g'} M \subseteq N$ or $m_h \in N$. This is not possible and hence Case (1) does not occur. Similarly, Case (4) is not possible. Next, Case (2) is considered. We have $a_g b_{g'} m_h \in N \cap K \subseteq K$ and since K is a graded prime subhypermodule of M, it follows that $a_g M \subseteq K$ or $b_{g'} m_h \in K$. If $a_g M \subseteq K$, then $a_g b_{g'} M \subseteq a_g M \subseteq K$ which contradicts $a_g b_{g'} M \nsubseteq K$, thus $b_{g'} m_h \in K$. From $a_g b_{g'} m_h \in N \cap K \subseteq N$ we have $a_g M \subseteq N$ or $b_{g'} m_h \in N$. Since $b_{g'} m_h \notin N$, $a_g M \subseteq N$ is not possible. Hence $b_{g'} m_h \in N \cap K$.

The proof of Case (3) is similar to that of Case (2). □

Proposition 5.8. *Let N and K be two graded subhypermodules of a graded Krasner R-hypermodule M and $N \subseteq K$. If N is a graded 2-absorbing subhypermodule of M, then N is a graded 2-absorbing subhypermodule of K.*

Proof. If $K = M$, then there is nothing to prove. Let $a_g b_{g'} m_h \in N$ where $a_g, b_{g'} \in h(R)$ and $m_h \in h(K)$. Since N is a graded 2-absorbing subhypermodule of M, so either $a_g m_h \in N$ or $b_{g'} m_h \in N$ or $a_g b_{g'} \in (N : M)$. Since $(N : M) \subseteq (N : K)$, implies either $a_g m_h \in N$ or $b_{g'} m_h \in N$ or $a_g b_{g'} \in (N : K)$. Therefore N is a graded 2-absorbing subhypermodule of K. □

Lemma 5.9. *Let N be a proper graded subhypermodule of a graded Krasner R-hypermodule M. Let $g \in G$. N_g is a g-2-absorbing R_e-subhypermodule of M_g if and only if $abK \subseteq N_g$ implies $ab \in (N_g :_{R_e} M_g)$ or $aK \subseteq N_g$ or $bK \subseteq N_g$ for each $a, b \in R_e$ and R_e-subhypermodule K of M_g.*

Proof. Let N_g be a g-2-absorbing subhypermodule of M_g and $abK \subseteq N_g$. Suppose that $ab \notin (N_g :_{R_e} M_g)$ and $aK \nsubseteq N_g$ and $bK \nsubseteq N_g$ for some $a, b \in R_e$ and a subhypermodule K of M_g. Then there exist $m_g, m'_g \in K$ such that $am_g \notin N_g$ and $bm'_g \notin N_g$. Since $abm_g \in abK \subseteq N_g$, $ab \notin (N_g :_{R_e} M_g)$ and $am_g \notin N$ we get $bm_g \in N_g$. Also, since $abm'_g \in abK \subseteq N_g$, $ab \notin (N_g :_{R_e} M_g)$ and $bm'_g \notin N_g$ we get $am'_g \in N_g$. Now, since $ab(m_g + m'_g) \in abK \subseteq N_g$ and $ab \notin (N_g :_{R_e} M_g)$ we have $a(m_g + m'_g) \in N_g$ or $b(m_g + m'_g) \in N_g$. If $a(m_g + m'_g) \in N_g$, i.e., $(am_g + am'_g) \in N_g$, then since $am'_g \in N_g$ we get $am_g \in N_g$ which is a contradiction. If $b(m_g + m'_g) \in N_g$, i.e., $(bm_g + bm'_g) \in N_g$, then since $bm_g \in N_g$ we get $bm'_g \in N_g$ which is a contradiction. Thus $ab \in (N_g :_{R_e} M_g)$ or $aK \subseteq N_g$ or $bK \subseteq N_g$. The converse is clear. □

Theorem 5.10. *If N is a proper graded subhypermodule of a graded Krasner R-hypermodule M. Let $g \in G$. If N_g is a g-2-absorbing R_e-subhypermodule of M_g and I and J are hyperideals of R_e and K an R_e-subhypermodule of M_g such that $IJK \subseteq N_g$, then $IK \subseteq N_g$ or $JK \subseteq N_g$ or $IJ \subseteq (N_g :_{R_e} M_g)$.*

