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Nilpotent soft polygroups

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Abstract

In this paper, we introduce nilpotent soft (sub)polygroups. In addition, nilpotency of intersection, extended intersection, restricted union of two nilpotent soft polygroups are studied. Espesialy, a necessary and suficient condition between nilpotency of a polygroup and soft polygroups is obtained. Finally, we define two new soft polygroups $(S_\alpha)_{A\cup\{c\}}$ and $(Q_\alpha)_A$ derived from a soft polygroup α_A and study on nilpotency of these structures. Also, we extend a soft homomorphism of groups to polygroups. This helps us to extend some properties of groups to polygroups.

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1 Introduction

Some problems in engineering, medical science and social science are uncertain. One way for dealing with them is soft set theory. It was proposed by Molodtsov [20]. In addition, it has applications in Rieman integration, probability theory, game theory and etc (see [20, 21]). After that time it became an interesting topic for many authors and so they work on soft set theory. Maji et al [17], introduced several operations on soft sets. Ali et al. [15], redefined compliment of a soft set. Soft sets were used in lattice theory by Qin et al. [23]. Also, soft set theory was applied on research in BCI/BCK-algebras [10]. The studying of soft sets in groups began with the work of Aktas and Cagman in [2], where the notion of soft groups were investigated and then Acar et al. in [1], extended the notion to rings. Recently, Wang et al. in [24], introduced soft polygroups.

In group theory, nilpotent group is an interesting subject and has been studied by many scholars. Abelian groups are an example of nilpotent groups. Hassanzadeh [13] introduced the

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concept of nilpotency for pair of groups. Also, Ozkan and et al. in [22], investigated some applications of Fibonancci sequences in a finite nilpotent group.
An important branch in algebra is hyperstructures. It has applications in geometry, automata, probabilities, and so on. In 1934, Marty [19] introduced the concept of polygroups as a special hypergroup. In addition, polygroups have been discussed by Corsini [6], Borzooei [5], Davvaz [8] and so on. Some results of group theory are translated on polygroups such as nilpotent polygroup that has been studied in [5, 8]).

Now, in this paper we study on nilpotent soft polygroup and investigate some properties of it. Espesially, we obtain a neccessary and sufficient condition between soft nilpotent polygroups and nilpotent polygroups. Finally, we define two new soft sets $(S_F, A \cup \{c\})$ and (Q_F, A) derived from a soft polygroup (F, A). Then, we investigate some properties of them.

2 Preliminary

We begin our discusion with some fundamental definitions and results.

A hyperoperation $\circ$ is a mapping from $H \times H$ into the family of non-empty subsets of H. A *hypergroupoid* $(H, \circ)$ is a non-empty set H with a hyperoperation $\circ$. If A and B are non-empty subsets of H, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. Also, we use $x \circ A$ instead of $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. The structure $(H, \circ)$ is called a *hypergroup* if $a \circ (b \circ c) = (a \circ b) \circ c$ and $a \circ H = H \circ a = H$ for any $a, b, c \in H$.

Definition 2.1. [8] *Let $\cdot$ be a hyperoperation, $e \in P$ and $^{-1}$ be an unitary operation on P. Then $(P, \cdot, e, ^{-1})$, is called a polygroup if for any $x, y, z \in P$ the following conditions hold:*
(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(ii) $e \cdot x = x \cdot e = x$,
(iii) $x \in y \cdot z \Leftrightarrow y \in x \cdot z^{-1} \Leftrightarrow z \in y^{-1} \cdot x$.

Let $(P_1, \cdot, e_1, ^{-1})$ and $(P_2, *, e_2, ^{-1})$ be two polygroups. Then $(P_1 \times P_2, \circ)$, where $\circ$ is defined as follows, is a polygroup (see [8]).

$$(x_1, y_1) \circ (x_2, y_2) = \{(x, y) \mid x \in x_1 \cdot x_2, \text{ and } y \in y_1 * y_2\}.$$

Note. From now on, let $(H, \cdot)$ be a hypergroup and $(P, \cdot, e, ^{-1})$ be a polygroup. For $x, y \in P$ we use xy instead of $x \cdot y$.

