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Prime double-framed soft bi-ideals of ordered semigroups

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Abstract

The notions of a prime (strongly prime, semiprime, irreducible, and strongly irreducible) double-framed soft bi-ideals (briefly, prime, (strongly prime, semiprime, irreducible and strongly irreducible) DFS bi-ideals) in ordered semigroups are introduced and related properties are investigated. Several examples of these notions are provided. The relationship between prime and strongly prime, irreducible and strongly irreducible DFS bi-ideals are considered and characterizations of these concepts are established. The Characterizations of regular and intra-regular ordered semigroups in terms of these notions are studied.

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1 Introduction

The notion of soft sets was introduced in 1999 by Molodtsov [24] as a new mathematical tool for dealing with uncertainties. Due to its importance, it has received much attention in the mean of algebraic structures such as groups (Cagman and Enginoglu [5]), semirings (Feng et al. [7]), rings (Acar et al. [1]), ordered semigroups (Jun et al. [11]). Also Feng [6] considered soft rough sets and applied it to group decision making problems. Jun et al. [11] applied the notion of soft set theory to ordered semigroups and introduced the notions of (trivial, whole) soft ordered semigroups, soft ordered subsemigroups, soft r-ideals, soft l-ideals and r-idealistic and l-idealistic soft ordered semigroup. They investigated various properties of ordered semigroups using these notions. In

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(Khan et al. [21]) further extended the notions of uni-soft and int-soft sets into double-framed soft set theory given by Jun and Ahn in [9] and introduced the notions of double-framed soft l-ideals and r-ideals in ordered semigroups. Jun et al. [9] introduced the notion of double-framed soft sets (briefly, DFS sets) and applied it to BCK/BCI-algebra. They discussed double-framed soft algebra (briefly DFS-algebra) and investigated related properties. Yousafzai et al. [29] applied the notion of double-framed soft sets to non-associative ordered semigroups and investigated various results. Moreover, double-framed soft sets are further elaborated in non-associative ordered semigroups [20]. Further, several researchers applied the notion of doubleframed soft sets in diverse fields of algebra. For instance, Asif et al. [3] discussed ideal theory in ordered AG-groupoid based on double-framed soft sets. Also, Asif and coauthors [2] determined fully prime double-framed soft ordered semigrouops. For further reading on ordered semigroups, soft sets and double-framed soft sets we refer the reader to references ([8, 10, 12, 13, 14, 15, 16, 17, 18, 19, 22, 23, 26, 27, 28, 30]).

In this paper, we apply the concept of DFS-set in ordered semigroups and the notions of a prime (strongly prime, semiprime, irreducible, and strongly irre-ducible) double-framed soft bi-ideals (briefly, prime, (strongly prime, semiprime, irreducible and strongly irreducible) DFS bi-ideals) in ordered semigroups are introduced and related properties are investigated. Several related examples of these notions are provided. The relationship between prime and strongly prime, irreducible and strongly irreducible DFS bi-ideals are considered and characterizations of these concepts are established. The Characterizations of regular and intra-regular ordered semigroups in terms of these notions are studied.

2 Preliminaries

By an ordered semigroup, we mean a system $(S, ., \leq)$ in which the following are satisfied:

(OS1) $(S, .)$ is a semigroup,

(OS2) $(S, \leq)$ is a poset,

(OS3) $x \leq y \Rightarrow ax \leq ay$ and $xa \leq ya$ for all $x, y, a \in S$.

Let $\emptyset \neq A \subseteq S$, we denote $(A]$ by $(A] := \{x \in S/x \leq a$ for some $a \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For any nonempty subsets A, B of S, we denote by $AB := \{AB/a \in A, b \in B\}$.

Definition 2.1. *An element e of an ordered semigroup S is called an identity element if $xe = ex = x$ for all $x \in S$.*

Definition 2.2. *A non-empty subset A of an ordered semigroup S is called a sub-semigroup of S if $A^2 \subseteq A$.*

In (Kehayopulu and Tesinglis [18, 19]), defined that a nonempty subset A of an ordered semigroup S is called a left (resp., right) ideal of S if:

(1) $SA \subseteq A$ (resp., $AS \subseteq A$),

(2) If $b \in B$ and $a \in S$ such that $a \leq b$, then $a \in B$..

If A is both a left and a right ideal of S, then A is called a two-sided ideal or simply an ideal of S.

Definition 2.3. *A subsemigroup A of an ordered semigroup S is called a bi-ideal of S if:*

(1) $ASA \subseteq A$,

(2) If $b \in B$ and $a \in S$ such that $a \leq b$, then $a \in B$.

Definition 2.4. *An ordered semigroup S is called regular* [19] *if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$, or equivalently,*

(i) $a \in (aSa] \ \forall \ a \in S$.
(ii) $A \subseteq (ASA] \ \forall \ A \subseteq S$.

Definition 2.5. *An ordered semigroup S is called intra-regular [19] if for every $a \in S$, there exist $y, z \in S$ such that $a \leq ya^2z$, or equivalently,*
(i) $a \in (Sa^2S] \ \forall \ a \in S$.
(ii) $A \subseteq (SA^2S] \ \forall \ A \subseteq S$.

Lemma 2.6. (cf: [21]) *Let S be an ordered semigroup. Then the following are equivalent:*
(i) *S is both regular and intra-regular.*
(ii) *$B = (B^2]$ for every bi-ideal B of S.*
(iii) *$B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1]$ for all bi-ideals B_1, B_2 of S.*
(iv) *$R \cap L = (RL] \cap (LR]$ for every right ideal R and every left ideal L of S.*
(v) *$R(a) \cap L(a) = (R(a)L(a)] \cap (L(a)R(a)]$ for every $a \in S$.*

In the following we recall the concept of a soft set given by Sezgin and Atagun in [4]. Throughout this article, S will represent an ordered semigroup unless otherwise stated. The initial universe set will be denoted by U, E is a set of parameters, $P(U)$ is the power set of U and $A, B \subseteq E$.

Definition 2.7. *Let U be an initial universe set, E a set of parameters, $P(U)$ the power set of U and $A \subseteq E$. Then a soft set f_A over U is a function defined by: $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$, if $x \notin A$.*

Here f_A is called an approximate function. A soft set over U can be represented by the set of ordered pairs give below:

$$f_A := \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

It is clear that a soft set is a parameterized family of subsets of U. The set of all soft sets is denoted by $S(U)$.

Definition 2.8. *Let f_A, $f_B \in S(U)$. Then f_A is a soft subset of f_B, denoted by $f_A \mathrel{\tilde{\subseteq}} f_B$ if $f_A(x) \mathrel{\tilde{\subseteq}} f_B(x)$ for all $x \in S$. Two soft sets f_A, f_B are said to be equal soft sets if $f_A \mathrel{\tilde{\subseteq}} f_B$ and $f_B \mathrel{\tilde{\subseteq}} f_A$ and is denoted by $f_A \mathrel{\tilde{=}} f_B$.*

Definition 2.9. *Let f_A, $f_B \in S(U)$. Then the union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined by $f_A \tilde{\cup} f_B = f_{A \cup B}$, where $\left(f_A \tilde{\cup} f_B\right)(x) = f_A(x) \tilde{\cup} f_B(x)$, for all $x \in E$.*

Definition 2.10. *Let f_A, $f_B \in S(U)$. Then the intersection of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$, is defined by $f_A \tilde{\cap} f_B = f_{A \cap B}$, where $\left(f_A \tilde{\cap} f_B\right)(x) = f_A(x) \tilde{\cap} f_B(x)$, for all $x \in E$.*

Throughout this paper, let $E = S$, where S is an ordered semigroup, unless otherwise stated.

Definition 2.11. (cf. [25]) *Let f_A, $f_B \in S(U)$. Then the soft product of f_A and f_B, denoted by $f_A \mathrel{\tilde{\circ}} f_B$, is defined by:*

$$(f_A \mathrel{\tilde{\circ}} f_B)(x) = \begin{cases} \bigcup\limits_{(y,z) \in A_x} \{f_A(y) \cap g_B(z)\} & \text{if } A_x \neq \emptyset \\ \emptyset & \text{if } A_x = \emptyset \end{cases}$$

where $A_x = \{(y,z) \in S \times S / x \leq yz\}$.

Definition 2.12. (Jun et al. [9]) *Let* $\langle (f_A^+,\ f_A^-); A\rangle$ *be a DFS-set. Let* γ, δ *be two subsets.U. Then, the* $\gamma-$*inclusive set and the* $\delta-$*exclusive set of* $\langle (f_A^+,\ f_A^-); A\rangle$, *denoted by* $i_A(f_A^+;\gamma)$ *and* $e_A(f_A^-;\delta)$, *respectively, are defined as follows:*

$$\begin{aligned} i_A(f_A^+;\gamma) &: = \{x \in A / f_A^+(x) \supseteq \gamma\}, \\ e_A(f_A^-;\delta) &: = \{x \in A / f_A^-(x) \subseteq \delta\}. \end{aligned}$$

Definition 2.13. *The set*

$$DF_A(f_A^+; f_A^-)_{(\gamma,\delta)} := \{x \in A /\ f_A^+(x) \supseteq \gamma,\ f_A^-(x) \subseteq \delta\}.$$

is called a double-framed soft including set (cf. [10]) *of* $\langle (f_A^+,\ f_A^-); A\rangle$. *It is clear that*

$$DF_A(f_A^+; f_A^-)_{(\gamma,\delta)} := i_A(f_A^+;\gamma) \cap e_A(f_A^-;\delta).$$

Definition 2.14. (cf. [2]) *Let* $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ *and* $g_A = \langle (g_A^+,\ g_A^-); A\rangle$ *be two double-framed soft sets of an ordered semigroup* $(S, ., \leq)$ *over* U. *Then the uni-int soft product, denoted by* $f_A \diamond g_B = \left\langle (f_A^+ \tilde{\circ}\, g_A^+,\ f_A^- \tilde{*}\, g_A^-); A\right\rangle$ *is defined by to be a double-framed soft set of* S *over* U, *in which* $f_A^+ \tilde{\circ}\, g_A^+$ *and* $f_A^- \tilde{*}\, g_A^-$ *are mappings from* S *to* $P(U)$, *given as follows:*

Let $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ *and* $g_A = \langle (g_A^+,\ g_A^-); A\rangle$ *be two double-framed soft sets of an ordered AG-groupoid* S *over* U. *Then the uni-int soft product, denoted by* $f_A \diamond g_A = \left\langle (f_A^+ \tilde{\circ}\, g_A^+,\ f_A^- \tilde{\star}\, g_A^-); A\right\rangle$ *is defined to be a double-framed soft set of* S *over* U, *in which* $f_A^+ \tilde{\circ}\, g_A^+$ *and* $f_A^- \tilde{\star}\, g_A^-$ *are mapping from* S *to* $P(U)$, *given as follows:*

$$f_A^+ \tilde{\circ}\, g_A^+ : S \longrightarrow P(U), x \longmapsto \begin{cases} \bigcup\limits_{(y,z)\in A_x} \{f_A^+(y) \cap g_A^+(z)\} & \text{if } A_x \neq \emptyset \\ \emptyset & \text{if } A_x = \emptyset, \end{cases}$$

$$f_A^- \tilde{*}\, g_A^- : S \longrightarrow P(U), x \longmapsto \begin{cases} \bigcap\limits_{(y,z)\in A_x} \{f_A^-(y) \cup g_A^-(z)\} & \text{if } A_x \neq \emptyset \\ U & \text{if } A_x = \emptyset. \end{cases}$$

Let $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ and $g_A = \langle (g_A^+,\ g_A^-); A\rangle$ be two double-framed soft sets over a common universe set U. Then $\langle (f_A^+,\ f_A^-); A\rangle$ is called a *double-framed soft subset* (*briefly, DFS-subset*) (cf. [9]) of $\langle (g_A^+,\ g_A^-); A\rangle$, denote by $\langle (f_A^+,\ f_A^-); A\rangle \sqsubseteq \langle (g_A^+,\ g_A^-); A\rangle$ if

(*i*) $A \subseteq B$,

(*ii*) $(\forall e \in A) \begin{pmatrix} f_A^+ \text{ and } g_A^+ \text{ are identical approximations } (f_A^+(e) \subseteq g_A^+(e)) \\ f_A^- \text{ and } g_A^- \text{ are identical approximations } (f_A^-(e) \supseteq g_A^-(e)) \end{pmatrix}$.

