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Applications of states to BI-algebras

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Abstract

This paper aims is to introduce states, Bosbach states and state-morphism operators on BI-algebras. We define state ideals on BI-algebras and give a characterization of the least state ideal of a BI-algebra. It is proved that the kernel of a Bosbach state on a BI-algebra X is an ideal of X. Further, by these concepts, we introduce the notions of state BI-algebras and state-morphism BI-algebras. The notion of complement pairs of a BI-algebra X is defined, and proves that under suitable conditions, there is a one-to-one correspondence between complement pairs of BI-algebras and state-morphism operators on BI-algebras.

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1 Introduction

As a generalization of dual implication algebras and implicative BCK-algebras, Borumand Saeid et al. introduced a new notion of logical algebras namely, BI-algebras, and gave basic properties of BI-algebras and investigated ideals and congruence relations on this new algebra. BI-algebras are weaker than some well-known algebras, such as implicative BCK-algebras and Boolean lattices. Indeed, these algebras are BI-algebra, but the converse does not hold (see, [2]). For more details and other comparisons with the other algebras, ideals, normal subalgebras in BI-algebras and the quotient BI-algebras are studied in [1].

It is known that the notion of *state* was firstly defined on an MV-algebra by Kõpka and Chovane in [32], and then has been studied and applied to other algebraic structures, since they have important roles in studying logical algebras (see, for instance, Borzooei et al. [3, 4, 5], Buşneag [8, 9], Ciungu [13, 14, 15], Ciungu and Dvurečenskij [17], Ciungu et al. [16, 18], Chen and Dudek [10], Cheng et al. [11], Di Nola and Dvurečenskij [21], Di Nola et al. [22], Dvurečenskij and Zahiri

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[26], Ghasemi et al. [30], Hua [31], Lee and Kim [33], Mertanen [34], Qing and Long [36], Rezaei et al. [38], Turunen and Mertanen [39], Xin et al. [40], Xin and Davvaz [41] and Xin et al. [42]).

Since states can be thought of in another way, the Bosbach state was defined in [6], [7], [12], and [28]. Georgescu and Mureşan, by replacing the MV-algebra [0,1] with an arbitrary residuated lattice L, defined a new concept of state were named generalized Bosbach state in [29], and extended it to type I and type II states. Then the Bosbach states defined on residuated lattices with values in residuated lattices were investigated by Ciungu et al. in [19] and [20]. Flaminio and Montagna introduced MV-state algebras in [27]. The notion of state-morphism MV-algebra, which is a stronger variation of a state MV-algebra, is introduced by Di Nola and Dvurečenskij in [21] and [22]. The notion of a state operator was extended to the cases of fuzzy structures, bounded (non-) commutative $R\ell$-monoids, and GMV-algebras (pseudo-MV-algebras) by Dvurečenckij and Rachunek in [23] and [24], Dvurečenckij et al. in [25], Rachunek and Salounova in [37].

In this paper, we introduce the notions of *states*, *Bosbach states* and *state-morphism operators* on *BI-algebras*. Also, we define *state ideals* on BI-algebras and give a characterization of the least state ideal of a BI-algebra. It is proven that the kernel of a Bosbach state on a BI-algebra X is an ideal of X. Further, by these concepts, we introduce the notions of *state BI-algebras* and *state-morphism BI-algebras*. The notion of *complement pairs* of a BI-algebra X is defined, and prove that under suitable conditions, there is one-to-one correspondence between complement pairs of BI-algebras and state-morphism operators on BI-algebras.

2 Preliminaries

We recalled some definitions and results which will be used in the sequel. Throughout this paper, we will denote $\mathbb{N}$ for the set of all positive integers and $\mathbb{R}$ for the set of real numbers.

Definition 2.1. [2] *An algebra* $(X;*,0)$ *of type* $(2,0)$ *is called a BI-algebra if satisfying the following axioms: for all* $x, y \in X$,

(B) $x * x = 0$,

(BI) $x * (y * x) = x$.

From now on, by X, we mean that it is a BI-algebra $(X;*,0)$.

We introduce the binary relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 0$. Notice that the relation $\leq$ is not a partial order, since it is only reflexive.

A BI-algebra X is said to be *right distributive* or *self distributive* (briefly, distributive) if

$$(x * y) * z = (x * z) * (y * z),$$

for all $x, y, z \in X$ (see, [2]).

Proposition 2.2. [2] *The following statements hold: for all* $x, y, z, u \in X$,

(i) $x * 0 = x$,

(ii) $0 * x = 0$,

(iii) $x * y = (x * y) * y$,

(iv) *if* $y * x = x$, *then* $X = \{0\}$,

(v) *if* $x * (y * z) = y * (x * z)$, *then* $X = \{0\}$, (*vi*) *if* $x * y = z$, *then* $z * y = z$ *and* $y * z = y$,

(*vii*) *if* $(x * y) * (z * u) = (x * z) * (y * u)$, *then* $X = \{0\}$.

Remark 2.3. *Notice that, if* $z * (x * y) = (z * x) * (z * y)$, *for all* $x, y, z \in X$, *then* $X = \{0\}$. *Since if we take* $y := x$, *for all* $z \in X$, *and using Proposition 2.2(i) and* (B), *we obtain* $z = z * 0 = z * (x * x) = (z * x) * (z * x) = 0$.

Proposition 2.4. [2] *Let* X *be distributive. Then for all* $x, y, z \in X$,

(*i*) $y * x \leq y$,

(*ii*) $x * (x * y) \leq y$,

(*iii*) $(x * z) * (y * z) \leq x * y$,

(*iv*) *if* $x \leq y$, *then* $x * z \leq y * z$,

(*v*) $(x * y) * z \leq x * (y * z)$,

(*vi*) *if* $x * y = z * y$, *then* $(x * z) * y = 0$.

A subset I of X is called an ideal of X if (I1) $0 \in I$ and (I2) $y \in I$ and $x * y \in I$ imply $x \in I$, for all $x, y \in X$ (see, [2]).

Denote the set of all ideals on X by $\mathcal{I}(X)$.

Theorem 2.5. [2] *Let* X *be distributive, and* $I \in \mathcal{I}(X)$. *Then the binary relation"*$\sim_I$*" where defined by*

$$x \sim_I y \text{ if and only if } x * y \in I \text{ and } y * x \in I$$

is a right congruence relation on X.

Analytic constructions for BI-algebras are considered in [1].
Let $X := \{x \in \mathbb{R} : x \geq 0\}$. Define the binary operation "$*$" on X as follows:

$$x * y = \max\{0, f(x, y)(x - y)\} = \max\{0, \lambda(x, y)\, x\},$$

where $f(x, y)$ and $\lambda(x, y)$ are non-negative real valued functions, with $\lambda(0, y) = 0$, for all $y \in X$. If we define

$$\lambda(x, y) = \begin{cases} 1 & \text{if } y = 0; \\ 0 & \text{if } y \neq 0, \end{cases}$$

then

$$x * y = \begin{cases} x & \text{if } y = 0; \\ 0 & \text{if } y \neq 0, \end{cases}$$

Thus $(X; *, 0)$ is a BI-algebra (see, [1]).

3 State operators on BI-Algebras

In this section, we introduce the notion of states on BI-algebras and investigate their properties.