Definition 2.2. [8] *Let K be a non-empty subset of P. Then for any $a, b \in K$, K is called a subpolygroup of P and we denote by $K \preceq P$ if $ab \subseteq K$ and $a^{-1} \in K$. Also, a subpolygroup N of P is called normal and we denote by $N \trianglelefteq P$ if for any $a \in P$, $a^{-1}Na \subseteq N$.*

For $K \preceq P$ and $x \in P$, let xK (Kx) be the left (right) coset of K and P/K be the set of all left (right) cosets of K in P. We recall that for $N \trianglelefteq P$, $x, y \in P$ and every $z \in xy$ we have $Nx = xN$ and $Nxy = Nz$. Also, $(P/N, \odot, N, ^{-1})$ is a polygroup, where

$$(Nx) \odot (Ny) = \{Nz \mid z \in xy\} \text{ and } (Nx)^{-1} = Nx^{-1}.$$

A polygroup is called commutative if for any $x, y \in P$, $xy = yx$. For two polygroups $(P, \bullet)$ and $(P, *)$, a map $f : (P, \bullet) \to (P, *)$ is called a *homomorphism* if for any $a, b \in P$, $f(a \bullet b) \subseteq f(a) * f(b)$.

Also, f is a *good homomorphism* if the equality holds. For an equvalence relation $\rho \subseteq P \times P$ and two non-empty subsets X, Y of P we have

$$X\overline{\overline{\rho}}Y \Leftrightarrow x\rho y, \quad \forall x \in X, \forall y \in Y.$$

The relation ρ is called *strongly regular* if for any $x, y, a \in P$ we have

$$x\overline{\overline{\rho}}y \Leftrightarrow a \cdot x\overline{\overline{\rho}}a \cdot y \text{ and } x \cdot a\overline{\overline{\rho}}y \cdot a.$$

We use SR(H) for the set of all strongly regular relations on H.

In [16], Koskas defined the relation $\beta = \bigcup_{n\geq 1} \beta_n$, where β_1 is the diagonal relation and

$$a\beta_n b \Leftrightarrow \exists(x_1, ..., x_n) \in H^n, \{a, b\} \subseteq \prod_{i=1}^{n} x_i.$$

In addition $\beta^* \in SR(H)$, where β^* is the transitive closure of β. In [11], Freni showed that if H is a hypergroup, then $\beta = \beta^*$. The kernel of the canonical map $\pi : H \longrightarrow \frac{H}{\beta^*}$, denote by ω_P or ω, is called the *core* of P.

Theorem 2.3. [8] *Let A be a non-empty subset of P. The intersection of any subpolygroups of P containing A, denoted by $\langle A\rangle$ is equall to $\cup\{x_1^{\epsilon_1}...x_k^{\epsilon_k} | x_i \in A, k \in \mathbb{N}, \epsilon_i \in \{1, -1\}\}$.*

Definition 2.4. [8] *The lower central series of P is the sequence $\cdots \subseteq \gamma_1(P) \subseteq \gamma_0(P)$, where $\gamma_0(P) = P$ and for $k > 0$,*

$$\gamma_{k+1}(P) =< \{h \in P | xy \cap hyx \neq \emptyset \;\; such \;\; that \;\; x \in \gamma_k(P), y \in P\} > .$$

Also, P is called a nilpotent polygroup(we write NP) if for some $n \in \mathbb{N}$, $\gamma_n(P) \subseteq \omega$. The smallest such n is called class of P.

In [4] it is proved that for any $x, y \in P$ we have

$$\{h \in P | xy \cap hyx \neq \emptyset\} = \{h \in P | h \in [x, y]\},$$

where $[x, y] = \{t | t \in xyx^{-1}y^{-1}\}$ is the commutator of x, y.

Theorem 2.5. [8] *Let P be an NP, $N \trianglelefteq P$ and $K \preceq P$. Then K and P/N are NP.*

Definition 2.6. [17] *A pair $(\alpha, A) = \alpha_A$ is called a soft set over U, where U refers to an initial universe set, E is a set of parameters, $A \subseteq E$ and α is a map from A to the power set $P(U)$.*

We use $S(U)$ to show the set of all soft sets over U.

Definition 2.7. [10] *For $\alpha_A, \gamma_B \in S(U)$ we have the following statments:*

(i) $\alpha_A \subseteq \gamma_B$, *if* $A \subseteq B$ *and for any* $a \in A$, $\alpha(a) \subseteq \gamma(a)$.

(ii) $\alpha_A = \gamma_B$, *if* $\alpha_A \subseteq \gamma_B$ *and* $\gamma_B \subseteq \alpha_A$.

(iii) If for any $a \in A$, $\alpha(a) = \emptyset$, *then* α_A *is said a null soft set.*

Theorem 2.8. [17] *Let $\alpha_A \in S(U)$ and $Supp(\alpha_A) = \{x \in A \mid \alpha(x) \neq \emptyset\}$. Then α_A is non-null if $Supp(\alpha_A) \neq \emptyset$.*

Definition 2.9. [17] *Let $\alpha_A, \gamma_B \in S(U)$. Then for any $x \in A \cap B$ and $(x, y) \in A \times B$ we have*

(*i*) *the soft intersection* $(\alpha_A \tilde{\cap} \gamma_B, (A \cap B))$ *is defined by* $(\alpha_A \tilde{\cap} \gamma_B)(x) = \alpha(x) \cap \gamma(x)$.