For two DFS-sets $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ and $g_A = \langle (g_A^+,\ g_A^-); A\rangle$ over U are said to be *equal*, denoted by $\langle (f_A^+,\ f_A^-); A\rangle = \langle (g_A^+,\ g_A^-); A\rangle$, if $\langle (f_A^+,\ f_A^-); A\rangle \sqsubseteq \langle (g_A^+,\ g_A^-); A\rangle$ and $\langle (g_A^+,\ g_A^-); A\rangle \sqsubseteq \langle (f_A^+,\ f_A^-); A\rangle$.

For two DFS-sets $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ and $g_A = \langle (g_A^+,\ g_A^-); A\rangle$ over U, the *DFS int-uni* set (cf. [9]) of $\langle (f_A^+,\ f_A^-); A\rangle$ and $\langle (g_A^+,\ g_A^-); A\rangle$, is defined to be a DFS-set $\langle (f_A^+ \cap g_A^+,\ f_A^- \cup g_A^-); A\rangle$, where $f_A^+ \cap g_A^+$ and $f_A^- \cup g_A^-$ are mapping given as follows:

$$\begin{aligned} f_A^+ \cap g_A^+ &: A \longrightarrow P(U),\ x \longmapsto f_A^+(x) \cap g_A^+(x), \\ f_A^- \cup g_A^- &: A \longrightarrow P(U),\ x \longmapsto f_A^-(x) \cup g_A^-(x). \end{aligned}$$

It is denoted by $\langle (f_A^+,\ f_A^-); A\rangle \sqcap \langle (g_A^+,\ g_A^-); A\rangle = \langle (f_A^+ \cap g_A^+,\ f_A^- \cup g_A^-); A\rangle$.

Definition 2.15. (cf. [7]) *A double-framed soft set* $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ *of* S *over* U *is called a double-framed soft semigroup (briefly, DFS semigroup) of* S *over* U *if it satisfies*:

$$f_A^+(xy) \supseteq f_A^+(x) \cap f_A^+(y) \text{ and } f_A^-(xy) \subseteq f_A^-(x) \cup f_A^-(y),$$

for all $x,\, y \in S$.

Definition 2.16. (cf. [2]) *A double-framed soft set* $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ *of* S *over* U *is called*

(i) a double-framed soft l-ideal (briefly, DFS-l-ideal of S over U if
(a) $f_A^+(xy) \supseteq f_A^+(y)$ and $f_A^-(xy) \subseteq f_A^-(y)$ and
(b) $x \leq y \Longrightarrow f_A^+(x) \supseteq f_A^+(y)$ and $f_A^-(x) \subseteq f_A^-(y)$, $\forall\ x,\, y \in S$.
(ii) a double-framed soft r-ideal (briefly, DFS-r-ideal of S over U if
(a) $(f_A^+(xy) \supseteq f_A^+(x)$ and $f_A^-(xy) \subseteq f_A^-(x)$
(b) $x \leq y \Longrightarrow f_A^+(x) \supseteq f_A^+(y)$ and $f_A^-(x) \subseteq f_A^-(y)$, $\forall\ x,\, y \in S$.
(iii) a double-framed soft ideal (briefly, DFS ideal) of S over U, if it is both a double-framed soft l- and r-ideal of S over U.

Definition 2.17. (cf. [2]) *A double-framed soft set* $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ *of* S *over* U *is called an idempotent if* $f_A \diamond f_A = f_A$ *i.e.,* $f_A^+ \,\widetilde{\circ}\, f_A^+ = f_A^+$ *and* $f_A^- \,\widetilde{*}\, f_A^- = f_A^-$.

Definition 2.18. (cf. [2]) *A double-framed soft set* $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ *of* S *over* U *is called double-framed bi-ideal (briefly, DFS-bi-ideal of* S *over* U *if*

(i) $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ is a double-framed soft semigroup (briefly, DFS semigroup) of S,
(ii) $(f_A^+(xyz) \supseteq f_A^+(x) \cap f_A^+(z)$ and $f_A^-(xyz) \subseteq f_A^-(x) \cup f_A^-(z)$,
(iii) $x \leq y \Longrightarrow f_A^+(x) \supseteq f_A^+(y)$ and $f_A^-(x) \subseteq f_A^-(y)$, $\forall\ x,\, y \in S$.

3 Prime and semiprime double-framed soft bi-ideals

In this section, we define prime (resp., strongly prime, irreducible and strongly irreducible) double-framed soft bi-ideals of an ordered semigroup S over U. We characterize ordered semigroups by the properties of these notions.

Definition 3.1. *A double-framed soft set* $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ *of* S *over* U *is called a prime (resp., strongly prime) double-framed soft bi-ideal (briefly, PDFS bi-ideal (resp., SPDFS bi-ideal)) of* S *over* U *if* $g_B \diamond h_C \sqsubseteq f_A$ *(resp.,* $(g_B \diamond h_C) \sqcap (h_C \diamond g_B) \sqsubseteq f_A$*) implies* $g_B \sqsubseteq f_A$ *or* $h_C \sqsubseteq f_A$ *(resp.,* $g_B \sqsubseteq f_A$ *or* $h_C \sqsubseteq f_A$*). That is,* $g_B^+ \,\widetilde{\circ}\, h_C^+ \subseteq f_A^+$ *and* $g_B^- \,\widetilde{*}\, h_C^- \supseteq f_A^-$ *(resp.,* $\left(g_B^+ \,\widetilde{\circ}\, h_C^+\right) \widetilde{\cap} \left(h_C^+ \,\widetilde{\circ}\, g_B^+\right) \widetilde{\subseteq} f_A^+$ *and* $\left(g_B^- \,\widetilde{*}\, h_C^-\right) \widetilde{\cup} \left(h_C^- \,\widetilde{*}\, g_B^-\right) \widetilde{\supseteq} f_A^-$*) imply* $g_B^+ \widetilde{\subseteq} f_A^+$ *or* $h_C^+ \widetilde{\subseteq} f_A^+$ *and* $g_B^- \widetilde{\supseteq} f_A^-$ *or* $h_C^- \widetilde{\supseteq} f_A^-$ *(resp.,* $g_B^+ \widetilde{\subseteq} f_A^+$ *or* $h_C^+ \widetilde{\subseteq} f_A^+$ *and* $g_B^- \widetilde{\supseteq} f_A^-$ *or* $h_C^- \widetilde{\supseteq} f_A^-$*) for all double-framed soft bi-ideals* $g_B = \langle (g_B^+,\ g_B^-); B\rangle$ *and* $h_C = \langle (h_C^+,\ h_C^-); C\rangle$ *of* S *over* U.

Definition 3.2. *A double-framed soft set* $f_A = \langle (f_A^+,\ f_A^-); A\rangle$ *of* S *over* U *is called a semiprime double-framed soft bi-ideal (briefly, SPDFS bi-ideal) of* S *over* U*, if* $g_B \diamond g_B \sqsubseteq f_A$ *implies* $g_B \sqsubseteq f_A$*. That is,* $g_B^+ \,\widetilde{\circ}\, g_B^+ \widetilde{\subseteq} f_A^+$ *and* $g_B^- \,\widetilde{*}\, g_B^- \widetilde{\supseteq} f_A^-$ *imply* $g_B^+ \widetilde{\subseteq} f_A^+$ *and* $g_B^- \widetilde{\supseteq} f_A^-$ *for all double-framed soft bi-ideal* $g_B = \langle (g_B^+,\ g_B^-); B\rangle$ *of* S *over* U.

Remark 3.3. *Note that every prime DFS bi-ideal of an ordered semigroup S is a semiprime DFS bi-ideal of S over U. But the converse is not true in general.*

Example 3.4. *There are six women patients in the initial universe set U given by*

$$U := \{p_1, p_2, p_3, p_4, p_5, p_6\}.$$

Let a set of parameters $E = \{e_0, e_1, e_2, e_3\}$ be a set of status of each patient in U with the following type of disease:
e_0 stands for the parameter "headache",
e_1 stands for the parameter "Chest pain",
e_2 stands for the parameter "mental depression",
e_3 stands for the parameter "periodic pain",
with the following binary operation given in the Cayley table.

$*$	e_0	e_1	e_2	e_3
e_0	e_0	e_0	e_0	e_0
e_1	e_0	e_0	e_0	e_0
e_2	e_0	e_0	e_1	e_0
e_3	e_0	e_0	e_1	e_1

We define the order relation "$\leq$" on E as follows.

$$\leq = \{(e_0, e_0), (e_0, e_1), (e_0, e_2), (e_0, e_3), (e_2, e_3), (e_1, e_1), (e_2, e_2), (e_3, e_3)\}.$$

We define the covering relation "$\prec$" as given below.

$$\prec := \{(e_0, e_1), (e_0, e_2), (e_0, e_3), (e_2, e_3)\}.$$

Then $(E, *, \leq)$ is an ordered semigroup. Here $A = \{e_0\}$, $B = \{e_0, e_1\}$, $D = \{e_0, e_1, e_2\}$, $F = \{e_0, e_1, e_3\}$ and S are the bi-ideals of E. Consider a double-framed soft set $f_A = \langle (f_A^+, f_A^-); A\rangle$ *of E over U as follows:*

$$f_A^+ : A \longrightarrow P(U), x \longmapsto \begin{cases} \{p_1, p_2\} & \text{if } x = e_0 \\ \{\} & \text{if } x \in \{e_1, e_2, e_3\} \end{cases}$$

$$f_A^- : A \longrightarrow P(U), x \longmapsto \begin{cases} \{p_4, p_5\} & \text{if } x = e_0 \\ \{p_3, p_4, p_5\} & \text{if } x \in \{e_1, e_2, e_3\}. \end{cases}$$

Then it is easy to verify that $f_A = \langle (f_A^+, f_A^-); A\rangle$ is DFS bi-ideal of S over U. Now let $h_D = \langle (h_D^+, h_D^-); D\rangle$ be a double-framed soft set over U defined as follows:

$$h_D^+ : D \longrightarrow P(U), x \longmapsto \begin{cases} \{p_1, p_2, p_3\} & \text{if } x = e_0 \\ \{p_1, p_3\} & \text{if } x = e_1 \\ \{p_1\} & \text{if } x = e_2 \\ \{\} & \text{if } x = e_3 \end{cases}$$

$$h_D^- : D \longrightarrow P(U), x \longmapsto \begin{cases} \{p_3, p_4, p_5, p_6\} & \text{if } x = e_0 \\ \{p_5, p_6\} & \text{if } x = e_1 \\ \{p_4\} & \text{if } x = e_2 \\ \{p_4, p_5, p_6\} & \text{if } x = e_3. \end{cases}$$

By an easy verification it can be seen that $h_D = \langle (h_D^+, h_D^-); D\rangle$ is a double-framed soft bi-ideal over U.

Consider a double-framed soft set $l_F = \langle (l_F^+, l_F^-); F\rangle$ over U, defined as follows:

$$l_F^+ : F \longrightarrow P(U), x \longmapsto \begin{cases} \{p_1, p_2, p_3\} & \text{if } x = e_0 \\ \{p_1, p_3\} & \text{if } x = e_1 \\ \{\} & \text{if } x = e_2 \\ \{p_2\} & \text{if } x = e_3 \end{cases}$$

$$l_F^- : F \longrightarrow P(U), x \longmapsto \begin{cases} \{p_2, p_3, p_5\} & \text{if } x = e_0 \\ \{p_2, p_4\} & \text{if } x = e_1 \\ \{p_6\} & \text{if } x = e_2 \\ \{p_5\} & \text{if } x = e_3. \end{cases}$$

The DFS bi-ideal $h_D = \langle (h_D^+, h_D^-); D\rangle$ of S over U is a prime DFS bi-ideal, which is also a semiprime DFS bi-ideal of S over U. The DFS bi-ideal $l_F = \langle (l_F^+, l_F^-); F\rangle$ of S over U is semiprime but not a prime DFS bi-ideal of S over U. Because

$$f_A^+ \mathbin{\tilde{\circ}} h_D^+ \subseteq l_F^+ \text{ implies that } f_A^+ \mathbin{\tilde{\nsubseteq}} l_F^+ \text{ and } h_D^+ \mathbin{\tilde{\nsubseteq}} l_F^+,$$

and

$$f_A^- \mathbin{\tilde{*}} h_D^- \supseteq l_F^- \text{ implies that } f_A^- \supseteq l_F^- \text{ and } h_D^- \supseteq l_F^-.$$

Remark 3.5. *Every strongly prime DFS bi-ideal of S over U is a prime DFS bi-ideal of S over U, but the converse is not true in general.*

Example 3.6. *Suppose that there are six houses in an initial universe set U, given by*

$$U := \{h_1, h_2, h_3, h_4, h_5, h_6\}.$$

Let a set of parameters $E = \{e_0, e_1, e_2, e_3\}$ be a set of status of houses in which
e_0 stands for the parameter "beautiful",
e_1 stands for the parameter "cheap",
e_2 stands for the parameter "in good location",
e_3 stands for the parameter "in green surrounding",
with the following binary operation given in the Cayley table.