Proof. Suppose $IJK \subseteq N_g$ and $IJ \nsubseteq (N_g : M_g)$. We show that $IK \subseteq N_g$ or $JK \subseteq N_g$. Suppose $IK \nsubseteq N_g$ and $JK \nsubseteq N_g$. There exist $a_1 \in I$ and $a_2 \in J$ such that $a_1 K \nsubseteq N_g$ and $a_2 K \nsubseteq N_g$. But $a_1 a_2 K \subseteq IJK \subseteq N_g$. Since N_g is a g-2-absorbing R_e-subhypermodule of M_g it follows from

Lemma 5.9 that $a_1a_2 \in (N_g : M_g)$. Since $IJ \nsubseteq (N_g : M_g)$, there exist $b_1 \in I$ and $b_2 \in J$ such that $b_1b_2M_g \nsubseteq N_g$. Now since N_g is a g-2-absorbing R_e-subhypermodule of M_g and $b_1b_2K \subseteq IJK \subseteq N_g$ and also $b_1b_2M_g \nsubseteq N_g$ it follows from Lemma 5.9 that $b_1K \subseteq N_g$ or $b_2K \subseteq N_g$. We have the following cases:

Case (1): $b_1K \subseteq N_g$ and $b_2K \nsubseteq N_g$. Since $a_1b_2K \subseteq IJK \subseteq N_g$ and $a_1K \nsubseteq N_g$ and $b_2K \nsubseteq N_g$ it follows from Lemma 5.9 that $a_1b_2 \in (N_g : M_g)$. Since $b_1K \subseteq N_g$ and $a_1K \nsubseteq N_g$, we conclude $(a_1+b_1)K \nsubseteq N_g$. On the other hand, $(a_1+b_1)b_2K \subseteq N_g$ and neither $(a_1+b_1)K \subseteq N_g$ nor $b_2K \subseteq N_g$, we get that $(a_1+b_1)b_2 \in (N_g : M_g)$ by Lemma 5.9. But since $(a_1+b_1)b_2 = (a_1b_2+b_1b_2) \in (N_g : M_g)$ and $a_1b_2 \in (N_g : M_g)$, we get $b_1b_2 \in (N_g : M_g)$ which is a contradiction.

Case (2): $b_2K \subseteq N_g$ and $b_1K \nsubseteq N_g$. By a similar argument to Case (1), we get a contradiction.

Case (3): $b_1K \subseteq N_g$ and $b_2K \subseteq N_g$. $b_2K \subseteq N_g$ and $a_2K \nsubseteq N_g$ gives $(a_2 + b_2)K \nsubseteq N_g$. But $a_1(a_2+b_2)b_2K \subseteq N_g$ and neither $a_1K \subseteq N_g$ nor $(a_2+b_2)K \subseteq N_g$, hence $a_1(a_2+b_2) \subset (N_g : M_g)$ by Lemma 5.9. Since $a_1a_2 \in (N_g : M_g)$ and $(a_1a_2+b_1b_2) \in (N_g : M_g)$, we have $a_1b_2 \in (N_g : M_g)$. Since $(a_1+b_1)a_2K \subseteq N_g$ and neither $a_2K \subseteq N_g$ nor $(a_1+b_1)K \subseteq N_g$, we conclude $(a_1+b_1)a_2 \subset (N_g : M_g)$ by Lemma 5.9. But $(a_1 + b_1)a_2 = a_1a_2 + b_1a_2$, so $(a_1a_2 + b_1a_2) \subset (N_g : M_g)$ and since $a_1a_2 \in (N_g : M_g)$, we get $b_1a_2 \in (N_g : M_g)$. Now, since $(a_1 + b_1)(a_2 + b_2)K \subseteq N_g$ and neither $(a_1 + b_1)K \subseteq N_g$ nor $(a_2 + b_2)K \subseteq N_g$, we have $(a_1 + b_1)(a_2 + b_2) = (a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2) \subset (N_g : M_g)$ by Lemma 5.9. But $a_1a_2, a_1b_2, b_1a_2 \in (N_g : M_g)$, so $b_1b_2 \in (N_g : M_g)$ which is a contradiction. Consequently, $IK \subseteq N_g$ or $JK \subseteq N_g$. □