(*ii*) *the soft* $\tilde{\wedge}$*-product* $(\alpha_A \tilde{\wedge} \gamma_B, (A \times B))$ *is defined by* $(\alpha_A \tilde{\wedge} \gamma_B)(x, y) = \alpha(x) \cap \gamma(y)$.

(*iii*) *the soft* $\tilde{\times}$*-product* $(\alpha_A \tilde{\times} \gamma_B, A \times B)$ *is defined by* $(\alpha_A \tilde{\times} \gamma_B)(x, y) = \alpha(x) \times \gamma(y)$.

Definition 2.10. [24] *Let α_A be a non-null soft set over P. Then α_A is called a soft polygroup over P if $\alpha(x) \preceq P$ for any $x \in Supp(\alpha_A)$*

Note. From now on, assume A is a non-empty subset of P and $\alpha_A \in SP(P)$, where SP(P) is the set of all soft polygroups over P. In addition, we use $K \preceq^n P$ when K is a nilpotent subpolygroup of P.

3 Nilpotent soft polygroups

In this section first we define a nilpotent soft polyroup (we write NSP). Then, some examples are added to clarify the notion. Basically, for two soft polygroups α_A and γ_B we study the nilpotency of derived soft sets such as $\alpha_A \cap_g \gamma_B$ and $\alpha_A \cap_R \gamma_B$ and so on. Finally, a relation between a nilpotent polygroup and its soft polygroups is obtained.

Definition 3.1. *The soft polygroup α_A is called a nilpotent soft polygroup over P, we write NSP, if there is $n \in \mathbb{N}$ such that for any $a \in \mathrm{Supp}(\alpha_A)$, $\alpha(a) \preceq^n P$.*

We use NSP(P) for the set of all nilpotent soft polygroups over P.

Example 3.2. *Let $P = \{a, b, c, e\}$. Then $(P, \diamond)$ is an NP (see [8]).*

$\diamond$	a	b	c	e
a	$\{e, a\}$	c	$\{b, c\}$	a
b	c	e	a	b
c	$\{b, c\}$	a	$\{e, a\}$	c
e	a	b	c	e

Assume $A = P$ and define the soft set $\alpha_A \in S(P)$ by $\alpha(a) = \alpha(e) = P$ and $\alpha(b) = \alpha(e) = \{a, e\}$. Since $\alpha(a), \alpha(e), \alpha(b), \alpha(c) \preceq^n P$ we conclude that $\alpha \in NSP(P)$.

In what follows we have a soft polygroup that is not an NSP.

Example 3.3. *Assume $P = \{a, b, c, d, f, g, e\}$ is a polygroup with the hyperoperation $\bullet$ such that*

$\bullet$	e	a	b	c	d	f	g
a	a	e	b	c	d	f	g
b	b	b	$\{e, a\}$	g	f	d	c
c	c	c	f	$\{e, a\}$	g	b	d
d	d	d	g	f	$\{e, a\}$	c	b
f	f	f	c	d	b	g	$\{e, a\}$
g	g	g	d	b	c	$\{e, a\}$	f
e	e	a	b	c	d	f	g

Assume $A = P$ and define the soft set $\alpha_A \in S(P)$ by $\alpha(e) = \alpha(a) = \alpha(b) = \{e, a, b\}$ and $\alpha(c) = \alpha(d) = \alpha(f) = \alpha(g) = P$. Then P is not an NP. Because $\omega_P = \{e, a\}$ and $\gamma_n(P) = \{e, a, f, g\}$ and so $\gamma_n(P) \nsubseteq \omega_P$. Therefore, $\alpha_A \notin NSP(P)$.

Theorem 3.4. *Assume $\alpha_A \in NSP(P)$ and $B \subseteq A$. If $(\alpha\mid_B)_B$ is non-null, then it is an NSP.*

Proof. For $b \in B$ since $B \subseteq A$, we have $\alpha\mid_B (b) = \alpha(b)$ and so by hypotheses $(\alpha\mid_B)_B \in NSP(P)$. □

By the following example, we define a subset $B \subseteq A$ such that α_A is not an NSP but $\alpha\mid_B$ is an NSP.