$*$	e_0	e_1	e_2	e_3
e_0	e_0	e_0	e_0	e_0
e_1	e_0	e_1	e_1	e_1
e_2	e_0	e_2	e_2	e_2
e_3	e_0	e_3	e_3	e_3

We define the order relation "$\leq$" on E as follows.

$$\leq = \{(e_0, e_0), (e_1, e_1), (e_2, e_2), (e_0, e_2), (e_0, e_3), (e_2, e_3), (e_3, e_3)\}.$$

We define the covering relation "$\prec$" as given below.

$$\prec := \{(e_0, e_2), (e_0, e_3), (e_2, e_3)\}.$$

Then $(E,*,\leq)$ is an ordered semigroup. Every subset of S containing e_0 is a bi-ideal of E. Let $A=\{e_0,e_1,e_2\}$. Let us define a DFS set $f_A=\langle(f_A^+,f_A^-);A\rangle$ of S over U as follows:
$f_A^+(e_0)=\{h_1,h_2,h_3,h_5,h_6\}, f_A^+(e_1)=\{h_1,h_2\}$, $f_A^+(e_2)=\{h_2,h_3\}$, $f_A^+(e_3)=\{\}$,
$f_A^-(e_0)=\{h_4,h_5,h_6\}, f_A^-(e_1)=\{h_3,h_4\}$, $f_A^-(e_2)=\{h_4,h_5\}$, $f_A^-(e_3)=\{h_1,h_3,h_5\}$,

$$i_A(f_A^+;\gamma)=\begin{cases}\{e_0,e_1\} \text{ if } \gamma=\{h_1\}\\ \{e_0,e_1,e_2\} \text{ if } \gamma=\{h_2\}\\ \{e_0,e_2\} \text{ if } \gamma=\{h_3\}\\ \{e_0,e_1\} \text{ if } \gamma=\{h_1,h_2\}\\ \{e_0\} \text{ if } \gamma=\{h_1,h_3\}\\ \{e_0\} \text{ if } \gamma=\{h_2,h_3\}\end{cases}$$

$$e_A(f_A^-;\gamma)=\begin{cases}\{e_0,e_1\} \text{ if } \delta=U\\ \{e_0,e_1,e_2\} \text{ if } \delta=\{h_3,h_4,h_5,h_6\}\\ \{e_0,e_2\} \text{ if } \delta=\{h_4,h_5,h_6\}\\ \{e_0,e_1\} \text{ if } \delta=\{h_3,h_4,h_5,h_6\}\\ \{e_0\} \text{ if } \delta=\{h_1,h_3,h_4,h_5,h_6\}\\ \{e_0\} \text{ if } \delta=\{h_2,h_3,h_4,h_5,h_6\}\end{cases}$$

Then $f_A=\langle(f_A^+,f_A^-);A\rangle$ is a DFS bi-ideal of S over U.
Let $B=\{e_0,e_1,e_3\}$ and define a DFS set $g_B=\langle(g_B^+,g_B^-);B\rangle$ of S over U, as follows:
$g_B^+(e_0)=\{h_1,h_2,h_3,h_5,h_6\}$, $g_B^+(e_1)=\{h_2,h_4,h_6\}$, $g_B^+(e_2)=\{h_1,h_3,h_5\}$, $g_B^+(e_3)=\{h_1,h_2,h_3,h_4\}$,
$g_B^-(e_0)=\{h_1,h_2\}$, $g_B^-(e_1)=\{h_2,h_3,h_4\}$, $g_B^-(e_2)=\{h_6\}$, $g_B^-(e_3)=\{h_3,h_5\}$,

$$i_A(g_B^+;\gamma)=\begin{cases}\{e_0\} \text{ if } \gamma=\{h_2,h_4,h_6\}\\ \{e_1\} \text{ if } \gamma=\{h_6\}\\ \{e_0,e_1\} \text{ if } \gamma=\{h_4,h_6\}\\ \{e_2\} \text{ if } \gamma=\{h_4\}\\ \{e_3\} \text{ if } \gamma=\{h_1,h_2,h_3\}\end{cases}$$

$$e_A(g_B^-;\gamma)=\begin{cases}\{e_0\} \text{ if } \delta=\{h_1,h_2,h_4,h_6\}\\ \{e_1\} \text{ if } \delta=\{h_2,h_3,h_4,h_6\}\\ \{e_0,e_1\} \text{ if } \delta=U\\ \{e_2\} \text{ if } \delta=\{h_6\}\\ \{e_3\} \text{ if } \delta=\{h_1,h_2,h_3,h_5\}\end{cases}$$

Then it follows that $g_B=\langle(g_B^+,g_B^-);B\rangle$ is a DFS bi-ideal of S over U. Also $f_A^+\,\widetilde{\circ}\,g_B^+=f_A^+$ and $f_A^-\,\widetilde{*}\,g_B^-=f_A^-$ for all DFS bi-ideals $f_A=\langle(f_A^+,f_A^-);A\rangle$ and $g_B=\langle(g_B^+,g_B^-);B\rangle$ of S over U. Then, every DFS bi-ideal of S over U is prime.
Let $F=\{e_0,e_1\}$ and define a DFS set $l_F=\langle(l_F^+,l_F^-);F\rangle$ on F over U as follows:
$l_F^+(e_0)=\{h_1,h_2,h_4,h_5,h_6\}$, $l_F^+(e_1)=\{h_2,h_4,h_6\}$, $l_F^+(e_2)=\{\ \}$, $l_F^+(e_3)=\{\ \}$,
$l_F^-(e_0)=\{h_1,h_2\}$, $l_F^-(e_1)=\{h_2,h_3,h_4\}$, $l_F^-(e_2)=\{h_6\}$, $l_F^-(e_3)=\{h_3,h_5\}$,
Then $l_F=\langle(l_F^+,l_F^-);F\rangle$ is prime DFS bi-ideal but this is not a strongly prime DFS bi-ideal. Because

$$\left(f_A^+\,\widetilde{\circ}\,g_B^+\right)\widetilde{\cap}\left(g_B^+\,\widetilde{\circ}\,f_A^+\right)\widetilde{\subseteq}\,l_F^+ \text{ and } \left(f_A^-\,\widetilde{*}\,g_B^-\right)\widetilde{\cup}\left(g_B^-\,\widetilde{*}\,f_A^-\right)\widetilde{\supseteq}\,l_F^-,$$

but $f_A^+\subsetneq l_F^+$, $g_B^+\subsetneq l_F^+$and $f_A^-\supsetneq l_F^-$, $g_B^-\supsetneq l_F^-$.

Let A be a nonempty subset of S. Then the *characteristic double-framed soft mapping* of A, denoted by $\left\langle (X_A^+,\ X_A^-); A\right\rangle = X_A$ is defined to be a double-framed soft set, in which X_A^+ and X_A^- are soft mappings over U, given as follows:

$$\begin{aligned} X_A^+ \quad &: \quad S \longrightarrow P(U), x \longmapsto \begin{cases} U & \text{if } x \in A \\ \emptyset & \text{if } x \notin A, \end{cases} \\ X_A^- \quad &: \quad S \longrightarrow P(U), x \longmapsto \begin{cases} \emptyset & \text{if } x \in A \\ U & \text{if } x \notin A. \end{cases} \end{aligned}$$

Note that the characteristic mapping of the whole set S, denoted by $X_S = \left\langle (X_S^+,\ X_S^-); S\right\rangle$, is called the *identity double-framed soft mapping*, where $X_S^+(x) = U$ and $X_S^-(x) = \emptyset$, $\forall\ x \in S$.

Lemma 3.7. *A DFS set $f_A = \langle (f_A^+, f_A^-); A\rangle$ of S over U is a DFS semigroup of S over U if and only if $f_A \diamond f_A \sqsubseteq f_A$ i.e., $f_A^+ \tilde{\circ} f_A^+ \mathrel{\tilde{\subseteq}} f_A^+$ and $f_A^- \tilde{*} f_A^- \mathrel{\tilde{\supseteq}} f_A^-$.*

Proof. Let $x, y \in S$. If $A_x = \emptyset$, then obviously $\left(f_A^+ \tilde{\circ} f_A^+\right)(x) = \emptyset \mathrel{\tilde{\subseteq}} f_A^+$ and $\left(f_A^- \tilde{*} f_A^-\right)(x) = U \mathrel{\tilde{\supseteq}} f_A^-$. Assume that $A_x \neq \emptyset$, then $(y, z) \in A_x$ and so $x \leq yz$ for $x, y \in S$. Hence, we have

$$\begin{aligned} \left(f_A^+ \tilde{\circ} f_A^+\right)(x) \quad &= \bigcup_{(y,z)\in A_x} \{f_A^+(y) \cap g_A^+(z)\} \\ &\supseteq \quad f_A^+(y) \cap g_A^+(z), \end{aligned}$$

and

$$\begin{aligned} \left(f_A^- \tilde{*} f_A^-\right)(x) \quad &= \bigcap_{(y,z)\in A_x} \{f_A^-(y) \cup g_A^-(z)\} \\ &\subseteq \quad f_A^-(y) \cup g_A^-(z). \end{aligned}$$

Since $(y, z) \in A_x$, we have $x \leq yz$ and $\langle (f_A^+, f_A^-); A\rangle$ is a DFS semigroup of S over U, we have $f_A^+(x) \supseteq f_A^+(yz) \supseteq f_A^+(y) \cap g_A^+(z)$ and $f_A^-(x) \subseteq f_A^-(yz) \subseteq f_A^-(y) \cup g_A^-(z)$.Thus,

$$\left(f_A^+ \tilde{\circ} f_A^+\right)(x) = \bigcup_{(y,z)\in A_x} \{f_A^+(y) \cap g_A^+(z)\} \subseteq \bigcup_{(y,z)\in A_x} f_A^+(x) = f_A^+(x),$$

and

$$\left(f_A^- \tilde{*} f_A^-\right)(x) = \bigcap_{(y,z)\in A_x} \{f_A^-(y) \cup g_A^-(z)\} \supseteq \bigcap_{(y,z)\in A_x} f_A^-(x) = f_A^-(x).$$

Therefore, $f_A^+ \tilde{\circ} f_A^+ \mathrel{\tilde{\subseteq}} f_A^+$ and $f_A^- \tilde{*} f_A^- \mathrel{\tilde{\supseteq}} f_A^-$, that is, $f_A \diamond f_A \sqsubseteq f_A$.

Conversely, let $x, y \in S$. Then

$$\begin{aligned} f_A^+(xy) \quad &\supseteq \quad \left(f_A^+ \tilde{\circ} f_A^+\right)(xy) = \bigcup_{(y,z)\in A_{xy}} \{f_A^+(y) \cap g_A^+(z)\} \\ &\supseteq \quad f_A^+(x) \cap g_A^+(y), \end{aligned}$$

and

$$\begin{aligned} f_A^-(xy) \quad &\subseteq \quad \left(f_A^- \tilde{*} f_A^-\right)(xy) = \bigcap_{(y,z)\in A_{xy}} \{f_A^+(y) \cup g_A^+(z)\} \\ &\subseteq \quad f_A^-(x) \cup g_A^-(y). \end{aligned}$$

Consequently, $\langle (f_A^+, f_A^-); A\rangle$ is a DFS semigroup of S over U. $\square$

Lemma 3.8. *(cf. [7]) For a nonempty subset A of S, the following assertions are equivalent:*

(1) A is a left (right or bi-ideal) of S.

(2) The double-framed soft set $\left\langle (X_A^+,\ X_A^-); A\right\rangle$ of S over U is a double-framed soft l-ideal (resp., double-framed soft r- or bi-ideal) of S over U.