Corollary 5.11. *Let I and J be two hyperideals of R_e and P a g-2-absorbing R_e-subhypermodule of M_g. If $m_g \in M_g$ such that $IJm_g \subseteq P$, then $Im_g \subseteq P$ or $Jm_g \subseteq P$ or $IJ \subseteq (P :_{R_e} M_g)$.*

Proof. Let $IJm_g \subseteq P$. Then $IJ(R_em_g) \subseteq P$ and consequently $Im_g \subseteq I(R_em_g) \subseteq P$ or $Jm_g \subseteq J(R_em_g) \subseteq P$ or $IJ \subseteq (P : M_g)$. □

Theorem 5.12. *Let I be a hyperideal of R_e and N_g be a g-2-absorbing subhypermodule of M_g. If $a \in R_e$, $m_g \in M_g$ and $Iam_g \subseteq N_g$, then $am_g \in N_g$ or $Im_g \subseteq N_g$ or $Ia \subseteq (N_g : M_g)$.*

Proof. Suppose that $am_g \notin N_g$ and $Ia \nsubseteq (N_g : M_g)$. Then there exists $b \in I$ such that $ba \notin (N_g : M_g)$. Now, $bam_g \in N_g$, implies that $bm_g \in N_g$, since N_g is a g-2-absorbing R_e-subhypermodule of M_g. We show that $Im_g \subseteq N_g$. Let $c \in I$. Thus $(b + c)am_g \in Iam_g \subseteq N_g$. Hence either $(b + c)m_g \in N_g$ or $(b + c)a \in (N_g : M_g)$. If $(b + c)m_g \in N_g$, then by bm_g it follows that $cm_g \in N_g$. If $(b+c)a \in (N_g : M_g)$, then $ca \notin (N_g : M_g)$, but $cam_g \in N_g$. Thus $cm_g \in N_g$. Hence we conclude that $Im_g \subseteq N_g$. □

Corollary 5.13. *Let N_g be a g-2-absorbing subhypermodule of M_g. Then $(N_g :_{M_g} I)$ is a g-2-absorbing subhypermodule of M_g for every hyperideal I of R_e.*

Proof. Let $a, b \in R_e$ and $m_g \in M_g$ be such that $abm_g \subseteq (N_g :_{M_g} I)$. Since $Iabm_g \subseteq N_g$ and N_g is g- 2-absorbing, so by Theorem 5.12 we have $abm_g \in N_g$ or $Im_g \subseteq N_g$ or $Iab \subseteq (N_g : M_g)$. If $abm_g \in N_g$, then $am_g \in N_g$ or $bm_g \in N_g$ or $ab \in (N_g : M_g)$. Hence for $am_g \in N_g$ it follows that $Iam_g \subseteq IN_g \subseteq N_g$ and we have $am_g \in (N_g : M_g)$. For $bm_g \in N_g$ it follows that $Ibm_g \subseteq IN_g \subseteq N_g$ and we have $bm_g \in (N_g : M_g)$. For $ab \in (N_g : M_g)$, we have $ab \in ((N_g :_{R_e} M_g) :_{R_e} I) = ((N_g :_{M_g} I) :_{R_e} M_g)$. For $Im_g \subseteq N_g$, we have $m_g \in (N_g :_{M_g} I)$ and thus $am_g \in (N_g :_{M_g} I)$. For $Iab \subseteq (N_g : M_g)$, we have $ab \in ((N_g :_{R_e} M_g) :_{R_e} I) = ((N_g :_{M_g} I) :_{R_e} M_g)$ and $(N_g :_{M_g} I)$ is a g-2-absorbing subhypermodule of M_g. □

Theorem 5.14. *Let N be a proper graded subhypermodule of a graded Krasner R-hypermodule M. Let $g \in G$. Let N_g be a g-2-absorbing R_e-subhypermodule of M_g. Then $(N_g :_{R_e} M_g)$ is a prime* *hyperideal of R_e if and only if $(N_g :_{R_e} P)$ is a prime hyperideal of R_e for every subhypermodule P of M_g containing N_g.*