Example 3.5. *Assume A and P are as Example 3.2, and $B = \{e, a\}$. Define the soft set $\alpha_A \in S(P)$ by*

$$\alpha(e) = \alpha(a) = \{e, a\},\ \alpha(b) = \alpha(c) = \{b, c\}.$$

$\{b, c\} \npreceq P$ implies that $\alpha_A \notin SP(P)$. But $\{e, a\} \preceq^n P$. It implies that $\alpha\mid_B \in NSP(P)$.

Example 3.6. *Consider P, A and α_A are as Example 3.3 and $B = \{e, a, b\}$. Since $\alpha(c) = P$ and P is not nilpotent we have $\alpha_A \notin NSP(P)$ but every proper polygroup of order less than 7 is an NP (see* [8]*), thus $(\alpha\mid_B)_B \in NSP(P)$.*

Definition 3.7. [24] *For $\alpha_A,\ \gamma_B \in SP(U)$ and $x \in A \cup B$,*
(i) the soft extended intersection $\alpha_A \cap_g \gamma_B$ is defined to be the soft set $(D, A \cup B)$, where

$$D(x) = \begin{cases} \alpha(x) & \text{if } x \in A - B, \\ \gamma(x) & \text{if } x \in B - A, \\ \alpha(x) \cap \gamma(x) & \text{if } x \in A \cap B. \end{cases}$$

Replacing $\alpha(x) \cap \gamma(x)$ with $\alpha(x) \cup \gamma(x)$ in $D(x)$ we have the soft set $\alpha_A \cup^{\sim} \gamma_B = (D, A \cup B)$.
(ii) the restricted intersection $\alpha_A \cap_R \gamma_B$ is the soft set (E, C) where $A \cap B \neq \phi$ and $C = A \cap B$ and for any $x \in C$, $E(x) = \alpha(x) \cap \gamma(x)$.

Theorem 3.8. *Let $\alpha_A, \gamma_B \in NSP(P)$. Then*

(*i*) *$\alpha_A \bigcap_g \gamma_B \in NSP(P)$ if it is non-null.*

(*ii*) *$\alpha_A \bigcap_R \gamma_B \in NSP(P)$ if it is non-null and $A \cap B \neq \emptyset$.*

(*iii*) *$\alpha_A \tilde{\cup} \gamma_B \in NSP(P)$ if $A \cap B = \emptyset$.*

(*iv*) *$\alpha_A \tilde{\wedge} \gamma_B \in NSP(P)$.*

Proof.

(*i*) Consider $\alpha_A \bigcap_g \gamma_B = (D, C)$ and $x \in \text{Supp}(D, C)$ and $x \in A - B$. By Definition 3.7, since $\alpha_A \in NSP(P)$ we obtain $D(x) = \alpha(x) \prec^n P$. For the case $x \in B - A$ by $\gamma_B \in NSP(P)$ we have $D(x) = \gamma(x) \prec^n P$. Finally, for $x \in A \cap B$ by Theorem 2.5, we have $D(x) = \alpha(x) \cap \gamma(x) \prec^n P$. Hence $(D, C) \in NSP(P)$.

(*ii*) By Definition 3.7, and the same manipulation of part (*i*), we have $\alpha_A \cap_R \gamma_B \in NSP(P)$.

(*iii*) By Definition 3.7 and $A \cap B = \emptyset$, we have

$$\text{Supp}(D, C) = \text{Supp}(\alpha_A) \cup \text{Supp}(\gamma_B) \neq \emptyset.$$

Then (D, C) is non-null. For $x \in A - B$ we have $D(x) = \alpha(x)$ and $\alpha_A \in NSP(P)$ implies that $D(x) \prec^n P$. Also, for the case $x \in B - A$ we have $D(x) = \gamma(x) \prec^n P$. Therefore, $(D, C) \in NSP(P)$.

(*iv*) Put $(H, A \times B)$ be the soft set $\alpha_A \tilde{\wedge} \gamma_B$ .By Definition 2.9 and Theorem 2.8, since α_A and γ_B are non-null we have

$$\mathrm{Supp}(H, A \times B) = \mathrm{Supp}(\alpha_A) \times \mathrm{Supp}(\gamma_B) \neq \emptyset.$$

Also, since $\alpha_A, \gamma_B \in NSP(P)$ we conclude that for any $(x, y) \in A \times B$, $\alpha(x) \cap \gamma(y) \preceq^n P$. Therefore, $(H, A \times B) \in NSP(P)$.

□

Assume I is an index set and $(\alpha_i)_{A_i i \in I} \in NSP(P)$. Then by extending Theorem 3.8, we have the following corollary.