Lemma 3.9. *Let $f_{A_i} = \left\{\langle (f_{A_i}^+, f_{A_i}^-); A_i\rangle / i \in I\right\}$ be a family of double-framed soft bi-ideals of S over U. Then $\bigcap\limits_{i\in I} f_{A_i} = \left\langle (\widetilde{\bigcap\limits_{i\in I}} f_{A_i}^+, \widetilde{\bigcup\limits_{i\in I}} f_{A_i}^-); A_i\right\rangle$ is a semiprime double-framed soft bi-ideal of S over U.*

Proof. Since the intersection of DFS bi-ideals is again a DFS bi-ideal of S over U. Hence $\bigcap\limits_{i\in I} f_{A_i} = \left\langle (\widetilde{\bigcap\limits_{i\in I}} f_{A_i}^+, \widetilde{\bigcup\limits_{i\in I}} f_{A_i}^-); A_i\right\rangle$ is DFS bi-ideal of S over U. Let $\left\langle (h_D^+,\ h_D^-); D\right\rangle$ be a DFS bi-ideal of S such that $h_D \diamond h_D \sqsubseteq \bigcap\limits_{i\in I} f_{A_i}$ i.e., $h_D^+ \tilde{\circ}\, h_D^+ \widetilde{\subseteq} \widetilde{\bigcap\limits_{i\in I}} f_{A_i}^+$ and $h_D^- \tilde{*}\, h_D^- \widetilde{\supseteq} \widetilde{\bigcup\limits_{i\in I}} f_{A_i}^-$ for $i \in I$. Then $h_D^+ \tilde{\circ}\, h_D^+ \widetilde{\subseteq} f_{A_i}^+$ and $h_D^- \tilde{*}\, h_D^- \widetilde{\supseteq} f_{A_i}^-$ for all $i \in I$ Since each $\langle (f_{A_i}^+, f_{A_i}^-); A_i\rangle$ is prime DFS bi-ideal of S over U. Hence $h_D^+ \widetilde{\subseteq} f_{A_i}^+$ and $h_D^- \widetilde{\supseteq} f_{A_i}^-$ for all $i \in I$. Thus, $\bigcap\limits_{i\in I} f_{A_i}$ is a semiprime DFS bi-ideal of S over U. $\square$

Lemma 3.10. *If $f_A = \langle (f_A^+, f_A^-); A\rangle$ and $g_B = \langle (g_B^+, g_B^-); B\rangle$ DFS bi-ideal of S over U. Then $f_A \diamond g_B$ is a DFS bi-ideal of S over U.*

Definition 3.11. *A DFS bi-ideal $f_A = \langle (f_A^+, f_A^-); A\rangle$ of an ordered semigroup S over U is called an irreducible (resp., strongly irreducible) DFS bi-ideal if for any double-framed soft bi-ideals $g_B = \langle (g_B^+, g_B^-); B\rangle$ and $h_C = \langle (h_C^+, h_C^-); C\rangle$ of S over U, $g_B \sqcap h_C = f_A$ (resp., $g_B \sqcap h_C \sqsubseteq f_A$) implies either $g_B = f_A$ or $h_C = f_A$ (resp., $g_B \sqsubseteq f_A$ or $h_C \sqsubseteq f_A$). That is, $g_B^+ \widetilde{\cap}\, h_C^+ \widetilde{=} f_A^+$ and $g_B^- \widetilde{\cup}\, h_C^- \widetilde{=} f_A^-$ (resp., $g_B^+ \widetilde{\cap}\, h_C^+ \widetilde{\subseteq} f_A^+$ and $g_B^- \widetilde{\cup}\, h_C^- \widetilde{\supseteq} f_A^-$) imply either $g_B^+ \widetilde{=} f_A^+$ or $h_C^+ \widetilde{=} f_A^+$, and $g_B^- \widetilde{=} f_A^-$, $h_C^- \widetilde{=} f_A^-$ (resp., $g_B^+ \widetilde{\subseteq} f_A^+$ or $h_C^+ \widetilde{\subseteq} f_A^+$, and $g_B^- \widetilde{\supseteq} f_A^-$, $h_C^- \widetilde{\subseteq} f_A^-$).*

Remark 3.12. *Every strongly irreducible DFS bi-ideal of S over U is an irre-ducible DFS bi-ideal but the converse is not true in general.*

Example 3.13. *Suppose that there are five houses in an initial universe set U, given by*

$$U := \{h_1, h_2, h_3, h_4, h_5\}.$$

Let a set of parameters $E = \{e_0, e_1, e_2, e_3, e_4, e_5\}$ be a set of status of houses in which
e_0 stands for the parameter ”beautiful”,
e_1 stands for the parameter ”cheap”,
e_2 stands for the parameter ”in good location”,
e_3 stands for the parameter ”in green surrounding”,
e_4 stands for the parameter ”with double exit”,
e_5 stands for the parameter ”in city area”,

$*$	e_0	e_1	e_2	e_3	e_4	e_5
e_0	e_0	e_0	e_0	e_0	e_0	e_0
e_1	e_0	e_1	e_1	e_1	e_1	e_1
e_2	e_0	e_1	e_2	e_3	e_1	e_1
e_3	e_0	e_1	e_1	e_1	e_2	e_3
e_4	e_0	e_1	e_4	e_5	e_1	e_1
e_5	e_0	e_1	e_1	e_1	e_4	e_5

We define the order relation "$\leq$" on E as follows.

$$\leq = \{(e_0,e_0),(e_1,e_1),(e_2,e_2),(e_3,e_3),(e_4,e_4),(e_5,e_5),(e_0,e_1),(e_0,e_2),(e_0,e_3),(e_0,e_4),(e_2,e_3)\}.$$

We define the covering relation "$\prec$" as given below.

$$\prec := \{(e_0,e_1),(e_0,e_2),(e_0,e_3),(e_0,e_4),(e_2,e_3)\}.$$

Then $(E,*,\leq)$ is an ordered semigroup. The sets

$$\{e_0\},\{e_0,e_1,e_2\},\{e_0,e_1,e_3\},\{e_0,e_1,e_4\},\{e_0,e_1,e_5\},\{e_0,e_1,e_2,e_4\},$$
$$\{e_0,e_1,e_3,e_5\},\{e_0,e_1,e_2,e_3\},E,$$

are bi-ideals of E. The bi-ideals

$$\{e_0,e_1,e_2,e_4\},\{e_0,e_1,e_3,e_5\},\{e_0,e_1,e_2,e_3\},$$

are irreducible but not strongly irreducible. The only strongly irreducible bi-ideals of E are $\{e_0\}$ and E. By Lemma 3.8, the characteristic double-framed soft set of the irreducible bi-ideals

$$\{e_0,e_1,e_2,e_4\},\{e_0,e_1,e_3,e_5\},\{e_0,e_1,e_2,e_3\},$$

are irreducible double-framed soft bi-ideal but not strongly irreducible double-framed soft bi-ideal of S over U.

Lemma 3.14. *Every strongly irreducible semiprime DFS bi-ideal of S over U is a strongly prime DFS bi-ideal of S over U.*

Proof. Suppose that $f_A = \langle (f_A^+, f_A^-); A\rangle$ be a strongly irreducible semiprime DFS bi-ideal of S over U. Let $g_B = \langle (g_B^+, g_B^-); B\rangle$ and $h_C = \langle (h_C^+, h_C^-); C\rangle$ be DFS bi-ideals of S over U such that

$$\left(g_B^+ \tilde{\circ}\, h_C^+\right) \tilde{\cap} \left(h_C^+ \tilde{\circ}\, g_B^+\right) \tilde{\subseteq} f_A^+ \text{ and } \left(g_B^- \tilde{*}\, h_C^-\right) \tilde{\cup} \left(h_C^- \tilde{*}\, g_B^-\right) \tilde{\supseteq} f_A^-.$$

Since $g_B^+ \tilde{\cap}\, h_C^+$ and $g_B^- \tilde{\cup}\, h_C^-$ are soft bi-ideals of S over U and

$$\left(g_B^+ \tilde{\circ}\, h_C^+\right) \tilde{\cap} \left(h_C^+ \tilde{\circ}\, g_B^+\right) \tilde{\subseteq} \left(g_B^+ \tilde{\circ}\, h_C^+\right),$$
$$\left(g_B^+ \tilde{\circ}\, h_C^+\right) \tilde{\cap} \left(h_C^+ \tilde{\circ}\, g_B^+\right) \tilde{\subseteq} \left(h_C^+ \tilde{\circ}\, g_B^+\right),$$

$$\left(g_B^- \tilde{*}\, h_C^-\right) \tilde{\cup} \left(h_C^- \tilde{*}\, g_B^-\right) \tilde{\supseteq} \left(g_B^- \tilde{*}\, h_C^-\right),$$
$$\left(g_B^- \tilde{*}\, h_C^-\right) \tilde{\cup} \left(h_C^- \tilde{*}\, g_B^-\right) \tilde{\supseteq} \left(h_C^- \tilde{*}\, g_B^-\right).$$

Thus,

$$\left(g_B^+ \widetilde{\cap}\, h_C^+\right) \widetilde{\circ} \left(h_C^+ \widetilde{\cap}\, g_B^+\right) \widetilde{\subseteq} \left(g_B^+ \widetilde{\circ}\, h_C^+\right) \widetilde{\cap} \left(h_C^+ \widetilde{\circ}\, g_B^+\right) \widetilde{\subseteq} f_A^+.$$

$$\left(g_B^- \widetilde{\cup}\, h_C^-\right) \widetilde{*} \left(h_C^- \widetilde{\cup}\, g_B^-\right) \widetilde{\supseteq} \left(g_B^- \widetilde{*}\, h_C^-\right) \widetilde{\cup} \left(h_C^- \widetilde{*}\, g_B^-\right) \widetilde{\supseteq} f_A^-.$$

Since $\langle (f_A^+, f_A^-); A\rangle$ is semiprime, we have $\left(g_B^+ \widetilde{\cap}\, h_C^+\right) \widetilde{\subseteq} f_A^+$ and $\left(g_B^- \widetilde{\cup}\, h_C^-\right) \widetilde{\supseteq} f_A^-$. Since $\langle (f_A^+, f_A^-); A\rangle$ is strongly irreducible DFS bi-ideal of S over U, we have either $g_B^+ \widetilde{\subseteq} f_A^+$ or $h_C^+ \widetilde{\subseteq} f_A^+$ and $g_B^- \widetilde{\supseteq} f_A^-$, $h_C^- \widetilde{\supseteq} f_A^-$.Thus, $\langle (f_A^+, f_A^-); A\rangle$ is strongly prime DFS bi-ideal of S over U. $\square$

Lemma 3.15. *(Khan et al. [2]) Let S be an ordered semigroup and A;B are non-empty subsets of S, then the following are equivalent:*

(i) $A \subseteq B$ *if and only if* $X_A \subseteq X_B$,

(ii) $X_A \sqcap X_B = X_{(A\cap B)}$,

(iii) $X_A \sqcup X_B = X_{(A\cup B)}$,

(iv) $X_A \diamond X_B = X_{(AB]}$.

Lemma 3.16. *Let* $\langle (f_A^+, f_A^-); A\rangle$ *be a DFS bi-ideal of* S *over* U *with* $f_A^+(a) = \gamma$ *and* $f_A^-(a) = \delta$, *where* a *is an element of* S *and* $\gamma, \delta \in P(U)$. *Then there exists an irreducible DFS bi-ideal* $\langle (g_B^+, g_B^-); B\rangle$ *of* S *over* U *such that* $g_A^+(a) = \gamma$ *and* $g_A^-(a) = \delta$.