Definition 3.1. *A map $\sigma : X \to X$ is called state operator on X if it satisfying the following conditions: for all $x, y \in X$,*

(SO1) $x \leq y$ *implies* $\sigma(x) \leq \sigma(y)$,

(SO2) $\sigma(x * y) = \sigma(x) * \sigma(x * (x * y))$,

(SO3) $\sigma(\sigma(x) * \sigma(y)) = \sigma(x) * \sigma(y)$.

A *state BI-algebra* is a pair (X, σ).

Denote $\ker \sigma = \{x \in X : \sigma(x) = 0\}$, that is the kernel of σ. A state operator σ is *faithful* if $\ker \sigma = \{0\}$.

Example 3.2. (i) Let X be a BI-algebra, and $\sigma : X \to X$ be a map defined by $\sigma(x) = 0$, for all $x \in X$. Then it is easy to see that σ is a state operator on X.
(ii) Let $X = \{0, a, b\}$. Define the binary operation "$*_1$" in Table 1 and define $\sigma : X \to X$ by

Table 1: BI-algebra $(X; *_1, 0)$

$*_1$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

$\sigma(0) = 0$ and $\sigma(a) = \sigma(b) = b$. Then (X, σ) is a state BI-algebra.
(iii) Let X be a BI-algebra. Define two operators "σ_1" and "σ_2" on the direct product BI-algebra $X \times X$ as follows:

$$\sigma_1(x, y) = (x, x) \text{ and } \sigma_2(x, y) = (y, y), \text{ for all } (x, y) \in X \times X.$$

Then σ_1 and σ_2 are two state operators on $X \times X$.

Denote the set of all state operators on X by $\mathcal{S}(X)$.

Now, we give some properties of state operators on BI-algebras.

Proposition 3.3. *Let $\sigma \in \mathcal{S}(X)$. Then the following hold: for all $x \in X$,*

(*i*) $\sigma(0) = 0$,

(*ii*) $\sigma(\sigma(x)) = \sigma(x)$,

(*iii*) *img* $\sigma = \{x \in X : \sigma(x) = x\}$,

(*iv*) *img* σ *is a subalgebra of* X,

(*v*) $\ker \sigma \in \mathcal{I}(X)$.

Proof. (i) If we take $x := y$ in (SO3) and using (B), we get

$$\sigma(0) = \sigma\left(\sigma\left(x\right) * \sigma\left(x\right)\right) = \sigma\left(x\right) * \sigma\left(x\right) = 0.$$

(ii) By (i) and Proposition 2.2(i), we have

$$\sigma\left(\sigma\left(x\right) * \sigma\left(0\right)\right) = \sigma\left(\sigma\left(x\right) * 0\right) = \sigma(\sigma\left(x\right)).$$

On the other hand, if we take $y := 0$ in (SO3), then

$$\sigma\left(\sigma\left(x\right) * \sigma\left(0\right)\right) = \sigma\left(x\right) * \sigma(0) = \sigma\left(x\right).$$

Therefore (ii) holds.
(iii) Clearly, $\{x \in X : \ \sigma\left(x\right) = x\} \subseteq img\ \sigma$. Now, suppose $x \in img\ \sigma$. Then there exists $x' \in X$ such that $\sigma\left(x'\right) = x$. Then by (ii), we have

$$x = \ \sigma\left(x'\right) = \sigma(\sigma\left(x'\right)) = \sigma\left(x\right).$$

This shows that $x \in img\ \sigma$, and so $img\ \sigma \subseteq \{x \in X : \sigma\left(x\right) = x\}$. Hence (iii) holds.
(iv) From (i), $0 \in img\ \sigma$. For all $x, y \in X$ by (SO3), we have $\sigma\left(x\right) * \sigma(y) \in img\ \sigma$. Thus $img\ \sigma$ is a subalgebra of X.
(v) Suppose $y \in \ker\sigma$ and $x * y \in \ker\sigma$. Then $\sigma\left(y\right) = \sigma\left(x * y\right) = 0$. On the other hand, using (SO2), we get

$$0 = \sigma\left(x * y\right) = \sigma\left(x\right) * \sigma\left(x * \left(x * y\right)\right).$$

Now, by Proposition 2.4(ii), since $x * (x * y) \leq y$, using (SO1), we get

$$\sigma\left(x * \left(x * y\right)\right) \leq \sigma\left(y\right) = 0.$$

Hence $\sigma\left(x * \left(x * y\right)\right) = 0$, and so $\sigma\left(x\right) = 0$. This means that $x \in \ker\sigma$, and so $\ker\sigma \in \mathcal{I}(X)$. □

Proposition 3.4. *Let X be distributive and $\sigma \in \mathcal{S}(X)$. Then the following statements hold: for all $x, y \in X$,*

(*i*) *if $x \leq y$ and for any $z \in X$, $z * y \leq z * x$, then $\sigma\left(y\right) * \sigma(x) \leq \sigma(y * x)$,*

(*ii*) $\ker\sigma \cap img\ \sigma = \{0\}$.

Proof. (i) Given $x, y \in X$. Using Proposition 2.4(ii), we have $y * (y * x) \leq x$. Hence $\sigma(y * (y * x)) \leq \sigma(x)$. Then $\sigma\left(y\right) * \sigma\left(x\right) \leq \sigma\left(y\right) * \sigma\left(y * \left(y * x\right)\right) = \sigma(y * x)$, by hypothesis and (SO2).
(ii) Suppose $x \in \ker\sigma \cap img\ \sigma$. It follows that $\sigma(x) = 0$. Moreover, $x \in img\ \sigma$, so there exists $x' \in X$ such that $\sigma\left(x'\right) = x$. Then by Proposition 3.3(ii), $0 = \sigma\left(x\right) = \sigma\left(\sigma\left(x'\right)\right) = \sigma\left(x'\right) = x$. Thus $x = 0$, and so (ii) holds. □

The following example shows that in Proposition 3.4(i), the distributive law and condition $z * y \leq z * x$, for any $z \in X$, are necessary.

Example 3.5. Let $X = \{0, a, b, c\}$. Define the binary operation "$*_2$" in Table 2. Then $(X; *_2, 0)$ is a BI-algebra (see, [2]), but is not distributive, since

$$(a *_2 b) *_2 c = a *_2 c = b \neq (a *_2 c) *_2 (b *_2 c) = b *_2 b = 0.$$

Define $\sigma : X \to X$ by $\sigma(0) = \sigma(b) = \sigma(c) = 0$ and $\sigma(a) = a$. Then (X, σ) is a state BI-algebra, but not satisfies in Proposition 3.4(i), since $0 \leq c$, but $a *_2 c = b \nleq a *_2 0 = a$. Further, $\sigma(a) *_2 \sigma(c) = a *_2 0 = a \nleq \sigma(a *_2 c) = \sigma(b) = 0$.

Table 2: BI-algebra $(X; *_2, 0)$

$*_2$	0	a	b	c
0	0	0	0	0
a	a	0	a	b
b	b	b	0	b
c	c	b	c	0

Proposition 3.6. *Let X be commutative (i.e., $x * (x * y) = y * (y * x)$ for all $x, y \in X$), $\sigma \in \mathcal{S}(X)$ and $y \leq x$. Then $\sigma(x * y) = \sigma(x) * \sigma(y)$.*

Proof. Given $x, y \in X$ with $y \leq x$. Using the commutative law and Proposition 2.2(i), we have

$$\begin{aligned}\sigma(x * y) &= \sigma(x) * \sigma(x * (x * y)) = \sigma(x) * \sigma(y * (y * x)) \\ &= \sigma(x) * \sigma(y * 0) = \sigma(x) * \sigma(y).\end{aligned}$$

This completes the proof. □

The following example shows that the commutative law in Proposition 3.6 is necessary.