Corollary 3.9. *The soft set* $(\bigcap_g)_{i \in I}(\alpha_i)_{A_i} \in NSP(P)$ *if it is non-null. Also, if* $\bigcap_{i \in I} A_i \neq \emptyset$*, then* $(\bigcap_R)_{i \in I}(\alpha_i)_{A_i} \in NSP(P)$*, whenever it is non-null.*

Corollary 3.10. *Let* $(\alpha_i)_{A_i i \in I} \in NSP(P)$ *such that for any* $i, j \in I$, $A_i \cap A_j = \emptyset$. *Then* $\tilde{\bigcup}_{i \in B}(\alpha_i)_{A_i} \in NSP(P)$. *Also,* $\tilde{\bigwedge}_{i \in I}(\alpha_i)_{A_i} \in NSP(P)$.

Proof. The proof is clear by Theorem 3.8. □

In what follows we show that $A \cap B = \emptyset$ is a vital condition in Theorem 3.8(iii).

Example 3.11. *Let* P *and* A *be as Example 3.2, and* $B = \{a\}$. *Define two soft sets* α_A, $\gamma_B \in SP(P)$ *by* $\alpha(e) = P$, $\alpha(a) = \alpha(b) = \alpha(c) = \{e, a\}$ *and* $\gamma(a) = \{e, b\}$*, respectively. Then* $\gamma_B \in NSP(P)$. *But* $D(a) = \alpha(a) \cup \gamma(a) = \{b, a, e\} \npreceq P$ *and so* $(D, C) \notin NSP(P)$.

Theorem 3.12. [8] *Let* $f : P_1 \to P_2$ *be a one to one and good homomorphism of polygroups* P_1 *and* P_2. *If* $A \preceq^n P_1$*, then* $\alpha(a) \preceq^n P_2$.

Theorem 3.13. *Let* $f : P_1 \to P_2$ *be a good homomorphism,* $\alpha_A \in SP(P_1)$. *Then the soft set* $f\alpha_A \in SP(P_2)$*, where* $f\alpha_A(x) = f(\alpha(x))$ *for any* $x \in A$.

Proof. Let $x \in A$ and $y_1, y_2 \in f\alpha_A(x)$. Then there exist $x_1, x_2 \in \alpha_A(x)$ such that $y_1 = f(x_1), y_2 = f(x_2)$. Since f is a good homomorphism we get that $y_1 y_2 \subseteq f\alpha_A(x)$ and ${y_1}^{-1} \in f\alpha_A(x)$. This complete the proof. □

Theorem 3.14. *Assume* $f : P_1 \to P_2$ *is a one to one and good homomorphism. If* $\alpha_A \in NSP(P_1)$*, then* $f\alpha_A \in NSP(P_2)$.

Proof. By Theorem 3.13, $f\alpha_A \in SP(P_2)$ and

$$\begin{aligned} \mathrm{Supp}(f\alpha_A) &= \{x \in A \mid (f\alpha_A)(x) \neq \emptyset\} \\ &= \{x \mid f(\alpha(x)) \neq \emptyset\} \\ &= \{x \mid \alpha(x) \neq \emptyset\} = \mathrm{Supp}(\alpha_A). \end{aligned}$$

Since $\alpha_A \in NSP(P_1)$ we conclude that for any $x \in \mathrm{Supp}(\alpha_A)$, $\alpha(x) \preceq^n P_1$. It follows by Theorem 3.12 and $(f\alpha_A)(x) = f(\alpha(x))$ that for any $x \in \mathrm{Supp}(f\alpha_A)$, $(f\alpha_A)(x) \preceq^n P_2$. Therefore, $f\alpha_A \in NSP(P_2)$. □

Definition 3.15. *Assume* $\alpha_A, \gamma_B \in SP(P)$. *Then* γ_B *is called a nilpotent soft subpolygroup of* α_A*, denote by* $\gamma_B \blacktriangleleft^{ns} \alpha_A$*, if* $B \subseteq A$ *and for any* $x \in \mathrm{Supp}(\gamma_B)$, $\gamma(x) \preceq^n \alpha(x)$ *for some* $n \in \mathbb{N}$.

Example 3.16. *Assume A, P are as Example 3.2. Define $\alpha_A \in SP(P)$ by $\alpha(e) = \alpha(b) = P$ and $\alpha(c) = \alpha(a) = \{b, e\}$. Let $B = \{a, b, e\}$ and define $\gamma_B \in SP(P)$ by $\gamma(e) = \{b, e\} = \gamma(b)$ and $\gamma(a) = \{e\}$. Since $B \subseteq A$ and*

$$\gamma(e) = \gamma(b) = \{b, e\} \preceq^n P = \alpha(e) = \alpha(b), \quad \gamma(a) = \{e\} \preceq^n \alpha(a) = \{e, b\},$$

we conclude that $\gamma_B \blacktriangleleft^{ns} \alpha_A$.