Proof. Let $X := \left\{ \begin{array}{c} h_C = \langle (h_C^+, h_C^-); C\rangle : \text{ such that } h_C = \langle (h_C^+, h_C^-); C\rangle \\ \text{be a DFS ideal of } S \text{ over } U, \\ h_A^+(a) = \gamma \text{ and } h_A^-(a) = \delta \text{ and } f_A^+ \widetilde{\subseteq} h_C^+,\ f_A^- \widetilde{\supseteq} h_C^- \end{array} \right\}$

Then $X \neq \emptyset$, since $\left\langle (f_A^+,\ f_A^-); A\right\rangle \in X$. The collection X is partially ordered set by set inclusion $\sqsubseteq$. Suppose that Y is a totally ordered subset of X, say $Y := \left\{\left\langle (h_{C_i}^+,\ h_{C_i}^-); C_i\right\rangle : i \in I\right\}$.
: Let $x, y \in S$ be such that $x \leq y$. Then,

$$\left(\bigcup_{i\in I} h_{C_i}^+\right)(x) = \bigcup_{i\in I}\left(h_{C_i}^+(x)\right) \supseteq \bigcup_{i\in I}\left(h_{C_i}^+(y)\right),$$

$$\left(\bigcap_{i\in I} h_{C_i}^-\right)(x) = \bigcap_{i\in I}\left(h_{C_i}^-(x)\right) \subseteq \bigcap_{i\in I}\left(h_{C_i}^-(y)\right).$$

Let x, y be any elements of S, then

$$\begin{aligned} \left(\bigcup_{i\in I} h_{C_i}^+\right)(xy) &= \bigcup_{i\in I}\left(h_{C_i}^+(xy)\right) \supseteq \bigcup_{i\in I}\left(h_{C_i}^+(x) \cap h_{C_i}^+(y)\right) \\ &= \left(\bigcup_{i\in I}\left(h_{C_i}^+(x)\right)\right) \cap \left(\bigcup_{i\in I}\left(h_{C_i}^+(y)\right)\right) \\ &= \left(\bigcup_{i\in I} h_{C_i}^+\right)(x) \cap \left(\bigcup_{i\in I} h_{C_i}^+\right)(y), \end{aligned}$$

and

$$\begin{aligned}\left(\bigcap_{i\in I} h^-_{C_i}\right)(xy) &= \bigcap_{i\in I}\left(h^-_{C_i}(xy)\right) \subseteq \bigcap_{i\in I}\left(h^-_{C_i}(x)\cup h^-_{C_i}(y)\right)\\ &= \left(\bigcap_{i\in I}\left(h^+_{C_i}(x)\right)\right)\cup\left(\bigcap_{i\in I}\left(h^+_{C_i}(y)\right)\right)\\ &= \left(\bigcap_{i\in I} h^-_{C_i}\right)(x)\cup\left(\bigcap_{i\in I} h^-_{C_i}\right)(y).\end{aligned}$$

For any $x, y, z \in S$,

$$\begin{aligned}\left(\bigcup_{i\in I} h^+_{C_i}\right)(xyz) &= \bigcup_{i\in I}\left(h^+_{C_i}(xyz)\right) \supseteq \bigcup_{i\in I}\left(h^+_{C_i}(x)\cap h^+_{C_i}(z)\right)\\ &= \left(\bigcup_{i\in I}\left(h^+_{C_i}(x)\right)\right)\cap\left(\bigcup_{i\in I}\left(h^+_{C_i}(z)\right)\right)\\ &= \left(\bigcup_{i\in I} h^+_{C_i}\right)(x)\cap\left(\bigcup_{i\in I} h^+_{C_i}\right)(z),\end{aligned}$$

and

$$\begin{aligned}\left(\bigcap_{i\in I} h^-_{C_i}\right)(xyz) &= \bigcap_{i\in I}\left(h^-_{C_i}(xyz)\right) \subseteq \bigcap_{i\in I}\left(h^-_{C_i}(x)\cup h^-_{C_i}(z\right)\\ &= \left(\bigcap_{i\in I}\left(h^+_{C_i}(x)\right)\right)\cup\left(\bigcap_{i\in I}\left(h^+_{C_i}(z)\right)\right)\\ &= \left(\bigcap_{i\in I} h^-_{C_i}\right)(x)\cup\left(\bigcap_{i\in I} h^-_{C_i}\right)(z).\end{aligned}$$

Therefore $\left\{\left\langle(\bigcup_{i\in I} h^+_{C_i}, \bigcap_{i\in I} h^-_{C_i}); C_i\right\rangle : i\in I\right\}$ is a DFS ideal of S over U. Since $f^+_A \,\tilde{\subseteq}\, h^+_{C_i}$ and $f^-_A \,\tilde{\supseteq}\, h^-_{C_i}$ for each $i \in I$. So $f^+_A \,\tilde{\subseteq} \bigcup_{i\in I} h^+_{C_i}$ and $f^-_A \,\tilde{\supseteq} \bigcap_{i\in I} h^-_{C_i}$. Also $\left(\bigcup_{i\in I} h^+_{C_i}\right)(a) = \bigcup_{i\in I}\left(h^-_{C_i}(a)\right) = \gamma$ and $\left(\bigcap_{i\in I} h^-_{C_i}\right)(a) = \bigcap_{i\in I}\left(h^-_{C_i}(a)\right) = \delta$. Thus, $\left\langle(\bigcup_{i\in I} h^+_{C_i}, \bigcap_{i\in I} h^-_{C_i}); C_i\right\rangle$ is the least upper bound of Y, by Zorn's Lemma, there exists a DFS ideal $\langle(g^+_B,\ g^-_B); B\rangle$ of S over U which is maximal with respect to the property $f^+_A \,\tilde{\subseteq}\, g^+_B$, $f^-_A \,\tilde{\supseteq}\, g^-_B$ and $g^+_B(a) = \gamma$, $g^-_B(a) = \delta$. Now we show that $\langle(g^+_B,\ g^-_B); B\rangle$ is irreducible DFS bi-ideal of S over U. Suppose that $g^+_B \,\tilde{=}\, l^+_F \tilde{\cap} t^+_D$and $g^-_B \,\tilde{=}\, l^-_F \tilde{\cup} t^-_D$, where $\langle(l^+_F,\ l^-_F); F\rangle$ and $g_D = \langle(g^+_D,\ g^-_D); D\rangle$ are DFS bi-ideals of S over U. Then $g^+_B \,\tilde{\subseteq}\, l^+_F$ or $g^+_B \,\tilde{\subseteq}\, g^+_D$ and $g^-_B \,\tilde{\supseteq}\, l^-_F$, $g^-_B \,\tilde{\supseteq}\, g^-_D$. We claim that $g^+_B \,\tilde{=}\, l^+_F$ or $g^+_B \,\tilde{=}\, g^+_D$ and $g^-_B \,\tilde{=}\, l^-_F$, $g^-_B \,\tilde{=}\, g^-_D$. Suppose on contrary that $g^+_B \,\tilde{\neq}\, l^+_F$ or $g^+_B \,\tilde{\neq}\, t^+_D$ and $g^-_B \,\tilde{\neq}\, l^-_F$, $g^-_B \,\tilde{\neq}\, t^-_D$. Since $\langle(g^+_B,\ g^-_B); B\rangle$ is maximal with respect to the property that $g^+_B(a) = \gamma$, $g^+_B(a) = \delta$. Since $g^+_B \,\tilde{\neq}\, l^+_F$, $g^+_B \,\tilde{\neq}\, g^+_D$ and $g^-_B \,\tilde{\neq}\, l^-_F$, $g^-_B \,\tilde{\neq}\, g^-_D$, it follows that $l^+_F(a) \,\tilde{\neq}\, \gamma \,\tilde{\neq}\, g^+_D(a)$ and $l^-_F(a) \,\tilde{\neq}\, \delta \,\tilde{\neq}\, g^-_D(a)$. Hence

$\gamma = g_B^+(\gamma) \cong l_F^+(a) \,\tilde{\cap}\, g_D^+(a) = \left(l_F^+ \,\tilde{\cap}\, g_D^+\right)(a) \neq \gamma$ and $\delta = g_B^-(a) \cong l_F^-(a) \,\tilde{\cup}\, g_D^-(a) = \left(l_F^- \,\tilde{\cup}\, g_D^-\right)(a)$ $\neq \delta$, which is a contradiction. Hence either $g_B^+ \cong l_F^+$ or $g_B^+ \cong g_D^+$ and $g_B^- \cong l_F^-$, $g_B^- \cong g_D^-$. Therefore, $\left\langle (g_B^+,\ g_B^-); B\right\rangle$ is an irreducible DFS bi-ideal of S over U. □

Theorem 3.17. *For an ordered semigroup $(S, ., \leq)$, the following assertions are equivalent:*

(i) S is both regular and intra-regular,

(ii) $\left(f_A^+ \,\tilde{\circ}\, f_A^+\right) \cong f_A^+$ and $\left(f_A^- \,\tilde{*}\, f_A^-\right) \cong f_A^-$, for every DFS bi-ideal $\langle (f_A^+, f_A^-); A\rangle$ of S over U.

(iii) $g_B^+ \,\tilde{\cap}\, h_C^+ \cong \left(g_B^+ \,\tilde{\circ}\, h_C^+\right) \tilde{\cap} \left(h_C^+ \,\tilde{\circ}\, g_B^+\right)$, $g_B^- \,\tilde{\cup}\, h_C^- \cong \left(g_B^- \,\tilde{*}\, h_C^-\right) \tilde{\cap} \left(h_C^- \,\tilde{*}\, g_B^-\right)$, for every DFS bi-ideals $\langle (g_B^+, g_B^-); B\rangle$ and $\langle (h_C^+, h_C^-); C\rangle$ of S over U,

(iv) Each DFS bi-ideal of S over U is semiprime,

(v) Each proper DFS bi-ideal of S over U is the intersection of all irreducible semiprime DFS bi-ideals of S over U which contain it.

Proof. $(i) \Rightarrow (ii)$. Suppose S is both regular and intra-regular ordered semigroup and $\langle (f_A^+, f_A^-); A\rangle$ a DFS bi-ideal of S over U. Then for each $a \in S$, we have $\left(f_A^+ \,\tilde{\circ}\, f_A^+\right)(a) \,\tilde{\supseteq}\, f_A^+(a)$ and $\left(f_A^- \,\tilde{*}\, f_A^-\right)(a) \cong f_A^-(a)$. Indeed: Since S is regular and intra-regular therefore there exist $x, y, z \in S$, such that $a \leq axa$ and $a \leq ya^2z$. Thus

$$a \leq axa \leq axaxa \leq ax(ya^2z)xa = (axya)(azxa).$$

Then $(axya, azxa) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned}
\left(f_A^+ \,\tilde{\circ}\, f_A^+\right)(a) &= \bigcup_{(y,z)\in A_a} \{f_A^+(y) \cap g_A^+(z)\} \\
&\supseteq f_A^+(axya) \cap g_A^+(azxa),
\end{aligned}$$

$$\begin{aligned}
\left(f_A^- \,\tilde{*}\, f_A^-\right)(a) &= \bigcap_{(y,z)\in A_a} \{f_A^-(y) \cup g_A^-(z)\} \\
&\subseteq f_A^-(axya) \cup g_A^-(azxa).
\end{aligned}$$

Since $\langle (f_A^+, f_A^-); A\rangle$ is a DFS bi-ideal of S over U, we have

$$\begin{aligned}
f_A^+(axya) &\supseteq f_A^+(a) \cap f_A^+(a) = f_A^+(a), \\
f_A^+(azxa) &\supseteq f_A^+(a) \cap f_A^+(a) = f_A^+(a),
\end{aligned}$$

$$\begin{aligned}
f_A^-(axya) &\subseteq f_A^-(a) \cup f_A^-(a) = f_A^-(a), \\
f_A^-(azxa) &\subseteq f_A^-(a) \cup f_A^-(a) = f_A^-(a).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(f_A^+ \cap f_A^+\right)(a) &\supseteq f_A^+(axya) \cap f_A^+(azxa) \\
&\supseteq f_A^+(a) \cap f_A^+(a) = f_A^+(a)
\end{aligned}$$

$$\begin{aligned}
\left(f_A^- \cap f_A^-\right)(a) &\subseteq f_A^-(axya) \cup f_A^-(azxa) \\
&\subseteq f_A^-(a) \cup f_A^-(a) = f_A^-(a),
\end{aligned}$$

and so $f_A^+ \mathbin{\widetilde{\circ}} f_A^+ \mathrel{\widetilde{\supseteq}} f_A^+$ and $f_A^- \mathbin{\widetilde{*}} f_A^- \mathrel{\widetilde{\subseteq}} f_A^-$. For the reverse inclusion, if $A_x = \emptyset$, then $\left(f_A^+ \mathbin{\widetilde{\circ}} f_A^+\right)(x) = \emptyset \mathrel{\widetilde{\subseteq}} f_A^+(x)$ and $\left(f_A^- \mathbin{\widetilde{*}} f_A^-\right)(x) = U \supseteq f_A^-(x)$. If $A_x \neq \emptyset$,

$$\begin{aligned}\left(f_A^+ \mathbin{\widetilde{\circ}} f_A^+\right)(x) &= \bigcup_{(y,z)\in A_x} \{f_A^+(y) \cap g_A^+(z)\},\\ \left(f_A^- \mathbin{\widetilde{*}} f_A^-\right)(x) &= \bigcap_{(y,z)\in A_x} \{f_A^-(y) \cup g_A^-(z)\}.\end{aligned}$$