Example 3.7. Consider Example 3.5. It is not commutative, since

$$a *_2 (a *_2 c) = a *_2 b = a \neq c *_2 (c *_2 a) = c *_2 b = c.$$

Define $\sigma : X \to X$ by $\sigma(a) = \sigma(a) = \sigma(b) = 0$ and $\sigma(c) = c$. Then (X, σ) is a state BI-algebra and we can see that

$$\sigma(c *_2 a) = \sigma(b) = 0 \neq \sigma(c) *_2 \sigma(a) = c *_2 0 = c.$$

Remark 3.8. *Notice that, if $\sigma(x * y) = \sigma(y)$ or $\sigma(x * y) = \sigma(x)$, for any $x, y \in X$, then σ is zero map. By contrary, if there is $x \in X$ such that $\sigma(x) \neq 0$, then we have $\sigma(0) = \sigma(x * 0) = \sigma(x)$. By Proposition 3.3(i), $\sigma(0) = 0$, this implies $\sigma(x) = 0$, a contradiction. Then σ is zero map. Further, $\sigma(0) = \sigma(x * x) = \sigma(x)$, using Proposition 3.3(i), we get $\sigma(x) = 0$, for all $x \in X$.*

Proposition 3.9. *Let $X = \{x \in \mathbb{R} : x \geq 0\}$. Then there exists non-zero non-negative real valued function $\lambda(x, y)$ such that $(X; *_\lambda, 0)$ becomes a BI-algebra, where*

$$x *_\lambda y = \max\{0, \lambda(x, y)\, x\}$$

and for every state operator σ on X, we have $\sigma(x *_\lambda y) = 0$, for all $0 \neq x, y \in X$.

Proof. Assume $X = \{x \in \mathbb{R} : x \geq 0\}$. Define $\lambda : X \times X \to X$ by

$$\lambda(x, y) = \begin{cases} 1 & \text{if } y = 0; \\ 0 & \text{if } y \neq 0, \end{cases}$$

for all $(x, y) \in X \times X$. Then $(X; *_\lambda, 0)$ is a BI-algebra (see, [1]). Let $x, y \in X$, since X is linearly ordered, we have $x \leq y$ or $y < x$. If $x \leq y$, then $x *_\lambda y = 0$. Thus the proof completes. Now, suppose $y < x$. Then

$$\begin{aligned}x *_\lambda (x *_\lambda y) &= x *_\lambda (\max\{0,\ \lambda(x, y)\, x\}) = x *_\lambda \left(\max\left\{0, \begin{cases} x & \text{if } y = 0; \\ 0 & \text{if } y \neq 0. \end{cases}\right\}\right) \\ &= x *_\lambda \left(\begin{cases} x & \text{if } y = 0; \\ 0 & \text{if } y \neq 0. \end{cases}\right) = \begin{cases} x *_\lambda x & \text{if } y = 0; \\ x *_\lambda 0 & \text{if } y \neq 0. \end{cases} = \begin{cases} 0 & \text{if } y = 0; \\ x & \text{if } y \neq 0. \end{cases}\end{aligned}$$

If $y = 0$, then $\sigma(x *_\lambda 0) = \sigma(x)$, and if $y \neq 0$, then

$$\sigma(x *_\lambda y) = \sigma(x) *_\lambda \sigma(x *_\lambda (x *_\lambda y)) = \sigma(x) *_\lambda \sigma(x) = 0.$$

This completes the proof. □

Definition 3.10. *Let $\sigma \in \mathcal{S}(X)$. An ideal I of X is called a state ideal if $\sigma(I) \subseteq I$.*

We denote the set of all state ideals on X by $\mathcal{SI}(X)$.

Example 3.11. Consider Example 3.2(ii) and take $I := \{0, b\}$ and $J := \{0, a\}$. Then $I \in \mathcal{SI}(X)$, but $J \notin \mathcal{SI}(X)$, since $\sigma(a) = b \notin J$.

Proposition 3.12. *Let $\sigma \in \mathcal{S}(X)$ and $\{I_i\}_{i\in\Lambda}$ be a family of states ideals of X, then $\bigcap_{i\in\Lambda} I_i$, is too.*

Proof. Assume $\sigma \in \mathcal{S}(X)$ and $\{I_i\}_{i\in\Lambda}$ is a family of states ideals of X. Since $I_i \in \mathcal{I}(X)$, we get $\bigcap_{i\in\Lambda} I_i \in \mathcal{I}(X)$. Now, let $x \in \bigcap_{i\in\Lambda} I_i$. Then $x \in I_i$, for all $i \in \Lambda$, and so $\sigma(x) \in \sigma(I_i) \subseteq I_i$, since I_i is a state ideal of X. Hence $\sigma(x) \in \bigcap_{i\in\Lambda} I_i$. It follows that $\sigma(\bigcap_{i\in\Lambda} I_i) \subseteq \bigcap_{i\in\Lambda} I_i$. Thus $\bigcap_{i\in\Lambda} I_i \in \mathcal{SI}(X)$. □

Since the set $\mathcal{SI}(X)$ is closed under arbitrary intersections, we have the following theorem.

Theorem 3.13. $(\mathcal{SI}(X); \subseteq)$ *is a complete lattice.*

The following example shows that the union of two state ideals may not be a state ideal, in genaral.

Example 3.14. Let $X = \{0, a, b, c\}$. Define the binary operation "$*_3$" in Table 3. Then $(X; *_3, 0)$

Table 3: BI-algebra $(X; *_3, 0)$

$*_3$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	a
c	c	c	a	0

is a BI-algebra. Define $\sigma : X \to X$ by $\sigma(0) = \sigma(a) = 0$ and $\sigma(b) = \sigma(c) = c$. Then $\sigma \in \mathcal{S}(X)$ and (X, σ) is a state BI-algebra. If we take $I_1 := \{0, a\}$ and $I_2 := \{0, c\}$, then $I_1, I_2 \in \mathcal{SI}(X)$, but $I_1 \cup I_2 = \{0, a, c\}$ is not an ideal of X, since $c, b *_3 c \in I_1 \cup I_2$, but $b \notin I_1 \cup I_2$. Thus $I_1 \cup I_2 \notin \mathcal{SI}(X)$.

Definition 3.15. *Let $\sigma \in \mathcal{S}(X)$ and $I \in \mathcal{I}(X)$. For any $x, y \in X$, define*

$$I_\sigma(x, y) := \{t \in X : (t * x) * \sigma(y) \in I\}.$$

Notice that, by Proposition 2.2(ii), since $0 * x = 0$, for all $x \in X$, we get $0 \in I_\sigma(x, y)$, for all $x, y \in X$. Hence $I_\sigma(x, y) \neq \emptyset$. Also, for all $x \in X$, $I_\sigma(0, x) := \{t \in X : t * \sigma(x) \in I\}$ and $I_\sigma(x, 0) := \{t \in X : t * x \in I\}$, since $\sigma(0) = 0$.