Theorem 3.17. *Assume $\alpha_A, \gamma_B \in NSP(P)$. If $B \subseteq A$ and for any $x \in \mathrm{Supp}(\gamma_B)$, $\gamma(x) \subseteq \alpha(x)$, then $\gamma_B \blacktriangleleft^{ns} \alpha_A$.*

Proof. It is straight forward. □

Theorem 3.18. *Assume $\alpha_A \in NSP(P)$ and $(\gamma_i)_{B_i i\in I} \blacktriangleleft^{ns} \alpha_A$. Then*

(i) $\bigcap_{i\in I}(\gamma_i)_{B_i} \blacktriangleleft^{ns} \alpha_A$.

(ii) If $\bigcap_{i\in I} B_i \neq \emptyset$, then $(\bigcap_R)_{i\in I}(\gamma_i)_{B_i} \blacktriangleleft^{ns} \alpha_A$ when it is non-null.

(iii) If for any $i, j \in I$, $B_i \cap B_j = \emptyset$, then $\tilde{\bigcup}_{i\in I}(\gamma_i)_{B_i} \blacktriangleleft^{ns} \alpha_A$.

(iv) $\tilde{\bigwedge}_{i\in I}(\gamma_i)_{B_i} \blacktriangleleft^{ns} \tilde{\bigwedge}_{i\in I}\alpha_A$.

Proof. By Theorems 3.8 and 3.17, we get (ii). Other parts are proved similarly. □

Definition 3.19. *The soft set α_A is called a whole soft polygroup over P if for any $x \in A$, $\alpha(x) = P$.*

Theorem 3.20. *P is an NP if and only if every soft polygroup of P is nilpotent.*

Proof. ($\Rightarrow$) By Theorem 2.5, we get the result.
($\Leftarrow$) Consider every soft polygroup of P is nilpotent. Put α_A be the whole soft polygroup. Then for any $x \in \mathrm{Supp}(\alpha_A)$, $P = \alpha(x)$ and so P is an NP. □

4 Soft homomorphism

In this section first we clarify the notion of soft homomorphism by an example. Also, we define two new soft sets $(S_\alpha)_{A\cup\{c\}}$ and $(Q_\alpha)_A$ derived from a soft polygroup α_A. Then, we investigate some properties of them.

Definition 4.1. [24] *Suppose $\alpha_A \in SP(P_1)$ and $\gamma_B \in SP(P_2)$. Then*
(i) (f, g) is called a soft homomorphism between α_A and γ_B if $f : P_1 \to P_2$ is a good epimorphism, $g : A \to B$ is a surjective map and for any $x \in A$, $f(\alpha(x)) = \gamma(g(x))$.
(ii) we write $\alpha_A \sim \gamma_B$ if there is a soft homomorphism.
(iii) we write $\alpha_A \simeq \gamma_B$ if $\alpha_A \sim \gamma_B$ such that f is a good isomorphism and g is a bijective map.

Theorem 4.2. [8] *Let $(G, .)$ be a group. Then $(P_G, \circ, e, ^{-1})$ is a polygroup, where $P_G = G \cup \{a\}$, $a \notin G$ and $\circ$ is defined as follows:*
(1) $a \circ a = e$,
(2) $e \circ x = x \circ e = x$, $\forall x \in P_G$,
(3) $a \circ x = x \circ a = x$, $\forall x \in P_G - \{e, a\}$,
(4) $x \circ y = x.y$, $\forall (x, y) \in G^2; y \neq x^{-1}$,
(5) $x \circ x^{-1} = x^{-1} \circ x = \{e, a\}$, $\forall x \in P_G - \{e, a\}$.
In addition, P_G is an NP if and only if G is a nilpotent group.

Example 4.3. *Assume G is the quaternion group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$. Since G is nilpotent by Theorem 4.2, we conclude that $(P_G, \circ, e,^{-1})$ is an NP.*

Example 4.4. *Consider $P = \mathbb{Z} \cup \{a\}$, $P' = (\{0\} \otimes \mathbb{Z}) \cup \{(0, r)\}$ be two polygroups as Definition 4.2. Take $A = 2\mathbb{Z} \cup \{a\}$, $B = (\{0\} \otimes 6\mathbb{Z}) \cup \{(0, r)\}$ and define $\delta_A \in SP(P)$ and $\eta_B \in SP(P')$ by*

$$\delta(x) = \begin{cases} x * 18\mathbb{Z} & x \in 2\mathbb{Z} \\ \{0, a\} & x = a, \end{cases} \quad \text{and} \quad \eta(0, y) = \begin{cases} \{0\} \otimes 6y\mathbb{Z} & y \in 6\mathbb{Z} \\ \{(0, r), (0, 0)\} & y = r \end{cases}$$