Since $x \leq yz$ and $\langle (f_A^+, f_A^-); A\rangle$ is a DFS bi-ideal of S over U, we have $f_A^+(x) \supseteq f_A^+(yz) \supseteq f_A^+(y) \cap f_A^+(z)$ and $f_A^-(x) \subseteq f_A^-(yz) \subseteq f_A^-(y) \cup f_A^-(z)$. Hence

$$\begin{aligned}\left(f_A^+ \mathbin{\widetilde{\circ}} f_A^+\right)(x) &= \bigcup_{(y,z)\in A_x} \{f_A^+(y) \cap g_A^+(z)\} \subseteq \bigcup_{(y,z)\in A_x} f_A^+(yz)\\ &\subseteq \bigcup_{(y,z)\in A_x} f_A^+(x) = f_A^+(x),\\ \left(f_A^- \mathbin{\widetilde{*}} f_A^-\right)(x) &= \bigcap_{(y,z)\in A_x} \{f_A^-(y) \cup g_A^-(z)\} \supseteq \bigcap_{(y,z)\in A_x} f_A^-(yz)\\ &\subseteq \bigcap_{(y,z)\in A_x} f_A^-(x) = f_A^-(x).\end{aligned}$$

Hence, we have $f_A^+ \mathbin{\widetilde{\circ}} f_A^+ \mathrel{\widetilde{\subseteq}} f_A^+$ and $f_A^- \mathbin{\widetilde{*}} f_A^- \mathrel{\widetilde{\supseteq}} f_A^-$. Therefore, $f_A^+ \mathbin{\widetilde{\circ}} f_A^+ \mathrel{\widetilde{=}} f_A^+$ and $f_A^- \mathbin{\widetilde{*}} f_A^- \mathrel{\widetilde{=}} f_A^-$.

$(ii) \Rightarrow (iii)$. Let $\langle (g_B^+, g_B^-); B\rangle$ and $\langle (h_C^+, h_C^-); C\rangle$ be DFS bi-ideals of S over U. Then $g_B \sqcap h_C$ is a DFS bi-ideal of S over U. By (ii), we have

$$\begin{aligned}\left(g_B^+ \mathbin{\widetilde{\cap}} h_C^+\right) \mathbin{\widetilde{\circ}} \left(g_B^+ \mathbin{\widetilde{\cap}} h_C^+\right) &\mathrel{\widetilde{\subseteq}} g_B^+ \mathbin{\widetilde{\circ}} h_C^+,\\ \left(g_B^- \mathbin{\widetilde{\cup}} h_C^-\right) \mathbin{\widetilde{*}} \left(g_B^- \mathbin{\widetilde{\cup}} h_C^-\right) &\mathrel{\widetilde{\supseteq}} g_B^- \mathbin{\widetilde{*}} h_C^-.\end{aligned}$$

Similarly,

$$\begin{aligned}\left(g_B^+ \mathbin{\widetilde{\cap}} h_C^+\right) \mathbin{\widetilde{\circ}} \left(g_B^+ \mathbin{\widetilde{\cap}} h_C^+\right) &\mathrel{\widetilde{\subseteq}} h_C^+ \mathbin{\widetilde{\circ}} g_B^+,\\ \left(g_B^- \mathbin{\widetilde{\cup}} h_C^-\right) \mathbin{\widetilde{*}} \left(g_B^- \mathbin{\widetilde{\cup}} h_C^-\right) &\mathrel{\widetilde{\supseteq}} h_C^- \mathbin{\widetilde{*}} g_B^-.\end{aligned}$$

Thus,

$$\begin{aligned}\left(g_B^+ \mathbin{\widetilde{\cap}} h_C^+\right) \mathbin{\widetilde{\circ}} \left(g_B^+ \mathbin{\widetilde{\cap}} h_C^+\right) &\mathrel{\widetilde{\subseteq}} \left(h_C^+ \mathbin{\widetilde{\circ}} g_B^+\right) \mathbin{\widetilde{\cap}} \left(g_B^+ \mathbin{\widetilde{\circ}} h_C^+\right),\\ \left(g_B^- \mathbin{\widetilde{\cup}} h_C^-\right) \mathbin{\widetilde{*}} \left(g_B^- \mathbin{\widetilde{\cup}} h_C^-\right) &\mathrel{\widetilde{\supseteq}} \left(h_C^- \mathbin{\widetilde{*}} g_B^-\right) \mathbin{\widetilde{\cup}} \left(g_B^- \mathbin{\widetilde{*}} h_C^-\right).\end{aligned}$$

For the reverse inclusion, since $g_B^+ \mathbin{\widetilde{\circ}} h_C^+$, $g_B^- \mathbin{\widetilde{*}} h_C^-$, $h_C^+ \mathbin{\widetilde{\circ}} g_B^+$, $h_C^- \mathbin{\widetilde{*}} g_B^-$ are soft bi-ideals of S over U (Lemma 3.10) and so $\left(g_B^+ \mathbin{\widetilde{\circ}} h_C^+\right) \mathbin{\widetilde{\cap}} \left(h_C^+ \mathbin{\widetilde{\circ}} g_B^+\right)$ and $\left(g_B^- \mathbin{\widetilde{*}} h_C^-\right) \mathbin{\widetilde{\cup}} \left(h_C^- \mathbin{\widetilde{*}} g_B^-\right)$ (Lemma 3.9). By (ii), we have

$$\begin{aligned}&\left(g_B^+ \mathbin{\widetilde{\circ}} h_C^+\right) \mathbin{\widetilde{\cap}} \left(h_C^+ \mathbin{\widetilde{\circ}} g_B^+\right) \mathrel{\widetilde{=}} \left(\left(g_B^+ \mathbin{\widetilde{\circ}} h_C^+\right) \mathbin{\widetilde{\cap}} \left(h_C^+ \mathbin{\widetilde{\circ}} g_B^+\right)\right) \mathbin{\widetilde{\circ}} \left(\left(g_B^+ \mathbin{\widetilde{\circ}} h_C^+\right) \mathbin{\widetilde{\cap}} \left(h_C^+ \mathbin{\widetilde{\circ}} g_B^+\right)\right)\\ &\mathrel{\widetilde{\subseteq}} \left(g_B^+ \mathbin{\widetilde{\circ}} h_C^+\right) \mathbin{\widetilde{\cap}} \left(h_C^+ \mathbin{\widetilde{\circ}} g_B^+\right) \mathrel{\widetilde{=}} g_B^+ \mathbin{\widetilde{\circ}} \left(h_C^+ \mathbin{\widetilde{\circ}} h_C^+\right) \mathbin{\widetilde{\circ}} g_B^+\\ &\mathrel{\widetilde{=}} g_B^+ \mathbin{\widetilde{\circ}} h_C^+ \mathbin{\widetilde{\circ}} g_B^+ \text{ (since } h_C^+ \mathbin{\widetilde{\circ}} h_C^+ \mathrel{\widetilde{=}} h_C^+ \text{ by (ii))}\\ &\mathrel{\widetilde{\subseteq}} g_B^+ \mathbin{\widetilde{\circ}} X_S^+ \mathbin{\widetilde{\circ}} g_B^+ \mathrel{\widetilde{\subseteq}} g_B^+,\end{aligned}$$

and

$$\begin{aligned}
&\left(g_B^- \,\widetilde{*}\, h_C^-\right) \widetilde{\cup} \left(h_C^- \,\widetilde{*}\, g_B^-\right) \widetilde{=} \left(\left(g_B^- \,\widetilde{*}\, h_C^-\right) \widetilde{\cup} \left(h_C^- \,\widetilde{*}\, g_B^-\right)\right) \widetilde{*} \left(\left(g_B^- \,\widetilde{*}\, h_C^-\right) \widetilde{\cup} \left(h_C^- \,\widetilde{*}\, g_B^-\right)\right) \\
&\widetilde{\supseteq} \left(g_B^- \,\widetilde{*}\, h_C^-\right) \widetilde{\cup} \left(h_C^- \,\widetilde{*}\, g_B^-\right) \widetilde{=} g_B^- \,\widetilde{*}\, \left(h_C^- \,\widetilde{*}\, h_C^-\right) \widetilde{*}\, g_B^- \\
&\widetilde{=} g_B^- \,\widetilde{*}\, h_C^- \,\widetilde{*}\, g_B^- \text{ (since } h_C^- \,\widetilde{*}\, h_C^- \widetilde{=} h_C^- \text{ by (ii))} \\
&\widetilde{\supseteq} g_B^- \,\widetilde{*}\, X_S^- \,\widetilde{*}\, g_B^- \,\widetilde{\supseteq}\, g_B^-.
\end{aligned}$$

-Similarly, we can prove that $\left(g_B^+ \,\widetilde{\circ}\, h_C^+\right) \widetilde{\cap} \left(h_C^+ \,\widetilde{\circ}\, g_B^+\right) \,\widetilde{\subseteq}\, h_C^+$, $\left(g_B^- \,\widetilde{*}\, h_C^-\right) \widetilde{\cup} \left(h_C^- \,\widetilde{*}\, g_B^-\right) \,\widetilde{\supseteq}\, h_C^-$. Thus, $\left(g_B^+ \,\widetilde{\circ}\, h_C^+\right) \widetilde{\cap} \left(h_C^+ \,\widetilde{\circ}\, g_B^+\right) \widetilde{\subseteq} g_B^+ \,\widetilde{\cap}\, h_C^+$ and $\left(g_B^- \,\widetilde{*}\, h_C^-\right) \widetilde{\cup} \left(h_C^- \,\widetilde{*}\, g_B^-\right) \widetilde{\supseteq} g_B^- \,\widetilde{\cup}\, h_C^-$.

Therefore, $\left(g_B^+ \,\widetilde{\circ}\, h_C^+\right) \widetilde{\cap} \left(h_C^+ \,\widetilde{\circ}\, g_B^+\right) \widetilde{=} g_B^+ \,\widetilde{\cap}\, h_C^+$ and $\left(g_B^- \,\widetilde{*}\, h_C^-\right) \widetilde{\cup} \left(h_C^- \,\widetilde{*}\, g_B^-\right) \widetilde{=} g_B^- \,\widetilde{\cup}\, h_C^-$.

(iii) $\Rightarrow$ *(i)*. Let $\langle (g_B^+, g_B^-); B\rangle$ and $\langle (h_C^+, h_C^-); C\rangle$ be DFS left and right ideals of S over U, respectively. Then, $\langle (g_B^+, g_B^-); B\rangle$ and $\langle (h_C^+, h_C^-); C\rangle$ are DFS bi-ideals of S over U. By hypothesis,

$$\left(g_B^+ \,\widetilde{\circ}\, h_C^+\right) \widetilde{\cap} \left(h_C^+ \,\widetilde{\circ}\, g_B^+\right) \widetilde{=} g_B^+ \,\widetilde{\cap}\, h_C^+ \text{ and } \left(g_B^- \,\widetilde{*}\, h_C^-\right) \widetilde{\cup} \left(h_C^- \,\widetilde{*}\, g_B^-\right) \widetilde{=} g_B^- \,\widetilde{\cup}\, h_C^-.$$

To prove that S is both regular and intra-regular, by Lemma 3.15, it is enough to prove that $R \cap L = (RL] \cap (LR]$ for every right ideal R and left ideal L of S.

Let R be a right and L a left ideal of S. Then, the characteristic mappings $\langle (X_R^+, X_R^-); R\rangle$ and $\langle (X_L^+, X_L^-); L\rangle$ are DFS left and right ideals of S over U,
respectively. By hypothesis,

$$\begin{aligned}
X_{R\cap L}^+ &= X_R^+ \,\widetilde{\cap}\, X_L^+ = \left(X_R^+ \,\widetilde{\circ}\, X_L^+\right) \widetilde{\cap} \left(X_R^+ \,\widetilde{\circ}\, X_L^+\right) \\
&= X_{(RL]}^+ \,\widetilde{\cap}\, X_{(RL]}^+ = X_{(RL]\cap(RL]}^+.
\end{aligned}$$

$$\begin{aligned}
X_{R\cup L}^- &= X_R^- \,\widetilde{\cup}\, X_L^- = \left(X_R^- \,\widetilde{*}\, X_L^-\right) \widetilde{\cup} \left(X_R^- \,\widetilde{*}\, X_L^-\right) \\
&= X_{(RL]}^- \,\widetilde{\cup}\, X_{(RL]}^- = X_{(RL]\cup(RL]}^-.
\end{aligned}$$

By Lemma 3.15, part (iv), we have $R \cap L = (RL] \cap (LR]$, and S is both regular and intra-regular (Lemma 3.15).