The following example shows that for $\sigma \in \mathcal{S}(X)$ and $I \in \mathcal{I}(X)$, $I_\sigma(x, y) \neq I_\sigma(y, x)$, and may $I_\sigma(x, y) \notin \mathcal{SI}(X)$, in general.

Example 3.16. (i) Consider Example 3.5. If we take $I := \{0, a, c\}$, then $I \in \mathcal{I}(X)$. One can easily see that $I_\sigma(c, a) = \{0, c\} \neq I_\sigma(a, c) = I$ and $I_\sigma(b, c) = X$.

(ii) In Example 3.14, take $I := \{0, c\}$. Then $I \in \mathcal{SI}(X)$ and $I_\sigma(a, b) = \{0, a, c\} \notin \mathcal{SI}(X)$, since $c, b *_3 c = a \in I_\sigma(a, b)$, but $b \notin I_\sigma(a, b)$.

Proposition 3.17. *Let X be distributive, $\sigma \in \mathcal{S}(X)$ and $I \in \mathcal{I}(X)$. Then $I_\sigma(x, y) \in \mathcal{I}(X)$.*

Proof. Assume $I \in \mathcal{I}(X)$ and $x, y \in X$. Using Proposition 2.2(ii) and (I1), we get

$$(0 * x) * \sigma(y) = 0 * \sigma(y) = 0 \in I.$$

Hence $0 \in I_\sigma(x, y)$, and so $I_\sigma(x, y) \neq \emptyset$.
Let $b, a * b \in I_\sigma(x, y)$. Then $(b * x) * \sigma(y) \in I$ and $((a * b) * x) * \sigma(y) \in I$. Using distributive law, we obtain

$$((a * x) * \sigma(y)) * ((b * x) * \sigma(y)) = ((a * x) * (b * x)) * \sigma(y) = ((a * b) * x) * \sigma(y) \in I.$$

Since $I \in \mathcal{I}(X)$ and $(b * x) * \sigma(y) \in I$, we get $(a * x) * \sigma(y) \in I$. It follows that $a \in I_\sigma(x, y)$. Thus $I_\sigma(x, y) \in \mathcal{I}(X)$. □

The following example shows that the distributive law in Proposition 3.17 is necessary.

Example 3.18. Consider Example 3.5. If we take $I := \{0, a, c\}$, then $I \in \mathcal{I}(X)$. We can see that $I_\sigma(b, a) = \{0, b, a\}$, where $I_\sigma(b, a) \notin \mathcal{I}(X)$, since a, $c *_2 a = b \in I_\sigma(b, a)$, but $c \notin I_\sigma(b, a)$.

Proposition 3.19. *Let $I \in \mathcal{SI}(X)$ and $\sigma \in \mathcal{S}(X)$. Then $I = \bigcup_{x \in I} I_\sigma(0, x)$.*

Proof. Assume $I \in \mathcal{SI}(X)$, $\sigma \in \mathcal{S}(X)$ and $t \in I$. Let $x \in I$. Hence $\sigma(x) \in \sigma(I) \subseteq I$. Then $t * \sigma(x) \in I$, and so $t \in I_\sigma(0, x) \subseteq \bigcup_{x \in I} I_\sigma(0, x)$ It follows that $t \in \bigcup_{x \in I} I_\sigma(0, x)$. Thus $I \subseteq \bigcup_{x \in I} I_\sigma(0, x)$. On the other hand, let $t \in \bigcup_{x \in I} I_\sigma(0, x)$. Then there exists $x \in I$ such that $t \in I_\sigma(0, x)$. Hence $t * \sigma(x) \in I$. Since $I \in \mathcal{SI}(X)$ and $x \in I$, we have $\sigma(x) \in \sigma(I) \subseteq I$, and so $\sigma(x) \in I$. Thus $t \in I$. It shows that $\bigcup_{x \in I} I_\sigma(0, x) \subseteq I$. □

Corollary 3.20. *Let X be distributive, $I \in \mathcal{I}(X)$, $\sigma \in \mathcal{S}(X)$ and $a \in X$. If we take $M_a := \{t \in X : (t * a) * \sigma(a) \in I\}$, then $M_a \in \mathcal{I}(X)$.*

Proof. Similar to the proof Proposition 3.17, if we take $M_a := I_\sigma(a, a)$. □

The following example shows that there is $I \in \mathcal{SI}(X)$ and $a \in X$, where $M_a \notin \mathcal{SI}(X)$.

Example 3.21. Consider the state ideal $I_2 = \{0, c\}$ in Example 3.14. One can easily see that $M_a = \{0, a, c\} \notin \mathcal{SI}(X)$, since c, $b *_3 c = a \in M_a$, but $b \notin M_a$.

Open problem. Consider status Proposition 3.17 or Corollary 3.20, if $\sigma \in \mathcal{S}(X)$ and $I \in \mathcal{SI}(X)$, then $I_\sigma(x, y) \in \mathcal{SI}(X)$ or $M_a \in \mathcal{SI}(X)$? Under what condition/conditions is/are it possible?

Theorem 3.22. *Let X be distributive, $I \in \mathcal{I}(X)$ and $\sigma \in \mathcal{S}(X)$. Then $\sim_I$, where defined in Theorem 2.5, is a congruence relation on X, and $[0]_I \in \mathcal{SI}(X)$.*

Proof. By [2, Lemma 5.6], $[0]_I \in \mathcal{I}(X)$. Let $x \in [0]_I$. Then $x \sim_I 0$. This implies that $\sigma(x) \sim_I \sigma(0)$. By Proposition 3.3(i), $\sigma(x) \sim_I 0$. It shows that $\sigma(x) \in [0]_I$. Thus $[0]_I \in \mathcal{SI}(X)$. □

Definition 3.23. *Let $\sigma \in \mathcal{S}(X)$ and $\emptyset \neq I \subseteq X$. Define the state ideal generated by I as follows:*

$$\langle I \rangle_S := \bigcap_{I \subseteq I_i} I_i,$$

where $\sigma \in \mathcal{S}(X)$ and $I \subseteq I_i \in \mathcal{SI}(X)$, for $i \in \Lambda$.

Notice that, in Definition 3.23, $\bigcap_{I \subseteq I_i} I_i \in \mathcal{SI}(X)$, by Proposition 3.12. Also, if $I \in \mathcal{SI}(X)$, then $\langle I \rangle_S = I$.

Borumand Saeid et al. defined the set $A(x, y) := \{t \in X : (t * x) * y = 0\}$, and it was shown that if X is distributive, then $A(x, y) \in \mathcal{I}(X)$, where $x, y \in X$.
Also, we can see that $A(x, 0) = A(0, x)$, for all $x \in X$. Further, it is shown that if $I \in \mathcal{I}(X)$, then (see for details, Proposition 4.10 and Theorem 4.11 (see, [2]))

$$I = \bigcup_{x \in I} A(0, x) = \bigcup_{x, y \in I} A(x, y).$$

The following example shows that if X is distributive and $\emptyset \neq I \subseteq X$, then

$$\langle I \rangle_S \neq \bigcap_{x, y \in I} A(x, y).$$

Example 3.24. Consider Example 3.2(ii). Then (X, σ) is a state BI-algebra. If we take $I = \{b\}$, then $I \notin \mathcal{I}(X)$ and $< I >_S = \{0, b\}$.
Also, we can see that $A(0, 0) = \{0\}$ and $A(0, b) = A(b, 0) = A(b, b) = \{0, b\}$, where

$$\langle I \rangle_S = \{0, b\} \neq \bigcap_{x, y \in I \cup \{0\}} A(x, y) = \{0\}.$$

Also, If we take $I := \{0, a\}$, then we can see that $I \in \mathcal{I}(X)$ and

$$\langle I \rangle_S = I = \{0, a\} \neq \bigcap_{x, y \in I \cup \{0\}} A(x, y) = \{0\}.$$

The following theorem show that a representation of $\langle I \rangle_S$.