Proof. Let I and J be hyperideals of R_e such that $IJ \subseteq (N_g :_{R_e} P)$. Hence $IJP \subseteq N_g$. Since N_g is a g-2-absorbing R_e-subhypermodule of M_g it follows from Theorem 5.10 that $IP \subseteq N_g$ or $JP \subseteq N_g$ or $IJ \subseteq (N_g :_{R_e} P)$. For $IJ \subseteq (N_g :_{R_e} P)$, by assumption that $(N_g :_{R_e} M_g)$ is a prime hyperideal of R_e, we get $IP \subseteq IM_g \subseteq N_g$ or $JP \subseteq JM_g \subseteq N_g$. Hence $I \subseteq (N_g :_{R_e} P)$ or $J \subseteq (N_g :_{R_e} P)$ and so $(N_g :_{R_e} P)$ is a prime hyperideal of R_e. □

Proposition 5.15. *Let N and K be graded subhypermodules of a graded Krasner R-hypermodule M with $K \nsubseteq N$. If N is a graded 2-absorbing subhypermodule of M, then $N \cap K$ is a graded 2-absorbing subhypermodule of K.*

Proof. Since N and K are graded subhypermodules of M and $K \nsubseteq N$, $K \cap N$ is a proper graded subhypermodule of K. Assume that N is a graded 2-absorbing subhypermodule of M. Let $a_g, b_{g'} \in h(R)$ and $x_h \in h(K)$ be such that $a_g b_{g'} m_h \in N$. Since K is a graded subhypermodule of M, $a_g b_{g'} K \subseteq K$ and $a_g x_h, b_{g'} x_h \in K$. Moreover, since $a_g b_{g'} m_h \in N \cap K \subseteq N$ and N is a graded 2-absorbing subhypermodule of M, $a_g b_{g'} M \subseteq N$ or $a_g x_h \in N$ or $b_{g'} x_h \in N$. Thus $a_g b_{g'} K \subseteq a_g b_{g'} K \cap a_g b_{g'} M \subseteq K \cap N$ or $a_g x_h \in K \cap N$ or $b_{g'} x_h \in K \cap N$. Therefore, $N \cap K$ is a graded 2-absorbing subhypermodule of K. □

Proposition 5.16. *Let N and K be graded subhypermodules of a graded Krasner R-hypermodule M with $K \subseteq N$. Then N is a graded 2-absorbing subhypermodule of M if and only if N/K is a graded 2-absorbing subhypermodule of M/K.*

Proof. Assume that N is a graded 2-absorbing subhypermodule of M. Then N/K is a proper graded subhypermodule of M/K. Let $a_g, b_{g'} \in h(R)$ and $(m_h + K) \in h(M/K)$ be such that $a_g b_{g'}(m_h + K) \in N/K$. Let $s, t \in R$. Hence $a_g s b_{g'} t m_h + K = a_g s b_{g'} t(m_h + K) \in N/K$. Then there exists $n \in N$ such that $a_g s b_{g'} t m_h + K = n + K$ so that $a_g s b_{g'} t m_h - n \in K \subseteq N$ and so $a_g s b_{g'} t m_h \in N$. This shows that $a_g b_{g'} m_h \in N$. As a result, $a_g m_h \in N$ or $b_{g'} m_h \in N$ or $a_g b_{g'} M \subseteq N$ because N is a graded 2-absorbing subhypermodule of M. Therefore, $a_g(m_h + K) \in N/K$ or $b_{g'}(m_h+K) \in N/K$ or $a_g b_{g'}(M/K) \subseteq N/K$. Hence N/K is a graded 2-absorbing subhypermodule of M/K. Conversely, assume that N/K is a graded 2-absorbing subhypermodule of M/K. Then N is a proper graded subhypermodule of M. Let $a_g, b_{g'} \in h(R)$ and $m_h \in h(M)$ be such that $a_g b_{g'} m_h \in N$. Then $a_g b_{g'}(m_h + K) \in N/K$. Since N/K is a graded 2-absorbing subhypermodule of M/K, we obtain $a_g(m_h + K) \in N/K$ or $b_{g'}(m_h + K) \in N/K$ or $a_g b_{g'}(M/K) \subseteq N/K$. That is $a_g m_h \in N$ or $b_{g'} m_h \in N$ or $a_g b_{g'} M \subseteq N$. This implies that N is a graded 2-absorbing subhypermodule of M. □

Let R_1 and R_2 be two G-graded hyperrings. Then $R = R_1 \times R_2$ becomes a G-graded hyperring with homogeneous elements $h(R) = \bigcup_{g \in G} R_g$, where $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Let M_1 be a graded R_1-hypermodule and M_2 be a graded R_2-hypermodule. Then $M = M_1 \times M_2$ is a graded $R = R_1 \times R_2$-hypermodule.