(iii) $\Rightarrow$ *(iv)*. Let $\langle (f_A^+, f_A^-); A\rangle$ and $\langle (g_B^+, g_B^-); B\rangle$ be DFS left and right ideals of S over U, respectively, such that $f_A^+ \,\widetilde{\circ}\, f_A^+ \,\widetilde{\subseteq}\, g_B^+$ and $f_A^- \,\widetilde{*}\, f_A^- \,\widetilde{\supseteq}\, g_B^-$. By hypothesis, $f_A^+ \,\widetilde{=}\, f_A^+ \,\widetilde{\cap}\, f_A^+ \,\widetilde{=}\, f_A^+ \,\widetilde{\circ}\, f_A^+ \,\widetilde{\cap}\, f_A^+ \,\widetilde{\circ}\, f_A^+ \,\widetilde{=}\, f_A^+ \,\widetilde{\circ}\, f_A^+$ and $f_A^- \,\widetilde{=}\, f_A^- \,\widetilde{\cup}\, f_A^- \,\widetilde{=}\, f_A^- \,\widetilde{*}\, f_A^- \,\widetilde{\cup}\, f_A^- \,\widetilde{*}\, f_A^- \,\widetilde{=}\, f_A^- \,\widetilde{*}\, f_A^-$. Thus, $f_A^+ \,\widetilde{\subseteq}\, g_B^+$ and $f_A^- \,\widetilde{\supseteq}\, g_B^-$ and $\langle (g_B^+, g_B^-); B\rangle$ is semiprime. Since $\langle (g_B^+, g_B^-); B\rangle$ is arbitrary, hence every DFS bi-ideal of S over U is semiprime.

(iv) $\Rightarrow$ *(v)*. Let $\langle (f_A^+, f_A^-); A\rangle$ be a proper DFS bi-ideal of S over U and $\left\{\langle (f_{A_i}^+, f_{A_i}^-); A_i\rangle / i \in I\right\}$ be a collection of irreducible DFS bi-ideal of S over U, which contain $\langle (f_A^+, f_A^-); A\rangle$. By Lemma 3.16, this collection is non-empty. Hence, so $f_A^+ \,\widetilde{\subseteq}\, \widetilde{\bigcap_{i\in I}} f_{A_i}^+$ and $f_A^- \,\widetilde{\supseteq}\, \widetilde{\bigcup_{i\in I}} f_{A_i}^-$. Let $a \in S$, then by Lemma 3.16, there exists an irreducible DFS bi-ideal $\langle (f_A^+, f_A^-); A\rangle$ of S over U such that $f_A^+ \,\widetilde{\subseteq}\, f_{A_i}^+$ and $f_A^- \,\widetilde{\supseteq}\, f_{A_i}^-$ and $f_A^+(a) = f_{A_\alpha}^+(a)$, $f_A^-(a) = f_{A_\alpha}^-(a)$. Thus, $\langle (f_{A_\alpha}^+, f_{A_\alpha}^-); A_\alpha\rangle \in$

$\left\{\langle(f^+_{A_i}, f^-_{A_i}); A_i\rangle / i \in I\right\}$. Hence $\widetilde{\bigcap\limits_{i\in I}} f^+_{A_i} \widetilde{\subseteq} f^+_{A_\alpha}$ and $\widetilde{\bigcup\limits_{i\in I}} f^-_{A_i} \widetilde{\supseteq} f^-_{A_\alpha}$.So $\widetilde{\bigcap\limits_{i\in I}} f^+_{A_i}(a) \widetilde{\subseteq} f^+_{A_\alpha}(a)$ and $\widetilde{\bigcup\limits_{i\in I}} f^-_{A_i}(a) \widetilde{\supseteq} f^-_{A_\alpha}(a)$. Thus $\widetilde{\bigcap\limits_{i\in I}} f^+_{A_i} \widetilde{\subseteq} f^+_A$ and $\widetilde{\bigcup\limits_{i\in I}} f^-_{A_i} \widetilde{\supseteq} f^-_A$. Consequently $\widetilde{\bigcap\limits_{i\in I}} f^+_{A_i} \widetilde{\cong} f^+_A$ and $\widetilde{\bigcup\limits_{i\in I}} f^-_{A_i} \widetilde{\cong} f^-_A$. By hypothesis, each DFS bi-ideal is semiprime. Hence, each DFS bi-ideal of S over U, is the intersection of all irreducible semiprime DFS bi-ideals of S over U which contain it.

$(v) \Rightarrow (ii)$ Let $\langle(f^+_A, f^-_A); A\rangle$ be a DFS bi-ideal of S over U. Then $\langle(f^+_A \widetilde{\circ} f^+_A, f^-_A \widetilde{*} f^-_A); A\rangle$ is a double-framed soft bi-ideal of S over U. Since $\langle(f^+_A, f^-_A); A\rangle$ is a DFS semigroup of S over U, hence by Lemma 3.7, $f^+_A \widetilde{\circ} f^+_A \widetilde{\subseteq} f^+_A$, and $f^-_A \widetilde{*} f^-_A \widetilde{\supseteq} f^-_A$. By hypothesis, $\widetilde{\bigcap\limits_{i\in I}} f^+_{A_i} \widetilde{\cong} f^+_A$ and $\widetilde{\bigcup\limits_{i\in I}} f^-_{A_i} \widetilde{\cong} f^-_A$ where $\left\{\langle(f^+_{A_i}, f^-_{A_i}); A_i\rangle / i \in I\right\}$ are irreducible semiprime DFS bi-ideals of S over U. Thus, $f^+_A \widetilde{\circ} f^+_A \widetilde{\subseteq} f^+_{A_i}$ and $f^-_A \widetilde{*} f^-_A \widetilde{\supseteq} f^-_{A_i}$ for all $i \in I$. $f^+_A \widetilde{\subseteq} f^+_{A_i}$ and $f^-_A \widetilde{\supseteq} f^-_{A_i}$ for all $i \in I$, because each $\langle(f^+_{A_i}, f^-_{A_i}); A_i\rangle$ is semiprime. Hence, $\widetilde{\bigcap\limits_{i\in I}} f^+_{A_i} \widetilde{\cong} f^+_A \widetilde{\cong} f^+_A \widetilde{\circ} f^+_A$and $\widetilde{\bigcup\limits_{i\in I}} f^-_{A_i} \widetilde{\cong} f^-_A \widetilde{\cong} f^-_A \widetilde{*} f^-_A$. Thus, $f^+_A \widetilde{\circ} f^+_A \widetilde{\cong} f^+_A$, and $f^-_A \widetilde{*} f^-_A \widetilde{\cong} f^-_A$.

In the following result, we study a relationship among strongly irreducible and and strongly prime DFS bi-ideals of S over U. □

Theorem 3.18. *Let S be both regular and intra-regular ordered semigroup. Then the following are equivalent:*

Proposition 3.19. *(i) Every DFS bi-ideal of S over U is strongly irreducible.*

(ii) Every DFS bi-ideal of S over U is strongly prime.

Proof. $(i) \Rightarrow (ii)$ Let S be both regular and intra-regular and $\langle(f^+_A, f^-_A); A\rangle$ a strongly irreducible DFS bi-ideal of S over U. Let $\langle(g^+_B, g^-_B); B\rangle$ and $\langle(h^+_C, h^-_C); C\rangle$ be DFS bi-ideals of S over U such that

$$\left(g^+_B \widetilde{\circ} h^+_C\right) \widetilde{\cap} \left(h^+_C \widetilde{\circ} g^+_B\right) \widetilde{\subseteq} f^+_A \text{ and } \left(h^-_C \widetilde{\cup} g^-_B\right) \widetilde{*} \left(g^-_B \widetilde{\cup} h^-_C\right) \widetilde{\supseteq} f^-_A.$$

Since S is both regular and intra-regular, hence by Theorem 3.17, $\left(g^+_B \widetilde{\circ} h^+_C\right) \widetilde{\cap} \left(h^+_C \widetilde{\circ} g^+_B\right) \widetilde{\cong} g^+_B \widetilde{\cap} h^+_C$ and $\left(g^-_B \widetilde{*} h^-_C\right) \widetilde{\cup} \left(h^-_C \widetilde{*} g^-_B\right) \widetilde{\cong} g^-_B \widetilde{\cup} h^-_C$. Thus, $g^+_B \widetilde{\cap} h^+_C \widetilde{\subseteq} f^+_A$ and $g^-_B \widetilde{\cup} h^-_C \widetilde{\supseteq} f^-_A$. Since $\langle(f^+_A, f^-_A); A\rangle$ isstronglyirrducible, so either $g^+_B \widetilde{\subseteq} f^+_A$ or $h^+_C \widetilde{\subseteq} f^+_A$ and $g^-_B \widetilde{\supseteq} f^-_A$, $h^-_C \widetilde{\supseteq} f^-_A$. Thus, $\langle(f^+_A, f^-_A); A\rangle$ is strongly prime DFS bi-ideal of S over U.

$(ii) \Rightarrow (i)$. Suppose that $\langle(f^+_A, f^-_A); A\rangle$ is a strongly prime DFS bi-ideal of S over U and $\langle(g^+_B, g^-_B); B\rangle$ and $\langle(h^+_C, h^-_C); C\rangle$ be DFS bi-ideals of S over U such that $g^+_B \widetilde{\cap} h^+_C \widetilde{\subseteq} f^+_A$ and $g^-_B \widetilde{\cup} h^-_C \widetilde{\supseteq} f^-_A$. Since

$$\left(g^+_B \widetilde{\circ} h^+_C\right) \widetilde{\cap} \left(h^+_C \widetilde{\circ} g^+_B\right) \widetilde{\subseteq} g^+_B \widetilde{\cap} h^+_C \widetilde{\subseteq} f^+_A,$$

and

$$\left(g^-_B \widetilde{*} h^-_C\right) \widetilde{\cup} \left(h^-_C \widetilde{*} g^-_B\right) \widetilde{\supseteq} g^-_B \widetilde{\cup} h^-_C \widetilde{\supseteq} f^-_A.$$

Since $\langle(f^+_A, f^-_A); A\rangle$ is a strongly prime DFS bi-ideal of S over U, so either $g^+_B \widetilde{\subseteq} f^+_A$ or $h^+_C \widetilde{\subseteq} f^+_A$ and $g^-_B \widetilde{\supseteq} f^-_A$, $h^-_C \widetilde{\supseteq} f^-_A$. Thus, $\langle(f^+_A, f^-_A); A\rangle$ is a strongly irreducible DFS bi-ideal of S over U. □

Theorem 3.20. *Each double-framed soft bi-ideal of an ordered semigroup S is strongly prime if and only if S is regular and intra-regular and the set of DFS bi-ideals of S over U is totally ordered by inclusion.*

Proof. Suppose that each DFS bi-ideal of S over U is strongly prime. Then each DFS bi-ideal of S over U is semiprime. Thus, by Theorem 3.17, S is both regular and intra-regular. We show that the set of DFS bi-ideals of S over U is totally ordered under inclusion. Let $\langle (g_B^+, g_B^-); B\rangle$ and $\langle (h_C^+, h_C^-); C\rangle$ be DFS bi-ideals of S over U. Then by Theorem 3.17, $g_B^+ \mathbin{\widetilde{\cap}} h_C^+ \cong \left(g_B^+ \mathbin{\widetilde{\circ}} h_C^+\right) \mathbin{\widetilde{\cap}} \left(h_C^+ \mathbin{\widetilde{\circ}} g_B^+\right)$ and $g_B^- \mathbin{\widetilde{\cup}} h_C^- \cong \left(g_B^- \mathbin{\widetilde{*}} h_C^-\right) \mathbin{\widetilde{\cup}} \left(h_C^- \mathbin{\widetilde{*}} g_B^-\right)$. Since each DFS bi-ideal of S over U is strongly prime, so $\langle (g_B^+, g_B^-); B\rangle \sqcap \langle (h_C^+, h_C^-); C\rangle$ is strongly prime. Hence either $g_B^+ \mathbin{\widetilde{\subseteq}} g_B^+ \mathbin{\widetilde{\cap}} h_C^+$ or $h_C^+ \mathbin{\widetilde{\subseteq}} g_B^+ \mathbin{\widetilde{\cap}} h_C^+$ and $g_B^- \mathbin{\widetilde{\supseteq}} g_B^- \mathbin{\widetilde{\cup}} h_C^-$, $h_C^- \mathbin{\widetilde{\supseteq}} g_B^- \mathbin{\widetilde{\cup}} h_C^-$. If $g_B^+ \mathbin{\widetilde{\subseteq}} g_B^+ \mathbin{\widetilde{\cap}} h_C^+$ and $g_B^- \mathbin{\widetilde{\supseteq}} g_B^- \mathbin{\widetilde{\cup}} h_C^-$, then $g_B^+ \mathbin{\widetilde{\subseteq}} h_C^+$ and $g_B^- \mathbin{\widetilde{\supseteq}} h_C^-$.