Theorem 3.25. *Let $\emptyset \neq I \subseteq X$. Then $\langle I \rangle_S = \bigcap_{I \subseteq I_i} \bigcup_{x \in I_i} I_{i\sigma}(0, x)$, where $\sigma \in \mathcal{S}(X)$ and $I_i \in \mathcal{SI}(X)$, for all $i \in \Lambda$.*

Proof. By Definition 3.23 and Proposition 3.19, the proof is obvious. □

4 Bosbach states on BI-algebras

In this section, we introduce the notion of Bosbach states and show that there exists a Bosbach state via $\frac{X}{\sim_I}$ where $\sim_I$ is a congruence relation induced by an ideal I of distributive BI-algebra X.

Definition 4.1. *Let $\sigma : X \to [0,1]$ be a map. We say that σ is a Bosbach state on X, if the following conditions hold: for all $x, y \in X$*

(BS1) $\sigma(0) = 0$,

(BS2) $\sigma(x) + \sigma(y * x) = \sigma(y) + \sigma(x * y)$.

Example 4.2. (i) Consider Example 3.5. Define $\sigma : X \to [0,1]$ as follows:

$$\sigma(0) = 0 \text{ and } \sigma(a) = \sigma(b) = \sigma(c) = \frac{1}{2}.$$

Then σ is a Bosbach state on X.
(ii) Let $X = \{0, a, b\}$. Define the binary operation "$*_4$" in Table 4. Then $(X; *_4, 0)$ is a BI-algebra

Table 4: BI-algebra $(X; *_4, 0)$

$*_4$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

(see, [2]). Define $\sigma : X \to [0,1]$ by $\sigma(0) = 0$, $\sigma(a) = \frac{1}{2}$ and $\sigma(b) = 1$. Then σ is a Bosbach state on X.

Denote the set of all Bosbach states on X by $\mathcal{BS}(X)$.

Proposition 4.3. *Let $\sigma \in \mathcal{BS}(X)$. Then*

(i) $x \leq y$ *implies* $\sigma(x) \leq \sigma(y)$ *and* $\sigma(y * x) \leq \sigma(y)$,

(ii) $\ker \sigma \in \mathcal{I}(X)$.

Proof. (i) Given $x, y \in X$, if $x \leq y$, then $x * y = 0$. Hence

$$\sigma(x) + \sigma(y * x) = \sigma(y) + \sigma(x * y) = \sigma(y) + \sigma(0) = \sigma(y) + 0 = \sigma(y).$$

Since $\sigma(t) \geq 0$, for all $t \in X$, and $\sigma(x) + \sigma(y * x) = \sigma(y)$, we get $\sigma(x) \leq \sigma(y)$ and $\sigma(y * x) \leq \sigma(y)$.
(ii) Clearly, $0 \in \ker \sigma$. If $y, x * y \in \ker \sigma$, then $\sigma(y) = \sigma(x * y) = 0$. Since $\sigma \in \mathcal{BS}(X)$, we have

$$0 = 0 + 0 = \sigma(y) + \sigma(x * y) = \sigma(x) + \sigma(y * x).$$

Since $\sigma(x)$, $\sigma(y * x) \in [0,1]$ and $\sigma(x) + \sigma(y * x) = 0$, we get $\sigma(x) = 0$ and $\sigma(y * x) = 0$. Thus $x \in \ker \sigma$. □

Definition 4.4. *Let $(X; *, 0)$ and $(Y; \diamond, 0)$ be two BI-algebras. A map $\theta : X \to Y$ is called a homomorphism if $\theta(x * y) = \theta(x) \diamond \theta(y)$, for all $x, y \in X$.*

Notice that, if we take $y := x$, than by (B), $\theta(0) = \theta(x * x) = \theta(x) \diamond \theta(x) = 0$.

Example 4.5. (i) The identity map from any BI-algebra is a homomorphism.
(ii) Consider BI-algebra X in Example 4.2(ii) and Y is the BI-algebra in Example 3.5. Define $\theta : X \to Y$ by $\theta(0) = 0$, $\theta(a) = b$ and $\theta(b) = c$. Then θ is a homomorphism.
(iii) Every map $\theta : X \to Y$ between BI-algebras defined by $\theta(x) = 0$, for all $x \in X$ is a homomorphism.

Lemma 4.6. *Let $(X; *, 0)$ and $(Y; \diamond, 0)$ be two BI-algebras and $\theta : X \to Y$ be a homomorphism. Then*

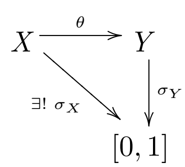
(*i*) $\theta(0) = 0$,

(*ii*) $x \leq y$ *implies* $\theta(x) \leq \theta(y)$,

(*iii*) $\ker\theta \in \mathcal{I}(X)$.

Proof. (i) From (BS1), we have $\theta(0) = \theta(0 * 0) = \theta(0) \diamond \theta(0) = 0$.
(ii) If $x \leq y$, then $x * y = 0$. Using (i), we have $0 = \theta(0) = \theta(x * y) = \theta(x) \diamond \theta(y)$. This means that $\theta(x) \leq \theta(y)$.
(iii) Clearly, $0 \in \ker\theta$. Now, let $y, x * y \in \ker\theta$. Then $\theta(x * y) = 0$ and $\theta(y) = 0$. Using Proposition 2.2(ii), we obtain $0 = \theta(x * y) = \theta(x) \diamond \theta(y) = \theta(x) \diamond 0 = \theta(x)$. This means that $x \in \ker\theta$, and so $\ker\theta \in \mathcal{I}(X)$. □

Theorem 4.7. *Let $(X; *, 0)$ and $(Y; \diamond, 0)$ be two BI-algebras, $\theta : X \to Y$ be a homomorphism and $\sigma_Y \in \mathcal{BS}(Y)$. Then there is a unique $\sigma_X \in \mathcal{BS}(X)$ such that the following diagram is commutative (i.e., $\sigma_X = \sigma_Y \circ \theta$).*



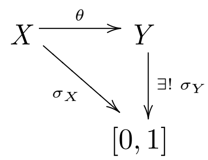
Proof. Define $\sigma_X : X \to [0, 1]$ by $\sigma_X(x) = \sigma_Y \circ \theta(x)$. Since σ_Y and θ are well-defined, σ_X is well-defined. By Lemma 4.6(i) and (BS1), we get $\sigma_X(0) = \sigma_Y(\theta(0)) = \sigma_Y(0) = 0$. Moreover, since $\sigma_Y \in \mathcal{BS}(Y)$, for all $x, y \in X$, we have

$$\begin{aligned}\sigma_X(x) + \sigma_X(y * x) &= \sigma_Y \circ \theta(x) + \sigma_Y \circ \theta(y * x) = \sigma_Y(\theta(x)) + \sigma_Y(\theta(y) \diamond \theta(x)) \\ &= \sigma_Y(\theta(y)) + \sigma_Y(\theta(x) \diamond \theta(y)) = \sigma_Y \circ \theta(y) + \sigma_Y \circ \theta(x * y) \\ &= \sigma_X(y) + \sigma_X(x * y).\end{aligned}$$

Thus $\sigma_X \in \mathcal{BS}(X)$. Now, let $\sigma' \in \mathcal{BS}(X)$ such that $\sigma' = \sigma_Y \circ \theta$. Then $\sigma'(x) = (\sigma_Y \circ \theta)(x) = \sigma_X(x)$, for all $x \in X$. This means that $\sigma' = \sigma_X$. Hence σ_X is a unique Bosbach state on X. □

Let $(X; *, 0)$ and $(Y; \diamond, 0)$ be two BI-algebras, and $\theta : X \to Y$ be a homomorphism. Then we say that θ is *injective*, if $\ker\theta = \{0\}$. The homomorphisms defined in Example 4.5(i)-(ii) are injective and the homomorphism defined in Example 4.5(iii) is not injective. As usual, a homomorphism is called bijective, if it is injective and surjective.