Theorem 5.17. *Let M_1 be a graded Krasner R_1-hypermodule, M_2 be a graded Krasner R_2-hypermodule, $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then*

(*i*) *N_1 is a graded 2-absorbing subhypermodule of M_1 if and only if $N_1 \times M_2$ is a graded 2-absorbing subhypermodule of M.*

(*ii*) *N_2 is a graded 2-absorbing subhypermodule of M_2 if and only if $M_1 \times N_2$ is a graded 2-absorbing subhypermodule of M.*

Proof. (*i*) Let N_1 be a graded 2-absorbing subhypermodule of M_1. Let $(a_g, b_g)(c_h, d_h)(m_k, m'_k) \in N_1 \times M_2$ where $(a_g, b_{g'}), (c_h, d_h) \in h(R)$ and $(m_k, m'_k) \in h(M)$. Then $(a_g c_h m_k, b_g d_h m'_k) = (a_g, b_g)(c_h, d_h)(m_k, m'_k) \in N_1 \times M_2$, i.e., $a_g c_h m_k \in N_1$ and $b_g d_h m'_k \in M_2$. Since N_1 is a graded 2-absorbing R_1-subhypermodule of M_1, it follows that $a_g c_h M_1 \subseteq N_1$ or $a_g m_k \in N_1$ or $c_h m_k \in N_1$. That is $(a_g, b_g)(c_h, d_h)M = (a_g c_h M_1, b_g d_h M_2) \subseteq N_1 \times M_2$ or $(a_g, b_g)(m_k, m'_k) = (a_g m_k, b_g m'_k) \in N_1 \times M_2$ or $(c_h, d_h)(m_k, m'_k) = (c_h m_k, d_h m'_k) \in N_1 \times M_2$. Therefore $N_1 \times M_2$ is a graded 2-absorbing subhypermodule of M. Conversely, assume that $N_1 \times M_2$ is a graded 2-absorbing subhypermodule of M. Let $a_g, b_h \in h(R_1)$ and $m_k \in h(M_1)$ be such that $a_g b_h m_k \in N_1$. Let $x_g, y_h \in h(R_2)$ and $m'_k \in h(M_2)$. Then $(a_g, x_{g'})(b_h, y_{h'})(m_k, m'_k) = (a_g b_h m_k, x_g y_h m'_k) \in N_1 \times M_2$. Since $N_1 \times M_2$ is a graded 2-absorbing R-subhypermodule of M, $(a_g, x_g)(b_h, y_h)M \subseteq N_1 \times M_2$ or $(a_g, x_g)(m_k, m'_k) \in N_1 \times M_2$ or $(b_h, y_h)(m_k, m'_k) \in N_1 \times M_2$. Hence $a_g b_h M_1 \subseteq N_1$ or $a_g m_k \in N_1$ or $b_h m_k \in N_1$. Thus N_1 is a graded 2-absorbing subhypermodule of M_1.

(*ii*) The proof is similar to that (*i*). □

6 Conclusions

In this article, we introduced the concepts of graded 2-absorbing hyperideals and graded n-absorbing hyperideals of a graded Krasner hyperring as a generalization of prime hyperideals. Also, we introduced and studied graded 2-absorbing subhypermodules of a graded krasner hyperring. We showed that 2-absorbing (n-absorbing) hyperideals and graded 2-absorbing (n-absorbing) hyperideals are totally different. Furthermore, several properties, examples and characterizations of graded 2-absorbing (n-absorbing) hyperideals have been investigated. Moreover, we investigated the properties and the behaviour of this structure under homogeneous components, graded hyperring homomorphisms, Cartesian product. Finally, we introduced the concept of graded 2-absorbing subhypermodules of a graded Krasner R-hypermodule and we investigate some properties of such graded subhypermodules.

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