Conversely, assume that S is regular, intra-regular and the set of DFS bi-ideal of S over U is totally ordered under inclusion. We have to show that each DFS bi-ideal of S over U is strongly prime. Let $\langle (f_A^+, f_A^-); A\rangle$ be a DFS bi-ideal of S over U. Let $\langle (g_B^+, g_B^-); B\rangle$ and $\langle (h_C^+, h_C^-); C\rangle$ be DFS bi-ideals of S over U such that

$$\left(g_B^+ \mathbin{\widetilde{\circ}} h_C^+\right) \mathbin{\widetilde{\cap}} \left(h_C^+ \mathbin{\widetilde{\circ}} g_B^+\right) \mathbin{\widetilde{\subseteq}} f_A^+ \text{ and } \left(g_B^- \mathbin{\widetilde{*}} h_C^-\right) \mathbin{\widetilde{\cup}} \left(h_C^- \mathbin{\widetilde{*}} g_B^-\right) \mathbin{\widetilde{\supseteq}} f_A^-.$$

Since S is both regular and intra-regular, by Theorem 3.17,

$$\left(g_B^+ \mathbin{\widetilde{\circ}} h_C^+\right) \mathbin{\widetilde{\cap}} \left(h_C^+ \mathbin{\widetilde{\circ}} g_B^+\right) \cong g_B^+ \mathbin{\widetilde{\cap}} h_C^+ \text{ and } \left(g_B^- \mathbin{\widetilde{*}} h_C^-\right) \mathbin{\widetilde{\cup}} \left(h_C^- \mathbin{\widetilde{*}} g_B^-\right) \cong g_B^- \mathbin{\widetilde{\cup}} h_C^-.$$

Thus, $g_B^+ \mathbin{\widetilde{\cap}} h_C^+ \mathbin{\widetilde{\subseteq}} f_A^+$ and $g_B^- \mathbin{\widetilde{\cup}} h_C^- \mathbin{\widetilde{\supseteq}} f_A^-$ Since the set of DFS bi-ideals of S over U is totally ordered, so either $g_B^+ \mathbin{\widetilde{\subseteq}} h_C^+$ or $h_C^+ \mathbin{\widetilde{\subseteq}} g_B^+$ and $g_B^- \mathbin{\widetilde{\supseteq}} h_C^-$, $h_C^- \mathbin{\widetilde{\supseteq}} g_B^-$. That is, either $g_B^+ \mathbin{\widetilde{\cap}} h_C^+ \cong g_B^+$ or $g_B^+ \mathbin{\widetilde{\cap}} h_C^+ \cong h_C^+$ and $g_B^- \mathbin{\widetilde{\cup}} h_C^- \cong g_B^-$, $g_B^- \mathbin{\widetilde{\cup}} h_C^- \cong h_C^-$. Therefore, either $g_B^+ \mathbin{\widetilde{\subseteq}} f_A^+$ or $h_C^+ \mathbin{\widetilde{\subseteq}} f_A^+$ and $g_B^- \mathbin{\widetilde{\supseteq}} f_A^-$, $h_C^- \mathbin{\widetilde{\supseteq}} f_A^-$ and $\langle (f_A^+, f_A^-); A\rangle$ is strongly prime. □

Theorem 3.21. *If the set of DFS bi-ideals of an ordered semigroup S over U is totally ordered under inclusion, then S is both regular and intra-regular if and only if each double-framed soft bi-ideal of S over U is prime.*

Proof. Suppose that S is both regular and intra-regular. Let $\langle (f_A^+, f_A^-); A\rangle$ be a DFS bi-ideal of S over U and $\langle (g_B^+, g_B^-); B\rangle$ and $\langle (h_C^+, h_C^-); C\rangle$ be DFS bi-ideals of S over U such that $g_B^+ \mathbin{\widetilde{\circ}} h_C^+ \mathbin{\widetilde{\subseteq}} f_A^+$ and $g_B^- \mathbin{\widetilde{*}} h_C^- \mathbin{\widetilde{\supseteq}} f_A^-$. Since the set of DFS bi-ideals of S is totally ordered, therefore, either $g_B^+ \mathbin{\widetilde{\subseteq}} h_C^+$ or $h_C^+ \mathbin{\widetilde{\subseteq}} g_B^+$ and $g_B^- \mathbin{\widetilde{\supseteq}} h_C^-$, $h_C^- \mathbin{\widetilde{\supseteq}} g_B^-$. Suppose that $g_B^+ \mathbin{\widetilde{\subseteq}} h_C^+$ and $g_B^- \mathbin{\widetilde{\supseteq}} h_C^-$, then $g_B^+ \mathbin{\widetilde{\circ}} g_B^+ \mathbin{\widetilde{\subseteq}} g_B^+ \mathbin{\widetilde{\circ}} h_C^+ \mathbin{\widetilde{\subseteq}} f_A^+$ and $g_B^- \mathbin{\widetilde{*}} g_B^- \mathbin{\widetilde{\supseteq}} g_B^- \mathbin{\widetilde{*}} h_C^- \mathbin{\widetilde{\supseteq}} f_A^-$. By Theorem 3.17, $\langle (f_A^+, f_A^-); A\rangle$ is semiprime. So, $g_B^+ \mathbin{\widetilde{\subseteq}} f_A^+$ and $g_B^- \mathbin{\widetilde{\supseteq}} f_A^-$. Hence, $\langle (f_A^+, f_A^-); A\rangle$ is prime DFS bi-ideal of S over U.

Conversely, assume that every DFS bi-ideal of S over U is prime. Since every prime DFS bi-ideal of S over U is semiprime, so by Theorem 3.17, S is both regular and intra-regular. □

Theorem 3.22. *For an ordered semigroup S, the following assertions are equivalent:*

(*i*) *The set of DFS bi-ideals of S over U is totally ordered under inclu-sion.*

(*ii*) *Each DFS bi-ideal of S over U is strongly irreducible.*

(*iii*) *Each DFS bi-ideal of S over U is irreducible.*

Proof. $(i) \Rightarrow (ii)$. Let $\langle(f_A^+, f_A^-); A\rangle$ be a DFS bi-ideal of S over U and $\langle(g_B^+, g_B^-); B\rangle$ and $\langle(h_C^+, h_C^-); C\rangle$ beDFSbi-idealsof S over U such that $g_B^+ \,\tilde{\cap}\, h_C^+ \,\tilde{\subseteq}\, f_A^+$ and $g_B^- \,\tilde{\cup}\, h_C^- \,\tilde{\supseteq}\, f_A^-$. Since the set of DFS bi-ideals of S over U is totally ordered $g_B^+ \,\tilde{\subseteq}\, h_C^+$ or $h_C^+ \,\tilde{\subseteq}\, g_B^+$ and $g_B^- \,\tilde{\supseteq}\, h_C^-$, $h_C^- \,\tilde{\supseteq}\, g_B^-$. Thus, either $g_B^+ \,\tilde{\cap}\, h_C^+ \cong g_B^+$ or $g_B^+ \,\tilde{\cap}\, h_C^+ \cong h_C^+$ and $g_B^- \,\tilde{\cup}\, h_C^- \cong g_B^-$, $g_B^- \,\tilde{\cup}\, h_C^- \cong h_C^-$. Thus, $g_B^+ \,\tilde{\cap}\, h_C^+ \,\tilde{\subseteq}\, f_A^+$ and $g_B^- \,\tilde{\cup}\, h_C^- \,\tilde{\supseteq}\, f_A^-$ imply that either $g_B^+ \,\tilde{\subseteq}\, f_A^+$ or $h_C^+ \,\tilde{\subseteq}\, f_A^+$ and $g_B^- \,\tilde{\supseteq}\, f_A^-$, $h_C^- \,\tilde{\supseteq}\, f_A^-$. Hence $\langle(f_A^+, f_A^-); A\rangle$ is strongly irreducible.

$(ii) \Rightarrow (iii)$. Let $\langle(f_A^+, f_A^-); A\rangle$ be a DFS bi-ideal of S over U and $\langle(g_B^+, g_B^-); B\rangle$ and $\langle(h_C^+, h_C^-); C\rangle$ be DFS bi-ideals of S over U such that $g_B^+ \,\tilde{\cap}\, h_C^+ \cong f_A^+$ and $g_B^- \,\tilde{\cup}\, h_C^- \cong f_A^-$. Then, $f_A^+ \,\tilde{\subseteq}\, g_B^+$ or $f_A^+ \,\tilde{\subseteq}\, h_C^+$ and $f_A^- \,\tilde{\supseteq}\, g_B^-$, $f_A^- \,\tilde{\supseteq}\, h_C^-$. By hypothesis, either $g_B^+ \,\tilde{\subseteq}\, f_A^+$ or $h_C^+ \,\tilde{\subseteq}\, f_A^+$, and $g_B^- \,\tilde{\supseteq}\, f_A^-$, $h_C^- \,\tilde{\supseteq}\, f_A^-$. Thus, $g_B^+ \cong f_A^+$ or $h_C^+ \cong f_A^+$, and $g_B^- \cong f_A^-$, $h_C^- \cong f_A^-$. That is, $\langle(f_A^+, f_A^-); A\rangle$ is irreducible DFS bi-ideal of S over U.

$(iii) \Rightarrow (i)$. Suppose that $\langle(g_B^+, g_B^-); B\rangle$ and $\langle(h_C^+, h_C^-); C\rangle$ be DFS bi-ideals of S over U. Then $g_B^+ \tilde{\cap} h_C^+$ and $g_B^- \tilde{\cup} h_C^-$ are soft bi-ideals of S over U. Also, $g_B^+ \tilde{\cap} h_C^+ \cong g_B^+ \tilde{\cap} h_C^+$ and $g_B^- \tilde{\cup} h_C^- \cong g_B^- \tilde{\cup} h_C^-$. So by hypothesis, either $g_B^+ \cong g_B^+ \,\tilde{\cap}\, h_C^+$ or $h_C^+ \cong g_B^+ \,\tilde{\cap}\, h_C^+$ and $g_B^- \cong g_B^- \,\tilde{\cup}\, h_C^-$, $h_C^- \cong g_B^- \,\tilde{\cup}\, h_C^-$. That is, either $g_B^+ \,\tilde{\subseteq}\, h_C^+$ or $h_C^+ \,\tilde{\subseteq}\, g_B^+$ and $g_B^- \,\tilde{\supseteq}\, h_C^-$, $h_C^- \,\tilde{\supseteq}\, g_B^-$. Hence the set of DFS bi-ideals of S over U is totally ordered. □

4 Conclusion

We have considered the following items.

1. To introduce the notions of prime (resp., strongly prime, irreducible, and strongly irreducible) DFS bi-ideals in ordered semigroups and to give several examples of these notions. To give the basic properties of these notions and to characterize ordered semigroups by means of these types of DFS bi-ideals.

2. To chracterize regular and intra-regular ordered semigroups by means of prime and semiprime DFS bi-ideals.

3. To study the relationship between prime and strongly prime DFS bi-ideals and to characterize the class of those ordered semigroups for which the notions of prime and semiprime DFS bi-ideals are coincide.

4. To introduce the concepts of irreducible and strongly irreducible DFS bi-ideals and to investigate the basic properties of these notions.

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