Theorem 4.8. *Let $(X;*,0)$ and $(Y;\diamond,0)$ be two BI-algebras, $\theta : X \to Y$ be a bijective homomorphism and $\sigma_X \in \mathcal{BS}(X)$. Then there is a unique $\sigma_Y \in \mathcal{BS}(Y)$ such that the following diagram is commutative (i.e., $\sigma_X = \sigma_Y \circ \theta$).*



Proof. Assume $y \in Y$ is an arbitrary element. Then from surjectivity of θ, there exists $x \in X$ such that $\theta(x) = y$. Thus for any $y \in Y$ there exists $x \in X$ such that x is depend on y. If we take $\sigma_Y(y) := \sigma(x)$, where x is depend on y, then $\sigma(x) = \sigma_Y(y) = \sigma_Y(\theta(x)) = \sigma_Y \circ \theta(x)$ and since θ is injective, we have $\sigma(x) = \sigma_Y \circ \theta(x)$, for all $x \in X$. Now, we show that $\sigma_Y \in \mathcal{BS}(Y)$.
(BS1) From Lemma 4.6(i), injectivity of θ and (BS1) property on σ_X, we have

$$\sigma_Y(0) = \sigma_Y(\theta(0)) = \sigma_X(0) = 0.$$

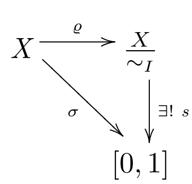
(BS2) Given $y, y' \in Y$, then there exist $x, x' \in X$ such that $\theta(x) = y$ and $\theta(x') = y'$. Thus

$$\begin{aligned}
\sigma_Y(y) + \sigma_Y(y' \diamond y) &= \sigma_Y(\theta(x)) + \sigma_Y(\theta(x') \diamond \theta(x)) \\
&= \sigma_Y(\theta(x)) + \sigma_Y(\theta(x' * x)) \\
&= \sigma_Y \circ \theta(x) + \sigma_Y \circ \theta(x' * x) \\
&= \sigma_X(x) + \sigma_X(x' * x) \\
&= \sigma_X(x') + \sigma_X(x * x') \\
&= \sigma_Y \circ \theta(x') + \sigma_Y \circ \theta(x * x') \\
&= \sigma_Y(y') + \sigma_Y(y \diamond y').
\end{aligned}$$

Then $\sigma_Y \in \mathcal{BS}(Y)$. Suppose $\sigma' \in \mathcal{BS}(Y)$ such that $\sigma_X(x) = \sigma' \circ \theta(x)$, for all $x \in X$. Let $y \in Y$. Then there exists $x \in X$ such that $\theta(x) = y$, and so $\sigma'(y) = \sigma'(\theta(x)) = \sigma' \circ \theta(x) = \sigma_X(x)$. On the other hand, according to the definition of σ_Y, we have $\sigma_Y(y) = \sigma_X$. Hence $\sigma'(y) = \sigma_X(x) = \sigma_Y(y)$, for all $y \in Y$. It follows that $\sigma' = \sigma_Y$. Thus σ_Y is unique and this completes the proof. □

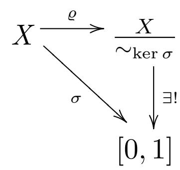
Let X be a distributive BI-algebra and $I \in \mathcal{I}(X)$. Consider relation "$\sim_I$" in Theorem 2.5, we denote by C_x the congruence class of x and let $\frac{X}{\sim_I} = \{C_x : x \in X\}$. Also, we define $\varrho : X \to \frac{X}{\sim_I}$ by $\varrho(x) = C_x$. Then $(\frac{X}{\sim_I}; \star, C_0)$ is a BI-algebra, where $C_x \star C_y = C_{x*y}$. Notice that, if $x \in I$, then $C_x = C_0$.

Corollary 4.9. *Let X be distributive BI-algebra, $I \in \mathcal{I}(X)$ and $\sigma \in \mathcal{BS}(X)$. Then there exists a unique Bosbach state $t : \frac{X}{\sim_I} \to [0,1]$ such that the following diagram is commutative (i.e., $\sigma = s \circ \varrho$), in fact, $\sim_I$ is a congruence relation induced by ideal I.*



Proof. Using Theorem 4.8, if we take $Y := \frac{X}{\sim_I}$, then the proof is complete. □

Corollary 4.10. *Let X be distributive BI-algebra and $\sigma \in \mathcal{BS}(X)$. Then there exists a unique Bosbach state $t : \frac{X}{\sim_{\ker \sigma}} \to [0, 1]$ such that the following diagram is commutative (i.e., $\sigma = s \circ \varrho$), in fact, $\sim_I$ is a congruence relation induced by $\ker \sigma$.*



Proof. Using Proposition 4.3(ii) and Corollary 4.9, if we take $I := \ker \sigma$, then the proof is complete. □

5 State-morphism operators on BI-algebras

In this section, we introduce the notion of state-morphism operators on BI-algebras. By this new notion, we introduce the notion of state-morphism BI-algebras.

Definition 5.1. *A homomorphism $\sigma : X \to X$ is called a state-morphism operator if $\sigma \circ \sigma = \sigma$, and the pair $(X; \sigma)$ is called a state-morphism BI-algebra.*

Example 5.2. (i) Let Id_X be the identity map on X. Then, clearly Id_X is a state-morphism operator. Notice that, Id_X is not a state operator on X.
(ii) Consider Example 3.2(ii), for any $x, y \in X$, we have $x *_1 y = (x *_1 y) *_1 y$. Define $f_b : X \to X$ by $f_b(x) = x *_1 b$, for all $x \in X$. Then by easy calculations, one can show that f_b is a homomorphism. Moreover,

$$(f_b \circ f_b)(x) = f_b(x *_1 b) = (x *_1 b) *_1 b = x *_1 b = f_b(x),$$

for all $x \in X$. Thus f_b is a state-morphism operator on X and (X, f_b) becomes a state-morphism BI-algebra.

From Example 5.2(i), we can see that any state-morphism operator may not be a state operator. Moreover, the converse may not be true, i.e., any state operator may not be a state-morphism operator. For example, consider the state σ in Example 3.2(ii). Then σ is not a state-morphism operator, since

$$b = \sigma(a) = \sigma(a *_1 b) \neq \sigma(a) *_1 \sigma(b) = b *_1 b = 0.$$

We denote the set of all state-morphism operators on X by $\mathcal{SMO}(X)$.