Then the functions

$$f : P \to P', \qquad g : A \to B$$

$$f(x) = \begin{cases} (0, x) & x \in \mathbb{Z} \\ (0, r) & x = a \end{cases}, \quad g(y) = \begin{cases} (0, 3y) & y \in 2\mathbb{Z} \\ (0, r) & y = a \end{cases}$$

are isomorphism and bijective map, respectively. Also, for any $x \in A$, $f(\delta(x)) = \eta(g(x))$. Consequently, $\delta_A \simeq \eta_B$.

Definition 4.5. *Assume α_A is a soft group over the group G with identity element e and $a \notin G$. We define the soft set $(S_\alpha)_{A\cup\{a\}} \in S(P_G)$ by*

$$S_\alpha(x) = \begin{cases} \alpha(x) & x \in A \\ \{e, a\} & x = a. \end{cases}$$

In what follows we extend a soft group to an NSP.

Theorem 4.6. *Consider α_A is a soft group over a nilpotent group G. Then $(S_\alpha)_{A\cup\{a\}} \in NSP(P_G)$.*

Proof. Since $\alpha(x), \{e, a\} \preceq P_G$ we conclude that $(S_\alpha)_{A\cup\{a\}} \in SP(P_G)$. Also, by the nilpotency of G and Theorems 4.2 and 2.5, we get $\alpha(x), \{a, e\} \preceq^n P_G$. Consequently, $(S_\alpha)_{A\cup\{a\}} \in NSP(P_G)$. □

Theorem 4.7. *Consider α_A and γ_B are two soft polygroups over P_1 and P_2, respectively. If $\alpha_A \simeq \gamma_B$ and $\alpha_A \in NSP(P_1)$, then $\gamma_B \in NSP(P_2)$.*

Proof. Since $\alpha_A \in NSP(P_1)$ we have for any $x \in \text{Supp}(\alpha_A)$, $\alpha(x) \preceq^n P_1$ and so by Theorem 3.12, $f(\alpha(x)) \preceq^n P_2$. On the other hand, for any $y \in \text{Supp}(\gamma_B)$, there exists $x \in \text{Supp}(\alpha_A)$ with $\gamma(x) = y$. Thus, $\alpha_A \simeq \gamma_B$ implies that $\gamma(y) = \gamma(g(x)) = f(\alpha(x)) \preceq^n P_2$. Therefore, $\gamma_B \in NSP(P_2)$. □

Definition 4.8. *Let $\alpha_A \in SP(P)$ and $N \trianglelefteq P$ such that for any $x \in A$, $N \subseteq \alpha(x)$. Then the soft set $Q_\alpha : A \to P(\frac{P}{N})$ defined by $Q_\alpha(x) = \frac{\alpha(x)}{N}$ is called the quotient soft polygroup of α_A.*

Example 4.9. *Assume P and A are an Example 3.2, $N = \{e, a\}$ and α_A is the whole soft polygroup of P. Then $Q_\alpha(x) = \frac{P}{N}$ is the whole soft polygroup of α_A.*

Theorem 4.10. *Assume $\alpha_A \in NSP(P)$. Then $(Q_\alpha)_A \in NSP(\frac{P}{N})$.*

Proof. By $\alpha_A \in NSP(P)$, for any $x \in \text{Supp}(\alpha_A)$, we have $\alpha(x) \preceq^n P$ of class say n. Since

$$\emptyset \neq \text{supp}(Q_\alpha) = \{x \in A \mid Q_\alpha(x) \neq \emptyset\} = \{x \in A \mid \frac{\alpha(x)}{N} \neq \emptyset\},$$

we conclude that $F(x) \neq \emptyset$, i.e $x \in \text{Supp}(\alpha_A)$. Then by Definition 4.8 and Theorem 3.4, for any $x \in Supp(Q_\alpha)$, $Q_\alpha(x) = \frac{\alpha(x)}{N} \preceq^n \frac{P}{N}$ and so $(Q_\alpha)_A \in NSP(\frac{P}{N})$. □

Theorem 4.11. [8] *Consider $\alpha_A \in SP(P_1)$, $\gamma_B \in SP(P_2)$ and $\alpha_A \sim \gamma_B$ with a soft homomorphism (f, g). If $N \trianglelefteq P_1$, $N \subseteq \alpha(x)$ for any $x \in \text{Supp}(\alpha_A)$ and g is a bijective map, then $(Q_\alpha)_A \simeq \gamma_B$, where $Q_\alpha(x) = \dfrac{\alpha(x)}{N}$.*