Proposition 5.3. *Let X be distributive. Then $\mathcal{SMO}(X) \neq \emptyset$.*

Proof. Assume X is distributive and $x, y \in X$. Define $\sigma_y : X \to X$ by $\sigma_y(x) = x * y$. Then for any $z \in X$,

$$\sigma_z(x * y) = (x * y) * z = (x * z) * (y * z) = \sigma_z(x) * \sigma_z(y).$$

Hence σ_z is a homomorphism. We show that $\sigma_z \circ \sigma_z = \sigma_z$. Using the distributive law, we get

$$(\sigma_z \circ \sigma_z)(x) = \sigma_z(\sigma_z(x)) = \sigma_z(x * z) = (x * z) * z = x * z = \sigma_z(x).$$

Thus $\sigma_z \in \mathcal{SMO}(X)$, and so $\mathcal{SMO}(X) \neq \emptyset$. □

It was shown that if $x \leq y$ and X satisfies the following condition:

$$(z * x) * (z * y) = y * x \qquad (\star)$$

Then $z * y \leq z * x$ (see, [2, Prop. 3.13]).

Proposition 5.4. *Let X be distributive and satisfies $(\star)$. Then $(x * y) * z \leq (x * z) * y$, for all $x, y, z \in X$.*

Proof. Using the distributive law, $(\star)$ and Proposition 2.2(ii), we get

$$\begin{aligned}((x * y) * z) * ((x * z) * y) &= ((x * y) * z) * ((x * y) * (z * y)) \\ &= (z * y) * z = (z * z) * (y * z) \\ &= 0 * (y * z) = 0.\end{aligned}$$

Thus $(x * y) * z \leq (x * z) * y$. □

The following example shows that the distributive law in Proposition 5.4 is necessary.

Example 5.5. Let $X = \{0, a, b, c, d\}$. Define the binary operation "$*_6$" in Table 6. Then $(X; *_6, 0)$

Table 5: BI-algebra $(X; *_6, 0)$

$*_6$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	d	d	c
b	b	0	0	b	b
c	c	0	c	0	c
d	d	0	d	d	0

is a BI-algebra and satisfies $(\star)$, but not distributive, since

$$(a *_6 d) *_6 b = c *_6 b = c \neq 0 = d *_6 d = (a *_6 b) *_6 (d *_6 b).$$

Also, $((a *_6 d) *_6 b) *_6 ((a *_6 b) *_6 d) = (c *_6 b) *_6 (d *_6 d) = c *_6 0 = c \neq 0$.

Proposition 5.6. *Let X be distributive and $\sigma \in \mathcal{SMO}(X)$, where satisfies $(\star)$, and $I \in \mathcal{I}(X)$. Then*

$$\langle I \rangle_S = \{x \in X : (((x * \sigma(x_1)) * \sigma(x_2)) * \cdots) * \sigma(x_n) \in I, \exists n \in \mathbb{N}, \exists x_1, \ldots, x_n \in X\}.$$

Proof. We denote the right hand by M. Clearly, $I \subseteq M$. We show that $M \in \mathcal{I}(X)$.
Assume $x, y * x \in M$. Then there exist $m, n \in \mathbb{N}$, and $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ such that

$$(((x * \sigma(x_1)) * \sigma(x_2)) * \cdots) * \sigma(x_n) \in I \text{ and } ((((y * x) * \sigma(y_1)) * \sigma(y_2)) * \cdots) * \sigma(y_m) \in I.$$

Then by Proposition 2.4(iii)-(iv) and Proposition 5.4,

$$
\begin{aligned}
&((((((y*\sigma(y_1))*\sigma(y_2))*\cdots)*\sigma(y_m))*\sigma(x_1))*\cdots*\sigma(x_n))\\
&*((((x*\sigma(x_1))*\sigma(x_2))*\cdots)*\sigma(x_n))*((((y*x)*\sigma(y_1))*\sigma(y_2))*\cdots)*\sigma(y_m))\\
&\leq ((((((y*\sigma(y_1))*\sigma(y_2))*\cdots)*\sigma(y_m))*\sigma(x_1))*\cdots*\sigma(x_{n-1}))\\
&*(((x*\sigma(x_1))*\sigma(x_2))*\cdots)*\sigma(x_{n-1})*((((y*x)*\sigma(y_1))*\sigma(y_2))*\cdots)*\sigma(y_m))\\
&\vdots\\
&\leq (((((y*\sigma(y_1))*\sigma(y_2))*\cdots)*\sigma(y_m)*x)*((((y*x)*\sigma(y_1))*\sigma(y_2))*\cdots)*\sigma(y_m))\\
&\leq (((((y*\sigma(y_1))*\sigma(y_2))*\cdots)*x)*\sigma(y_m))*((((y*x)*\sigma(y_1))*\sigma(y_2))*\cdots)*\sigma(y_m))\\
&\vdots\\
&\leq (((((y*\sigma(y_1))*x)*\sigma(y_2))*\cdots)*\sigma(y_m))*(((((y*x)*\sigma(y_1))*\sigma(y_2))*\cdots)*\sigma(y_m))\\
&\leq (((((y*x)*\sigma(y_1))*\sigma(y_2))*\cdots)*\sigma(y_m))*(((((y*x)*\sigma(y_1))*\sigma(y_2))*\cdots)*\sigma(y_m))\\
&= 0 \in I.
\end{aligned}
$$

This means that $y \in M$, and so $M \in \mathcal{I}(X)$. Now, let $x \in M$. Then there exist $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$ such that $y = (((x*\sigma(x_1))*\sigma(x_2))*\cdots)*\sigma(x_n) \in I$. Then

$$\sigma(y) = (((\sigma(x)*\sigma(x_1))*\sigma(x_2))*\cdots)*\sigma(x_n).$$

Hence

$$((((\sigma(x)*\sigma(x_1))*\sigma(x_2))*\cdots)*\sigma(x_n))*\sigma(y) = \sigma(0) = 0 \in I.$$

Thus there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n, x_{n+1} \in X$, where $x_{n+1} = y$, such that

$$(((\sigma(x)*\sigma(x_1))*\sigma(x_2))*\cdots)*\sigma(x_n) \in I.$$

This means that $\sigma(x) \in M$, and hence M is a state ideal of X. Now, let K be a state ideal of X containing I and $x \in M$. Then according to definition of M, we conclude that $x \in K$. Hence $M \subseteq K$. Thus M is the least ideal of X containing I. This means that $M = \langle I \rangle_S$. □

Proposition 5.7. *Let X be distributive and $\sigma \in \mathcal{SMO}(X)$. Then the following statments hold:*

(i) $\ker\sigma = \{x*\sigma(x) : x \in X\} = \{\sigma(x)*x : x \in X\}$,

(ii) $X = \langle \ker\sigma \cup \text{img }\sigma \rangle_S$.

Proof. (i) Clearly, $\{x*\sigma(x) : x \in X\} \subseteq \ker\sigma$. Let $x \in \ker\sigma$. Then

$$x = x*0 = x*\sigma(x) \in \{x*\sigma(x) : x \in X\}.$$

Thus $\ker\sigma \subseteq \{x*\sigma(x) : x \in X\}$, and so $\ker\sigma = \{x*\sigma(x) : x \in X\}$. By a similar argument, we have $\ker\sigma = \{\sigma(x)*x : x \in X\}$.
(ii) Clearly, $\langle \ker\sigma \cup \text{img }\sigma \rangle_S \subseteq X$. Let $x \in X$, we show that $x \in \langle \ker\sigma \cup \text{img }\sigma \rangle_S$. By (i), $x*\sigma(x) \in \ker\sigma$, for any $x \in X$. Moreover, $\sigma(x) \in \text{img }\sigma$, for any $x \in X$. Then $x \in \langle \ker\sigma \cup \text{img }\sigma \rangle_S$. Thus $X \subseteq \langle \ker\sigma \cup \text{img }\sigma \rangle_S$. This shows that (ii) holds. □

Definition 5.8. *Let $I \in \mathcal{I}(X)$, and T be a subalgebra of X. We say T and I are complement sets of X if,* (C1) $T \cap I = \{0\}$,

(C2) $\langle T \cup I \rangle_S = X$,

(C3) *for any* $x \in X$, *there exists* $a_x \in T$ *such that* $x \sim_I a_x$.

Example 5.9. Consider Example 3.5. Define $\sigma : X \to X$ by $\sigma(0) = \sigma(b) = \sigma(c) = 0$ and $\sigma(a) = a$. Then (X, σ) is a state BI-algebra. If we take $I := \{0, c\}$ and $T := \{0, a, b\}$, then we can see that $I \in \mathcal{I}(X)$ and (C1)-(C3) hold.

If T and I are complement pair sets of X, then we denote these by (T, I) and we call it *complement pair* of X. We denote the set of all complement pairs of X by $\mathcal{C}\,(X)$.

Proposition 5.10. *Let* $(T, I) \in \mathcal{C}\,(X)$. *Then* a_x *is a unique element of* T, *for any* $x \in X$.

Proof. Let $x \in X$ and $a, b \in T$ such that $x \sim_I a$ and $x \sim_I b$. Since $\sim_I$ is an equivalence relation on X, we have $a \sim_I b$. This means that $a * b, b * a \in I$. On the other hand, $a * b, b * a \in T$, since T is a subalgebra of X. Hence $a * b, b * a \in I \cap T$. But from (C1), we have $I \cap T = \{0\}$. This implies that $a = b$. Thus a_x is a unique element of T, for any $x \in X$. □

Theorem 5.11. *Let* X *be distributive such that for any ideal* I, $\sim_I$ *is a right congruence relation. Then there is a one-to-one correspondence between complement pairs of* X *and state-morphism operators on* X.

Proof. Assume $\sigma \in \mathcal{SMO}(X)$. Set $I = \ker \sigma$ and $T = \text{img } \sigma$. Then $I \in \mathcal{I}(X)$ and T is a subalgebra of X. Now, we show that $(T, I) \in \mathcal{C}\,(X)$. Clearly, (C1) holds and by Proposition 5.7(ii), (C2) holds. Let $x \in X$. Then $\sigma\,(x) \in \text{img } \sigma = T$. Moreover, by Proposition 5.7(i), $x * \sigma\,(x)\,,\ \sigma\,(x) * x \in \ker \sigma = I$. Thus $x \sim_I \sigma(x)$. Therefore, for any $x \in X$, there exists $\sigma(x) \in T$ such that $x \sim_I \sigma(x)$. This shows that $(T, I) \in \mathcal{C}\,(X)$.

Conversely, we show that for any complement pair of X, one can define a state-morphism. Let $(T, I) \in \mathcal{C}\,(X)$. Define $\sigma_{T,I} : X \to X$ by $\sigma_{T,I}(x) = a_x$, for all $x \in X$. Proposition 5.10 follows that $\sigma_{T,I}$ well defined. Let $x, y \in X$. Then $\sigma(x) = a_x$ and $\sigma_{T,I}(y) = a_y$. Thus $x \sim_I a_x$ and $y \sim_I a_y$. Since $\sim_I$ is a congruence relation, we have $x * y \sim_I a_x * a_y$. Moreover, $a_x * a_y \in T$, since T is a subalgebra of X, then by Proposition 5.10, $\sigma_{T,I}(x * y) = a_{x*y}$. Since $x * y \sim_I a_x * a_y$, again by Proposition 5.10, $a_x * a_y$ is unique, and so $a_{x*y} = a_x * a_y$. This implies that

$$\sigma_{T,I}\,(x * y) = a_{x*y} = a_x * a_y = \sigma_{T,I}\,(x) * \sigma_{T,I}(y).$$

Hence $\sigma_{T,I}$ is a homomorphism on X. Moreover, for any $a \in T$, $a*a = 0 \in I$, so by Proposition 5.10, $\sigma_{T,I}\,(a) = a_a = a$. This follows that $\sigma_{T,I}\,(\sigma_{T,I}(x)) = \sigma_{T,I}(x)$, for all $x \in X$. Thus $\sigma_{T,I} \in \mathcal{SMO}(X)$. Now, define $\alpha : \mathcal{C}\,(X) \to \mathcal{SMO}\,(X)$, by $\alpha\,(T, I) = \sigma_{T,I}$, and $\beta : \mathcal{SMO}\,(X) \to \mathcal{C}\,(X)$ by $\beta\,(\sigma) = (\text{img } \sigma, \ker \sigma)$. Also, we have

$$\begin{aligned} \ker \sigma_{T,I} &= \{x \in X : \sigma_{T,I}\,(x) = 0\} \\ &= \{x \in X : a_x = 0\}\,. \end{aligned}$$

It is obvious that $I \subseteq \ker \sigma_{T,I}$. On the other hand, assume $x \in \ker \sigma_{T,I}$. Hence $a_x = 0$. Since $x * a_x \in I$ and $a_x = 0 \in I$, we obtain $x \in I$, and so $\ker \sigma_{T,I} \subseteq I$. Thus $\ker \sigma_{T,I} = I$. Moreover, it is easy to cheek that $\sigma_{T,I}\,(x) = \text{img } \sigma_{T,I} = T$. Then

$$(\alpha \circ \beta)\,(\sigma_{T,I}) = \alpha\,(\text{img } \sigma_{T,I}, \ker \sigma_{T,I}) = \alpha\,(T, I) = \sigma_{T,I}$$

and

$$(\beta \circ \alpha)\,(T, I) = \beta\,(\sigma_{T,I}) = (\text{img } \sigma_{T,I}, \ker \sigma_{T,I}) = (T, I)\,.$$

These complete the proof. □

6 Conclusions and future works

In this paper, we have studied various versions of maps that we called *Bosbach states* and *state-morphism operators* in a BI-algebra. Essential properties of the above mentioned mappings and examples for clarifying these new notions are given. Besides, we defined *state ideals* on BI-algebras and gave a characterization of the least state ideal of a BI-algebra. It is proved that, the kernel of a Bosbach state on a BI-algebra X is an ideal of X. Further, by these concepts, we have introduced the notions of *complement pairs* of a BI-algebra. It is proved that under suitable conditions, there is a one-to-one correspondence between complement pairs of a BI-algebra and state-morphism operators in a BI-algebra. In our next research, we will consider the notions of measures, generalized states, Riečan states, modal operators, and internal states on BI-algebras. Hyper BI-algebras were defined by Niazian in [35]. As another direction of research, we will extend and investigate these results to hyper BI-algebras.

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