Corollary 4.12. *Assume α_A and γ_B, N and $(Q_\alpha)_A$ are as Theorem 4.11. If $\gamma_B \in NSP(P_2)$, then $(Q_\alpha)_A \in NSP(\dfrac{P_1}{N})$.*

Proof. By Theorem 4.11, $(Q_\alpha)_A \simeq \gamma_B$. Since $\gamma_B \in NSP(P_2)$ by Theorems 4.7 and 4.10, we conclude that $(Q_\alpha)_A \in NSP(\dfrac{P_1}{N})$. □

By the following theorem we extend a soft homomorphism of groups to polygroups.

Theorem 4.13. *If α_{A_1}, γ_{A_2} are two soft groups of G_1, G_2, $c_i \notin G_i$ (i=1,2) and $\alpha_{A_1} \sim \gamma_{A_2}$, then*

$$(S_\alpha)_{A_1 \cup \{c_1\}} \sim (S_\gamma)_{A_2 \cup \{c_2\}}.$$

Proof. The proof of Theorem 4.6, implies that $(S_\alpha)_{A_1 \cup \{c_1\}} \in SP(P_{G_1}), (S_\gamma)_{A_2 \cup \{c_2\}} \in SP(P_{G_2})$. Since $\alpha_{A_1} \sim \gamma_{A_2}$ by Definition 4.1, $f : G_1 \to G_2$ is a homomorphism of groups, $g : A_1 \to A_2$ is a surjective map and for any $x \in A_1$, $f(\alpha_{A_1})(x) = (\gamma_{A_2})(g(x))$. Define $g_1 : A_1 \cup \{c_1\} \to A_2 \cup \{c_2\}$ and $f_1 : P_{G_1} \to P_{G_2}$, by

$$g_1(x) = \begin{cases} \gamma(x) & x \in A_1, \\ c_2 & x = c_1, \end{cases} \text{ and } f_1(x) = \begin{cases} \alpha(x) & x \in G_1, \\ c_2 & x = c_1. \end{cases}$$

Now, it is easy to see that f_1 is a good epimorphism of polygroups and g_1 is a surjective map. In addition, $(f_1(S_\alpha)_{A_1 \cup \{c_1\}})(c_1) = (S_\gamma)_{A_2 \cup \{c_2\}}(g_1(c_1))$ and so for any $x \in B_1$,

$$(f_1(S_\alpha)_{A_1 \cup \{c_1\}})(x) = (S_\gamma)_{A_2 \cup \{c_2\}}(g_1(x)).$$

Therefore, (f_1, g_1) is a soft homomorphism between $(S_\alpha)_{A_1 \cup \{c_1\}}$ and $(S_\gamma)_{A_2 \cup \{c_2\}}$. Consequently, $(S_\alpha)_{A_1 \cup \{c_1\}} \sim (S_\gamma)_{A_2 \cup \{c_2\}}$. □

Corollary 4.14. *Consider α_{A_1} and γ_{A_2} are as Theorem 4.13. If G_1 is a nilpotent group and $(S_\alpha)_{A_1} \simeq (S_\gamma)_{A_2}$, then $(S_\gamma)_{A_2 \cup \{c_2\}} \in NSP(P_{G_2})$.*

Proof. By the same manipulation of Theorem 4.13, we have if $(\alpha_{A_1}) \simeq (\gamma)_{A_2}$, then $(S_\alpha)_{A_1 \cup \{c_1\}} \simeq (S_\gamma)_{A_2 \cup \{c_2\}}$. Also, by Theorem 4.2, P_{G_1} is an NP and so Theorem 3.20, implies that $(S_\alpha)_{A_1 \cup \{c_1\}} \in NSP(P_{G_1})$. Therefore, by Theorem 4.7, we have $(S_\gamma)_{A_2 \cup \{c_2\}} \in NSP(P_{G_2})$.

□

5 Conclusion

In this paper, for a polygroup P and a soft set α_A the notion of nilpotent soft (sub)polygroups were defined. Some examples have been used to clarify the concept of nilpotent soft polygroup. In addition, a connection between nilpotentcy of soft polygroup and polygroup was obtained. Espesially, the quotient of a soft polygroup was defined and a relation between nilpotency of a soft polygroup and its quotient was obtained. Also, by the notion of soft homomorphism we extend a soft homorphic of groups to get a soft homomorphic of polygroups. Then, some new nilpotent soft polygroups were atained. This work can be used on Engel and solvabel soft polygroups, too.

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